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Monotone Comparative Statics without Lattices

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Abstract

The theory of Monotone Comparative Statics (MCS) has traditionally required a lattice structure, excluding certain multi-dimensional environments like mixed-strategy games where this property fails. We show this structure is not essential. We introduce a weaker notion, pseudo lattice property, and preserve the theory's core results by generalizing the MCS theorems for individual choice and Tarski's fixed-point theorem. Our framework expands comparative statics to pseudo quasi-supermodular games. Crucially, it enables the first MCS analysis of mixed strategy Nash equilibria and (trembling-hand) perfect equilibria.

JEL Classification Numbers: C61, C72, D47.

Keywords: monotone comparative statics, pseudo lattice, fixed-point theo-

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rem, pseudo quasi-supermodular games, mixed-strategy Nash equilibria, perfect equilibria.

1 Introduction

Comparative statics analysis is a cornerstone of economic inquiry. It concerns how economic behavior—whether individual choices, collective outcomes, or game-theoretic equilibria—responds to changes in the underlying environment. As such, comparative statics provides the principal means by which economic models generate falsifiable predictions, forming a critical link between theory and empirical work.

The modern methodological framework for this analysis is **Monotone Comparative Statics (MCS)**. Pioneered by Topkis (1979, 1998), Milgrom and Roberts (1990), and Milgrom and Shannon (1994), it is built upon a powerful order-theoretic foundation. The universal requirement of this framework is that the underlying domain of choice must be a **lattice**—a partially ordered set where any two elements have a least upper bound and greatest lower bound. This structure is essential for defining the notions of complementarity and supermodularity that are central to the theory, both with respect to parameters and among the dimensions of a choice variable.

The reliance on the lattice property, however, raises critical conceptual and practical questions. Conceptually, is this rigid mathematical structure truly necessary for monotone comparative statics? Practically, many important economic environments fail to satisfy the lattice property. A leading example is the domain of stochastic decisions. The set of lotteries $\Delta(X)$ over a choice space, X, fails to form a lattice under standard stochastic orders, even when X is itself a lattice. This limitation has severely constrained the application of MCS to settings involving risk, uncertainty, or strategic randomization.

This paper demonstrates that the analytical power of monotone comparative statics can be preserved without the rigid lattice requirement. We show that the lattice property can be replaced by a much weaker condition, a **pseudo lattice**, which merely requires the existence of, possibly non-unique, *minimal* upper bounds and *maximal* lower bounds. This property is remarkably general; for instance, any finite or compact set with largest and smallest elements is a **complete pseudo lattice**.[1]

[1] While a formal definition is provided later, this is a mild requirement: for instance, any compact set that contains the largest and smallest elements is a complete pseudo lattice.

This relaxation significantly expands the scope of MCS, allowing us to analyze environments previously beyond its reach, including mixed strategy Nash equilibria and, most strikingly, (trembling-hand) perfect equilibria.

Our analysis begins by rebuilding the theory for the canonical individual choice problem, substituting the standard lattice assumption with our weaker pseudo lattice structure. This section delivers two foundational results that parallel the cornerstones of the classic theory. First, we provide a full characterization—a necessary and sufficient condition—for the set of optimal choices by an individual to be monotone with respect to changes in the economic environment. This result directly generalizes the celebrated Monotonicity Theorem of Milgrom and Shannon (1994), demonstrating that the core comparative statics conclusions can be obtained with virtually no loss of analytical power, with appropriately generalized ordinal conditions. Second, we show that the set of maximizers inherits the complete pseudo lattice structure of the domain. This finding is analogous to the well-known result of Milgrom and Roberts (1990) that the optimizer set forms a complete sublattice in the standard framework, a property that is often useful for equilibrium analysis.

The analysis of equilibrium in strategic environments requires a tool for establishing existence and characterizing the structure of the solution set. To this end, we develop a fixed-point theorem that serves as the analytical engine for our paper. This result generalizes the celebrated theorems by Tarski (1955) and Zhou (1994) by replacing the restrictive assumption of a complete lattice with that of a complete pseudo lattice. The theorem establishes that any monotonic correspondence from a complete pseudo lattice into itself has a nonempty set of fixed points that itself forms a complete pseudo lattice, thus admitting largest and smallest elements. We accompany this existence result with a comparative statics theorem (Theorem 6) which shows that as the correspondence shifts upwards, the set of fixed points also shifts upwards in the weak-set order.

With the results for individual choice and the fixed-point theorem in hand, we turn to the analysis of Nash equilibria in games with strategic complementarities. We apply our conditions to player payoffs to identify a broad class of pseudo quasi-supermodular games. In these games, the results from our individual choice analysis ensure that each player's best-response correspondence is monotonic in the manner required by our generalized fixed-point theorem. Applying the theorem to the joint best-response correspondence immediately establishes our main equilibrium results:

the set of pure-strategy Nash equilibria is nonempty and forms a complete pseudo lattice. Furthermore, when an exogenous parameter shifts the game in a way that strengthens each player's incentive to take higher actions, the entire set of Nash equilibria shifts upward monotonically.

The power and scope of this generalized framework are best illustrated through its applications. We first analyze a generalized Bertrand game with product substitutes, showing that our machinery accommodates a wider range of competitive environments than the existing literature. Notably, our approach is capable of handling the pure Bertrand game, whose discontinuous payoff functions prevent it from being analyzed by standard supermodularity methods like those in Milgrom and Shannon (1994). Second, and more fundamentally, we address the long-standing challenge of applying MCS to mixed strategy Nash equilibria. The domain of mixed strategies is a canonical example of a non-lattice space, making traditional methods inapplicable. We show that the set of mixed strategy equilibria is bounded by the extremal pure-strategy equilibria and therefore inherits their monotone comparative statics properties. While this result is implied by earlier work under the assumption of payoff continuity, our approach provides a more direct proof and, crucially, establishes the result without requiring payoff continuity.

The ability of our framework to handle non-lattice domains, particularly the space of mixed strategies, culminates in our paper's most novel contribution: the first general monotone comparative statics analysis of (trembling-hand) perfect equilibria. Our analysis applies to a broad class of supermodular games. We extend Selten (1975)'s classic concept to potentially infinite games by defining a perfect equilibrium as the limit of Nash equilibria from a sequence of perturbed games. In each perturbed game, players are constrained to play from a parameterized set of full-support mixed strategies, ensuring that each open set of pure strategies is played with some positive minimal probability.

Our proof strategy is to first establish existence and comparative statics properties for the Nash equilibria along the sequence of these perturbed games, and then to show that these properties are preserved in the limit. Specifically, we show that each perturbed game possesses extremal "constrained-pure" Nash equilibria that bound the entire equilibrium set and shift monotonically as the environment changes. The limit of these extremal equilibria yields the existence of perfect equilibria in pure strategies and, crucially, ensures that the set of perfect equilibria inherits the monotone comparative statics property of equilibria of the perturbed games. This entire line of argument would be impossible with standard methods. The space of mixed strategies in each perturbed game forms a pseudo lattice but not a lattice, rendering traditional fixed-point theorems inapplicable. Thus, our generalized fixed-point theorem is the essential tool that enables the analysis at each step of the sequence.

Related Literature. This paper contributes to the large and influential literature on the general methodology of monotone comparative statics. The workhorse methods in modern economic analysis were developed and refined in foundational contributions by Topkis (1979, 1998), Vives (1990), Milgrom and Roberts (1990), Milgrom and Shannon (1994), and Quah and Strulovici (2009). A unifying feature of this entire body of work is its reliance on the mathematical structure of a lattice for the domain of choice, which provides the foundation for defining complementarities and comparing sets of optima. Our primary methodological contribution is to show that this structural assumption can be substantially weakened. We demonstrate that the core analytical power of the theory is preserved when we dispense with the lattice property in favor of the more general pseudo lattice.

Several papers analyze monotone comparative statics and fixed points of monotone operators in non-lattice environments. Quah (2007) considers individual choice problems in which a constraint set lacks a lattice structure. Similarly, Abian and Brown (1961), Smithson (1971), and Li (2014) establish fixed-point existence under monotonicity without lattice assumptions. Compared with these papers, we require a more structure on the domain but obtain stronger results, including the extremal predictions. More detailed comments will follow.

The remainder of the paper is organized as follows. Section 2 introduces the preliminary mathematical concepts central to our analysis, including the formal definition of a pseudo lattice and the associated set orders. Section 3 develops the monotone comparative statics results for the individual choice problem in this new domain. Section 4 presents our generalized fixed-point theorem to establish the existence, structure, and comparative statics of (pure and mixed) Nash equilibria. Section 5 extends the analysis to perfect equilibria. Finally, Section 6 offers concluding remarks. All proofs omitted from the main text are provided in the Appendix and a Supplementary Appendix.

2 Preliminaries

This section introduces a set of notions and terminologies and establishes a number of preliminary results that will be used throughout the paper. Our theory weakens the structural properties of the domain (e.g., choice or strategy sets for players) as well as the set order.

2.1 The structural properties of domain.

Throughout, the choice domain X is assumed to be a *partially ordered set* with regard to a *primitive partial order* $\geqslant$, namely a binary relation that is *reflexive, transitive* and *anti-symmetric* on X. For any set S, let $U_S := \{x \in X : x \geqslant x', \forall x' \in S\}$ and $L_S := \{x \in X : x \leqslant x', \forall x' \in S\}$, that is, upper and lower contour sets of S, respectively. When $S = \{x\}$, we will simply write U_x and L_x. We assume that X is endowed with a topology under which the order $\geqslant$ is *closed*, meaning that the set $\{(x, y) \in X \times X : x \geqslant y\}$ is closed in the associated product topology.[2] Also, throughout the paper, we endow any space of probability measures with the weak topology.

Existing literature imposes additional order properties. X is a *lattice* if for any $x, x' \in X$, $x \vee x' \in X$ and $x \wedge x' \in X$, where $x \vee x' := \inf U_{\{x,x'\}}$ is their *join*, or the least (common) upper bound, of $\{x, x'\}$ and $x \wedge x' := \sup L_{\{x,x'\}}$ is their *meet*, or the greatest (common) lower bound, of $\{x, x'\}$. (We will write $\vee_S$ and $\wedge_S$ when the sup or the inf is taken over a set $S \neq X$.) X is a *complete lattice* if, for any $S \subset X$, $\inf U_S \in X$ and $\sup L_S \in X$, that is, its supremum and infimum exist in X. A subset $S \subset X$ is a *sublattice* of X if, for any $x, x' \in S$, $x \vee x' \in S$ and $x \wedge x' \in S$. A subset $S \subset X$ is a *complete sublattice* of X if $\inf U_{S'} \in S$ and $\sup L_{S'} \in S$ for all $S' \subseteq S$.[3]

Throughout, we require much weaker structural properties for the partial order $(X, \geqslant)$. For any $x, x' \in X$, let $x \curlyvee x' := \{y \in U_{\{x,x'\}} : z \in U_{\{x,x'\}} \Rightarrow y \ngtr z\}$ be their **pseudo join**—the set of minimal upper bounds of $\{x, x'\}$ in X—and let $x \curlywedge x' := \{y \in L_{\{x,x'\}} : z \in L_{\{x,x'\}} \Rightarrow y \nless z\}$ be their **pseudo meet**—the set of maximal lower

[2]This assumption is required to ensure that the induced stochastic order $\geqslant^{sd}$ on $\Delta(X)$ is a partial order—particularly, that it satisfies antisymmetry—when we analyze mixed-strategy Nash equilibria and perfect equilibria. In other parts of the analysis, specifically Lemma 1 and Corollary 1, it suffices to assume that $\geqslant$ is *semi-closed*, in the sense that U_x and L_x are closed for each $x \in X$.

[3]Some other terminologies are used for the same notion: Topkis (1998) uses subcomplete sublattice and Zhou (1994) uses closed sublattice. In particular, the "closedness" of Zhou (1994) should not be confused with the topological "closedness" used in this paper.

bounds of x and x' in X. If X is a lattice, $x \curlyvee y$ and $x \curlywedge y$ reduce to singleton sets $\{x \vee y\}$ and $\{x \wedge y\}$, respectively. In that sense, $\curlyvee$ and $\curlywedge$ are generalizations of the usual join and meet operations to the non-lattice sets. We consider a non-lattice set where $\curlyvee$ and $\curlywedge$ are well-defined.

We say that X is a **pseudo lattice** if $x \curlyvee y$ and $x \curlywedge y$ are nonempty for every $x, y \in X$. We also say X is a **complete pseudo lattice** if it is chain complete and, for all $S \subset X$, both $\bigcurlyvee_X S := \{z \in U_S : x \in U_S \Rightarrow x \nless z\}$ and $\bigcurlywedge_X S := \{z \in L_S : x \in L_S \Rightarrow x \ngtr z\}$ are nonempty.[4] A subset $S \subset X$ is a **weak pseudo sublattice** of X if, for any $x, x' \in S$, $(x \curlyvee x') \cap S$ and $(x \curlywedge x') \cap S$ are both nonempty, and a **pseudo sublattice** if, for any $x, x' \in S$, $x \curlyvee x' \subset S$ and $x \curlywedge x' \subset S$. Clearly, if S is a pseudo sublattice of X, then it is a weak pseudo sublattice of X; but the converse need not hold. A subset S of X is a **complete pseudo sublattice** if it is chain complete and, for every nonempty $S' \subseteq S$, $\bigcurlyvee_X S'$ and $\bigcurlywedge_X S'$ are nonempty subsets of S.

A pseudo lattice and a complete pseudo lattice are considerably weaker than a lattice and a complete lattice. As we will see later (Corollary 1), any compact set is a complete pseudo lattice if (and only if) it contains the largest and smallest elements. We provide several examples of X that is a pseudo lattice but not a lattice.

Example 1. Every finite set X is compact, and hence it becomes a complete pseudo lattice whenever it contains both the largest and smallest elements. To see such a set need not be a lattice, consider $X = \{(0,0), (1,0), (0,1), (2,1), (1,2), (3,3)\}$.

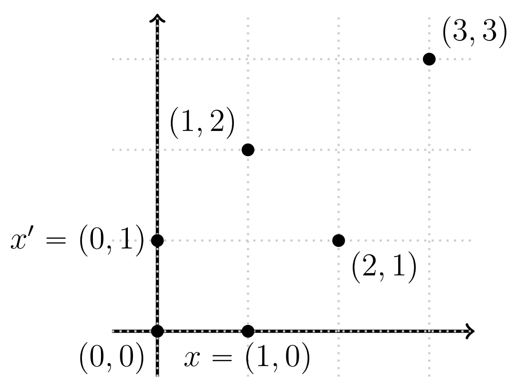

Figure 1: A complete pseudo lattice that is not a lattice.

[4] A partially ordered X is *chain complete* if every chain $C \subset X$ has a supremum and infimum in X. (This notion is sometimes called "chain complete in both directions.") Note that a chain is a totally ordered subset $C \subset X$; that is, for any $x, y \in C$, either $x \leqslant y$ or $y \leqslant x$.

This set has the largest and smallest elements, $(3,3)$ and $(0,0)$, and thus forms a complete pseudo lattice. However, it is not a lattice: for example, for $x = (1,0)$ and $x' = (0,1)$, $x \curlyvee x' = \{(2,1),(1,2)\}$. Since this set is non-singleton, $(1,0) \vee (0,1)$ is not well-defined, so X fails to be a lattice.

Example 2 (Stochastic dominance order). Consider the set $X = \Delta(S)$ of all Borel probability measures on a compact, partially ordered Polish space S containing largest and smallest elements $\overline{s}$ and $\underline{s}$. We endow X with the (first-order) stochastic dominance order, $\geqslant^{sd}$: $x \geqslant^{sd} y$ if $\int f dx \geqslant \int f dy$ for all bounded nondecreasing function $f : S \to \mathbb{R}$.[5] If S contains largest and smallest elements ($\overline{s}$ and $\underline{s}$), then X is a complete pseudo lattice by Corollary 1, as it is compact and has largest ($\delta_{\overline{s}}$) and smallest ($\delta_{\underline{s}}$) elements (where δ_s is the Dirac measure at s).

However, X is not a lattice in general, particularly if S is multidimensional, as an example from Kamae, Krengel, and O'Brien (1977) shows. Let $S = \{0,1\} \times \{0,1\}$, as depicted below.

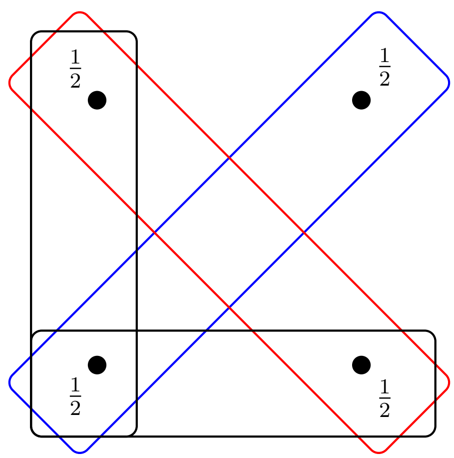

Figure 2: Failure of the lattice property

To see why $X = \Delta(S)$ fails to be a lattice, consider the two lotteries $x = \frac{1}{2}\delta_{(1,1)} + \frac{1}{2}\delta_{(0,0)}$ and $x' = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(0,1)}$. Both lotteries $\underline{a} = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,0)}$ and $\underline{b} = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(0,1)}$ are maximal lower bounds of x and x'. Since $x \curlywedge x'$ is a non-singleton set, X fails to be a lattice. This space is a canonical example for our later analysis of mixed strategies.

[5] This is equivalent to requiring that $x(S') \geqslant y(S')$ for every upward closed set $S' \subset S$, where S' is upward closed if $s \in S'$ and $s' \geqslant s$ imply $s' \in S'$. That $\geqslant^{sd}$ is a partial order is shown by Kamae and Krengel (1978).

Example 3 (**Mean-preserving spread/convex order**). Another important example is the set of information structures (distributions of posteriors) ordered by the mean-preserving spread, or convex, order ($\geqslant^{cx}$). Consider a compact, convex set $\Theta \subset \mathbb{R}^d$. For a fixed prior $\mu \in \Delta(\Theta)$, the set X_μ of all distributions $x \in \Delta(\Delta(\Theta))$ over posterior beliefs with mean μ constitutes a feasible set.[6] Note that this is a compact set and contains a largest and a smallest element, $\overline{x}_\mu$ and $\underline{x}_\mu$: $\underline{x}_\mu$ represents the no-information experiment; $\overline{x}_\mu$ represents the experiment that fully reveals every $\theta \in \Theta$. Hence, by Corollary 1, X_μ is a complete pseudo lattice.[7] However, X_μ generally fails to be a lattice as shown in Example 5 of the Online Appendix.

