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On strategy-proofness and the salience of single-peakedness in a private goods allotment problem ☆

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ABSTRACT

We consider strategy-proof rules operating on a rich domain of preference profiles in a set up where multiple private goods have to be assigned to a set of agents with entitlements and where preferences display satiation. We show that if the rule is in addition tops-only, same-sided and individually rational with respect to the entitlements, then the preferences in the domain have to satisfy a variant of single-peakedness (referred to as multi-dimensional single-peakedness relative to the entitlements). We also provide a converse of this main finding. It turns out that this domain coincides with the one already identified in a general set up with public goods. We relate the domain of multi-dimensional single-peaked preferences relative to the entitlements to well-known restricted domains with private goods under which non-trivial and strategy-proof rules do exist.

1. Introduction

The notion of single-peakedness has played a fundamental role in the design of rules with appealing incentive properties in various economic and political models with public or private components. We seek to identify the underlying fundamental property of a domain of preferences that admits a non-trivial strategy-proof rule. In particular, we enquire whether single-peakedness is indeed indispensable to the design of such rules in a set up where multiple private goods are assigned to a set of agents with entitlements and where preferences display satiation.

Our methodology postulates preference domains that admit the design of a strategy-proof rule that satisfies some additional axioms,[1] and investigates the implications of this postulate on the structure of preferences in the domain. Our main result is that if

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[1] Specifically, tops-onlyness, same-sidedness and individual rationality with respect to the entitlements. Tops-onlyness requires that the rule should depend only on the profile of the most-preferred assignments of the agents. Same-sidedness requires that the rule rations agents in the same direction in cases of imbalances. Individual rationality with respect to the entitlements requires that the assignments prescribed by the rule should be at least as good as agents' entitlements.

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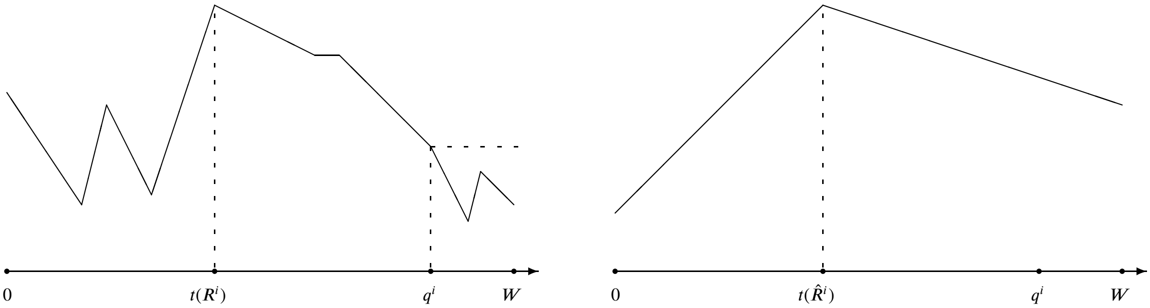

Fig. 1.a Fig. 1.b

Fig. 1. Single-peakedness relative to entitlements and single-peakedness.

the domain satisfies a “richness” condition, then the existence of a rule satisfying these axioms implies that the domain must satisfy a particular weakening of single-peakedness, called multi-dimensional single-peakedness relative to the entitlements. We also show that any domain of preferences that is multi-dimensional single-peaked relative to the entitlements admits a strategy-proof rule satisfying the same axioms.

The domain of multi-dimensional single-peaked preferences relative to the entitlements is known to be salient for the design of non-trivial and “simple” strategy-proof rules in the public good model (see Chatterji and Massó, 2018).[2] In spite of the significant differences between the public good model and the private goods setting, the domain implications of the existence of strategy-proof rules (satisfying other appealing properties) turn out to be identical. This may be seen as evidence to support the view that some form of single-peakedness lies at the heart of possibility results in the literature.

Our model extends two formulations of private goods allocation models already studied in the literature. Barberà et al. (1997) consider the problem of allocating a fixed amount W of a perfectly divisible private good among n potentially satiated agents, where each agent i has the entitlement q^i.[3] This requires that if i asks for q^i she must receive it. Individual rationality with respect to q^i would therefore require each agent to always receive an assignment at least as preferred as q^i. Sprumont (1991) studies the model without entitlements, and without explicitly requiring individual rationality.[4] He characterizes the uniform rule as the unique one satisfying strategy-proofness, efficiency and anonymity. His axioms, in particular anonymity, effectively guarantee the equal division to each agent i, converting $\frac{W}{n}$ to i's entitlement q^i, where n is the number of agents. These models assume agents have single-peaked preferences on the set of individual assignments $[0, W]$ and characterize specific families of rules that are strategy-proof.[5]

Private good allocation models, as studied by Sprumont (1991) and Barberà et al. (1997), among many others, draw inspiration from Benassy's (1982) general equilibrium model with fixed prices, where original quasi-concave preferences induce single-peaked preferences on the individual shares of the good. To address imbalances between demands and supplies in this context, the uniform rule is introduced. Therefore, it is both natural and compelling to expand these one-good models to encompass multiple goods and incorporate entitlements associated with each good. This extension would enable the application of the model to settings akin to Benassy's original model.[6]

In contrast to Sprumont (1991) and Barberà et al. (1997), rather than assuming single-peakedness from the outset, we postulate in a multi-dimensional version of these models[7] that the domain satisfies a “richness” condition and admits a strategy-proof rule that is tops-only, same-sided and individually rational with respect to $q = (q^1, \dots, q^n)$. We assume richness because in its absence strategy-proofness may be ineffectual. We show that the domain possesses a weak version of the single-peaked structure and observe that this structure suffices for the design of a strategy-proof rule with these properties. Multi-dimensional single-peakedness relative to the entitlement is a generalization of the preference R^i depicted in Fig. 1.a to $\mathbb{R}^m_+$ using the L_1-norm (*i.e.*, $\| x \|_{L_1} = \sum_{\ell=1}^m | x_\ell |$ for each $x \in \mathbb{R}^m$), where m is the number of goods to be allotted (see Section 2 for details). One-dimensional single-peakedness relative to q^i requires (i) preferences between $t(R^i)$, the top assignment of R^i, and q^i are declining in the usual sense of single-peakedness, and (ii) assignments “beyond” q^i are less preferred to q^i. Fig. 1.b depicts a single-peaked preference $\hat{R}^i$ relative to the linear order $<$ on the set of positive real numbers, where q^i does not play any role in the definition of the domain restriction. These two preferences

[2] Chatterji and Massó (2018) refer to preferences in this domain as semilattice single-peaked. See Subsection 5.1 for a discussion of their approach in the context of a public good, and its connection with our approach here in the context of private goods.

[3] This is a particular version of their general model without entitlements. Klaus et al. (1998) conforms precisely to this description.

[4] This is also the general set up in Barberà et al. (1997).

[5] For instance, Barberà et al. (1997) are also interested in situations where anonymity is not a reasonable requirement, and consequently the uniform allocation rule is not appropriate. Agents may have a wide range of priorities, seniorities or rights (different to entitlements) that the rule ought to respect, at least partially. They characterize the class of all strategy-proof, efficient and replacement monotonic rules as the family of sequential allotment rules (in Subsection 3.2 we describe in detail one rule within this family).

[6] See Example 1 in Morimoto et al. (2013) which clearly explains how in a model with money the induced preferences over commodities satisfies multi-dimensional single-peakedness under classical preferences.

[7] Considered first by Amorós (2002) and studied by Morimoto et al. (2013) and Cho and Thomson (2017). These contributions are discussed in Subsection 5.2.

in the one-dimensional model already give a hint of the level of weakening that multi-dimensional single-peakedness relative to the entitlement represents with respect to single-peakedness.

The main intuition of why a preference in a domain that admits a rule f satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q^i has to be one-dimensional single-peaked relative to q^i can be obtained by looking at the set of assignments that f can assign to i at R^i, together with some profile of the other agents' preferences. This set in Fig. 1.a is the interval $[t(R^i), q^i]$, provided that the domain of f is sufficiently rich.