The next result characterizes a complete pseudo lattice X in terms of the existence of its extremal elements. In particular, X is a complete pseudo lattice whenever it is compact (or, more generally, chain complete) and possesses the largest and smallest elements.

Theorem 1. *A chain complete set X is a complete pseudo lattice if and only if it admits both the largest and smallest elements.*

Chain completeness is a mild condition. It is automatically satisfied under compactness, which is often assumed for other purposes.

Lemma 1. *If X is compact, then it is chain complete.*[8]

Corollary 1. *A compact set X is a complete pseudo lattice if and only if it admits both a largest and a smallest element.*

2.2 Set orders

Consider a nonempty pseudo lattice $(X, \geqslant)$. One can define several set orders induced by $\geqslant$, and two of them are of particular interest to us: *pseudo strong-set (pSS) order* $\geqslant_{pss}$ and *weak-set (WS) order* $\geqslant_{ws}$.

[6]The order $\geqslant^{cx}$ is defined as: $x \geqslant^{cx} y$ if $\int f dx \geqslant \int f dy$ for all continuous, convex function $f : \Delta(\Theta) \to \mathbb{R}$. This defines a partial order. In particular, Corollary 3.24 of Elton and Hill (1992) ensures the antisymmetry of $\geqslant^{cx}$ since the set is a compact metrizable convex subset of a locally convex topological vector space.

[7]Often, we restrict attention to the distribution over posterior *means* rather than posterior *beliefs*. The feasible set of distributions of the posterior mean then becomes the convex-order interval $\{\, \nu \in \Delta(\mathbb{R}^d) : \delta_{\bar{m}} \leqslant^{cx} \nu \leqslant^{cx} \mu \,\}$, where $\bar{m} = \int_\Theta \theta \, d\mu(\theta)$. This set is also a complete pseudo lattice.

[8]The converse does not hold. For example, $X = [0, 1) \cup \{2\}$ is chain complete but not compact.

We say $S' \subset X$ **pSS dominates** $S \subset X$, and write $S' \geqslant_{pss} S$, if $\forall x \in S, \forall x' \in S'$, $x \curlyvee x' \subset S'$ and $x \curlywedge x' \subset S$. $S' \subset X$ **weak pSS dominates** $S \subset X$, and write $S' \geqslant_{wpss} S$, if $\forall x \in S, \forall x' \in S'$, $(x \curlyvee x') \cap S'$ and $(x \curlywedge x') \cap S$ are both nonempty. We say S' **weak-set dominates** S, and write $S' \geqslant_{ws} S$, if, for each $x \in S$, there exists $x' \in S'$ such that $x' \geqslant x$; and for each $x' \in S'$, there exists $x \in S$ such that $x \leqslant x'$.

As is easily seen, $S' \geqslant_{pss} S$ implies $S' \geqslant_{ws} S$. The following result further clarifies the relationship between these two orders by decomposing pseudo strong-set order into weak-set order and a couple of "extra properties" when the choice domain is a pseudo lattice (and the compared sets are pseudo sublattices):[9]

Theorem 2. *Consider a pseudo lattice X and its subsets S and S'. Then, $S' \geqslant_{pss} S$ if (i) $S' \geqslant_{ws} S$; (ii) $S \cup S'$ is a pseudo sublattice; and (iii) (sandwich property) for any $x \in S$ and $y, z \in S'$ (resp., any $x \in S'$ and $y, z \in S$), $x \in [y, z]$ implies $x \in S'$ (resp., $x \in S$). Conversely, if S and S' are nonempty pseudo sublattices, then $S' \geqslant_{pss} S$ implies the properties (i)–(iii).*

Proof. To prove the first statement, let us consider any $x \in S$ and $x' \in S'$. To show $x \curlyvee x' \subset S'$, suppose not. Then, there exists $\hat{x} \in x \curlyvee x'$ such that [JW: $\hat{x} \notin S'$] and $\hat{x} \in S$ by (ii). By (i), there exists $z \in S'$ such that $z \geqslant \hat{x}$. So we have $x' \leqslant \hat{x} \leqslant z$ while $x', z \in S'$ and $\hat{x} \in S$. Thus, by (iii), $\hat{x} \in S'$, a contradiction. The proof of $x \curlywedge x' \subset S$ is analogous and omitted.

Suppose now that $S' \geqslant_{pss} S$ where S and S' are nonempty pseudo sublattices. Clearly, (i) holds. To see that (ii) holds, consider any $x, x' \in S \cup S'$. If either $x, x' \in S$ or $x, x' \in S'$, then clearly $x \curlyvee x'$ and $x \curlywedge x'$ are subsets of $S \cup S'$ since S and S' are pseudo sublattices. If $x \in S$ and $x' \in S'$, then $S' \geqslant_{pss} S$ implies that both $x \curlyvee x'$ and $x \curlywedge x'$ are subsets of $S \cup S'$. To verify (iii), observe that for any $x \in S$ and $y, z \in S'$ with $x \in [y, z]$, we have $\{x\} = x \curlyvee y$ and thus $x \in S'$ since $S' \geqslant_{pss} S$. Also, for any $x \in S'$ and $y, z \in S$ with $x \in [y, z]$, we have $\{x\} = x \curlywedge z$ and thus $x \in S$. □

This characterization clarifies precisely what is "lost" when we use the weak-set order instead of the (pseudo) strong-set order—namely, properties (ii) and (iii).

[9]One can easily construct examples showing that each property is indispensable for this characterization.

3 Individual Choices

Consider an individual who chooses an action x from a feasible set $S \subset X$, facing a parameter $t \in T$, by maximizing an objective function $u : X \times T \to \mathbb{R}$, where X is a pseudo lattice and T is a partially ordered set. We are interested in how the set of optimal choices,

$$M_S(t) := \arg\max_{x \in S} u(x,t),$$

responds to changes in the *environment* from (t, S) to (t', S'). We write $M(t)$ when $S = X$ and M_S when t is fixed. The next subsection provides conditions that characterize the monotone comparative statics for the set $M_S(t)$ in the weak pseudo strong-set order.

3.1 MCS characterization of individual choices

We begin with some definitions. First, we say that $f : X \to \mathbb{R}$ is **pseudo quasi-supermodular** if, for any $x, y \in X$, $\overline{x} \in x \curlyvee y$, and $\underline{x} \in x \curlywedge y$,

$$f(x) - f(\underline{x}) \geqslant (>)\ 0 \Rightarrow f(\overline{x}) - f(y) \geqslant (>)\ 0, \tag{pQSUP}$$

and **pseudo supermodular** if, for any $x, y \in X$, $\overline{x} \in x \curlyvee y$, and $\underline{x} \in x \curlywedge y$,

$$f(x) - f(\underline{x}) \leqslant f(\overline{x}) - f(y). \tag{pSUP}$$

As is well-known, the prefix "quasi" weakens the notion from a cardinal concept to an ordinal one. More importantly for our purposes, "pseudo" extends the property to a pseudo lattice. If X is a lattice (i.e., not just a pseudo lattice), then pseudo quasi-supermodularity reduces to *quasi-supermodularity*, and pseudo supermodularity reduces to *supermodularity*, well known in the literature. These conditions capture the sense of complementarities across multiple dimensions of action under the payoff function f.

Next, let us consider a parametrized family of functions $u : X \times T \to \mathbb{R}$. We say that $u(x,t)$ satisfies *single-crossing in* (x,t) if, for each $x, x' \in X$ with $x' \geqslant x$ and $t, t' \in T$ with $t' \geqslant t$,

$$u(x', t) - u(x,t) \geqslant (>)\ 0 \Rightarrow u(x', t') - u(x, t') \geqslant (>)\ 0, \tag{SC}$$

and *increasing differences in* (x, t) if, for each $x, x' \in X$ with $x' \geqslant x$ and $t, t' \in T$ with $t' \geqslant t$,

$$u(x', t) - u(x, t) \leqslant u(x', t') - u(x, t'). \tag{ID}$$

These conditions capture the idea that an individual faces a stronger incentive to raise her action under $u(\cdot, t')$ than under $u(\cdot, t)$. As is clear, increasing differences imply single-crossing. Also, we say that u is pseudo (quasi-)supermodular in x if $u(x, t)$ is pseudo (quasi-)supermodular as a function of x for each $t \in T$.

Milgrom and Shannon (1994) characterize monotone comparative statics of the optima in the strong-set order for lattice domains with quasi-supermodular and single-crossing payoffs. We extend the characterization when X is (only) a pseudo lattice.[10]

Theorem 3 (Characterization). *$M_{S'}(t') \geqslant_{wpss} M_S(t)$ for all $t' \geqslant t$ and $S' \geqslant_{wpss} S$ if and only if u is pseudo quasi-supermodular in x and satisfies single-crossing in (x, t).*

This result reduces to the aforementioned result by Milgrom and Shannon (1994), their Theorem 4, if X is a lattice, since the qualifier "pseudo" then becomes immaterial. The pseudo quasi-supermodularity and single-crossing conditions, just like quasi-supermodularity and single-crossing under the lattice environment, play a key role in our subsequent applications. These conditions deliver MCS in the "weak" pSS order, but they require a correspondingly weaker condition for the feasible sets. We can obtain MCS in the "strong" pSS order, for free, under a stronger condition on the feasible sets:

Proposition 1 (Monotonicity). *If u is pseudo quasi-supermodular in x and satisfies single-crossing in (x, t), then $M_{S'}(t') \geqslant_{pss} M_S(t)$ for all $t' \geqslant t$ and $S' \geqslant_{pss} S$.*

Proof. For any $x \in M_S(t)$ and $x' \in M_{S'}(t')$, choose $z \in x \curlywedge x'$ and $z' \in x \curlyvee x'$. Then, $z \in S$ and $z' \in S'$ (from $S' \geqslant_{pss} S$), so $u(x, t) \geqslant u(z, t)$ and, by pseudo quasi-supermodularity, it follows that $u(z', t) \geqslant u(x', t)$. By single-crossing, $u(z', t') \geqslant u(x', t')$, so $z' \in M_{S'}(t')$. That $z \in M_S(t)$ follows analogously from the strict inequality parts of (pQSUP) and (SC). □

[10] Quah (2007) considers the individual choice problem in which the constraint set (defined in a lattice) lacks a sublattice property and identifies conditions under which optimal choices rise in weak-set order when a constraint set also rises in weak-set order. The current theorem strengthens both the hypothesis (a monotonic pSS shift of the constraint set) and the conclusion (a monotonic pSS shift of the optimal choices), and hence complements Quah (2007).

Since the cardinal conditions imply the ordinal ones, the following is trivial:

Corollary 2. *If u is pseudo supermodular in x and satisfies increasing differences in (x,t), then $M_{S'}(t') \geqslant_{pss} M_S(t)$ for all $t' \geqslant t$ and $S' \geqslant_{pss} S$.*

As in Milgrom and Shannon (1994), a strengthening of single crossing yields monotonicity regardless of how an optimal choice is selected.

Corollary 3 (Monotone Selection)**.** *Suppose u is pseudo quasi-supermodular in x and satisfies strict single-crossing in (x,t): for any $x' > x$, $t' > t$,*

$$u(x',t) - u(x,t) \geqslant 0 \Rightarrow u(x',t') - u(x,t') > 0,$$

and assume $S' \geqslant_{pss} S$. Then, $\forall z \in M_S(t)$, $\forall z' \in M_{S'}(t')$, we have $z' \geqslant z$.

Proof. Fix $z \in M_S(t)$ and $z' \in M_{S'}(t')$. By Proposition 1, any $\tilde{z} \in z \curlywedge z'$ satisfies $\tilde{z} \in M_S(t)$, so $u(z,t) - u(\tilde{z},t) = 0$. If $z' \not\geqslant z$, then $\tilde{z} < z$, and strict single crossing implies $u(z,t') - u(\tilde{z},t') > 0$. Pseudo quasi-supermodularity then implies $u(\hat{z},t') - u(z',t') > 0$ for some $\hat{z} \in z \curlyvee z' \subset M_{S'}(t')$, contradicting $z' \in M_{S'}(t')$. □

Directional weakening. One can provide a weaker "directional" version of Theorem 3 by splitting the conditions into upper and lower versions. We say $S' \subset X$ **upper** (resp. **lower**) **weak pSS dominates** $S \subset X$, and write $S' \geqslant_{uwpss} S$ (resp. $S' \geqslant_{lwpss} S$), if $\forall x \in S, \forall x' \in S'$, $x \curlyvee x' \cap S'$ (resp. $x \curlywedge x' \cap S$) is nonempty. Next, we say that a function u is **upper** (resp. **lower**) **pseudo quasi-supermodular** if the weak (resp. strict) inequality part of (pQSUP) holds, and that u is upper (resp. lower) single-crossing if the weak (resp. strict) inequality part of (SC) holds. Then, we have:

Theorem 3′. *$M_{S'}(t') \geqslant_{uwpss} M_S(t)$ for all $t' \geqslant t$ and $S' \geqslant_{wpss} S$ if and only if u is upper pseudo quasi-supermodular in x and satisfies upper single-crossing in (x,t). An analogous statement holds for the lower case.*[11]

The upper and lower versions of Theorem 3′ jointly imply Theorem 3, which is how the latter theorem is proven in Section B.

[11] That is, the same statement holds when $\geqslant_{uwpss}$, upper pseudo quasi-supermodularity, and upper single-crossing are replaced by $\geqslant_{lwpss}$, lower pseudo quasi-supermodularity, and lower single-crossing, respectively.

3.2 The structure of the maximizer set

As in the lattice case, the complementarity property of the payoff function over a pseudo lattice domain induces a distinctive structure for the set of maximizers. To ensure that this set is well defined, we assume that the payoff function $u : X \to \mathbb{R}$ is **order upper semicontinuous**: for any chain $C = \{x_\alpha\}_\alpha \subset X$,

$$\inf_\alpha \sup_{\beta \geqslant \alpha} u(x_\beta) \leqslant u(x),$$

where x is the infimum or supremum of C, if it exists.

Theorem 4. *Assume X is a pseudo lattice and $u : X \to \mathbb{R}$ is pseudo quasi-supermodular.*

(i) $\arg\max_{x \in X} u(x)$ *is a pseudo sublattice of X whenever it is nonempty.*

(ii) *If, in addition, X is a complete pseudo lattice and u is order upper semicontinuous, then* $\arg\max_{x \in X} u(x)$ *is a nonempty, complete pseudo sublattice, admitting the largest and the smallest point.*

That the maximizers inherit the complete pseudo lattice structure is interesting in its own right, but it will also be crucial for the subsequent analyses of games. This result generalizes Theorem 2 of Milgrom and Roberts (1990) and Theorem A4 of Milgrom and Shannon (1994). However, the lack of a lattice structure of the domain makes the argument distinct and more involved.

3.3 Decision under uncertainty

We now apply our framework to optimal behavior under uncertainty—a setting central to mixed-strategy equilibrium analysis—where an agent chooses a potentially random action to maximize expected utility. Let $u : A \times \Theta \to \mathbb{R}$ be the utility function, with A and Θ denoting partially ordered sets of actions and states, respectively. The agent's uncertainty is represented by a probability distribution $\eta \in \Delta(\Theta)$ over the state space; in game-theoretic contexts, η often represents the mixed strategy profile of opponents. For any randomized action $x \in X = \Delta(A)$ and state distribution $\eta \in \Delta(\Theta)$, we define the expected utility as

$$\overline{u}(x, \eta) := \int_A \left(\int_\Theta u(a, \theta) \eta(d\theta) \right) x(da) \tag{1}$$

and the associated choice correspondence as

$$M(\eta) := \arg\max_{x \in X} \overline{u}(x, \eta). \tag{2}$$

We endow both $\Delta(A)$ and $\Delta(\Theta)$ with the stochastic dominance order induced by the underlying orders on A and Θ.

Our aim is to establish the monotone comparative statics of $M(\eta)$ as η shifts. However, applying Theorem 3 directly to this lottery space is difficult. As illustrated next, the induced expected utility function $\overline{u}(x, \eta)$ may fail to satisfy pseudo quasi-supermodularity even when the underlying utility $u(a, \theta)$ is supermodular.

Example 4 (**Failure of pQSUP**)**.** Consider the space described in Example 2, where $X = \Delta(\{0,1\} \times \{0,1\})$ is endowed with the stochastic dominance order, alongside a utility function $u : \{0,1\} \times \{0,1\} \to \mathbb{R}$ defined by $u(0,0) = u(1,0) = 2$, $u(0,1) = 0$, and $u(1,1) = 1$. While u is supermodular, the associated expected utility $\overline{u}$ fails (pQSUP). To see this, let $x = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,1)}$, $x' = \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(0,1)}$, $\underline{x} = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(0,1)}$, and $\overline{x} = \frac{1}{2}\delta_{(0,1)} + \frac{1}{2}\delta_{(1,1)}$. Note that $\underline{x} \in x \curlywedge x'$ and $\overline{x} \in x \curlyvee x'$. We have $\overline{u}(x) = 3/2 > 1 = \overline{u}(\underline{x})$, but $\overline{u}(\overline{x}) = 1/2 < 1 = \overline{u}(x')$, violating (pQSUP).

Since pseudo quasi-supermodularity is necessary for the characterization in Theorem 3, its failure implies that optimal choices do not generally exhibit monotone comparative statics on arbitrary subsets $S \subset \Delta(A)$, such as the set $\{x, x', \underline{x}, \overline{x}\}$ in the example. Nevertheless, we can recover monotone comparative statics in "standard" cases where the choice set includes pure actions. Our analytical approach is to exploit the linearity of the expected payoff $\overline{u}$ in $x \in X$. In such cases, monotone comparative statics of $M(\eta)$ can be obtained in the weak-set order directly from standard conditions on the underlying utility u.

Proposition 2. *Assume that A is a complete pseudo lattice. Assume also that $u(a, \theta)$ is bounded, order upper semicontinuous, pseudo supermodular in a, and satisfies increasing differences in (a, θ). Then, for each $\eta \in \Delta(\Theta)$, $M(\eta) := \arg\max_{x \in \Delta(A)} \overline{u}(x, \eta)$ has largest and smallest elements, both of which belong to A and are nondecreasing in η. Hence, $M(\eta') \geqslant_{ws} M(\eta)$ for all $\eta, \eta' \in \Delta(\Theta)$ with $\eta' \geqslant^{sd} \eta$.*

The result consists of two main observations. Consider first a maximization where the choice set is restricted to only pure actions A. By Theorem 4, this restricted problem admits extremal optima, and Theorem 3 implies that they are pSS monotonic in η with respect to first-order stochastic dominance. This is because pseudo supermodularity and increasing-difference—the cardinal conditions—are preserved under expectation *over pure actions*.

Now, when we extend the choice space to the lottery space $\Delta(A)$, these pure actions remain optimal because $\overline{u}(\cdot, \eta)$ is linear in lotteries. Moreover, they remain extremal because any optimal lottery must assign its entire probability mass within the interval bounded by the two extremal pure optima. Note, however, that we still pay a price for the failure of pseudo quasi-supermodularity: the monotone comparative statics property holds only in the weak-set order. As will be seen in the following sections, Proposition 2 is sufficiently powerful to establish the desired comparative statics for mixed-strategy Nash equilibria and perfect equilibria.

4 Fixed Points and Nash Equilibria

This section develops the analytical machinery for equilibrium analysis on non-lattice domains. We first establish a fixed-point theorem for pseudo monotonic correspondences along with associated comparative statics results. We then apply these tools to generalize the theory of games with strategic complementarities, providing a unified framework that encompasses both pure and mixed strategies.

4.1 Existence and comparative statics of fixed points

Consider a self correspondence $F : X \rightrightarrows X$ defined over a complete pseudo lattice X endowed with a partial order $\geqslant$. We say that an element $x \in X$ is a *fixed point* of F if $x \in F(x)$.

Let us call a self-correspondence $F : X \rightrightarrows X$ **pseudo monotonic** if (i) $F(x)$ is a nonempty complete pseudo sublattice for each $x \in X$ and (ii) F is pSS monotonic, i.e., $F(x') \geqslant_{pss} F(x)$ for all $x', x \in X$ with $x' \geqslant x$.

Theorem 5. *If $F : X \rightrightarrows X$ is a pseudo monotonic correspondence on a complete pseudo lattice X, then its fixed-point set is a nonempty complete pseudo lattice, thus admitting the largest and smallest points.*

When F is a function, the following generalization of Tarski (1955)'s theorem obtains:

Corollary 4. *If $F : X \to X$ is a nondecreasing function on a complete pseudo lattice X, then its fixed-point set is a nonempty complete pseudo lattice.*

The theorems by Zhou (1994) and Tarski (1955) are the closest antecedents of Theorem 5 and Corollary 4, respectively. The crucial differences are that they require X to be a complete lattice and F, in case of a correspondence, to satisfy additional lattice properties.[12] Our results relax these lattice properties to pseudo-lattice counterparts. Recall these properties are substantially weaker. For example, a set X is a complete pseudo lattice if X is compact and contains the largest and smallest points (Corollary 1).[13] The only price paid is that the fixed points form a complete pseudo lattice rather than a complete lattice. Crucially, a complete pseudo lattice always admits extremal elements; thus, our framework remains sufficiently powerful to guarantee the existence of largest and smallest fixed points, which provides the foundation for the equilibrium analysis in the subsequent sections.

Crucially for our purposes, pseudo monotonic correspondences are readily amenable to monotone comparative statics analysis. Let $\mathcal{F}(t)$ denote the set of fixed points of a parametrized self-correspondence $F(\cdot, t) : X \rightrightarrows X$.

Theorem 6. *For a family of pseudo monotonic self-correspondences $F(\cdot, t)$ on a complete pseudo lattice X, if $F(x, t') \geqslant_{ws} F(x, t)$ for all $x \in X$, then $\mathcal{F}(t') \geqslant_{ws} \mathcal{F}(t)$.*

Directional weakening. As in the individual-choice setup, one may introduce directional weakenings of the monotonicity requirement for the fixed-point correspondence.[14] A subset S of X is a **complete upper** (resp. **lower**) **pseudo sublattice** if it is chain complete and, for every nonempty $S' \subseteq S$, $\curlyvee_X S'$ (resp. $\curlywedge_X S'$) is

[12] Specifically, the correspondence F is required to be complete sublattice-valued and monotonic in the strong-set dominance sense; see Zhou (1994).

[13] Abian and Brown (1961), Smithson (1971), and Li (2014)'s fixed-point theorems require even weaker conditions. For instance, the version of Li's theorem invoked by Che, Kim, and Kojima (2019) requires X to be only a compact partially-ordered set, together with some regularity conditions ensuring the existence of an upper diagonal or lower diagonal point. The complete pseudo lattice condition is not much stronger than this; it is weaker than the compactness with extremal points (see Lemma 1), which only strengthens the regularity condition. However, the results are considerably more powerful: fixed points are a complete pseudo lattice, so they contain the largest and smallest points, which is not guaranteed by that theorem (see Che, Kim, and Kojima (2019)). As will be seen, these properties will be used crucially for the later application, particularly the monotone comparative statics of perfect equilibria.

[14] See the generalized Bertrand game in Section 4.2 for an instance where such a weakening is needed.

a nonempty subset of S. We say $S' \subset X$ **upper** (resp. **lower**) **pSS dominates** $S \subset X$, and write $S' \geqslant_{upss} S$ (resp. $S' \geqslant_{lpss} S$), if $\forall x \in S, \forall x' \in S'$, $x \curlyvee x' \subset S'$ (resp. $x \curlywedge x' \subset S$).

If $F(x)$ is a nonempty complete upper pseudo sublattice for each $x \in X$, and $F(x') \geqslant_{upss} F(x), \forall x' \geqslant x$, then we say F is **upper pseudo monotonic**—and **lower pseudo monotonicity** is defined analogously.

Theorem 5′. *If $F : X \rightrightarrows X$ is an upper (resp. lower) pseudo monotonic correspondence on a complete pseudo lattice X, then its fixed-point set is nonempty and admits the largest (resp. smallest) point.*

To establish the comparative statics result, let us weaken the weak-set order similarly: we say that S' **upper weak-set dominates** S, and write $S' \geqslant_{uws} S$, if for every $x \in S$, there exists $x' \in S'$ with $x' \geqslant x$; S' **lower weak-set dominates** S, written $S' \geqslant_{lws} S$, if for every $x' \in S'$, there exists $x \in S$ with $x \leqslant x'$.

Theorem 6′. *Let $F(\cdot, t)$ and $F(\cdot, t')$ be self-correpondences defined on a complete pseudo lattice X. If $F(\cdot, t')$ is upper pseudo monotonic and $F(x, t') \geqslant_{uws} F(x, t)$ for all $x \in X$, then $\mathcal{F}(t') \geqslant_{uws} \mathcal{F}(t)$.*[15] *If $F(\cdot, t)$ is lower pseudo monotonic and $F(x, t') \geqslant_{lws} F(x, t)$ for all $x \in X$, then $\mathcal{F}(t') \geqslant_{lws} \mathcal{F}(t)$.*[16]

4.2 Pseudo quasi-supermodular games

Consider a normal-form game $\Gamma = (I, S, u)$, where $S = \times_{i \in I} S_i$. We assume that each player's strategy set S_i is a complete pseudo lattice and that S is endowed with the product order. A strategy profile $s = (s_i)_{i \in I}$ is a (pure-strategy) *Nash equilibrium* if $u_i(s) \geqslant u_i(s_i', s_{-i})$ for every $i \in I$ and $s_i' \in S_i$.

We say that Γ is a **pseudo quasi-supermodular game** if, for all $i \in I$,

(P1) u_i is bounded, and order upper semicontinuous in s_i for each $s_{-i} \in S_{-i}$;

(P2) u_i is pseudo quasi-supermodular in s_i and satisfies single-crossing in (s_i, s_{-i}).

[15] Notice that the monotonicity restriction is imposed only on $F(\cdot, t')$, but *not* on $F(\cdot, t)$. This generality will play an important role in our later analysis.

[16] This result is related to Theorem 3 in Acemoglu and Jensen (2015), in that both the fixed-point operator and the fixed-point set shift in the weak-set order. Their analysis, however, does not rely on pseudo-monotonicity; instead, it requires X to be compact and F to be upper hemicontinuous.

This class generalizes the quasi-supermodular games of Milgrom and Shannon (1994). It relaxes the crucial requirements that strategy spaces be lattices and that best-response correspondences form sublattices. Drawing on our individual choice results (Section 3), conditions (P1) and (P2) together ensure that each player's best-response correspondence is pSS monotonic. This allows us to apply our fixed-point results (Section 4.1) to establish the existence and comparative statics of equilibria.

As a cardinal specialization of a pseudo quasi-supermodular game, we say that Γ is a **pseudo supermodular game** if, for all $i \in I$, (P1) holds and

(P2′) u_i is pseudo supermodular in s_i and satisfies increasing-differences in (s_i, s_{-i}).

Pseudo supermodular games prove useful in later sections when we study mixed-strategy Nash equilibria and perfect equilibria. This is because, as a cardinal concept, the property (P2′) is preserved under randomization.

We next establish the existence, structure, and monotone comparative statics of Nash equilibria for pseudo quasi-supermodular games. For this purpose, we parametrize the players' payoff functions as $u_i(\cdot, t) : S \to \mathbb{R}$ and $u(\cdot, t) = (u_i(\cdot, t))_{i \in I}$. Letting $\Gamma(t) = (I, S, u(\cdot, t))$, we denote the set of pure Nash equilibria of $\Gamma(t)$ by $\mathcal{E}(t)$.

Proposition 3. *For a family of pseudo quasi-supermodular games* $\Gamma(t)$*,*

(i) *the set of (pure) Nash equilibria* $\mathcal{E}(t)$ *is a nonempty complete pseudo lattice;*

(ii) *if* u_i *satisfies single-crossing in* (s_i, t) *for all* $i \in I$*, then* $\mathcal{E}(t') \geqslant_{ws} \mathcal{E}(t)$ *for all* $t' > t$*.*

The proof builds directly on our earlier results. By Theorem 4, each player's best-response correspondence $B_i(s_{-i}, t)$ is a nonempty complete pseudo sublattice. By Proposition 1, it is also pSS monotonic in s_{-i}. The joint best-response correspondence $B(\cdot, t)$ is therefore pseudo monotonic, so applying Theorem 5 establishes that the equilibrium set $\mathcal{E}(t)$ is a nonempty complete pseudo lattice. The comparative statics result in part (ii) follows from Theorem 6.

Application to generalized Bertrand games. We now apply our equilibrium framework to a class of generalized Bertrand games. Consider a finite set of firms I, where each firm $i \in I$ chooses a price p_i from a finite set $P_i \subset \mathbb{R}_+$. Given a price profile

$p \in P := \prod_{j\in I} P_j$, firm i sells $D_i(p_i, p_{-i})$ units of its product at a cost determined by an increasing, convex function $C_i : \mathbb{R}_+ \to \mathbb{R}_+$. Firm i's payoff is its profit

$$U_i(p) := p_i D_i(p) - C_i(D_i(p)). \tag{3}$$

We assume that each firm's demand function $D_i : P \to \mathbb{R}_+$ satisfies:

(D1) D_i is weakly decreasing in p_i and weakly increasing in p_{-i};

(D2) $\frac{D_i(p_i', p_{-i})}{D_i(p_i, p_{-i})} \leqslant \frac{D_i(p_i', p_{-i}')}{D_i(p_i, p_{-i}')}$ for any $p_i < p_i'$ and $p_{-i} < p_{-i}'$ such that $D_i(p_i, p_{-i}) > 0$.

The second part of (D1) indicates that the firms' products are substitutes. Condition (D2) strengthens this property by implying that firm i's demand becomes more inelastic as rivals' prices increase.

This framework generalizes the Bertrand games studied by Milgrom and Shannon (1994), who assume a stronger version of (D2) requiring D_i to be strictly positive and continuously differentiable.[17] In contrast, our condition applies only when demand is strictly positive. Notably, our weaker requirement encompasses pure Bertrand games, which their analysis excludes.[18] Beyond pure Bertrand competition, this class includes various games where demand may drop to zero at certain price profiles or where demand functions are discontinuous—scenarios typically excluded from the existing monotone comparative statics literature.

Notably, a generalized Bertrand game is not necessarily pseudo quasi-supermodular, so our earlier theorems do not apply directly. To see this, consider a standard Bertrand game with two firms and focus on firm 1's payoff when it has constant marginal cost c_1. If firm 2 sets $p_2 < c_1$, then any $p_1 > p_2$ yields zero demand and thus maximizes firm 1's profit at zero. By contrast, if firm 2 raises its price to some $p_2' > c_1$, then firm 1 can earn strictly positive profit, and no price $p_1 > p_2'$ maximizes its profit (assuming P_1 is sufficiently dense). Consequently, firm 1's payoff fails the single-crossing property, meaning that the game is not pseudo quasi-supermodular.

[17]Strictly speaking, Milgrom and Shannon (1994) do not assume the finiteness of P_i. It ensures condition (P1) required of (lower) pseudo quasi-supermodular games and ensures the existence of a Nash equilibrium in pure strategies.

[18]More formally, a generalized Bertrand game is a pure Bertrand game if, for each i, $C_i(q) = c_i q$ for some $c_i \in [0, \max_{p_i \in P_i} p_i]$, and $D_i(p) = 1/|\arg\min_{j\in I} p_j|$ if $p_i = \min_{j\in I} p_j$ and $D_i(p) = 0$ otherwise. Then, one can show that a pure Bertrand game is a generalized Bertrand game; see Lemma S4 in Section F.2 of the Supplementary Appendix.

Despite this failure, the game satisfies the "lower" requirements of a pseudo quasi-supermodular game—the violation occurs only in the "upper" single-crossing condition. Thus, our directional results (Proposition 2′ in the Appendix) apply, ensuring both the existence and comparative statics of equilibria.[19]

For our comparative statics analysis, consider a family of games $\Gamma(t) = (I, P, (U_i(\cdot, t))_{i \in I})$ satisfying:

(B1) If $D_i(p_i, p_{-i}, t) > 0$, then $D_i(p_i, p_{-i}, t') > 0$ and $\frac{D_i(p_i', p_{-i}, t')}{D_i(p_i, p_{-i}, t')} \geqslant \frac{D_i(p_i', p_{-i}, t)}{D_i(p_i, p_{-i}, t)}$ for all $t' > t$ and $p_i' > p_i$.

(B2) $C_i(q', t) - C_i(q, t)$ is weakly increasing in t for all $q' > q$.

Intuitively, a higher parameter t corresponds to more inelastic demand or higher marginal costs. To derive payoff implications, we consider a slight strengthening of condition (B2):

(B2′) For $t < t'$, $C_i(q, t) = c_i q \leqslant c_i' q = C_i(q, t')$ with $\max_{p_i \in P_i} p_i \geqslant c_i$, and $D_i(p, t) \leqslant D_i(p, t')$ for all $q \in \mathbb{R}_+$ and $p \in P$.

Condition (B2′) implies condition (B2). With these preparations, we state our results for generalized Bertrand games below.

Corollary 5. *For a family of generalized Bertrand games $\Gamma(t)$,*

(i) *if (B1) and (B2) hold for each $i \in I$, then $\mathcal{E}(t) \neq \varnothing$ and $\mathcal{E}(t') \geqslant_{lws} \mathcal{E}(t)$ for all $t' > t$;*

(ii) *if (B1) and (B2′) hold for each $i \in I$, then the set of equilibrium profits for each firm i with $c_i' = c_i$ in $\Gamma(t')$ lower weak-set dominates that in $\Gamma(t)$ for all $t' > t$.*

4.3 Mixed-strategy Nash equilibria

A long-standing challenge in the theory of monotone comparative statics is its extension to mixed strategies. While mixed strategies are essential for game-theoretic analysis, the set of mixed strategies $\Delta(S_i)$, ordered by first-order stochastic dominance,

[19] To intuitively see why this directional generalization holds, note that our equilibrium existence and comparative statics results are derived from the properties of best-response correspondences and the associated fixed-point theorems. As established in Section 3 and Section 4.1, both the individual choice results and the fixed-point theory admit directional counterparts that remain valid in this setting. A formal statement and proof are provided in the Appendix.

generally fails to form a lattice—even when the underlying pure-strategy space S_i is a lattice; recall Example 2.[20] Echenique (2003) explicitly identifies this non-lattice structure in multidimensional settings as the primary reason why the standard complementarity framework has struggled to accommodate mixed strategies.

By dispensing with the lattice requirement and utilizing our fixed-point results for pseudo lattices, we provide a unified framework for the MCS analysis of mixed strategies. This section also serves as a crucial intermediate step for our analysis of perfect equilibria in Section 5. Establishing tools to handle mixed strategies enables us to conduct MCS on refinements, such as perfect equilibria, which require perturbing to fully mixed strategies.

We restrict our attention to a pseudo supermodular game $\Gamma = (I, S, u)$, as cardinal payoff properties such as pseudo supermodularity are preserved under randomization. We assume each pure-strategy set S_i is a compact, partially ordered Polish space containing the largest and smallest elements. Let $\mathcal{G} = (I, \Sigma, \overline{u})$ be the mixed extension of Γ, with $\Sigma := \times_{i \in I} \Sigma_i$ and $\bar{u} := (\bar{u}_i)_{i \in I}$, where $\Sigma_i = \Delta(S_i)$ is the set of player i's mixed strategies and $\overline{u}_i : \Sigma \to \mathbb{R}$ is the expected payoff defined by:

$$\overline{u}_i(\sigma) := \int u_i(s)\sigma(ds), \sigma \in \Sigma. \tag{4}$$

Each Σ_i is partially ordered by the first-order stochastic dominance relation $\geqslant^{sd}$. Further, since each Σ_i is compact and contains extremal elements, it forms a complete pseudo lattice by Corollary 1.

A mixed strategy profile $\sigma = (\sigma_i)_{i \in I}$ is a mixed-strategy Nash equilibrium if $\bar{u}_i(\sigma) \geqslant \bar{u}_i(\sigma'_i, \sigma_{-i})$ for every $i \in I$ and $\sigma'_i \in \Sigma_i$. For a parametrized game $\Gamma(t) = (I, S, u(\cdot, t))$ and its mixed extension $\mathcal{G}(t) = (I, \Sigma, \bar{u}(\cdot, t))$, we let $\mathcal{N}(t)$ denote the set of all Nash equilibria of $\mathcal{G}(t)$—that is, the set of all pure and mixed Nash equilibria of $\Gamma(t)$.

Theorem 7. *For a family of pseudo supermodular games $\Gamma(t)$ and their mixed extensions $\mathcal{G}(t)$,*

(i) *$\mathcal{N}(t)$ possesses the largest and smallest elements, which are both pure;*

(ii) *if each u_i satisfies increasing differences in (s_i, t), then $\mathcal{N}(t') \geqslant_{ws} \mathcal{N}(t)$ for all*

[20] In this example, one can take $X = \Delta(\{0, 1\}^2)$ to represent player i's mixed-strategy space over the pure strategy set $\{0, 1\}^2$.