We sketch out why the shape of R^i has to be as in Fig. 1.a. First, assume that $t(R^i) < x^i < y^i \le q^i$. Then, there exists a profile of the other agents' preferences at which, together with R^i, f selects x^i while f selects y^i if i submits any preference with top at y^i. Strategy-proofness implies that $x^i \, R^i \, y^i$, and this argument applies also to single-peaked domains. Second, assume that $q^i < x^i \le W$. Then, there exists a profile of the other agents' preferences (each with assignment $\frac{W-q^i}{n-1}$ as top) at which, together with R^i, f selects q^i while f selects x^i if i submits any preference with top at x^i. Strategy-proofness implies that $q^i \, R^i \, x^i$, and no restriction can be obtained on the preference ordering between any pair of i's assignments above q^i. Third, assume $x^i < t(R^i)$. Then, $t(R^i) P^i x^i$ holds trivially, and no restriction can be obtained on the preference ordering between any pair of i's assignments below $t(R^i)$. Our Theorem 1 provides a precise formulation of this intuition in a setting with potentially many private goods.

Moulin (1980) considers a public good model where the level of the public good has to belong to $[0, W]$. A two-agents, anonymous, efficient and strategy-proof rule can be defined by selecting a fixed ballot at some level $q^i \in [0, W]$; the rule selects at every profile of preferences the median of the set of two tops and q^i. Chatterji and Massó (2018) show that, in this simple setting, the preference restriction that is implied by the strategy-proofness of this median voter rule is exactly the one-dimensional version of single-peakedness relative to the entitlement q^i displayed in Fig. 1.a above.

Specifically, Chatterji and Massó (2018) consider the general version of the public good problem with no structure on the set of alternatives and show that the same notion of multi-dimensional single-peakedness relative to the entitlement emerges as a consequence of strategy-proofness along with tops-onlyness, unanimity and anonymity, provided that the domain is rich (they refer to this preference restriction as semilattice single-peakedness).[8] There are related papers that identify, for the public good problem, necessary features of any domain that admits rules satisfying strategy-proofness in combination with some other desirable properties.[9] The methodology in all these earlier papers strongly relies on the finiteness of the set of alternatives, on strict preferences, and on the assumption that the number of agents is even.[10]

The main contribution of this paper is to highlight the role of multi-dimensional single-peakedness relative to the entitlements as the fundamental underlying structure of preferences in a domain that permits the design of strategy-proof rules in the disparate private and public goods models.

The rest of the paper is organized as follows. Section 2 introduces basic definitions and notation, the desirable properties of rules and properties of preferences and domains. Section 3 contains the results of the paper. In Section 4 we present some corollaries of our main result. In Section 5 we discuss our axioms and richness condition and we relate our results to existing results in the literature on domain restrictions for strategy-proof rules for private goods. Section 6 concludes. An Appendix collects the proof of a complementary result omitted in the main text.

2. Preliminaries

2.1. Notation and basic definitions

Our general setup closely follows Morimoto et al. (2013). Let $N = \{1, \ldots, n\}$ be the finite set of *agents*, with $n \ge 2$, and let $M = \{1, \ldots, m\}$ be the set of perfectly divisible goods, with $m \ge 1$. For each $\ell \in M$, let $W_\ell \in \mathbb{R}_{++}$ be the strictly positive amount of good ℓ that has to be allotted among agents in N, and let $W = (W_1, \ldots, W_m) \in \mathbb{R}^M_{++}$. For each $\ell \in M$, let $X_\ell = [0, W_\ell]$ and $X_{-\ell} = \prod_{\ell' \in M \setminus \{\ell\}} X_{\ell'}$. For each agent $i \in N$, let

$$X = \prod_{\ell \in M} X_\ell = \{x^i = (x^i_1, \ldots, x^i_m) \in \mathbb{R}^M_+ \mid 0 \le x^i_\ell \le W_\ell \text{ for each } \ell \in M\}$$

be agent i's set of possible *assignments*, which is the same for everyone. To emphasize agent i's assignment, we often write $x = (x^i, x^{-i}) \in X^N$ and given $\bar{r} \in \mathbb{R}_+$ we write $(\bar{r})^{j \ne i}$ as the $n-1$ dimensional vector with all components different to i equal to $\bar{r}$.

Let

$$Z = \{x = (x^1, \ldots, x^n) \in X^N \mid \sum_{i \in N} x^i = W\}$$

be the set of *allotments*.

[8] It is known that only for $n = 2$, the private good case can be formulated as a public good case: this is not the case for $n \ge 3$ and multiple goods.

[9] Chatterji et al. (2013) provide a formulation of such a converse statement. See Chatterji and Massó (2018) and Barberà et al. (2020) for a detailed discussion of this literature.

[10] Bonifacio et al. (2023) contribute to these papers by using a formulation that does not require the set of alternatives to be finite, preferences to be strict, or the number of agents to be even. Furthermore, this formulation does not necessarily exclude those rules that are not equivalent to two-agents rules, unlike the formulations in Chatterji et al. (2013) and Chatterji and Massó (2018), which implicitly do so. For a precise description of this non-equivalence, see the discussion and example in Section 6 of Bonifacio et al. (2023).

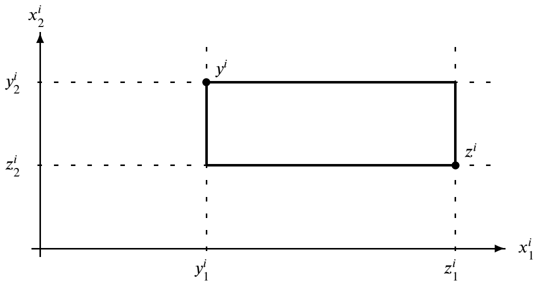

Fig. 2. A minimal box.

Given $y^i, z^i \in X$, we define the *minimal box* $MB(y^i, z^i)$ of y^i and z^i as the set of assignments for agent i that lie between y^i and z^i in the L_1-norm, denoted by $\|\cdot\|_{L_1}$, where, as already said, for any $x \in \mathbb{R}^M$, $\|x\|_{L_1} = \sum_{\ell \in M} |x_\ell|$; namely,

$$MB(y^i, z^i) = \{x^i \in X \mid \|y^i - z^i\|_{L_1} = \|y^i - x^i\|_{L_1} + \|x^i - z^i\|_{L_1}\}.$$

Fig. 2 depicts a minimal box $MB(y^i, z^i)$ of y^i and z^i for the case of two goods ($m = 2$).

Remark 1. The minimal box between any pair $y^i, z^i \in X$ can be written as a Cartesian product of intervals; namely,

$$MB(y^i, z^i) = \prod_{\ell \in M} [\min\{y^i_\ell, z^i_\ell\}, \max\{y^i_\ell, z^i_\ell\}]. \tag{1}$$

Each agent $i \in N$ has a preference $R^i \in \mathcal{D}^i \subseteq \mathcal{R}$ over X, where $\mathcal{D}^i$ is a subset of $\mathcal{R}$, the set of all complete and transitive binary relations over X. We impose neither the continuity nor the monotonicity of preferences. Note that different agents may have different sets of preferences. For any $x^i, y^i \in X$, $x^i \, R^i \, y^i$ means that agent i considers assignment x^i to be at least as preferred as assignment y^i. Let P^i and I^i denote the strict and indifference relations induced by R^i over X, respectively.

We refer to the set $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ as a *domain* of preferences, and often denote it as $\mathcal{D}$. A *profile* $R = (R^1, \ldots, R^n) \in \mathcal{D}$ is a n-tuple of preferences, one for each agent. To emphasize R^i in profile R we often write $R = (R^i, R^{-i})$.