$t' > t$.

The proof leverages the complete pseudo lattice structure of the mixed-strategy space. A primary difficulty, as discussed in Section 3.3 and Example 4, is that the player's best-response correspondence

$$B_i(\sigma_{-i}) := \arg\max_{\sigma_i \in \Sigma_i} \overline{u}_i(\sigma_i, \sigma_{-i}) \tag{5}$$

need not be pSS monotonic in rivals' mixed strategies. However, by Proposition 2, each $B_i(\sigma_{-i})$ possesses largest and smallest elements that are pure strategies. These extremal best responses are nondecreasing in σ_{-i} and therefore pSS monotonic. We can thus "sandwich" the full best-response correspondence between these monotonic extremal selections. Applying Theorem 5 and Theorem 6 then establishes the existence and comparative statics of the largest and smallest equilibria, both of which are pure.[21]

5 Perfect Equilibria

The existing monotone comparative statics analyses of games have been largely confined to Nash equilibria.[22] Extending the analysis to refinements such as (trembling-hand) perfect equilibria is important. For example, classical games motivating perfection are supermodular.[23]

However, such an extension presents a fundamental challenge: it requires dealing with perturbations in fully mixed strategies. As discussed earlier, however, the space

[21] This "sandwiching" of mixed equilibria between extremal pure equilibria was identified by Milgrom and Roberts (1990, Theorem 5) for lattice-based supermodular games. However, their approach relies on a serial undomination argument that requires full continuity of payoffs. In contrast, our approach utilizes monotone shifts in best-response correspondences on a pseudo lattice, establishing existence and MCS under weaker continuity assumptions and on more general domains.

[22] See, for instance, Echenique (2004), which develops a notion of supermodularity for dynamic games and applies it to subgame-perfect equilibrium.

[23] For instance, the two games below are supermodular.

Coordination Trap

	L	R
T	$0,0$	$1,1$
B	$0,0$	$0,0$

Entry Deterrence

	Fight	*Accom.*
Enter	$-1,-1$	$1,1$
Not	$0,2$	$0,2$

The profiles (B, L) and (Not, Fight) of the respective games are Nash but not perfect.

of mixed strategies does not form a lattice—let alone a complete lattice—making the standard lattice-based machinery inapplicable. Our approach, which accommodates a more general domain structure, overcomes this limitation and establishes both the existence of perfect equilibria in pure strategies and their monotone comparative statics properties.

Let us adopt the same framework as Section 4.3 and consider a pseudo supermodular game $\Gamma = (I, S, u)$. Let $\mathcal{G} = (I, \Sigma, \bar{u})$ denote the mixed extension of Γ, where each $\Sigma_i = \Delta(S_i)$ is partially ordered by the first-order stochastic dominance relation $\geqslant^{sd}$ and $\bar{u} = (\bar{u}_i)_{i \in I}$ with $\bar{u}_i$ defined in (4).

Following Selten (1975), we consider *perturbed* games in which players are constrained to play *fully mixed* strategies. To formalize this idea in general (not necessarily finite) games, for each player i, let $\mathscr{S}_i$ be the Borel σ-algebra on S_i, and let $\mathscr{M}_i$ denote the set of all nonnegative measures μ_i on $\mathscr{S}_i$ satisfying $\mu_i(S_i) \leqslant 1$. For two measures $\mu_i, \mu_i' \in \mathscr{M}_i$, let us write $\mu_i' \sqsupseteq \mu_i$ if $\mu_i'(S_i') \geqslant \mu_i(S_i')$ for all $S_i' \in \mathscr{S}_i$. Let $\mathscr{M}_i^0 \subset \mathscr{M}_i$ denote the set of full-support measures, i.e., the set of all measures $\mu_i \in \mathscr{M}_i$ such that $\mu_i(S_i') > 0$ for every nonempty open set $S_i' \in \mathscr{S}_i$. Let $\mathscr{M} := \times_i \mathscr{M}_i$ and $\mathscr{M}^0 := \times_i \mathscr{M}_i^0$.

For $\mu = (\mu_i)_{i \in I} \in \mathscr{M}$, define

$$\Sigma_i^\mu := \{\sigma_i \in \Sigma_i : \sigma_i \sqsupseteq \mu_i\},$$

the set of player i's mixed strategies that place at least the measure μ_i on every Borel subset of S_i. As shown in the Online Appendix, Σ_i^μ inherits the complete pseudo lattice structure of S_i.[24] When $\mu \in \mathscr{M}^0$, the strategy space Σ^μ captures the idea that players are constrained to play fully mixed strategies—that is, each player i must assign positive probability to every open subset of S_i. Let $\mathcal{G}^\mu := (I, \Sigma^\mu, \bar{u})$ denote the μ-constrained game of $\mathcal{G}$, where $\Sigma^\mu = \times_i \Sigma_i^\mu$.

To quantify the size of perturbations, define for each $\mu_i \in \mathscr{M}_i$ the total variation norm

$$\|\mu_i\| = \sup_{S_i' \in \mathscr{S}_i} |\mu_i(S_i')| = \mu_i(S_i),$$

and for each profile $\mu = (\mu_i)_i$, set $\|\mu\| = \max_i \|\mu_i\|$.

We are now ready to define perfect equilibrium: A strategy profile σ of the game

[24] See Lemma S5 in the Online Appendix for the proof.

$\mathcal{G} = (I, \Sigma, \bar{u})$ is a **perfect equilibrium** if there exists a sequence of μ^n-constrained games $\mathcal{G}^{\mu^n}$ with $\mu^n \in \mathscr{M}^0$ and $\|\mu^n\| \to 0$ such that each $\mathcal{G}^{\mu^n}$ admits a Nash equilibrium σ^n converging weakly to σ.

Our formulation of constrained games and perfect equilibrium parallels Selten's and coincides with his original definition when strategy spaces are finite. For games with infinite strategy spaces, our definition corresponds to Simon and Stinchcombe (1995)'s notion of strong perfect equilibrium.[25]

In the setting of infinite normal-form games, it refines the notion of Nash equilibrium when payoff functions are continuous:[26]

Lemma 2. *If each u_i is continuous, then any perfect equilibrium is also a Nash equilibrium.*

The payoff continuity is used only for Lemma 2 in this section; the subsequent results do not require this assumption.

To analyze the constrained game $\mathcal{G}^\mu$, it is useful to study a particular class of strategies in this game. We say a (mixed) strategy $\sigma_i \in \Sigma_i^\mu$ is **constrained-pure at $s_i \in S_i$** if σ_i puts maximal feasible mass on s_i: for each $S_i' \in \mathscr{S}_i$,

$$\sigma_i(S_i') = \begin{cases} 1 - \mu_i(S_i \backslash S_i') & \text{if } s_i \in S_i' \\ \mu_i(S_i') & \text{otherwise.} \end{cases} \tag{6}$$

It is straightforward to see that this strategy belongs to Σ_i^μ.[27] We call any strategy that is constrained-pure at some pure strategy a **constrained-pure** strategy.

[25] These authors advocate this notion because it preserves the "hallmark" property of *limit admissibility*, meaning that a strategy places no mass in the interior of the set of weakly dominated strategies. In finite games, limit admissibility coincides with admissibility (that requires weakly dominated strategies to be played with zero probability). Simon and Stinchcombe (1995) also define a weaker concept, called weak perfect equilibrium, which fails to satisfy limit admissibility.

[26] To see that payoff continuity cannot be dispensed with, consider a game in which each player $i = 1, 2$ chooses $x_i \in [0, 1]$ simultaneously, and the payoffs are $(1, 1)$ for all (x_1, x_2), except when $x_1 = x_2 = 1$, in which case both players get zero payoffs. Every pair $(x_1, x_2) \neq (1, 1)$ is a perfect equilibrium, so a limit point $(x_1, x_2) = (1, 1)$ is perfect as well. But it is not a Nash equilibrium.

[27] To see that σ_i is a probability measure, observe that for any $S_i' \subset S_i$,

$$\sigma_i(S_i') = (1 - \mu_i(S_i \backslash \{s_i\}))\delta_{s_i}(S_i') + \mu_i(S_i' \backslash \{s_i\}),$$

where δ_{s_i} denotes the Dirac measure at s_i. This expression represents a nonnegative linear combination of the two nonnegative measures $\delta_{s_i}(\cdot)$ and $\mu_i(\cdot \backslash \{s_i\})$, and hence defines a nonnegative measure. Moreover, since $\sigma_i(S_i) = (1 - \mu_i(S_i \backslash \{s_i\})) + \mu_i(S_i \backslash \{s_i\}) = 1$, it is a probability measure.

The following result is central to our comparative statics analysis of perfect equilibria. It establishes the existence of the largest and smallest Nash equilibria of the constrained games and their comparative statics. To do so, we consider a family of pseudo supermodular games $\Gamma(t) = (I, S, u(\cdot, t))$ and their μ-constrained games $\mathcal{G}^{\mu}(t) = (I, \Sigma^{\mu}, \bar{u}(\cdot, t))$. The set of all Nash equilibria of $\mathcal{G}^{\mu}(t)$ is denoted by $\mathcal{N}^{\mu}(t)$.

Proposition 4. *For a family of pseudo supermodular games* $\Gamma(t)$ *and for each* $\mu \in \mathscr{M}$*,*

(i) $\mathcal{N}^{\mu}(t)$ *contains the largest and smallest elements, which are constrained-pure;*

(ii) *if each* u_i *satisfies increasing-differences in* (s_i, t)*, then* $\mathcal{N}^{\mu}(t') \geqslant_{ws} \mathcal{N}^{\mu}(t)$ *for all* $t' > t$*.*

The logic for this result parallels that for mixed-strategy Nash equilibria (Theorem 7), except that here we work with the constrained mixed-strategy space Σ^{μ}. In this setting, the extremal best responses take the form of constrained-pure strategies. Apart from considering these constrained-pure best responses, the reasoning proceeds analogously: we "sandwich" the full best-response correspondence between the extremal constrained-pure best responses and apply our fixed-point results (Corollary 4 and Theorem 6) to establish the existence and monotone comparative statics of the extremal equilibria.

We now establish our main results of the current section, namely, the existence and comparative statics of perfect equilibria:

Theorem 8. *For a family of pseudo supermodular games* $\Gamma(t)$*,*

(i) $\Gamma(t)$ *has a perfect equilibrium in pure strategies;*

(ii) *the set of perfect equilibria of* $\Gamma(t)$ *is compact and contains maximal/minimal elements, which are all pure;*

(iii) *if each* u_i *satisfies increasing differences in* (s_i, t)*, perfect equilibria of* $\Gamma(t')$ *weak-set dominate those of* $\Gamma(t)$ *for all* $t' > t$*.*

We briefly sketch the proof; the full argument appears in the Appendix. Existence follows directly from Proposition 4(i), which guarantees that each constrained game $\mathcal{G}^{\mu}(t)$ admits a constrained-pure Nash equilibrium. For any sequence of full-support measures μ^n converging to zero, the corresponding constrained-pure equilibria converge (along a subsequence) to a pure strategy profile that forms a perfect equilibrium.

Part (ii) also builds on Proposition 4(i). Any perfect equilibrium can be obtained as the limit of Nash equilibria of constrained games for some vanishing sequence μ^n. Each such equilibrium is sandwiched between the smallest and largest constrained-pure equilibria of $\mathcal{G}^{\mu^n}(t)$, and their limits yield pure perfect equilibria that bound the given equilibrium. We do not, however, claim the existence of globally smallest or largest perfect equilibria, since different sequences μ^n may generate incomparable limits.

The comparative statics result in part (iii) follows from Proposition 4(ii). The constrained-pure equilibria of perturbed game $\mathcal{G}^{\mu^n}(t')$ weak-set dominate those of $\mathcal{G}^{\mu^n}(t)$ for each n, and this comparison is preserved after taking limits.

The existence of perfect equilibria in general infinite games is also established by Simon and Stinchcombe (1995) under the full continuity of payoffs. In contrast, our result ensures the existence of *pure* perfect equilibria and, more importantly, delivers a monotone comparative statics of perfect equilibria absent in Simon and Stinchcombe (1995).

6 Concluding Remarks

This paper has revisited the order-theoretic foundations of monotone comparative statics. The existing theory has long relied on the assumption that the domain of choice forms a lattice. We have shown that this structural requirement is not essential in many settings. By introducing the weaker notion of a *pseudo lattice*—which requires only the existence of minimal upper bounds and maximal lower bounds—we have generalized the core machinery of the theory, including the Monotonicity Theorem for individual choice and Tarski's fixed-point theorem.

The primary contribution of this generalization is its capacity to handle environments that fail the lattice property, most notably the space of probability distributions. This flexibility has allowed us to provide a unified framework for analyzing mixed-strategy Nash equilibria and, significantly, to conduct the first general monotone comparative statics analysis of (trembling-hand) perfect equilibria. By treating perfect equilibria as limits of Nash equilibria in constrained games—where strategy spaces are pseudo lattices but not lattices—we established the existence of pure perfect equilibria and their monotonicity with respect to the underlying environment.

Our framework opens several avenues for future research. First, while we have established the existence of maximal and minimal perfect equilibria, our current results do not guarantee the existence of a unique *largest* or *smallest* perfect equilibrium. Determining the conditions under which the set of perfect equilibria admits these global extremal elements remains an open question.

Second, the applicability of our framework to probability measures suggests natural extensions to *Bayesian games*. Since the space of distributional strategies often lacks a lattice structure under standard orders, our pseudo-lattice approach could facilitate monotone comparative statics analysis in games of incomplete information.

Finally, our results may prove useful in the field of *information design*. As illustrated in our examples, the set of information structures ordered by the mean-preserving spread (convex order) forms a complete pseudo lattice but generally fails to be a lattice. Applying our optimization and fixed-point theorems to this domain could yield new insights into how optimal information structures respond to changes in the economic environment.

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A Proofs for Section 2

Proof of Theorem 1. [**“$\Leftarrow$” direction:**] For any nonempty set $S \subset X$, we will show that $\bigwedge_X S \neq \varnothing$ (To prove $\bigvee_X S \neq \varnothing$ is analogous and thus omitted). Let $\underline{x}$ be the smallest point of X, which exists by assumption. For S, their common lower bound, denoted by L_S, is nonempty because $\underline{x} \in L_S$. For any chain $C \subset L_S$, there exists a supremum z of C in X since X is chain complete, by assumption. Then, $x \geqslant c$ for any $x \in S$ and $c \in C$ since $C \subset L_S$. This and the fact that z is the supremum of C imply that $x \geqslant z$ for each $x \in S$. Therefore, $z \in L_S$. Thus, by Zorn's lemma, the set L_S has a maximal element, that is, $\curlywedge_X S \neq \varnothing$, as desired.

[**"⇒" direction:**] Assume X is a complete pseudo lattice. Then, $\curlyvee_X X$ is nonempty. Let $x \in \curlyvee_X X$. Then, since $\curlyvee_X X \subset U_X$, we have $x \geqslant x'$ for every $x' \in X$, which shows that x is the largest element. To prove the existence of the smallest element is analogous and thus omitted. □

Proof of Lemma 1 . We prove that every nonempty chain $C \subset X$ admits a supremum (and analogously an infimum). To this end, fix any nonempty chain $C \subset X$ and let $U := \bigcap_{x \in C} U_x$ be the set of all upper bounds for C, where we recall $U_x = \{y \in X : y \geqslant x\}$.

Claim 1. *$U \neq \varnothing$.*

Proof. Suppose not. Then, $\bigcap_{x \in C} U_x = \varnothing$. Hence, $\bigcup_{x \in C}(X \backslash U_x) = X$. Recall, by the definition of natural topology, $X \backslash U_x$ is open. Since X is compact, there exists a finite set $C_f \subset C$ such that

$$\bigcup_{x \in C_f} (X \backslash U_x) = X. \tag{7}$$

Meanwhile, since C_f is a finite chain, it admits a maximum $\bar{x}$. Since $X \backslash U_{\bar{x}} \supset X \backslash U_x$ for all $x \in C_f$, we have

$$\bigcup_{x \in C_f} (X \backslash U_x) = X \backslash U_{\bar{x}}. \tag{8}$$

It follows from (7) and (8) that $U_{\bar{x}} = \varnothing$, which, however, contradicts $\bar{x} \in U_{\bar{x}}$. □

For each $c \in C$ and $u \in U$, let $[c, u] := \{x \in X : c \leqslant x \leqslant u\}$. Claim 1 ensures that this set is well defined.

Claim 2. *$\bigcap_{c \in C, u \in U} [c, u] \neq \varnothing$.*

Proof. Suppose not. Then, $X \backslash (\bigcap_{c \in C, u \in U} [c, u]) = X$, so

$$\bigcup_{c \in C, u \in U} (X \backslash [c, u]) = X.$$

Given our topology, $[c, u]$ is closed, so $X \backslash [c, u]$ is open for each $c \in C, u \in U$. By the compactness of X, we have $(c^1, u^1), ..., (c^K, u^K)$ in $C \times U$, for some $K \in \mathbb{N}$, such that

$$\bigcup_{k=1}^{K} (X \backslash [c^k, u^k]) = X,$$

so

$$\bigcap_{k=1}^{K} [c^k, u^k] = \varnothing.$$

Meanwhile, since C is a chain, there exists $\bar{c} := \max\{c^1, ..., c^K\}$, so

$$\bigcap_{k=1}^{K} [c^k, u^k] = \bigcap_{k=1}^{K} [\bar{c}, u^k] = \varnothing. \tag{9}$$

However, since U consists of upper bounds of all $c \in C$, $\bar{c} \leqslant u^k$ for all $k = 1, ..., K$, a contradiction. □

To complete the proof, let $a \in \bigcap_{c \in C, u \in U} [c, u]$, which is possible by Claim 2. Since $c \leqslant a$ for all $c \in C$ and since $a \leqslant u$ for all $u \in U$, we conclude that $a = \sup C$. □

B Proof for Section 3

Proof of Theorem 3′. In the following, we prove the "upper" version of the result: the "lower" version is analogous and thus omitted.