Before proceeding, we specify two basic properties that underlie all the individual preference domains that we consider in this paper. First, each preference R^i has a unique top ranked assignment which is not necessarily equal to W. We view our research as contributing to a research program that recognizes satiation as an important and inherent part of preferences that deserves more thorough treatment in equilibrium theory, in line with Mas-Colell (1992) who suggested, "It may be worthwhile, however, to explore how far we can go in developing an equilibrium theory with satiation." Specifically, our model delves into a (dis)equilibrium assignment theory that integrates the *incentives* of satiated agents, with a particular focus on strategy-proofness. In this context, the assumption of satiation becomes indispensable, providing the foundational framework upon which our analysis of tops-only rules (outlined in Subsection 2.2 below) is built.[11] We assume furthermore a form of "minimal richness" which requires that for every assignment of an agent, the individual preference domain of that agent includes a preference whose top is at this assignment. To sum up, we study minimally rich preferences with satiation, which rules out that all preferences are monotonic, an assumption typically used to analyze strategy-proofness in exchange economies.[12] We refer to a domain satisfying these two properties as an *Admissible Domain*.

Admissible Domain: The set of preferences $\mathcal{D}^i$ for $i \in N$ is *admissible* if it satisfies (i) for every preference $R^i \in \mathcal{D}^i$, there is a unique top ranked assignment $t(R^i) \in X$, namely, $t(R^i) \, P^i \, x^i$ for all $x^i \in X \setminus \{t(R^i)\}$, and (ii) for every assignment $x^i \in X$, there exists a preference $R^i \in \mathcal{D}^i$ such that $t(R^i) = x^i$. The domain $\mathcal{D} \equiv \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ is *admissible* if $\mathcal{D}^i$ is admissible for each i.

For $x^i \in X$, we will denote by $R^i_{x^i}$ a generic preference $R^i_{x^i} \in \mathcal{D}^i$ with $t(R^i_{x^i}) = x^i$.

2.2. Properties of rules

Since individual preferences are private information, they must be elicited through a rule. A *rule* is a mapping $f : \mathcal{D} \to Z$ that assigns to every profile $R \in \mathcal{D}$ an allotment $f(R) \in Z$.

We are interested in rules that induce agents to report preferences truthfully. A rule $f : \mathcal{D} \to Z$ is *strategy-proof* if for all $i \in N$, all $R \in \mathcal{D}$, and all $\widehat{R}^i \in \mathcal{D}^i$,

$$f^i(R^i, R^{-i}) \, R^i \, f^i(\widehat{R}^i, R^{-i}).$$

[11] The extension to preferences with multiple top ranked assignments, which we do not pursue here, is interesting but difficult and challenging even for the setting with only one public good; see, for instance, Moulin (1984) and Berga (1998).

[12] We take up this literature in Subsection 5.2.

A rule $f : \mathcal{D} \to Z$ is *tops-only* if for all $R, \widehat{R} \in \mathcal{D}$ such that $t(R^i) = t(\widehat{R}^i)$ for all $i \in N$, $f(R) = f(\widehat{R})$. Hence, a tops-only rule $f : \mathcal{D} \to Z$ (on an admissible domain $\mathcal{D}$) can be written as a function $f : X^N \to Z$. Accordingly, and since all the rules we study are tops-only, we will often use the notation $f(t^1, \ldots, t^n)$ interchangeably with $f(R^1, \ldots, R^n)$, where $t^i = t(R^i)$ for all $i \in N$.

Let $q \in Z$ be a feasible allotment, which we refer to as the *entitlements*. A rule $f : \mathcal{D} \to Z$ satisfies *individual rationality with respect to q* if for all $R \in \mathcal{D}$ and all $i \in N$, $f^i(R)\, R^i\, q^i$.[13]

A rule $f : \mathcal{D} \to Z$ is *same-sided* if for all $R \in \mathcal{D}$ and all $\ell \in M$,
(i) if $\sum_{j\in N} t_\ell(R^j) \geq W_\ell$, then $f^i_\ell(R) \leq t_\ell(R^i)$ for each $i \in N$, and
(ii) if $\sum_{j\in N} t_\ell(R^j) \leq W_\ell$, then $f^i_\ell(R) \geq t_\ell(R^i)$ for each $i \in N$.

A rule $f : \mathcal{D} \to Z$ is *efficient* if for all $R \in \mathcal{D}$, the allotment $f(R)$ is Pareto efficient; namely, there exists no $y \in Z$ such that $y^i\, R^i\, f^i(R)$ for all $i \in N$ and $y^j\, P^j\, f^j(R)$ for at least one $j \in N$.

Among our axioms, strategy-proofness and individual rationality are standard properties. We now provide some justification for tops-onlyness and same-sidedness.

One natural argument for the tops-only property is that many prominent rules in the literature (for example, the uniform rule and sequential allotment rules) are tops-only rules. Our model studies multiple private goods where the information contained in a preference is more complex than in the case of a single good, making the general analysis of rules that incorporate very detailed preference information intractable, and therefore, a natural starting point in such a model would be to focus on tops-only rules. A prominent motivation for our model is Benassy's (1982) model with fixed prices where, as mentioned in Section 1, demands and supplies are generally imbalanced. Here, the allocation of private goods relies solely on the information of demands and supplies. In our model, top-ranked assignments are the counterparts of demands and supplies, making tops-onlyness a natural requirement for focusing on informationally tractable rules.

We now turn to the axiom of same-sidedness. In models where demands and supplies may be imbalanced, goods are allocated using the *short side principle*, where only the short side of demand and supply is satisfied while the other side is rationed. In our model, same-sidedness corresponds to the trait of the short side principle that requires that all agents on the long side are rationed in the same direction, and thus constitutes a natural requirement as well. Additionally, same-sidedness can be justified as a condition implying unanimity. Next, we observe that for the domain of multi-dimensional single-peaked preferences (defined in Subsection 2.3 below), strategy-proofness, individual rationality with respect to q, and efficiency are incompatible.[14] Thus, it becomes necessary to sacrifice one of the three conditions. Although efficiency and same-sidedness are independent properties for general domains, for multi-dimensional single-peaked preferences, same-sidedness is weaker than efficiency. Moreover, in the domain of single-peaked preferences in the one-dimensional case, same-sidedness and efficiency are equivalent. Therefore, we opt to relinquish efficiency and adopt same-sidedness as an alternative efficiency condition. Additionally, we can justify the property of same-sidedness as a fairness axiom: if there is scarcity (or abundance) of a good at a profile, agents should be rationed in the same direction, all receiving an assignment smaller (or larger) than their tops. However, this interpretation relies on a restriction on preferences, top-separability, which we introduce and discuss in the next section.[15]

2.3. Properties of preferences and domains

We assume that preferences satisfy the requirement of *top-separability*.

TOP-SEPARABILITY: A preference $R^i \in \mathcal{D}^i$ is *top-separable* if, for every $x^i \in X \setminus \{t^i\}$ and every $\ell \in M$, we have that $(t^i_\ell, x^i_{-\ell})\, P^i\, (x^i_\ell, x^i_{-\ell})$ holds. A set of preferences $\mathcal{D}^i$ is *top-separable* if each $R^i \in \mathcal{D}^i$ is top-separable. The domain $\mathcal{D} = \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ is *top-separable* if $\mathcal{D}^i$ is top-separable for every i.

A preference $R^i \in \mathcal{D}^i$ is *separable* if for every pair $x^i, y^i \in X$ and for every $\ell \in M$, it holds that $(x^i_\ell, x^i_{-\ell})\, R^i\, (y^i_\ell, x^i_{-\ell})$ if and only if $(x^i_\ell, y^i_{-\ell})\, R^i\, (y^i_\ell, y^i_{-\ell})$.

Fig. 3.a illustrates the assignment t^i and the two assignments $(t^i_\ell, x^i_{-\ell})$ and $x^i = (x^i_\ell, x^i_{-\ell})$ involved in the top-separability condition. Fig. 3.b illustrates the four assignments $(x^i_\ell, x^i_{-\ell})$, $(y^i_\ell, x^i_{-\ell})$, $(x^i_\ell, y^i_{-\ell})$ and $(y^i_\ell, y^i_{-\ell})$ involved in the separability condition. We note that top-separability is a considerably weaker requirement than separability. Consequently, the domain of separable preferences is *strictly contained* within the domain of top-separable preferences.

Remark 2. We will restrict attention to top-separable preferences in the paper. This condition, first introduced by Le Breton and Weymark (1999), has received extensive attention in the literature on multi-dimensional models. For instance, it is both necessary

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[13] Individual rationality with respect to the egalitarian entitlements plays a similar role as anonymity. Moreover, in Sprumont (1991)'s characterization of the uniform rule, anonymity can be replaced by envy-freeness. Hence, individual rationality with respect to the egalitarian entitlements could echo some form of envy-freeness. Recall that envy-freeness is a requirement that each agent finds its assignment as good as anyone else's assignment.

[14] See Serizawa (2002) for a proof of this result. He shows that for the domain of monotonic, strictly convex and homothetic preferences, no rule satisfies strategy-proofness, individual rationality with respect to q, and efficiency. This conclusion extends to the domain of multi-dimensional single-peaked preferences since his domain is included in it.

[15] Note that Cho and Thomson (2017) assume same-sidedness even in the absence of top-separability. They demonstrate that top-separability follows from same-sidedness, strategy-proofness, and envy-freeness (refer to their Theorems 3 and 4). For the case of $n = 2$, envy-freeness in their results is substituted by *equal treatment of equals in physical terms*, implying that two agents with identical preferences should receive the same assignments.

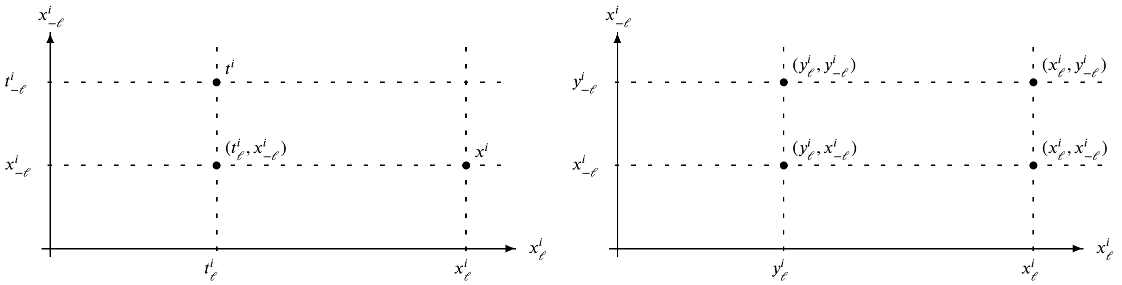

Fig. 3.a Fig. 3.b

Fig. 3. Top separable and separable preferences.

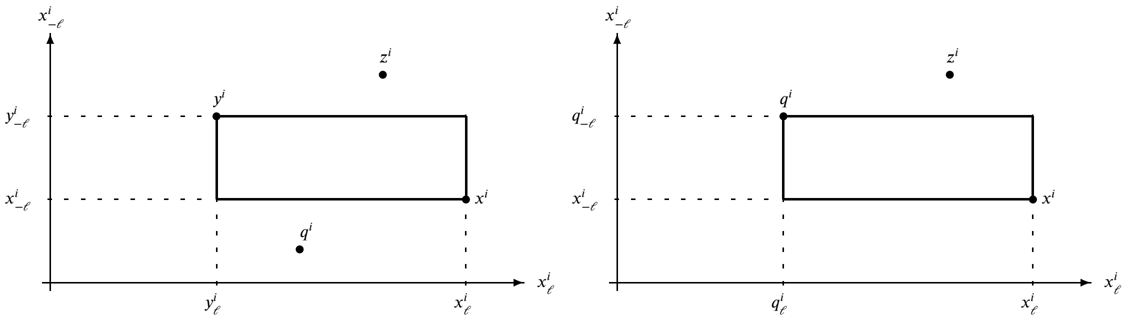

Fig. 4.a Fig. 4.b

Fig. 4. Richness relative to q^i.

and sufficient for the strategy-proofness of rules that take the form of generalized dictatorships in the voting model, as demonstrated by Chatterji and Zeng (2019). In our framework, we argue that this preference restriction is appropriate since we impose the axioms of tops-onlyness and same-sidedness on rules.[16] The tops-only condition says that the planner can only utilize the top assignment of each agent's preferences $t(R^i)$, when selecting the allotment at R. On the other hand, same-sidedness operates on each component $\ell \in M$ by summing $t_\ell(R^i)$ (representing the amount of that component specified in each agent's top-ranked assignment) across all agents. This aggregate is then compared to the total entitlement of that component, ensuring that all agents are rationed in the same manner out of a concern for fairness. The legitimacy of this interpretation is based on the presumption that $t_\ell(R^i)$ is indeed the most desirable assignment of component ℓ to agent i, a guarantee provided by top-separability. Top-separability however does not play any (explicit) role in the proof of Theorem 1, which therefore remains valid even if preferences are not assumed to be top-separable.

We further assume that each $\mathcal{D}^i$, besides being admissible and top-separable, satisfies the condition of *richness relative to* q^i.

Richness Relative to q^i: A set of preferences $\mathcal{D}^i$ is *rich relative to* $q^i \in X$ if, for all $x^i, y^i, z^i \in X$ such that $x^i \neq y^i$, $q^i \notin int\,MB(x^i, y^i)$, and $z^i \notin MB(x^i, y^i)$, there exist $R^i_{x^i}, R^i_{y^i} \in \mathcal{D}^i$ such that $y^i\, P^i_{x^i}\, z^i$ and $x^i\, P^i_{y^i}\, z^i$. A domain $\mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ is *rich relative to* $q = (q^1, \ldots, q^n) \in Z$ if, for each $i \in N$, $\mathcal{D}^i$ is rich relative to q^i.

The property of richness relative to q^i in this Euclidean setting corresponds to the richness condition introduced by Chatterji and Massó (2018) for semilattices in the context of a public good. However, the present formulation of richness relative to $q^i \in X$ is weaker because the existence of the two preferences $R^i_{x^i}, R^i_{y^i} \in \mathcal{D}^i$ is only required if $q^i \notin int\,MB(x^i, y^i)$. Fig. 4.a depicts assignments $x^i, y^i, z^i, q^i \in X$ that satisfy the hypothesis of the property of richness relative to q^i, while Fig. 4.b illustrates the hypothesis when $y^i = q^i$.[17]

A preference $R^i \in \mathcal{D}^i$ is *Euclidean* if, for every pair $x^i, y^i \in X$, $x^i\, R^i\, y^i$ if and only if $\| t(R^i) - x^i \| \leq \| t(R^i) - y^i \|$, where $\| \cdot \|$ denotes the Euclidean norm. Let $\mathcal{E}$ denote the set of all Euclidean preferences. It is evident that all Euclidean preferences are both separable and top-separable.

A preference $R^i \in \mathcal{D}^i$ is *Elliptical* if, for each $\ell \in M$, there exists $a_\ell > 0$ such that, for every pair $x^i, y^i \in X$, $x^i\, R^i\, y^i$ if and only if $\sum_{\ell \in M} a_\ell \cdot (t_\ell(R^i) - x^i_\ell)^2 \leq \sum_{\ell \in M} a_\ell \cdot (t_\ell(R^i) - y^i_\ell)^2$. Let $\mathcal{E}l$ denote the set of all elliptical preferences. It is also evident that all elliptical preferences are both separable and top-separable.

[16] We do believe that preferences are a primitive. However, we want to take a step back in the following sense. Our approach is to identify preference restrictions that are compatible with the design of rules that are strategy-proof and obey certain axioms. In this indirect sense, the question can be rephrased as what preferences restrictions are implied by the existence of a rule that obeys certain axioms, and hence in a sense *justified* by axioms on a rule.

[17] See Subsection 5.1 for a general discussion of the connection between the two approaches.