[**"⇐" direction:**] To show $M_{S'}(t') \geqslant_{uwpss} M_S(t)$ for every $t, t' \in T$ with $t' \geqslant t$ and $S, S' \subseteq X$ with $S' \geqslant_{wpss} S$, consider any $x \in M_S(t)$ and $x' \in M_{S'}(t')$. Consider any $z' \in (x \vee x') \cap S'$ and $z \in (x \wedge x') \cap S$; such z and z' exist since $S' \geqslant_{wpss} S$. We have

$$u(x, t) \geqslant u(z, t) \Rightarrow u(z', t) \geqslant u(x', t) \Rightarrow u(z', t') \geqslant u(x', t'),$$

where the first implication follows from upper pseudo quasi-supermodularity of u and the second implication follows because u satisfies upper single-crossing and $z' \geqslant x'$. Because $x' \in M_{S'}(t')$ by assumption, it follows that $z' \in M_{S'}(t')$, as desired.

[**"⇒" direction:**] To prove that u is upper single-crossing in (x, t), consider any $t' \geqslant t$ and $x' \geqslant x$ such that $u(x', t) \geqslant u(x, t)$. Choose $S = S' = \{x, x'\}$. Clearly, $S' \geqslant_{wpss} S$, and $x' \in M_S(t)$. Fix any $x'' \in M_{S'}(t') \subset \{x, x'\}$. That $M_{S'}(t') \geqslant_{uwpss} M_S(t)$ and

$x' \curlyvee x'' = \{x'\}$ implies $x' \in M_{S'}(t')$, which in turn implies $u(x', t') \geqslant u(x, t')$, as desired.

Next, to prove u is upper pseudo quasi-supermodular in x, let $S = \{x, z\}$ and $S' = \{x', z'\}$, for arbitrary $z \in x \curlywedge x'$ and $z' \in x \curlyvee x'$. Suppose that x and x' are incomparable (since otherwise the result holds trivially). Assume that $u(x, t) \geqslant u(z, t)$. Then, $x \in M_S(t)$. Since $S' \geqslant_{wpss} S$ and $S' \cap (x \curlyvee x') = \{z'\}$, $M_{S'}(t) \geqslant_{uwpss} M_S(t)$ requires $z' \in M_{S'}(t)$. It follows that $u(z', t) \geqslant u(x', t)$, proving the upper pseudo quasi-supermodularity of u in x. □

Proof of Theorem 4. This result follows immediately from a directional version in Theorem 4′ below. □

We say that $S \subset X$ is an *upper (resp. lower) pseudo sublattice* if, for any $x, x' \in S$, $x \curlyvee x' \subset S$ (resp. $x \curlywedge x' \subset S$), and that S is a *complete upper (resp. lower) pseudo sublattice* if, for every nonempty $S' \subseteq S$, $\curlyvee_X S'$ (resp. $\curlywedge_X S'$) is a nonempty subset of S.

Theorem 4′. *Assume X is a pseudo lattice and $u : X \to \mathbb{R}$ is upper (resp. lower) pseudo quasi-supermodular.*

(i) $\arg\max_{x \in X} u(x)$ *is an upper (resp. lower) pseudo sublattice of X whenever it is nonempty.*

(ii) *In addition, if X is a complete pseudo lattice and u is order upper semicontinuous, then* $\arg\max_{x \in X} u(x)$ *is a nonempty, complete upper (resp. lower) pseudo sublattice, admitting the largest (resp. smallest) point.*

Proof. Throughout the proof, we only establish the upper case since the proof for the lower case is analogous.

For (i), if $s, s' \in \arg\max_{x \in X} u(x)$, then $u(s) \geqslant u(z)$ for any $z \in s \curlywedge s'$, which implies by upper pseudo quasi-supermodularity that $u(z') \geqslant u(s')$ for any $z' \in s \curlyvee s'$, meaning $z' \in \arg\max_{x \in X} u(x)$, as desired.

For (ii), let us establish a couple of claims (see the Online Appendix for proof):

Claim 3. *For any subset $X' \subset X$ and each $x \in X'$, there is a maximal chain in X' containing x.*

Claim 4. *If X is chain complete, then for any x, y and z with $x, y \in U_z$, $(x \curlywedge y) \cap U_z$ is nonempty. Also, for any x, y and z with $x, y \in L_z$, $(x \curlyvee y) \cap L_z$ is nonempty.*

To first prove $\arg\max_{x\in X} u(x) \neq \varnothing$, let

$$M := \sup_{x\in X} u(x),$$

(where M is possibly infinite) and observe that there exists a sequence $\{x_n\}$ such that

$$\lim_{k\to\infty} \sup_{n\geqslant k} u(x_n) = M.$$

We construct an increasing sequence (hence a chain) $\{y_n\}$ satisfying $u(y_n) \geqslant u(x_n)$ for every n. Letting y denote the supremum of $\{y_n\}$, the order upper semicontinuity of u implies

$$u(y) \geqslant \lim_{k\to\infty} \sup_{n\geqslant k} u(y_n) \geqslant \lim_{k\to\infty} \sup_{n\geqslant k} u(x_n) = M,$$

which shows that $y \in \arg\max_{x\in X} u(x)$.

To construct $\{y_n\}$, we begin by defining $z^0 = \{z^0_n\}_{n\in\mathbb{N}}$ such that $z^0_n = x_n$. Given $z^{m-1} = \{z^{m-1}_n\}_{n\in\mathbb{N}}$, we recursively construct, for each $m \geqslant 1$, a sequence $z^m = \{z^m_n\}_{n\in\mathbb{N}}$ satisfying:

(a) $z^m_n = z^{m-1}_n$ for all $n < m$;

(b) $z^m_1 \leqslant z^m_2 \leqslant \cdots \leqslant z^m_m \leqslant z^m_n$ for all $n > m$;

(c) $u(z^m_n) \geqslant u(x_n)$ for all $n \geqslant 1$.

Note that z^0 satisfies (a)–(c) trivially.

Once such sequences are constructed, it suffices to define $y_n := z^n_n$. Indeed, for every n, conditions (a) and (b) imply that $z^{n+1}_{n+1} \geqslant z^n_n$, so $\{y_n\}$ is increasing, and condition (c) implies that $u(z^n_n) \geqslant u(x_n)$.

Fix $m \geqslant 1$ and suppose that z^{m-1} satisfies (a)–(c). To satisfy (a), define $z^m_n := z^{m-1}_n$ for all $n < m$. It remains to define z^m_n for $n \geqslant m$. Let $w_0 := z^{m-1}_m$. For $k \geqslant 1$, given the pair (w_{k-1}, z^{m-1}_{m+k}), we inductively construct (w_k, z^m_{m+k}) as follows, with the goal of defining z^m_m as the limit of $\{w_k\}$:

Case I: If there exists $x \in w_{k-1} \curlyvee z^{m-1}_{m+k}$ such that $u(x) \geqslant u(z^{m-1}_{m+k})$, then set

$$w_k := w_{k-1}, \qquad z^m_{m+k} := x.$$

Case II: If $u(x) < u(z^{m-1}_{m+k})$ for all $x \in w_{k-1} \curlyvee z^{m-1}_{m+k}$, then the upper pseudo quasi-supermodularity of u implies that

$$u(x) \geqslant u(w_{k-1}) \quad \text{for all } x \in w_{k-1} \curlywedge z^{m-1}_{m+k}.$$

By Claim 4, there exists some

$$x' \in (w_{k-1} \curlywedge z^{m-1}_{m+k}) \cap \{x : x \geqslant z^{m-1}_{m-1} = z^m_{m-1}\}.$$

Define

$$w_k := x', \qquad z^m_{m+k} := z^{m-1}_{m+k}.$$

In either case, for each $k \geqslant 1$, we have

$$z^m_{m-1} = z^{m-1}_{m-1} \leqslant w_k \leqslant w_{k-1} \quad \text{and} \quad w_k \leqslant z^m_{m+k}, \tag{10}$$

$$u(w_k) \geqslant u(w_{k-1}) \quad \text{and} \quad u(z^m_{m+k}) \geqslant u(z^{m-1}_{m+k}) \geqslant u(x_{m+k}). \tag{11}$$

Since $\{w_k\}$ is a chain, it admits an infimum; define z^m_m to be this infimum. By (10), we have $z^m_m \leqslant w_k \leqslant z^m_{m+k}$ for all $k \geqslant 1$, so condition (b) holds for z^m. Moreover, by the order upper semicontinuity of u,

$$u(z^m_m) \geqslant \lim_{n\to\infty} \sup_{k \geqslant n} u(w_k) \geqslant u(w_0) = u(z^{m-1}_m) \geqslant u(x_m).$$

Together with (11), this shows that z^m satisfies condition (c), completing the induction.

We now prove that $M_X(u)$ is a complete upper pseudo sublattice. Consider any $S \subset M_X(u)$ and any $\bar{s} \in \curlyvee_X S$. We need to show $\bar{s} \in M_X(u)$, which will imply that $\bar{s}$ is a supremum of S if S is a chain (since X is chain complete), so $M_X(u)$ is chain complete. (That $\curlywedge_X S \subset M_X(u)$ follows from an analogous argument.) Let $\hat{M} = M_X(u) \cap L_{\bar{s}}$. Suppose for contradiction that $\bar{s} \notin \hat{M}$. By Claim 3, there is a collection $(C_x)_{x\in\hat{M}}$ such that each C_x is a maximal chain in $\hat{M}$ that contains x. Letting $z_x = \sup C_x$, we have $z_x \in \hat{M}$ by the order upper semicontinuity of u, which implies $z_x < \bar{s}$. Also, there must exist some $x, y \in \hat{M}$ with $z_x \neq z_y$ since otherwise we would have some $\bar{x} = \sup C_x, \forall x \in \hat{M}$ with $\bar{x} < \bar{s}$, which would imply $\bar{x} \in U_S$ (since $S \subset \hat{M}$) and contradict $\bar{s} \in \curlyvee_X S$. Also, we cannot have $z_x < z_y$ or $z_x > z_y$

since we could then add z_y to C_x or z_x to C_y to form a larger chain, contradicting the maximality of C_x or C_y, respectively. Thus, z_x and z_y must be incomparable. By Claim 4, one can then find some $\hat{x} \in (z_x \curlyvee z_y) \cap L_{\bar{s}}$ with $\hat{x} > z_x$. Since $u(z_x) \geqslant u(x)$ for any $x \in z_x \curlywedge z_y$, the upper pseudo quasi supermodularity of u implies $u(\hat{x}) \geqslant u(z_y)$ and thus $\hat{x} \in \hat{M}$. This contradicts the maximality of C_x since $C_x \cup \{\hat{x}\}$ is a larger chain in $\hat{M}$. Thus, $\bar{s} \in \hat{M}$. □

Proof of Proposition 2. To begin, let us define $\hat{u} : A \times \Delta(\Theta) \to \mathbb{R}$ by

$$\hat{u}(a, \eta) := \int u(a, \theta)\eta(d\theta). \tag{12}$$

First, $\hat{u}(\cdot, \eta)$ is pseudo supermodular, and thus pseudo quasi-supermodular, in a since the pseudo supermodularity of u is preserved under a convex combination. By Lemma S1 in the Online Appendix, $\hat{u}(\cdot, \eta)$ is order upper semicontinuous. Thus, by Theorem 4(ii), $\arg\max_{a \in A} \hat{u}(a, \eta)$ is a complete pseudo sublattice, admitting the largest and smallest points, $\overline{a}(\eta)$ and $\underline{a}(\eta)$. Hence, any a is not optimal if $a \nleqslant \overline{a}(\eta)$ or $a \ngeqslant \underline{a}(\eta)$. Letting $\overline{a} = \overline{a}(\eta)$ and $\underline{a} = \underline{a}(\eta)$, we show that $\delta_{\overline{a}(\eta)}$ and $\delta_{\underline{a}(\eta)}$ are the greatest and smallest elements in $\arg\max_{\tilde{x} \in X} \bar{u}(\tilde{x}, \eta)$, respectively. Fix any upward closed set $A' \subset A$ and any $x \in \arg\max_{\tilde{x} \in X} \bar{u}(\tilde{x}, \eta)$. Then, $\delta_{\underline{a}}(A') = x(A') = \delta_{\overline{a}}(A') = 1$ if $\underline{a} \in A'$; $\delta_{\underline{a}}(A') = 0 \leqslant x(A') \leqslant 1 = \delta_{\overline{a}}(A')$ if $\underline{a} \notin A'$ but $\overline{a} \in A'$; and $\delta_{\underline{a}}(A') = x(A') = \delta_{\overline{a}}(A') = 0$ if $\overline{a} \notin A'$. Thus, the desired conclusion follows.

Observe next that for any $a' \geqslant a$ and $\eta' \geqslant^{sd} \eta$,

$$\begin{aligned}
\hat{u}(a', \eta') - \hat{u}(a, \eta') &= \int (u(a', \theta) - u(a, \theta))\eta'(d\theta) \\
&\geqslant \int (u(a', \theta) - u(a, \theta))\eta(d\theta) \\
&= \hat{u}(a', \eta) - \hat{u}(a, \eta),
\end{aligned}$$

where the inequality follows since $\eta' \geqslant^{sd} \eta$ and $u(a', \cdot) - u(a, \cdot)$ is monotonic (due to the increasing differences property of u). Thus, $\hat{u}$ satisfies single-crossing in (a, η). By Proposition 1, $\arg\max_{a \in A} \hat{u}(a, \eta') \geqslant_{pss} \arg\max_{a \in A} \hat{u}(a, \eta)$, implying that $\overline{a}(\eta)$ and $\underline{a}(\eta)$ are nondecreasing in η. Combined with the observation that $\delta_{\overline{a}(\eta)}$ and $\delta_{\underline{a}(\eta)}$ are the largest and smallest elements of $M(\eta)$, this implies that $M(\eta') \geqslant_{ws} M(\eta)$. □

C Proofs for Section 4

Proof of Theorem 5. Letting X_F denote the set of fixed points of F, we first prove $X_F \neq \varnothing$. To this end, consider any point $x \in X_+ = \{x' \in X : x'' \geqslant x' \text{ for some } x'' \in F(x')\}$.[28] By Claim 3, there is a maximal chain C in X_+ that contains x. Letting $\bar{x}_C = \sup C$, we have $\bar{y}_C := \sup F(\bar{x}_C) \geqslant \sup F(x') \geqslant x'$ for all $x' \in C$, which implies $\bar{y}_C \in U_C$ and thus $\bar{y}_C \geqslant \bar{x}_C$. Then, $F(\bar{y}_C) \geqslant_{pss} F(\bar{x}_C)$ and thus $y' \geqslant \bar{y}_C$ for some $y' \in F(\bar{y}_C)$ (since $\bar{y}_C \in F(\bar{x}_C)$). This implies $\bar{y}_C \in X_+$. If $\bar{y}_C > \bar{x}_C$, then $C \cup \{\bar{y}_C\}$ would be a larger chain in X_+ than C, contradicting the maximality of C. Thus, $\bar{y}_C = \bar{x}_C$, meaning $\bar{x}_C \in X_F$.

To show that X_F is a complete pseudo lattice, we need to prove: (i) for any $S \subset X_F$, $\curlyvee_{X_F} S \neq \varnothing$ and $\curlywedge_{X_F} S \neq \varnothing$; (ii) X_F is chain complete.

For (i), we only prove $\curlyvee_{X_F} S \neq \varnothing$ (since proving $\curlywedge_{X_F} S \neq \varnothing$ is analogous). Let

$$T = U_S \cap \{x' \in X : x' \geqslant x'' \text{ for some } x'' \in F(x')\}$$

and consider a maximal chain C in T (which is nonempty since it contains $\sup X$). Letting $z := \inf C$, we aim to show $z \in X_F$, which will imply $z \in \curlyvee_{X_F} S$ since, if there were any $z' < z$ such that $z' \in U_S \cap X_F$, then $C \cup \{z'\}$ would form a larger chain in T than C, a contradiction. Suppose now, for contradiction, that $z \notin X_F$. Observe that $y := \inf F(z) \leqslant \inf F(x) \leqslant x$ for all $x \in C$, implying $y \in L_C$ and thus $y \leqslant z$. We must have $y < z$ since $z \notin X_F$ and $y \in F(z)$. Given this, we show below that there is some $\tilde{y} \in T$ with $\tilde{y} < z$, which will lead to the desired contradiction since $C \cup \{\tilde{y}\}$ would be a larger chain in T than C.

By the well-ordering theorem, there exists an ordinal γ such that $S = \{s_\alpha\}_{\alpha<\gamma}$. We construct a chain $\tilde{C} = \{x_\alpha\}_{\alpha<\gamma}$ inductively, whose supremum gives the desired $\tilde{y}$.

Initial step. Choose x_1 to be any element of $(y \curlyvee s_1) \cap L_z$, which is nonempty by Claim 4 and the fact that $y, s_1 \in L_z$. Since $z \geqslant s_1$, $y \in F(z)$, and $s_1 \in F(s_1)$, the pSS monotonicity of F implies $x_1 \in F(z)$, because $F(z) \geqslant_{pss} F(s_1)$.

Inductive step. Let $\beta > 1$ be any ordinal smaller than γ, and suppose that we have constructed an increasing chain

$$(x_\alpha)_{\alpha<\beta} \subset L_z \quad \text{with} \quad x_\alpha \in F(z) \text{ for all } \alpha < \beta.$$

[28] Note that X_+ is nonempty since it contains the smallest point of X, which exists since X is a complete pseudo lattice.

Define

$$\tilde{x} := \sup_{\alpha<\beta} x_\alpha.$$

This supremum exists and satisfies $\tilde{x} \in F(z)$ since $F(z)$ is a complete pseudo sublattice. As before, the set $(\tilde{x} \curlyvee s_\beta) \cap L_z$ is nonempty, and we choose any element of this set to be x_β.

Since $z \geqslant s_\beta$, $\tilde{x} \in F(z)$, and $s_\beta \in F(s_\beta)$, the pSS monotonicity of F implies $x_\beta \in F(z)$, because $F(z) \geqslant_{pss} F(s_\beta)$. Adding x_β to $(x_\alpha)_{\alpha<\beta}$ preserves the chain property.

By construction, $\tilde{C} \subset F(z)$ and $x_\alpha \geqslant s_\alpha$ for all $\alpha < \gamma$. Since $\tilde{C}$ is a chain and X is chain complete, there exists

$$\tilde{y} := \sup \tilde{C} = \curlyvee_X \tilde{C}.$$

Moreover, $\tilde{y} \in F(z)$ because $F(z)$ is a complete pseudo sublattice. We also have $\tilde{y} \in U_S$, since $\tilde{y} \geqslant x_\alpha \geqslant s_\alpha$ for all $\alpha < \gamma$.

Finally, since $z \in U_{\tilde{C}}$ and $\tilde{y} = \sup \tilde{C}$, it follows that $\tilde{y} \leqslant z$. In fact, $\tilde{y} < z$, because $z \notin F(z)$. By pSS monotonicity, this implies that there exists $\tilde{z} \in F(\tilde{y})$ such that $\tilde{z} \leqslant \tilde{y}$, since $\tilde{y} \in F(z)$ and $F(\tilde{y}) \leqslant_{pss} F(z)$. Hence, $\tilde{y} \in T$, as desired.

For (ii), we prove only that any chain $C \subset X_F$ has a supremum in X_F, as the argument for the infimum is analogous. Let

$$x_C := \sup_X C,$$

which exists since X is chain complete. Define a self-correspondence G on $Y := U_{x_C}$ by

$$G(x) := F(x) \cap Y \quad \text{for each } x \in Y.$$

We first show that G is nonempty-valued. It suffices to show that

$$G(x_C) = F(x_C) \cap U_{x_C} \neq \varnothing.$$

Indeed, once this holds, the pSS monotonicity of F implies that for any $x \geqslant x_C$, there exist $x' \in F(x)$ and $x'' \in F(x_C)$ with $x'' \geqslant x_C$ such that, for some $\tilde{x} \in x' \curlyvee x''$, we have $\tilde{x} \in F(x)$. Since $\tilde{x} \geqslant x_C$, this implies $\tilde{x} \in F(x) \cap U_{x_C}$, and hence $G(x) \neq \varnothing$.

To show that $G(x_C) \neq \varnothing$, note that for each $x \in C$, we have $x \in F(x)$ and $F(x_C) \geqslant_{pss} F(x)$. It follows that $\sup F(x_C) \geqslant x$ for all $x \in C$. Therefore,

$$\sup F(x_C) \in F(x_C) \cap U_C \subset F(x_C) \cap U_{x_C} = G(x_C).$$

Observe next that for each $S \subset Y$, $\bigwedge_Y S = (\bigwedge_X S) \cap Y$ and $\bigvee_Y S = (\bigvee_X S) \cap Y$. Using this observation, it is straightforward to see that G is complete-pseudo-sublattice-valued and pSS monotonic. Thus, letting Y_G denote the set of fixed points of G [JW: and applying the part (i) to G and Y_G, it follows that $\bigwedge_{Y_G} Y_G \neq \varnothing$, so] Y_G contains a smallest point, say $\underline{x}_G$. Clearly, $\underline{x}_G$ is also a fixed point of F and corresponds to $\sup_{X_F} C$ since any fixed point of F weakly greater than x_C is also a fixed point of G and thus weakly greater than $\underline{x}_G$. □

Proof of Theorem 6. Fix any $x \in \mathcal{F}(t)$ and let $Z = U_x$. Define a self-correspondence on Z as follows: for each $\tilde{x} \in Z$, $H(\tilde{x}) = F(\tilde{x}, t') \cap Z$. Fix any $\tilde{x} \in Z$. Since $x \in F(x,t)$ and $F(\tilde{x}, t') \geqslant_{pss} F(x, t') \geqslant_{uws} F(x,t)$, there is $y \in F(\tilde{x}, t')$ with $y \geqslant x$, so H is nonempty-valued. It is straightforward to see that as $F(\tilde{x}, t')$ is a complete pseudo sublattice, so is $H(\tilde{x}) = F(\tilde{x}, t') \cap Z$. It is also straightforward that H is pSS monotonic on Z. Thus, by Theorem 5, the set of fixed points of H, denoted X_H, is nonempty. Since $X_H \subset \mathcal{F}(t')$ and any $\tilde{x} \in X_H$ is weakly greater than x, we have just proved that $\mathcal{F}(t') \geqslant_{uws} \mathcal{F}(t)$. The proof for $\mathcal{F}(t') \geqslant_{lws} \mathcal{F}(t)$ is analogous and hence omitted. □

Proof of Proposition 3. Denote the (pure-strategy) best-response correspondence for player i in game $\Gamma(t)$ by

$$B_i(s_{-i}, t) := \arg\max_{s_i \in S_i} u_i(s_i, s_{-i}, t), \tag{13}$$

and let $B(s,t) = \prod_{i \in I} B_i(s_{-i}, t)$. Note that a strategy profile $s = (s_i)_{i \in I}$ is a (pure-strategy) Nash equilibrium if and only if $s \in B(s,t)$.

To prove part (i), suppose that $\Gamma(t)$ is pseudo quasi-supermodular. Note first that S_i is a complete pseudo lattice, and u_i is order upper semi-continuous and pseudo quasi-supermodular in s_i for each i by assumption. So, by Theorem 4 (ii), $B_i(s_{-i}, t)$ is a nonempty, complete pseudo sublattice. Moreover, for each $s_{-i}, s'_{-i} \in S_{-i}$ with $s'_{-i} \geqslant s_{-i}$, because u_i satisfies single-crossing in (s_i, s_{-i}) by assumption, by Proposition 1, $B_i(s'_{-i}, t) \geqslant_{pss} B_i(s_{-i}, t)$. Thus, it follows that $B(\cdot, t)$ has the property that

$B(s',t) \geqslant_{pss} B(s,t)$ for every $s, s' \in S$ with $s' \geqslant s$, that is, $B(\cdot,t)$ is pSS monotonic (with respect to the product order on S). Therefore, it follows that B is pseudo monotonic. Thus, by Theorem 5, the set of fixed points of B is a nonempty, complete pseudo lattice, and thus has a largest element. We complete the proof by observing that the set of fixed points of B is equivalent to the set of pure Nash equilibria of $\Gamma(t)$, $\mathcal{E}(t)$.

To prove part (ii), observe that since u_i satisfies single-crossing in (s,t) for each $i \in I$ by assumption, by Proposition 1, we have $B(s,t') \geqslant_{pss} B(s,t)$ for each $s \in S$, which implies that $B(\cdot,t')$ weak-set dominates $B(\cdot,t)$. So, by Theorem 6, it follows that $\mathcal{E}(t') \geqslant_{ws} \mathcal{E}(t)$, as desired. □

Proof of Theorem 7. This result follows immediately from setting $\mu \equiv 0$ in Proposition 4 in Section 5. □

D Proofs for Section 5

First, we establish useful properties of the best response in the constrained games:

Lemma 3. *Consider a family of pseudo supermodular games $(I, S, u(\cdot,t))$. For any nonnegative measures $\mu = (\mu_i)_{i \in I} \in \mathscr{M}$, define the best response correspondence as*

$$B_i^{\mu}(\sigma_{-i};t) := \arg\max_{\sigma_i \in \Sigma_i^{\mu}} \int u_i(s_i, s_{-i}, t)\sigma_i(ds_i)\sigma_{-i}(ds_{-i}).$$

Then,

(i) $B_i^{\mu}(\sigma_{-i};t)$ *has largest and smallest elements,* $\overline{\sigma}_i^{\mu}(\sigma_{-i};t)$ *and* $\underline{\sigma}_i^{\mu}(\sigma_{-i};t)$*, which are both constrained-pure and nondecreasing in* σ_{-i}*;*

(ii) $\overline{\sigma}_i^{\mu}(\sigma_{-i};t') \geqslant^{sd} \overline{\sigma}_i^{\mu}(\sigma_{-i};t)$ *and* $\underline{\sigma}_i^{\mu}(\sigma_{-i};t') \geqslant^{sd} \underline{\sigma}_i^{\mu}(\sigma_{-i};t)$ *for any* $t' \geqslant t$ *if* u_i *satisfies increasing differences in* (s_i, t)*.*

Proof. Observe first that the unconstrained best response B_i, defined in (5), has the largest and smallest elements in pure strategies—denoted $\bar{b}_i(\sigma_{-i};t)$ and $\underline{b}_i(\sigma_{-i};t)$, respectively—that are nondecreasing in σ_{-i}. This follows directly from Proposition 2 by mapping the payoff function and best response in (4) and (5) to those in (1) and (2), respectively, with $a = s_i$ and $\theta = s_{-i}$ (so that $x = \sigma_i$ and $\eta = \sigma_{-i}$).

Next, let $\overline{\sigma}_i^\mu$ denote a strategy constrained-pure at $\overline{b}_i := \overline{b}_i(\sigma_{-i};t)$: that is, for each $S_i' \subset S_i$,

$$\overline{\sigma}_i^\mu(S_i') = \begin{cases} 1 - \mu_i(S_i \backslash S_i') & \text{if } \overline{b}_i \in S_i' \\ \mu_i(S_i') & \text{if } \overline{b}_i \notin S_i'. \end{cases}$$

Let us define for each $\sigma = (\sigma_i)_{i \in I}$ and $s_i \in S_i$,

$$\bar{u}_i(\sigma, t) = \int_{s \in S} u_i(s,t)\sigma(ds) \text{ and } \hat{u}_i(s_i, \sigma_{-i}, t) = \int_{s_{-i} \in S_{-i}} u_i(s_i, s_{-i}, t)\sigma_{-i}(ds_{-i}).$$

Then, for any $\sigma_i \in \Sigma_i^\mu$,

$$\begin{aligned}
&\bar{u}_i(\overline{\sigma}_i^\mu, \sigma_{-i}, t) - \bar{u}_i(\sigma_i, \sigma_{-i}, t) \\
=&\int \hat{u}_i(s_i, \sigma_{-i}, t)\overline{\sigma}_i^\mu(ds_i) - \int \hat{u}_i(s_i, \sigma_{-i}, t)\sigma_i(ds_i) \\
=&\hat{u}_i(\overline{b}_i, \sigma_{-i}, t)[\overline{\sigma}_i^\mu(\{\overline{b}_i\}) - \sigma_i(\{\overline{b}_i\})] - \int_{S_i \backslash \{\overline{b}_i\}} \hat{u}_i(s_i, \sigma_{-i}, t)[\sigma_i - \overline{\sigma}_i^\mu](ds_i) \\
=&\int_{S_i \backslash \{\overline{b}_i\}} [\hat{u}_i(\overline{b}_i, \sigma_{-i}, t) - \hat{u}_i(s_i, \sigma_{-i}, t)][\sigma_i - \overline{\sigma}_i^\mu](ds_i) \\
=&\int_{S_i \backslash \{\overline{b}_i\}} [\hat{u}_i(\overline{b}_i, \sigma_{-i}, t) - \hat{u}_i(s_i, \sigma_{-i}, t)][\sigma_i - \mu_i](ds_i) \\
\geqslant&0.
\end{aligned}$$

The third equality follows since $\overline{\sigma}_i^\mu(\{\overline{b}_i\}) - \sigma_i(\{\overline{b}_i\}) = \int_{S_i \backslash \{\overline{b}_i\}} [\sigma_i - \overline{\sigma}_i^\mu](ds_i)$ and the fourth equality follows from the construction of $\overline{\sigma}^\mu$. Finally, the inequality follows from the fact that $\overline{b}_i$ is a best reponse to σ_{-i} and $\sigma_i \sqsupseteq \mu_i$ for any $\sigma_i \in \Sigma_i^\mu$. Moreover, the inequality is strict if $[\sigma_i - \mu_i](S_i \backslash \{s_i : s_i \leqslant \overline{b}_i\}) > 0$. This proves that $\overline{\sigma}_i^\mu \in B_i^\mu(\sigma_{-i};t)$ and that any $\sigma_i \in \Sigma_i^\mu$ which is not stochastically dominated by $\overline{\sigma}_i^\mu$ cannot be a best response. We thus conclude that $\overline{\sigma}_i^\mu$ is the largest element of $B_i^\mu(\sigma_{-i};t)$. Likewise, a strategy constrained-pure at $\underline{b}_i(\sigma_{-i};t)$, denoted as $\underline{\sigma}_i^\mu$, is the smallest point of $B_i^\mu(\sigma_{-i};t)$. This completes the proof of part (i).

To prove part (ii), observe first that as u_i satisfies increasing differences in (s_i, t), so does $\hat{u}_i$ (since $\hat{u}_i$ is a convex combination of u_i with respect to σ_{-i}). Thus, $\overline{b}_i(\sigma_{-i};t') \geqslant \overline{b}_i(\sigma_{-i};t)$ and $\underline{b}_i(\sigma_{-i};t') \geqslant \underline{b}_i(\sigma_{-i};t)$. To prove $\overline{\sigma}_i^\mu(\sigma_{-i};t') \geqslant^{sd} \overline{\sigma}_i^\mu(\sigma_{-i};t)$, we need to show that $\overline{\sigma}_i^\mu(\sigma_{-i};t')(S_i') \geqslant \overline{\sigma}_i^\mu(\sigma_{-i};t)(S_i')$ for any upward closed set $S_i' \subset S_i$. There are two cases. Suppose first $\overline{b}_i(\sigma_{-i};t) \notin S_i'$. Then, $\overline{\sigma}_i^\mu(\sigma_{-i};t')(S_i') \geqslant \mu_i(S_i') =$

$\overline{\sigma}_i^\mu(\sigma_{-i};t)(S_i')$. Suppose next $\overline{b}_i(\sigma_{-i};t) \in S_i'$. Then, $\overline{b}_i(\sigma_{-i};t') \in S_i'$ (since S_i' is an upward closed set) so $\overline{\sigma}_i^\mu(\sigma_{-i};t')(S_i') = \overline{\sigma}_i^\mu(\sigma_{-i};t)(S_i') = 1 - \mu_i(S_i \backslash S_i')$, as desired. Likewise, we have $\underline{\sigma}_i^\mu(\sigma_{-i};t') \geqslant^{sd} \underline{\sigma}_i^\mu(\sigma_{-i};t)$, completing our proof. □

Proof of Proposition 4. Throughout this proof, for any mapping F, let X_F denote its fixed-point set.

For part (i), consider the following self-maps on $\Sigma^\mu = \times_{i\in I}\Sigma_i^\mu$:

$$\overline{F}(\sigma) = (\overline{\sigma}_i^\mu(\sigma_{-i};t))_{i\in I}, \quad \underline{F}(\sigma) = (\underline{\sigma}_i^\mu(\sigma_{-i};t))_{i\in I}, \quad H(\sigma) = (B_i^\mu(\sigma_{-i};t))_{i\in I}, \tag{14}$$

where $\overline{\sigma}_i^\mu$, $\underline{\sigma}_i^\mu$, and B_i^μ are as defined in Lemma 3. Note that any point in $X_{\underline{F}}$, $X_{\overline{F}}$, or X_H is a Nash equilibrium of $\mathcal{G}^\mu$. Both $\underline{F}$ and $\overline{F}$ are pseudo monotonic, as they are singleton-valued and $\underline{\sigma}_i^\mu(\cdot;t)$ and $\overline{\sigma}_i^\mu(\cdot;t)$ are weakly increasing by Lemma 3(i). By Theorem 5, $X_{\underline{F}}$ and $X_{\overline{F}}$ each admit extremal points, which must be constrained-pure by construction. Let $\underline{\sigma}$ and $\overline{\sigma}$ denote the smallest and largest elements of $X_{\underline{F}}$ and $X_{\overline{F}}$, respectively. Moreover, by Lemma 3(i), $\overline{\sigma}_i^\mu(\sigma_{-i};t) \geqslant^{sd} \sigma_i \geqslant^{sd} \underline{\sigma}_i^\mu(\sigma_{-i};t)$ for any $\sigma_i \in B_i^\mu(\sigma_{-i};t)$, implying $\overline{F}(\sigma) \geqslant_{ws} H(\sigma) \geqslant_{ws} \underline{F}(\sigma)$ at each $\sigma \in \Sigma^\mu$. Hence, by Theorem 6′, $\overline{\sigma} \geqslant \sigma \geqslant \underline{\sigma}$ for every $\sigma \in X_H = \mathcal{N}^\mu(t)$, as desired.

For part (ii), define $\overline{F}$ and $\underline{F}$ as in (14) and define

$$\overline{G}(\sigma) = (\overline{\sigma}_i^\mu(\sigma_{-i};t'))_{i\in I}, \quad \underline{G}(\sigma) = (\underline{\sigma}_i^\mu(\sigma_{-i};t'))_{i\in I}. \tag{15}$$

By Lemma 3(ii), $\overline{G}(\sigma) \geqslant_{ws} \overline{F}(\sigma)$ and $\underline{G}(\sigma) \geqslant_{ws} \underline{F}(\sigma)$ at each $\sigma \in \Sigma^\mu$. Then, by Theorem 6, the largest point of $X_{\overline{G}}$ is weakly greater than that of $X_{\overline{F}}$; the smallest point of $X_{\underline{G}}$ is weakly greater than that of $X_{\underline{F}}$. Given part (i), this implies $\mathcal{N}^\mu(t') \geqslant_{ws} \mathcal{N}^\mu(t)$.□

Proof of Theorem 8. For parts (i) and (ii), we focus on the game $G(t)$. Consider any sequence of constrained games $\{\mathcal{G}^{\mu^n}(t)\}_n$ with each μ^n belonging to $\mathscr{M}^0$ and $\|\mu^n\| \to 0$. By Proposition 4(i), each constrained game $\mathcal{G}^{\mu^n}(t)$ has a constrained-pure Nash equilibrium σ^n. Then, part (i) follows from the next claim:

Claim 5. *A (sub)sequence of constrained-pure strategy profiles $\{\sigma^n\}_n$ must weakly converge to a pure strategy profile.*

Proof. Let s_i^n denote a pure strategy on which σ_i^n puts the maximum weight. By the compactness of S_i, $\{s_i^n\}_n$ (or its subsequence) converges to some limit $\tilde{s}_i \in S_i$. To show that σ_i^n weakly converges to $\tilde{s}_i$ (i.e., the mixed strategy putting all the weight on $\tilde{s}_i$), it suffices to show that for any bounded continuous function f defined on S_i,

$$\int_{s_i \in S_i} (f(s_i) - f(\tilde{s}_i)) d\sigma_i^n(s_i) \to 0. \tag{16}$$

To show this, for any $\epsilon > 0$, we can find N such that for any $n > N$, $|f(s_i^n) - f(\tilde{s}_i)| < \frac{\epsilon}{2}$ and $\mu_i^n(S_i) < \frac{\epsilon}{2(M-m)}$, where $M = \sup_{s_i} f(s_i)$ and $m = \inf_{s_i} f(s_i)$.[29] Note that the latter inequality implies $\sigma_i^n(S_i \backslash \{s_i^n\}) < \frac{\epsilon}{2(M-m)}$. Observe now that

$$\begin{aligned} \left| \int_{s_i \in S_i} (f(s_i) - f(\tilde{s}_i)) d\sigma_i^n(s_i) \right| &\leqslant |f(s_i^n) - f(\tilde{s}_i)| \sigma_i^n(\{s_i^n\}) + \int_{s_i \in S_i \backslash \{s_i^n\}} |f(s_i) - f(\tilde{s}_i)| d\sigma_i^n(s_i) \\ &< \frac{\epsilon}{2} + (M-m) \frac{\epsilon}{2(M-m)} = \epsilon, \end{aligned}$$

establishing the desired convergence. □

For part (ii), we first establish compactness. Since Σ is a metrizable space, it suffices to prove sequential compactness. Consider a sequence of perfect equilibria $\{\sigma^n\}_n$. Because Σ is compact, this sequence admits a convergent subsequence. Denoting its limit by σ, we show that σ is a perfect equilibrium.