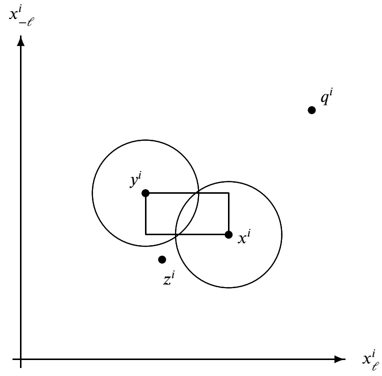

Fig. 5. Euclidean preferences violate richness.

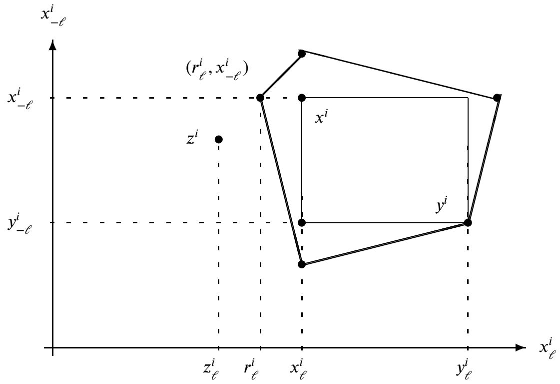

Fig. 6. Multidimensional single peaked preferences are rich relative to q^i.

The set of Euclidean preferences $\mathcal{E}$ is top-separable but does not satisfy this richness property. This is because for each assignment $x^i \in X$, there exists a unique Euclidean preference with x^i as its top. Fig. 5 illustrates this by depicting two circles, centered at x^i and y^i, representing indifference classes for the two unique Euclidean preferences $R^i_{x^i}$ and $R^i_{y^i}$ with tops at x^i and y^i respectively. Consequently, $z^i\, P^i_{x^i}\, y^i$ and $z^i\, P^i_{y^i}\, x^i$ necessarily hold, as z^i is closer to x^i than y^i is to x^i, and z^i is closer to y^i than x^i is to y^i.

The set of elliptical preference $\mathcal{E}l$ is top-separable and separable, yet it is not rich relative to any entitlement. This is despite the existence of infinitely many elliptic preferences with x^i as their top assignment for each x^i. To illustrate that $\mathcal{E}l$ is not rich relative to any q^i, consider the case where $m = 2$, $x^i = (5,1)$, $y^i = (3,3)$, and $z^i = (6,2)$, and observe that $z^i \notin MB(x^i, y^i) = [\min\{x^i_1, y^i_1\}, \max\{x^i_1, y^i_1\}] \times [\min\{x^i_2, y^i_2\}, \max\{x^i_2, y^i_2\}] = [3,5] \times [1,3]$. Since $(x^i_\ell - y^i_\ell)^2 > (x^i_\ell - z^i_\ell)^2$ for each $\ell \in \{1,2\}$, it follows that for any $a = (a_1, a_2) \in \mathbb{R}^2_{++}$, $\sum_{\ell=1}^{2} a_\ell \cdot (x^i_\ell - z^i_\ell)^2 < \sum_\ell a_\ell \cdot (x^i_\ell - y^i_\ell)^2$. Consequently, $z^i\, P^i_{x^i}\, y^i$ holds for all $R^i_{x^i} \in \mathcal{E}l$, indicating that the set of elliptical preferences is not rich relative to any entitlement.

A preference $R^i \in \mathcal{D}^i$ is *multi-dimensional single-peaked* if, for all $x^i, y^i \in X$ such that $x^i \in MB(y^i, t^i)$, $x^i\, R^i\, y^i$. The set of all multi-dimensional single-peaked preferences is denoted by $\mathcal{MSP}$. It is easy to see that all multi-dimensional single-peaked preferences are top-separable and separable. To see that the domain $\mathcal{MSP}$ is rich relative to any entitlement, let $x^i, y^i, z^i \in X$ be three assignments that satisfy the hypothesis of richness, as illustrated in Fig. 6, and let $q^i \notin int MB(x^i, y^i)$ be an arbitrary entitlement, not depicted in the figure. Specifically, $z^i \notin MB(x^i, y^i)$. Consequently, there exists a dimension $\ell \in M$ such that $z_\ell \notin [\min\{x^i_\ell, y^i_\ell\}, \max\{x^i_\ell, y^i_\ell\}]$. Without loss of generality, assume $z_\ell \notin [x^i_\ell, y^i_\ell]$. It is always possible to find $z^i_\ell < r^i_\ell < x^i_\ell$ such that $(r^i_\ell, x^i_{-\ell})$ and y^i are indifferent according to a preference $R^i_{x^i} \in \mathcal{MSP}$. Furthermore, z^i does not belong to the upper contour set of $R^i_{x^i}$ at y^i, and the set enclosed by the bold lines is included in this upper contour set. Therefore, there exists $R^i_{x^i} \in \mathcal{MSP}$ such that $y^i\, P^i_{x^i}\, z^i$. Note that this argument is independent of q^i, as long as $q^i \notin int MB(x^i, y^i)$. Thus, the set $\mathcal{MSP}$ is rich relative to any q^i.

To define the key concept of this paper, we need an additional piece of notation. Given assignments $x^i, q^i, t^i \in X$ such that $x^i \notin MB(q^i, t^i)$, we denote the assignment $\widetilde{x}^i$ in $MB(q^i, t^i)$ that is closest to x^i as $\widetilde{x}^i = \arg\min_{r^i \in MB(q^i,t^i)} \| x^i - r^i \|_{L_1}$. Observe that $\widetilde{x}^i$ is unique. Moreover, for every $\ell \in M$, the following conditions hold:

If $\min\{q^i_\ell, t^i_\ell\} \le x^i_\ell \le \max\{q^i_\ell, t^i_\ell\}$, then $\widetilde{x}^i_\ell = x^i_\ell$.
If $x^i_\ell < \min\{q^i_\ell, t^i_\ell\}$, then $\widetilde{x}^i_\ell = \min\{q^i_\ell, t^i_\ell\}$.
If $\max\{q^i_\ell, t^i_\ell\} < x^i_\ell$, then $\widetilde{x}^i_\ell = \max\{q^i_\ell, t^i_\ell\}$.

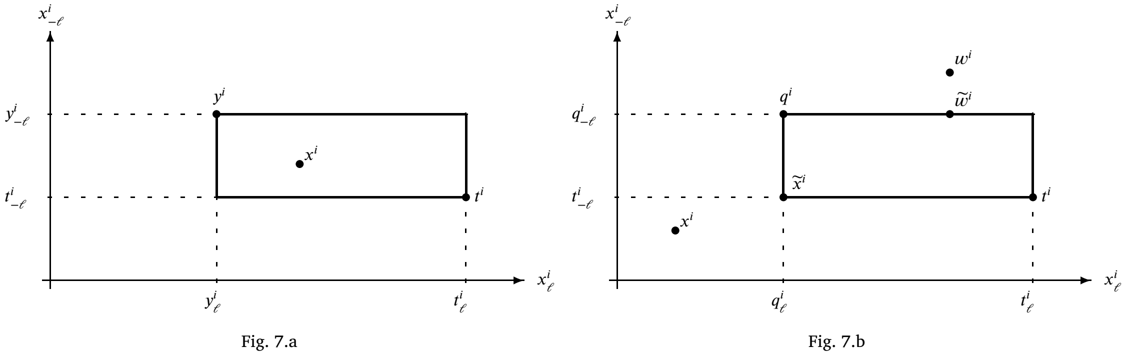

Fig. 7. Identification of $\widetilde{x}^i$ and $\widetilde{w}^i$.

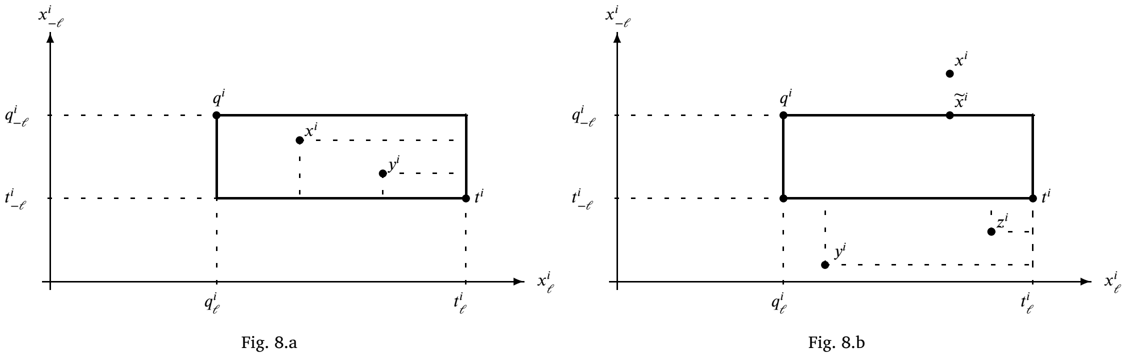

Fig. 8. Multidimensional single-peaked preference relative to the entitlement.

Fig. 7.a depicts a situation where $x^i\ R^i\ y^i$ according to multi-dimensional single-peakedness, while Fig. 7.b illustrates geometrically the corresponding $\widetilde{x}^i$ and $\widetilde{w}^i$ for x^i and w^i.

We now introduce the key concept of multi-dimensional single-peakedness relative to an entitlement.[18]

Multi-dimensional Single-peakedness Relative to the Entitlement: A preference $R^i \in \mathcal{D}^i$ is *multi-dimensional single-peaked relative to* $q^i \in X$ if it satisfies the following two conditions:

(MSP.1) For all $x^i, y^i \in X$ such that $x^i \in MB(q^i, t^i)$ and $y^i \in MB(x^i, t^i)$, we have $y^i\ R^i\ x^i$.

(MSP.2) For all $x^i \notin MB(q^i, t^i)$, $\widetilde{x}^i\ R^i\ x^i$.[19]

Fig. 8.a illustrates the hypotheses of property (MSP.1), under which $y^i\ R^i\ x^i$ has to hold. Fig. 8.b illustrates the hypothesis of property (MSP.2), under which $\widetilde{x}^i\ R^i\ x^i$ has to hold.

Given $q^i \in X$, denote the set of all multi-dimensional single-peaked preferences relative to q^i by $\mathcal{MSP}^*(q^i)$. Now, for $q \in Z$, define $\mathcal{MSP}^*(q) = \mathcal{MSP}^*(q^1) \times \cdots \times \mathcal{MSP}^*(q^n)$. Note that $\mathcal{MSP}^*(q^i)$ is larger than the set $\mathcal{MSP}$ because there are preferences $R^i \in \mathcal{MSP}^*(q^i)$ and $y^i, z^i \in X$ such that $t(R^i) = t^i$ but $z^i \in MB(y^i, t^i)$ and $y^i\ P^i\ z^i$ (see Fig. 8.b).

The set $\mathcal{MSP}^*(q^i)$ might contain preferences that are neither separable nor top-separable. Fig. 9 illustrates this by showing the possibility that there exists a preference $R^i \in \mathcal{MSP}^*(q^i)$ with $t(R^i) = t^i$ where (i) $(y^i_\ell, x^i_{-\ell})\ P^i\ (t^i_\ell, x^i_{-\ell})$ holds, indicating that R^i is not top-separable, and (ii) $(x^i_\ell, x^i_{-\ell})\ P^i\ (y^i_\ell, x^i_{-\ell})$ and $(y^i_\ell, t^i_{-\ell})\ P^i\ (x^i_\ell, t^i_{-\ell})$ hold, indicating that R^i is not separable. Neither condition (MSP.1) nor (MSP.2) excludes these two possibilities.

We denote the set of all top-separable and multi-dimensional single-peaked preferences relative to q^i by $\mathcal{MSP}(q^i)$. Since $\mathcal{MSP} \subsetneq \mathcal{MSP}(q^i)$ for any q^i, and the set $\mathcal{MSP}$ is rich relative to any q^i, as argued using Fig. 6, it follows that the set $\mathcal{MSP}(q^i)$ is also rich relative to any q^i.

A preference $R^i \in \mathcal{D}^i$ is *convex* (*strictly convex*) if, for each pair $x^i, y^i \in X$, and any $\lambda \in [0,1]$ (any $\lambda \in (0,1)$), whenever $x^i\ R^i\ y^i$, $(\lambda \cdot x^i + (1-\lambda) \cdot y^i)\ R^i\ y^i$ ($(\lambda \cdot x^i + (1-\lambda) \cdot y^i)\ P^i\ y^i$). We denote the set of all convex, top-separable and multi-dimensional single-peaked preferences as $\mathcal{MSP}^C$, the set of all convex, top-separable and multi-dimensional single-peaked preferences relative to q^i as

[18] This concept was originally formulated by Chatterji and Massó (2018) in the context of a public good by using a semilattice obtained from a given strategy-proof, unanimous, anonymous, and tops-only rule. They referred to this notion as semilattice single-peakedness. In our multi-dimensional Euclidean framework, we do not explicitly rely on a semilattice. Instead, we directly define multi-dimensional single-peakedness. In Subsection 5.1, we will describe how to obtain a semilattice over X in our private goods case, which yields the same domain of semilattice single-peaked preferences (with respect to this semilattice).

[19] Recall that given x^i, the assignment $\widetilde{x}^i$ is the assignment in $MB(q^i, t^i)$ that is closest to x^i.

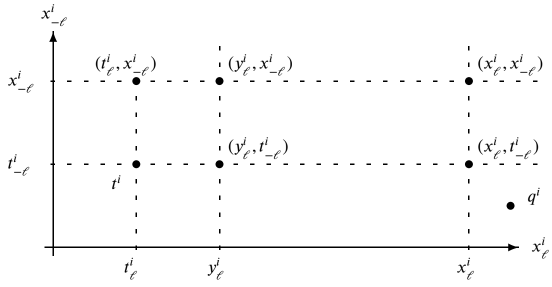

Fig. 9. A preference in $\mathcal{MSP}^*(q^i)$.

$\mathcal{MSP}^C(q^i)$, the set of all strictly convex, top-separable and multi-dimensional single-peaked preferences as $\mathcal{MSP}^{SC}$, and the set of all strictly convex, top-separable and multi-dimensional single-peaked preferences relative to q^i as $\mathcal{MSP}^{SC}(q^i)$.

Concluding and wrapping up this discussion, the following statements and remark apply to each $q^i \in X$.

(1) $\mathcal{MSP}$, $\mathcal{MSP}(q^i)$, $\mathcal{MSP}^C$, $\mathcal{MSP}^C(q^i)$, $\mathcal{MSP}^{SC}$, and $\mathcal{MSP}^{SC}(q^i)$ are rich relative to q^i.

(2) $\mathcal{E} \subsetneq \mathcal{E}l \subsetneq \mathcal{MSP} \subsetneq \mathcal{MSP}(q^i)$ and $\bigcap_{q^i \in X} \mathcal{MSP}(q^i) = \mathcal{MSP}$.

(3) $\mathcal{E}$ and $\mathcal{E}l$ are top-separable and separable, but they are not rich relative to q^i.