Since each σ^n is a perfect equilibrium, there exists a sequence of constrained games $\{\mathcal{G}^{\mu^{n,m}}(t)\}_m$ and an associated sequence of Nash equilibria $\{\sigma^{n,m}\}_m$ such that $\|\mu^{n,m}\| \to 0$ and $\sigma^{n,m} \to \sigma^n$ as $m \to \infty$. For each n, choose $m(n)$ sufficiently large so that

$$\max\left\{ \|\mu^{n,m(n)}\|, \ d(\sigma^n, \sigma^{n,m(n)}) \right\} < \frac{1}{n}, \tag{17}$$

where $d(\cdot, \cdot)$ denotes the metric on Σ, which generates the weak topology.

For each n, define $\tilde{\mu}^n := \mu^{n,m(n)}$ and $\tilde{\sigma}^n := \sigma^{n,m(n)}$. Observe first that $\tilde{\sigma}^n$ is a Nash equilibrium of the constrained game $\mathcal{G}^{\tilde{\mu}^n}(t)$. Moreover, by (17), we have $\|\tilde{\mu}^n\| \to 0$ as $n \to \infty$, and $\tilde{\sigma}^n \to \sigma$. Indeed,

$$d(\tilde{\sigma}^n, \sigma) = d(\sigma^{n,m(n)}, \sigma) \leqslant d(\sigma^{n,m(n)}, \sigma^n) + d(\sigma^n, \sigma),$$

and both terms on the right-hand side converge to zero. This establishes that σ is a

[29] We may assume without loss that $M > m$ since otherwise (16) holds trivially.

perfect equilibrium.

To prove the existence of a maximal perfect equilibrium, note that by Lemma 1, the set of perfect equilibria is chain complete since it is compact as shown above. Let σ be any perfect equilibrium of $\Gamma(t)$, whose existence follows from part (i). By Claim 3, there exists a maximal chain of perfect equilibria containing σ. Chain completeness then implies that this maximal chain admits a supremum, which must be a maximal perfect equilibrium.

To prove that all maximal perfect equilibria are pure, it suffices to show that for any perfect equilibrium σ of $\Gamma(t)$, there exists a pure perfect equilibrium that weakly dominates σ under $\geqslant^{sd}$. Consider a sequence of games $\{\mathcal{G}^{\mu^n}(t)\}_n$ whose Nash equilibria $\{\sigma^n\}_n$ converge weakly to σ. By Proposition 4(i), for each $\mathcal{G}^{\mu^n}(t)$, there exists a constrained-pure strategy Nash equilibrium $\bar{\sigma}^n$ that dominates σ^n under $\geqslant^{sd}$. By Claim 5, $\{\bar{\sigma}^n\}_n$ (or its subsequence) converges to a pure strategy profile. We denote this limit by $\bar{\sigma}$, and it constitutes a perfect equilibrium.

To show that $\bar{\sigma} \geqslant^{sd} \sigma$, we use the following result from Kamae, Krengel, and O'Brien (1977):

Fact 1. *Let $\{P_n\}_n$ and $\{Q_n\}_n$ be sequences of probability measures on a Polish space that converge weakly to P and Q, respectively. If $Q_n \geqslant^{sd} P_n$ for all n, then $Q \geqslant^{sd} P$.*

Since $\bar{\sigma}^n \geqslant^{sd} \sigma^n$ for all n, and $\bar{\sigma}$ and σ are their respective weak limits, it follows that $\bar{\sigma} \geqslant^{sd} \sigma$, as desired.

The proof for minimal perfect equilibria is analogous and is therefore omitted.

For part (iii), we only establish the upper weak-set dominance since the lower weak-set dominance can be established analogously. Consider any perfect equilibrium σ for $\Gamma(t)$ which is a weak limit of Nash equilibria $\{\sigma^n\}_n$ for $\{\mathcal{G}^{\mu^n}(t)\}_n$. By Proposition 4, there exists a constrained-pure strategy Nash equilibrium $\tilde{\sigma}^n$ for $\mathcal{G}^{\mu^n}(t')$ that dominates σ^n in $\geqslant^{sd}$. Letting $\tilde{\sigma}$ denote a weak limit of $\{\tilde{\sigma}^n\}_n$, $\tilde{\sigma}$ is a perfect equilibrium of $\Gamma(t')$ such that $\tilde{\sigma} \geqslant^{sd} \sigma$ due to Fact 1. □

Supplementary Appendix for: Monotone Comparative Statics without Lattices

E Omitted Results for Section 2 and Section 3

Example 5 (Failure of Lattice Property under Convex Order)**.** Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$, and let $\sigma = (\sigma_1, \sigma_2, \sigma_3) \in \Delta(\Theta)$ denote a belief. To simplify notation, we write (σ_1, σ_2) for the belief $(\sigma_1, \sigma_2, \sigma_3)$ with $\sigma_3 = 1 - \sigma_1 - \sigma_2 \geqslant 0$. Fix the prior belief at $\mu = (\frac{2}{5}, \frac{2}{5})$. The set

$$X_\mu = \{\, x \in X : \underline{x}_\mu \leqslant^{cx} x \leqslant^{cx} \overline{x}_\mu \,\}$$

then includes, among others, the following measures:

$$\begin{aligned}
P &= \tfrac{1}{2}\delta_{(\frac{3}{10},\frac{3}{10})} + \tfrac{1}{2}\delta_{(\frac{1}{2},\frac{1}{2})}, \\
Q &= \tfrac{1}{2}\delta_{(\frac{3}{10},\frac{1}{2})} + \tfrac{1}{2}\delta_{(\frac{1}{2},\frac{3}{10})}, \\
R &= \tfrac{1}{4}\big(\delta_{(\frac{1}{5},\frac{2}{5})} + \delta_{(\frac{2}{5},\frac{1}{5})} + \delta_{(\frac{3}{5},\frac{2}{5})} + \delta_{(\frac{2}{5},\frac{3}{5})}\big), \\
S &= \tfrac{5}{11}\,\delta_{(\frac{11}{20},\frac{11}{20})} + \tfrac{3}{11}\,\delta_{(\frac{11}{20},0)} + \tfrac{3}{11}\,\delta_{(0,\frac{11}{20})}.
\end{aligned}$$

It is straightforward to verify that each of these measures is a mean-preserving spread of δ_μ. Following Proposition 4.5 of Müller and Scarsini (2006) (from which this example is adapted), one can check that: (i) both R and S are mean-preserving spreads of P and Q; (ii) R and S are incomparable with each other; and (iii) there exists no $x \in X_\mu$ that is simultaneously a mean-preserving spread of P and Q and a mean-preserving contraction of R and S. Hence, the least upper bound of P and Q does not exist under $\geqslant^{cx}$ (so X_μ is not a lattice).

Proof of Claim 3. Fix any $X' \subset X$ and each $x \in X'$. Consider the family $\mathcal{C} := \{C \subset X' : C \text{ is a chain with } x \in C\}$, and order the elements in $\mathcal{C}$ by the set inclusion order. Take any chain of chains $\mathcal{D} = \{C_\alpha\}_{\alpha \in A}$ for totally ordered set A, that is, $C_\alpha \in \mathcal{C}$, and $C_\alpha \subseteq C_\beta$ or $C_\beta \subseteq C_\alpha$ for all $\alpha, \beta \in A$. Set $C^\sharp = \cup_{\alpha \in A} C_\alpha$, and observe that $C^\sharp \in \mathcal{C}$

and that it is an upper bound of $\mathcal{D}$ in $\mathcal{C}$. Thus, by Zorn's lemma, $\mathcal{C}$ must contain a maximal element. □

Proof of Claim 4. To prove the first statement, observe that by Claim 3, there is a maximal chain $\hat{C}$ in $S := U_z \cap L_{\{x,y\}}$ that contains z. Letting $\hat{x} = \sup \hat{C}$ by the chain-completeness, we have $\hat{x} \in U_z$. To argue that $\hat{x} \in x \curlywedge y$, note first that since both x and y are upper bounds of $\hat{C}$, $\hat{x} \leqslant x$ and $\hat{x} \leqslant y$, implying $\hat{x} \in S$. If $\hat{x} \notin x \curlywedge y$, then there exists $\bar{x} > \hat{x}$ such that $\bar{x} \in L_{\{x,y\}}$ so $\bar{x} \in S$. Then, $\hat{C} \cup \{\bar{x}\}$ would be a larger chain, contradicting the maximality of $\hat{C}$. The proof of the second statement is analogous and hence omitted. □

Lemma S1. *For $\eta \in \Delta(\Theta)$, $\hat{u}(\cdot, \eta)$ is order upper semicontinuous on A.*

Proof. Let $C = \{a_\alpha\}_\alpha \subset A$ be any chain, and suppose that $a' \in (\curlywedge_A C) \cup (\curlyvee_A C)$ exists (so a' is either $\inf C$ or $\sup C$).

For each α, define

$$h_\alpha(\theta) := \sup_{\beta \geqslant \alpha} u(a_\beta, \theta).$$

Then $(h_\alpha)_\alpha$ is pointwise decreasing in α, and

$$\inf_\alpha h_\alpha(\theta) = \inf_\alpha \sup_{\beta \geqslant \alpha} u(a_\beta, \theta) \quad \text{for all } \theta.$$

We obtain

$$\begin{aligned} \inf_\alpha \sup_{\beta \geqslant \alpha} \hat{u}(a_\beta, \eta) &= \inf_\alpha \sup_{\beta \geqslant \alpha} \int u(a_\beta, \theta)\, \eta(d\theta) \\ &\leqslant \inf_\alpha \int \sup_{\beta \geqslant \alpha} u(a_\beta, \theta)\, \eta(d\theta) \\ &= \inf_\alpha \int h_\alpha(\theta)\, \eta(d\theta). \end{aligned} \tag{18}$$

Since the family $(h_\alpha)_\alpha$ is bounded, and $(h_\alpha)_\alpha$ is pointwise decreasing, the Monotone Convergence Theorem yields

$$\inf_\alpha \int h_\alpha(\theta)\, \eta(d\theta) = \int \inf_\alpha h_\alpha(\theta)\, \eta(d\theta).$$

Combining this with (18),

$$\inf_{\alpha} \sup_{\beta \geqslant \alpha} \hat{u}(a_\beta, \eta) \leqslant \int \inf_{\alpha} \sup_{\beta \geqslant \alpha} u(a_\beta, \theta)\, \eta(d\theta). \tag{19}$$

By order upper semicontinuity of $u(\cdot, \theta)$ and the fact that a' is $\inf C$ or $\sup C$,

$$\inf_{\alpha} \sup_{\beta \geqslant \alpha} u(a_\beta, \theta) \leqslant u(a', \theta) \quad \text{for all } \theta.$$

Integrating both sides and using (19) yield

$$\inf_{\alpha} \sup_{\beta \geqslant \alpha} \hat{u}(a_\beta, \eta) \leqslant \int u(a', \theta)\, \eta(d\theta) = \hat{u}(a', \eta).$$

Thus $\hat{u}(\cdot, \eta)$ is order upper semicontinuous. □

F Omitted Proofs for Section 4

F.1 Proofs for Directional Results

Proof of Theorem 5′. For each $x \in X$, let $H(x)$ denote the largest element of $F(x)$, which exists since $F(x)$ is a complete upper pseudo sublattice. The upper pseudo monotonicity of F implies that $H(x)$, as a singleton-valued correspondence, is nondecreasing. Hence, by Corollary 4, the fixed-point set of H is a nonempty complete pseudo lattice and therefore contains a largest element, denoted by $\bar{x}_H$.

To see that $\bar{x}_H$ is also the largest fixed point of F, let $F(\cdot, t) = F(\cdot)$ and $F(\cdot, t') = H(\cdot)$. Then $F(x, t') \geqslant_{ws} F(x, t)$ for all $x \in X$, which implies by Theorem 6′ that $\mathcal{F}(t') \geqslant_{ws} \mathcal{F}(t)$. It follows that $\bar{x}_H$ is the largest fixed point of F as well.

The proof for the case in which the correspondence F is lower pseudomonotonic is analogous and therefore omitted. □

Proof of Theorem 6′. We prove only the first statement, as the proof of the second statement is analogous.

Fix any $x \in \mathcal{F}(t)$ and let $Z := U_x$. Define two self-correspondences on Z. The correspondence H is defined as in the proof of Theorem 5′, while $G(\tilde{x}) = x$ for all $\tilde{x} \in Z$. Clearly, both H and G are pSS monotonic, and $H(\tilde{x}) \geqslant_{ws} G(\tilde{x})$ for all $\tilde{x} \in Z$.

By Theorem 6, the fixed-point set of H weak-set dominates that of G, which implies that H has a fixed point $x' \geqslant x$ (since x is the unique fixed point of G). Observe that $x' \in \mathcal{F}(t')$, which establishes the desired result. □

F.2 Omitted Proofs for Generalized Bertrand Games

First, we define directional versions of pseudo quasi-supermodular games. Specifically, we say that Γ is an **upper (resp. lower) pseudo quasi-supermodular game** if, for all $i \in I$, condition (P1) of pseudo quasi-supermodular games holds and

(P2″) u_i is upper (resp. lower) pseudo quasi-supermodular in s_i and satisfies upper (resp. lower) single-crossing in (s_i, s_{-i}).

The following is a directional variant of Proposition 3.

Proposition 2′. *For a family of upper (resp. lower) pseudo quasi-supermodular games* $\Gamma(t)$*,*

(i) *the set of (pure) Nash equilibria* $\mathcal{E}(t)$ *is nonempty and has the largest (resp. smallest) element;*

(ii) *if* u_i *satisfies upper (resp. lower) single-crossing in* (s_i, t) *for all* $i \in I$*, then* $\mathcal{E}(t') \geqslant_{uws} \mathcal{E}(t)$ *(resp.* $\mathcal{E}(t') \geqslant_{lws} \mathcal{E}(t)$*) for all* $t' > t$*.*

Proof. First, we present a directional version of Proposition 1:

Corollary 2′. *If* u *is upper (resp. lower) pseudo quasi-supermodular in* x *and satisfies upper (resp. lower) single-crossing in* (x, t)*, then* $M_{S'}(t') \geqslant_{upss} M_S(t)$ *(resp.* $M_{S'}(t') \geqslant_{lpss} M_S(t)$*) for all* $t' \geqslant t$ *and* $S' \geqslant_{pss} S$*.*

Proof. For any $x \in M_S(t)$ and $x' \in M_{S'}(t')$, choose any $z \in x \curlywedge x'$ and $z' \in x \curlyvee x'$. Since $S' \geqslant_{pss} S$, $z \in S$ and $z' \in S'$. Suppose that x and x' are incomparable (or else, the result would be trivial). Letting $Z = \{x, z\}$ and $Z' = \{x', z'\}$, we have $Z' \geqslant_{wpss} Z$, with $Z' \cap (x \curlyvee x') = \{z'\}$, and $Z \cap (x \curlywedge x') = \{z\}$. Since $M_{Z'}(t') \geqslant_{uwpss} M_Z(t)$ by Theorem 3′ and since $x \in M_Z(t)$ and $x' \in M_{Z'}(t')$, we must have $z' \in M_{Z'}(t')$, which implies $z' \in M_{S'}(t')$. Since z and z' were chosen arbitrarily, we have shown $M_{S'}(t') \geqslant_{upss} M_S(t)$. □

Next, we proceed to prove the "upper" versions of the statements of Proposition 2′ (the proof for the "lower" version is symmetric and thus omitted).

To prove part (i), suppose that $\Gamma(t)$ is upper pseudo quasi-supermodular. Note first that S_i is a complete pseudo lattice, and u_i is order upper semi-continuous and upper pseudo quasi-supermodular in s_i for each i by assumption. So, by Theorem 4′ (ii), $B_i(s_{-i}, t)$ is a nonempty, complete upper pseudo sublattice. Moreover, for each $s_{-i}, s'_{-i} \in S_{-i}$ with $s'_{-i} \geqslant s_{-i}$, because u_i satisfies upper single-crossing in (s_i, s_{-i}) by assumption, by Corollary 2′, $B_i(s'_{-i}, t) \geqslant_{upss} B_i(s_{-i}, t)$. Thus, it follows that $B(\cdot, t)$ has the property that $B(s', t) \geqslant_{upss} B(s, t)$ for every $s, s' \in S$ with $s' \geqslant s$, that is, $B(\cdot, t)$ is upper pSS monotonic (with respect to the product order on S). Therefore, it follows that B is upper pseudo monotonic. Thus, by Theorem 5′, the set of fixed points of B is nonempty and admits a largest element. We complete the proof by observing that the set of fixed points of B is equivalent to the set of pure Nash equilibria of $\Gamma(t)$, $\mathcal{E}(t)$.

To prove part (ii), observe that since u_i satisfies upper single-crossing in (s_i, t) for each $i \in I$ by assumption, by Corollary 2′, we have $B(s, t') \geqslant_{upss} B(s, t)$ for each $s \in S$, which implies that $B(s, t') \geqslant_{uws} B(s, t)$ for all s. So, by Theorem 6′, it follows that $\mathcal{E}(t') \geqslant_{uws} \mathcal{E}(t)$, as desired. □

Now we are ready to analyze generalized Bertrand games. We begin by establishing that a generalized Bertrand game is in the class of a directional version of pseudo quasi-supermodular games.

Lemma S2. *A generalized Bertrand game is lower pseudo quasi-supermodular.*

Proof. First, because the strategy space for each player is finite and totally ordered, it is a complete pseudo lattice. Second, each payoff function is order upper semicontinuous in the player's own strategy because the strategy space is finite. Moreover, it also trivially follows that the payoff function of each player is lower pseudo quasi-supermodular because $p_i \vee \bar{p}_i = \bar{p}_i$ and $p_i \wedge \bar{p}_i = p_i$, for any p_i and $\bar{p}_i$ with $p_i < \bar{p}_i$.