Remark 3. As stated earlier, the set of convex, top-separable and multi-dimensional single-peaked preferences relative to any q^i satisfies richness relative to q^i. Consequently, our main result, Theorem 1 is applicable in economic settings where convexity is a natural requirement. For instance, let $\overline{R}^i$ denote a continuous and monotonic preference relation over $\mathbb{R}^M_+ \times \mathbb{R}$, where the first m coordinates denote the consumption levels of the divisible goods and the last one denotes the consumption level of money. Assume that $\overline{R}^i$ is linear with respect to money, and separable and strictly convex with respect to the divisible goods, that is, for each $\ell \in M$, there is a continuous and strictly concave function v^i_ℓ on $\mathbb{R}_+$ such that for any $(x,t) \in \mathbb{R}^M_+ \times \mathbb{R}$ and $(x',t') \in \mathbb{R}^M_+ \times \mathbb{R}$,

$$(x,t)\,\overline{R}^i\,(x',t') \Leftrightarrow \sum_{\ell \in M} v^i_\ell(x_\ell) + t \geq \sum_{\ell \in M} v^i_\ell(x'_\ell) + t',$$

where t and t' denote the consumption levels of money. Let $p \in \mathbb{R}^M_{++}$ denote a fixed price vector. Then, for each such preference relation $\overline{R}^i$, the preference relation R^i is induced as follows: for each $x \in \mathbb{R}^M_+$ and each $x' \in \mathbb{R}^M_+$,

$$x\,R^i\,x' \Leftrightarrow (x,t)\,\overline{R}^i\,(x',t'),$$

where $t = -\sum_{\ell \in M} p_\ell \cdot x_\ell$ and $t' = -\sum_{\ell \in M} p_\ell \cdot x'_\ell$. Then, the class of induced preference relations on $\mathbb{R}^M_+$ satisfies continuity and strict convexity in addition to the conditions of Theorem 1.

3. Results

3.1. The main result

Theorem 1. *Let $\mathcal{D}$ be an admissible and top-separable domain that is rich relative to $q \in Z$ and let $f : \mathcal{D} \to Z$ be a rule satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of multi-dimensional single-peaked preferences relative to q^i.*

The two proofs showing that conditions (MSP.1) and (MSP.2) in the definition of multi-dimensional single-peakedness relative to the entitlement hold use similar arguments. They require, after fixing an arbitrary agent $i \in N$ and entitlement $q^i \in X$, to prove that for any preference $R^i \in \mathcal{D}^i$ and assignments $x^i, y^i \in X$ (as those represented in Fig. 8.a and Fig. 8.b), $y^i\,R^i\,x^i$ holds for (MSP.1) and $\widetilde{x}^i\,R^i\,x^i$ holds for (MSP.2). Fix then $i \in N$ and a triplet $x^i, y^i, q^i \in X$ satisfying the hypothesis in the definition of multi-dimensional single-peakedness relative to the entitlement for R^i, and let $t^i = t(R^i)$. The critical step to show that (MSP.1) holds is to find a vector of tops for the remaining agents t^{-i} with the property that $f^i(t^i, t^{-i}) = y^i$ and $f^i(x^i, t^{-i}) = x^i$ (and to do so, we use that the domain $\mathcal{D}$ is rich relative to q and that f satisfies tops-onlyness, same-sidedness and individual rationality with respect to q); because then, by strategy-proofness, $y^i\,R^i\,x^i$ must hold. The identification of t^{-i} is done component by component. Similarly for the proof that (MSP.2) holds, replacing y^i by $\widetilde{x}^i$ and identifying $\widetilde{t}^{-i}$ instead of t^{-i} in such a way that $f^i(t^i, \widetilde{t}^{-i}) = \widetilde{x}^i$ and $f^i(x^i, \widetilde{t}^{-i}) = x^i$; because then, by strategy-proofness, $\widetilde{x}^i\,R^i\,x^i$ must hold. We now move to the formal and complete proof of Theorem 1. As we already said in Remark 2, it does not use explicitly that the domain is top-separable. We impose top-separability to substantiate and give meaning to the properties of tops-onlyness and same-sidedness.

Proof of Theorem 1. Let R^i be agent i's preference with top $t^i \equiv t(R^i)$ and let q^i be i's entitlement. Let $M_+ \equiv \{\ell \in M \mid q^i_\ell < t^i_\ell\}$, $M_0 \equiv \{\ell \in M \mid q^i_\ell = t^i_\ell\}$ and $M_- \equiv \{\ell \in M \mid q^i_\ell > t^i_\ell\}$.

Proof of (MSP.1). Let $x^i \in MB(q^i, t^i)$ and $y^i \in MB(x^i, t^i)$. We show $y^i\, R^i\, x^i$. By $x^i \in MB(q^i, t^i)$ and $y^i \in MB(x^i, t^i)$, we have:

$$\left.\begin{array}{ll} q^i_\ell \le x^i_\ell \le y^i_\ell \le t^i_\ell & \text{for each } \ell \in M_+ \\ q^i_\ell = x^i_\ell = y^i_\ell = t^i_\ell & \text{for each } \ell \in M_0 \\ q^i_\ell \ge x^i_\ell \ge y^i_\ell \ge t^i_\ell & \text{for each } \ell \in M_- \end{array}\right\}. \tag{2}$$

For each $\ell \in M_+$, let λ_ℓ be such that $W_\ell - y^i_\ell = \sum_{j\neq i} \min\{q^j_\ell, \lambda_\ell\}$,[20] and for each $\ell \in M_-$, let λ_ℓ be such that $W_\ell - y^i_\ell = \sum_{j\neq i} \max\{q^j_\ell, \lambda_\ell\}$.[21] For each $j \neq i$, let R^j be such that

$$\begin{array}{ll} t_\ell(R^j) \equiv t^j_\ell = \min\{q^j_\ell, \lambda_\ell\} & \text{for each } \ell \in M_+ \\ t_\ell(R^j) \equiv t^j_\ell = q^j_\ell & \text{for each } \ell \in M_0 \\ t_\ell(R^j) \equiv t^j_\ell = \max\{q^j_\ell, \lambda_\ell\} & \text{for each } \ell \in M_-. \end{array}$$

Note that since $\lambda_\ell \in [0, \max\{q^j_\ell \mid j \neq i\}]$ for each $\ell \in M_+$, and $\lambda_\ell \in [\min\{q^j_\ell \mid j \neq i\}, W_\ell]$ for each $\ell \in M_-$, $t^j_\ell \in [0, W_\ell]$ for each $j \neq i$ and each $\ell \in M$. By their definitions,

$$\sum_{j\neq i} t^j = W - y^i, \tag{3}$$

and

$$\left.\begin{array}{ll} & \text{for each } j \neq i, \\ q^j_\ell \ge t^j_\ell & \text{for each } \ell \in M_+ \\ q^j_\ell = t^j_\ell & \text{for each } \ell \in M_0 \\ q^j_\ell \le t^j_\ell & \text{for each } \ell \in M_- \end{array}\right\}. \tag{4}$$

By (2) and (3),

$$\begin{array}{ll} \sum_{j\in N} t^j_\ell = t^i_\ell + W_\ell - y^i_\ell \ge W_\ell & \text{for each } \ell \in M_+ \\ \sum_{j\in N} t^j_\ell = t^i_\ell + W_\ell - y^i_\ell = W_\ell & \text{for each } \ell \in M_0 \\ \sum_{j\in N} t^j_\ell = t^i_\ell + W_\ell - y^i_\ell \le W_\ell & \text{for each } \ell \in M_-. \end{array}$$

Thus, by same-sidedness,

$$\left.\begin{array}{ll} & \text{for each } j \in N, \\ f^j_\ell(t^i, t^{-i}) \le t^j_\ell & \text{for each } \ell \in M_+ \\ f^j_\ell(t^i, t^{-i}) = t^j_\ell & \text{for each } \ell \in M_0 \\ f^j_\ell(t^i, t^{-i}) \ge t^j_\ell & \text{for each } \ell \in M_- \end{array}\right\}. \tag{5}$$

We show that for each $j \neq i$ and each $\ell \in M$, $f^j_\ell(t^i, t^{-i}) = t^j_\ell$. Suppose $f^j_\ell(t^i, t^{-i}) \neq t^j_\ell$ for some $j \neq i$ and some $\ell \in M$. Then, by (4), $\ell \in M_+$ or $\ell \in M_-$.