Recall that the payoff function of firm i is her profit given p_i, p_{-i}, as defined by (3), and fix any $p_{-i} < p'_{-i}$ and pick any p_i and $\bar{p}_i$ with $p_i < \bar{p}_i$. We shall show that

$$U_i(p_i, p'_{-i}) \geqslant U_i(\bar{p}_i, p'_{-i}) \Rightarrow U_i(p_i, p_{-i}) \geqslant U_i(\bar{p}_i, p_{-i}), \tag{20}$$

which is the lower single-crossing condition we are left with to complete the proof.

Assume first that $D_i(p_i, p_{-i}) = 0$. Then, (D1) implies $D_i(\bar{p}_i, p_{-i}) = 0$, so $U_i(p_i, p_{-i}) = 0 = U_i(\bar{p}_i, p_{-i})$, as desired.

Assume next that $D_i(p_i, p_{-i}) > 0$ and thus $D_i(p_i, p'_{-i}) > 0$ by (D1). Assume

$$U_i(p_i, p'_{-i}) \geqslant U_i(\bar{p}_i, p'_{-i}). \tag{21}$$

One can define $c(p_{-i})$ and $K(p_{-i})$ such that

$$C_i(q) = qc(p_{-i}) + K(p_{-i}) \text{ for } q \in \{D_i(\bar{p}_i, p_{-i}), D_i(p_i, p_{-i})\}. \tag{22}$$

Define similarly $c(p'_{-i})$ and $K(p'_{-i})$ by replacing p_{-i} in (22) with p'_{-i}. By the convexity of C_i, we have $c(p'_{-i}) \geqslant c(p_{-i})$.[30] Observe that (21) can be rewritten as

$$(p_i - c(p'_{-i}))D_i(p_i, p'_{-i}) \geqslant (\bar{p}_i - c(p'_{-i}))D_i(\bar{p}_i, p'_{-i}). \tag{23}$$

We next argue that $p_i - c(p'_{-i}) \geqslant 0$. This is immediate from (23) if $D_i(\bar{p}_i, p'_{-i}) = 0$ (recall $D_i(p_i, p'_{-i}) > 0$). Suppose thus that $D_i(\bar{p}_i, p'_{-i}) > 0$. If $p_i - c(p'_{-i}) < 0$, then (23) would imply

$$\bar{p}_i - c(p'_{-i}) \leqslant (p_i - c(p'_{-i}))\frac{D_i(p_i, p'_{-i})}{D_i(\bar{p}_i, p'_{-i})} \leqslant p_i - c(p'_{-i})$$

since $\frac{D_i(p_i, p'_{-i})}{D_i(\bar{p}_i, p'_{-i})} \geqslant 1$, which contradicts $\bar{p}_i > p_i$. Thus $\bar{p}_i - c(p'_{-i}) > p_i - c(p'_{-i}) \geqslant 0$. Using this and $c(p'_{-i}) \geqslant c(p_{-i})$, we obtain

$$\frac{p_i - c(p_{-i})}{\bar{p}_i - c(p_{-i})} \geqslant \frac{p_i - c(p'_{-i})}{\bar{p}_i - c(p'_{-i})} \geqslant \frac{D_i(\bar{p}_i, p'_{-i})}{D_i(p_i, p'_{-i})} \geqslant \frac{D_i(\bar{p}_i, p_{-i})}{D_i(p_i, p_{-i})}, \tag{24}$$

where the second inequality follows from (23) while the last inequality from (D2). It follows from (24) that $(p_i - c(p_{-i}))D_i(p_i, p_{-i}) \geqslant (\bar{p}_i - c(p_{-i}))D_i(\bar{p}_i, p_{-i})$, which implies

[30] To see this, let $q = D_i(p_i, p_{-i})$ and $\bar{q} = D_i(\bar{p}_i, p_{-i})$ while letting $q' = D_i(p_i, p'_{-i})$ and $\bar{q}' = D_i(\bar{p}_i, p'_{-i})$. Note that $q \geqslant \bar{q}$ and $q' \geqslant \bar{q}'$. If $q = \bar{q}$, then $c(p_{-i})$ can be chosen sufficiency small to satisfy $c(p_{-i}) \leqslant c(p'_{-i})$. Also if $q' = \bar{q}'$, then $c(p'_{-i})$ can be chosen sufficiency large to satisfy $c(p_{-i}) \leqslant c(p'_{-i})$. Suppose thus that $q > \bar{q}$ and $q' > \bar{q}'$. Observe now that, by (D1), $q \leqslant q'$ and $\bar{q} \leqslant \bar{q}'$. Given this, the convexity of C_i implies

$$c(p_{-i}) = \frac{C_i(q) - C_i(\bar{q})}{q - \bar{q}} \leqslant \frac{C_i(q') - C_i(\bar{q}')}{q' - \bar{q}'} = c(p'_{-i}).$$

$U_i(p_i, p_{-i}) \geqslant U_i(\bar{p}_i, p_{-i})$, completing the proof. $\square$

Next, we proceed to study the comparative statics of generalized Bertrand games.

Lemma S3. *Suppose that a family of generalized Bertrand games $\Gamma(t)$ satisfies (B1) and (B2). Then, for any $i \in I$, U_i satisfies lower single-crossing in (p_i, t).*

Proof. Recall that the shift from $\Gamma(t)$ to $\Gamma(t')$ with $t' > t$ involves the two changes, (B1) and (B2). It suffices to establish the result under each change separately. First, consider the case in which (B2) holds, while the demand $D_i(p, t)$ is constant in t. Given (B2), we shall show that U_i satisfies lower single-crossing in (p_i, t): that is, for any $p_i' \geqslant p_i$ and p_{-i}, $U_i(p_i', p_{-i}, t) - U_i(p_i, p_{-i}, t) > 0$ implies $U_i(p_i', p_{-i}, t') - U_i(p_i, p_{-i}, t') > 0$. To show it, note that

$$
\begin{aligned}
& U_i(p_i', p_{-i}, t) - U_i(p_i, p_{-i}, t) > 0 \\
\Leftrightarrow\ & p_i' D_i(p_i', p_{-i}, t) - C_i(D_i(p_i', p_{-i}, t), t) - [p_i D_i(p_i, p_{-i}, t) - C_i(D_i(p_i, p_{-i}, t), t)] > 0 \\
\Leftrightarrow\ & p_i' D_i(p_i', p_{-i}, t) - p_i D_i(p_i, p_{-i}, t) > C_i(D_i(p_i', p_{-i}, t), t) - C_i(D_i(p_i, p_{-i}, t), t). \quad (25)
\end{aligned}
$$

Because $C_i(q', t) - C_i(q, t) \leqslant C_i(q', t') - C_i(q, t')$ for every $q' > q$ by assumption (B2) and $D_i(p_i', p_{-i}, t) \leqslant D_i(p_i, p_{-i}, t)$ as $p_i' \geqslant p_i$, we have

$$C_i(D_i(p_i', p_{-i}, t), t) - C_i(D_i(p_i, p_{-i}, t), t) \geqslant C_i(D_i(p_i', p_{-i}), t') - C_i(D_i(p_i, p_{-i}), t').$$

Therefore, it follows from (25) that that

$$p_i' D_i(p_i', p_{-i}, t) - p_i D_i(p_i, p_{-i}, t) > C_i(D_i(p_i', p_{-i}, t), t') - C_i(D_i(p_i, p_{-i}, t), t'),$$

which is equivalent to

$$U_i(p_i', p_{-i}, t') - U_i(p_i, p_{-i}, t') > 0,$$

as desired.

Next, consider the case in which (B1) holds while $C_i(q, t)$ is constant in t. Showing the lower single-crossing of U_i in (p_i, t) for this case is analogous to the proof of Lemma S2 and hence omitted. $\square$

Proof of Corollary 5. The existence of Nash equilibrium is a direct consequence of

Proposition 2′ (i) and Lemma S2. The comparative statics result between the sets of equilibria in $\Gamma(t)$ and $\Gamma(t')$ follows directly from Proposition 2′ (ii) and Lemma S3.

To prove the comparative statics result on the firm profit, suppose that $\Gamma(t)$ and $\Gamma(t')$ satisfy (B1) and (B2′) and consider any Nash equilibrium $p' = (p'_i)_{i \in I}$ in $\Gamma(t')$. By Lemma S2, Lemma S3 and Proposition 2′(ii), there exists an equilibrium $p^* \leqslant p'$ in $\Gamma(t)$. Now, consider any firm i with $c'_i = c_i$.

First, suppose that $p'_i < c_i$. Then, it follows that $D_i(p', t') = 0$ because otherwise $U_i(p', t') = (p'_i - c_i) D_i(p', t') < 0 \leqslant (\max P_i - c_i) D_i(\max P_i, p'_{-i}, t') = U_i(\max P_i, p'_{-i}, t')$ because $\max P_i \geqslant c_i$ by assumption, contradicting the assumption that p' is a Nash equilibrium in $\Gamma(t')$.[31] This implies that $U_i(p', t') = 0$. Now, we will show that $U_i(p^*, t) = 0$. To show this, suppose for contradiction that $U_i(p^*, t) \neq 0$. Because p^* is an equilibrium and $\max P_i \geqslant c_i$, this implies that $D_i(p^*, t) > 0$, $p^*_i > c_i$, and $U_i(p^*, t) > 0$. Hence, it follows that

$$
\begin{aligned}
0 &< U_i(p^*, t) \\
&= (p^*_i - c_i) D_i(p^*, t) \\
&\leqslant (p^*_i - c_i) D_i(p^*, t') \\
&\leqslant (p^*_i - c_i) D_i(p^*_i, p'_{-i}, t') \\
&\leqslant U_i(p', t'),
\end{aligned}
$$

where the first inequality is as established earlier, the equality is by the definition of U_i, the second inequality follows from condition (B2′) and $p^*_i > c_i$, the third inequality follows from $p^* \leqslant p'$, condition (D1), and $p^*_i > c_i$, and the last inequality follows from the assumption that p' is a Nash equilibrium in $\Gamma(t')$. Thus we obtain $U_i(p', t') > 0$, a contradiction. Thus we have shown that $U_i(p^*, t) = 0 = U_i(p', t')$.

Second, suppose that $p'_i \geqslant c_i$. Then, first note that, for any $p_i \in P_i$ with $p_i \geqslant c_i$, $U_i(p_i, p_{-i}, t) = (p_i - c_i) D_i(p_i, p_{-i}, t)$ is weakly increasing in p_{-i}. Define $\Pi_i(p_{-i}, t) := \max_{p_i \in P_i} U_i(p_i, p_{-i}, t)$, and define $\Pi_i(p_{-i}, t')$ similarly. Note that the above monotonicity of U_i for $p_i \geqslant c_i$ implies $\Pi_i(\cdot, t)$ is weakly increasing and that for any p_{-i} and $p_i \geqslant c_i$, $(p_i - c_i) D_i(p_i, p_{-i}, t') \geqslant (p_i - c_i) D_i(p_i, p_{-i}, t)$ and thus $\Pi_i(p_{-i}, t') \geqslant \Pi_i(p_{-i}, t)$. Therefore, it follows that for each i with $c'_i = c_i$, $\Pi_i(p^*_{-i}, t) \leqslant \Pi_i(p'_{-i}, t) \leqslant \Pi_i(p'_{-i}, t')$, where the first inequality follows because $p^* \leqslant p'$ and $\Pi_i(\cdot, t)$ is weakly increasing as

[31] We denote $U_i(p, t') = (p_i - c_i) D_i(p, t')$ for each $i \in I$ and $p \in P$.

established earlier, and the second inequality follows because $\Pi_i(p_{-i}, t') \geqslant \Pi_i(p_{-i}, t)$ for all p_{-i} as established earlier as well.

The preceding two cases complete the proof. $\square$

Finally, we demonstrate that the class of our generalized Bertrand games indeed subsumes pure Bertrand games as special cases.

Lemma S4. *A pure Bertrand game is a generalized Bertrand game.*

Proof. It suffices to check (D2) since it is straightforward to check (D1). Fix any $p_i < p'_i, p_{-i} < p'_{-i}$ such that $D_i(p_i, p_{-i}) > 0$. It must be that $p_i \leqslant p^m_{-i} := \min_{j \neq i} p_j$. There are two cases.

Consider first $p_i = p^m_{-i}$. Then, $D_i(p'_i, p_{-i}) = 0$ and $D_i(p_i, p'_{-i}) > 0$. Hence,

$$\frac{D_i(p'_i, p_{-i})}{D_i(p_i, p_{-i})} = 0 \leqslant \frac{D_i(p'_i, p'_{-i})}{D_i(p_i, p'_{-i})}.$$

Consider next $p_i < p^m_{-i}$, so $D_i(p_i, p_{-i}) = 1$. By (D1), $D_i(p_i, p'_{-i}) = 1$. Hence,

$$\frac{D_i(p'_i, p_{-i})}{D_i(p_i, p_{-i})} \leqslant \frac{D_i(p'_i, p'_{-i})}{D_i(p_i, p'_{-i})} \Leftrightarrow D_i(p'_i, p_{-i}) \leqslant D_i(p'_i, p'_{-i}).$$

The latter inequality is a direct consequence of (D1). $\square$

G Omitted Proof for Section 5

Proof of Lemma 2. We first establish the following claim that shows the Hausdorff distance between Σ_i and $\Sigma_i^{\mu_i^n}$ goes to zero as $n \to \infty$:

Claim 6. *Fix any $\epsilon > 0$. For sufficiently large n, $\sup_{\sigma'_i \in \Sigma_i} \inf_{\tilde{\sigma}_i \in \Sigma_i^{\mu_i^n}} d(\tilde{\sigma}_i, \sigma'_i) < \epsilon$.*

Proof. Given any $\sigma'_i \in \Sigma_i$, let $\tilde{\sigma}_i = (1 - \mu_i^n(S_i))\sigma'_i + \mu_i^n$. Clearly, $\tilde{\sigma}_i \in \Sigma_i^{\mu_i^n}$. Also,

$$d(\tilde{\sigma}_i, \sigma'_i) = \sup_{S'_i \in \mathscr{S}_i} |-\mu_i^n(S_i)\sigma'_i(S'_i) + \mu_i^n(S'_i)| \leqslant \sup_{S' \in \mathscr{S}_i} \mu_i^n(S_i)\sigma'_i(S'_i) + \mu_i^n(S'_i) \leqslant 2\mu_i^n(S_i).$$

The proof is complete by choosing n sufficiently large so that $\mu_i^n(S_i) < \frac{\epsilon}{2}$. $\square$

Consider a perfect equilibrium σ of game $\mathcal{G} = (I, \Sigma, \bar{u})$. Thus, there exists a sequence σ^n of Nash equilibrium of $\mathcal{G}^{\mu^n}$ that weakly converges to σ. Fix any $i \in I$

and $\sigma'_i \in \Sigma_i$ and use Claim 6 to find a sequence $\tilde{\sigma}_i^n \in \Sigma_i^{\mu_i^n}$ such that $d(\tilde{\sigma}_i^n, \sigma'_i) \to 0$ so $\tilde{\sigma}_i^n$ weakly converges to σ'_i. Since σ^n is a Nash equilibrium of $\mathcal{G}^{\mu^n}$, we have $\bar{u}_i(\sigma_i^n, \sigma_{-i}^n) \geqslant \bar{u}_i(\tilde{\sigma}_i^n, \sigma_{-i}^n)$. Since u_i is continuous, $\bar{u}_i(\sigma_i, \sigma_{-i}) \geqslant \bar{u}_i(\sigma'_i, \sigma_{-i})$ by the weak convergence of σ^n and $\tilde{\sigma}_i^n$ to σ and σ'_i, respectively.[32] Thus, σ is a Nash equilibrium of $\mathcal{G}$ as desired. □

Lemma S5. *If S_i is a complete pseudo lattice, so is Σ_i^μ.*

Proof. Thanks to Lemma 1 and Corollary 1, it suffices to show that Σ_i^μ is a compact set with the largest and smallest elements and that the stochastic dominance order on Σ_i^μ is closed under the weak topology on Σ_i^μ.

To first show the compactness, we prove that Σ_i^μ is closed. Consider any sequence $(\sigma_i^n)_n$ in Σ_i^μ so that $\sigma_i^n(S') \geqslant \mu_i(S'), \forall n, \forall S' \in \mathscr{S}_i$. Letting $\succeq'$ denote the weak limit of the sequence (which must exist since Σ_i is compact under the weak topology), we must show $\sigma'_i(S') \geqslant \mu_i(S'), \forall S' \in \mathscr{S}_i$. To show this, observe first that for any closed set $\tilde{S} \subset S_i$,

$$\sigma'_i(\tilde{S}) \geqslant \limsup_n \sigma_i^n(\tilde{S}) \geqslant \mu_i(\tilde{S}) \tag{26}$$

(by the weak convergence of $(\sigma_i^n)_n$ to σ'_i and Portmanteau theorem). Observe next that every Borel measure σ'_i on Polish space S_i is regular so that for any $S' \in \mathscr{S}_i$,

$$\sigma'_i(S') = \sup\{\sigma'_i(\tilde{S}) : \tilde{S} \subset S' \text{ and } \tilde{S} \text{ is compact}\}. \tag{27}$$

Since every compact set $\tilde{S}$ is closed, we have $\sigma'_i(\tilde{S}) \geqslant \mu_i(\tilde{S})$ by (26), which implies by (27) that for any $S' \subset \mathscr{S}_i$, $\sigma'_i(S') \geqslant \mu_i(S')$.

The existence of smallest point $\underline{\sigma}_i^\mu$ is proved by construction as follows: letting $\underline{s}_i$ denote the smallest element of S_i (which exists since S_i is a complete pseudo lattice),

$$\underline{\sigma}_i^\mu(S'_i) := \begin{cases} 1 - \mu_i(S_i \backslash S'_i) & \text{if } \underline{s}_i \in S'_i \\ \mu_i(S'_i) & \text{otherwise.} \end{cases}$$

To show that $\sigma_i \geqslant^{sd} \underline{\sigma}_i^\mu$ for any $\sigma_i \in \Sigma_i^\mu$, consider any upward closed set $S'_i \in \mathscr{S}_i$. If $\underline{s}_i \in S'_i$, then $S'_i = S_i$ and thus $\underline{\sigma}_i^\mu(S'_i) = 1 = \sigma_i(S'_i)$. Otherwise, $\underline{\sigma}_i^\mu(S'_i) = \mu_i(S'_i) \leqslant \sigma_i(S'_i)$, as desired. The largest point obtains analoguously.

[32] Note that since Σ_i is endowed with the weak topology, $\bar{u}_i(\cdot)$ remains continuous.

Lastly, the closedness of the stochastic dominance order follows immediately from Fact 1. □