Case 1: $\ell \in M_+$. By (5), $f^j_\ell(t^i, t^{-i}) < t^j_\ell$, and by (4), $t^j_\ell \le q^j_\ell$. Thus, $f^j(t^i, t^{-i}) \notin MB(q^j, t^j)$. Thus, by richness relative to q^j, there is $\widehat{R}^j \in \mathcal{D}^j$ such that $t(\widehat{R}^j) = t^j$ and $q^j\, \widehat{P}^j\, f^j(t^i, t^{-i})$.[22] By tops-onlyness, $q^j\, \widehat{P}^j\, f^j(t^i, t(\widehat{R}^j), t^{-\{i,j\}})$, contradicting individual rationality with respect to q.

Case 2: $\ell \in M_-$. By (5), $f^j_\ell(t^i, t^{-i}) > t^j_\ell$, and by (4), $t^j_\ell \ge q^j_\ell$. Hence, $f^j(t^i, t^{-i}) \notin MB(q^j, t^j)$. Thus, by richness relative to q^j, there is $\widehat{R}^j \in \mathcal{D}^j$ such that $t(\widehat{R}^j) = t^j$ and $q^j\, \widehat{P}^j\, f^j(t^i, t^{-i})$. By tops-onlyness, $q^j\, \widehat{P}^j\, f^j(t^i, t(\widehat{R}^j), t^{-\{i,j\}})$, contradicting individual rationality with respect to q.

Therefore, for each $j \neq i$ and each $\ell \in M$, $f^j_\ell(t^i, t^{-i}) = t^j_\ell$. Thus, by feasibility and (3),

$$f^i(t^i, t^{-i}) = W - \sum_{j\neq i} f^j(t^i, t^{-i}) = W - \sum_{j\neq i} t^j = y^i. \tag{6}$$

[20] To see that for each $\ell \in M_+$, such λ_ℓ exists, pick $\ell \in M_+$ and let $h_\ell(s) \equiv \sum_{j\neq i} \min\{q^j_\ell, s\}$. Note that $h_\ell(\cdot)$ is continuous, $h_\ell(0) = 0$, and $h_\ell(\max\{q^j_\ell \mid j \neq i\}) = \sum_{j\neq i} q^j_\ell$. Also note that by $y^i \in MB(x^i, t^i)$, $\ell \in M_+$ and (1), $0 \le W_\ell - y^i_\ell \le W_\ell - q^i_\ell = \sum_{j\neq i} q^j_\ell$. Thus, by the intermediate value theorem, there is $\lambda_\ell \in [0, \max\{q^j_\ell \mid j \neq i\}]$ such that $h_\ell(\lambda_\ell) = W_\ell - y^i_\ell$.

[21] To see that for each $\ell \in M_-$, such λ_ℓ exists, pick $\ell \in M_-$ and let $h_\ell(s) \equiv \sum_{j\neq i} \max\{q^j_\ell, s\}$. Note that $h_\ell(\cdot)$ is continuous, $h_\ell(\min\{q^j_\ell \mid j \neq i\}) = \sum_{j\neq i} q^j_\ell$, and $h_\ell(W_\ell) = (n-1)\cdot W_\ell$. Moreover by $y^i \in MB(x^i, t^i)$, $\ell \in M_-$ and (1), $\sum_{j\neq i} q^j_\ell = W_\ell - q^i_\ell \le W_\ell - y^i_\ell \le (n-1)\cdot W_\ell$. Thus, by the intermediate value theorem, there is $\lambda_\ell \in [\min\{q^j_\ell \mid j \neq i\}, W_\ell]$ such that $h_\ell(\lambda_\ell) = W_\ell - y^i_\ell$.

[22] By $t^j \neq q^i$, in the definition of richness relative to q, we may set $x^j \equiv t(\widehat{R}^j) = t^j$ and $y^j \equiv q^j$. Then, $q^i \notin int\, MB(x^i, y^i)$. By $f^j(t^i, t^{-i}) \notin MB(q^j, t^j)$, we may set $z^j \equiv f^j(t^i, t^{-i})$. Then, richness guarantees that there is $\widehat{R}^j \in \mathcal{D}^j$ such that $t(\widehat{R}^j) = x^j = t^j$ and $q^j = y^j\, \widehat{P}^j\, z^j = f^j(t^i, t^{-i})$.

Next, let $\widehat{R}^i \in \mathcal{D}^i$ be such that $t(\widehat{R}^i) = x^i$. Then, by (2) and (3),

$$\begin{aligned} &\textstyle\sum_{j\neq i} t^j_\ell + t_\ell(\widehat{R}^i) = W_\ell - y^i_\ell + x^i_\ell \leq W_\ell \quad \text{for each } \ell \in M_+ \\ &\textstyle\sum_{j\neq i} t^j_\ell + t_\ell(\widehat{R}^i) = W_\ell - y^i_\ell + x^i_\ell = W_\ell \quad \text{for each } \ell \in M_0 \\ &\textstyle\sum_{j\neq i} t^j_\ell + t_\ell(\widehat{R}^i) = W_\ell - y^i_\ell + x^i_\ell \geq W_\ell \quad \text{for each } \ell \in M_-. \end{aligned}$$

Thus, by same-sidedness,

$$\left.\begin{aligned} &f^i_\ell(t(\widehat{R}^i), t^{-i}) \geq t_\ell(\widehat{R}^i) = x^i_\ell \quad \text{for each } \ell \in M_+ \\ &f^i_\ell(t(\widehat{R}^i), t^{-i}) = t_\ell(\widehat{R}^i) = x^i_\ell \quad \text{for each } \ell \in M_0 \\ &f^i_\ell(t(\widehat{R}^i), t^{-i}) \leq t_\ell(\widehat{R}^i) = x^i_\ell \quad \text{for each } \ell \in M_- \end{aligned}\right\}. \tag{7}$$

We show that for each $\ell \in M$, $f^i_\ell(t(\widehat{R}^i), t^{-i}) = t_\ell(\widehat{R}^i) = x^i_\ell$. Suppose $f^i_\ell(t(\widehat{R}^i), t^{-i}) \neq t_\ell(\widehat{R}^i)$ for some $\ell \in M$. Then, by (7), $\ell \in M_+$ or $\ell \in M_-$.

Case 1: $\ell \in M_+$. By (2) and (7), $f^i_\ell(t(\widehat{R}^i), t^{-i}) > t_\ell(\widehat{R}^i) = x^i_\ell \geq q^i_\ell$. Thus, by richness relative to q^i, there is $\widetilde{R}^i \in \mathcal{D}^i$ such that $t(\widetilde{R}^i) = t(\widehat{R}^i)$ and $q^i\ \widetilde{P}^i\ f^i(t(\widehat{R}^i), t^{-i})$. By tops-onlyness, $q^i\ \widetilde{P}^i\ f^i(t(\widetilde{R}^i), t^{-i})$, contradicting individual rationality with respect to q.

Case 2: $\ell \in M_-$. By (2) and (7), $f^i_\ell(t(\widehat{R}^i), t^{-i}) < t_\ell(\widehat{R}^i) = x^i_\ell \leq q^i_\ell$. Thus, by richness relative to q^i, there is $\widetilde{R}^i \in \mathcal{D}^i$ such that $t(\widetilde{R}^i) = t(\widehat{R}^i)$ and $q^i\ \widetilde{P}^i\ f^i(t(\widehat{R}^i), t^{-i})$. By tops-onlyness, $q^i\ \widetilde{P}^i f^i(t(\widetilde{R}^i), t^{-i})$, contradicting individual rationality with respect to q.

Therefore, for each $\ell \in M$, $f^i_\ell(t(\widehat{R}^i), t^{-i}) = t_\ell(\widehat{R}^i) = x^i_\ell$; namely, $f^i(t(\widehat{R}^i), t^{-i}) = x^i$.

By (6) and $f^i(t(\widehat{R}^i), t^{-i}) = x^i$, strategy-proofness implies that $y^i\ R^i\ x^i$. Thus, (MSP.1) holds.

Proof of (MSP.2). Let $x^i \notin MB(q^i, t^i)$. We show $\widetilde{x}^i\ R^i\ x^i$.

Let $M'_+ \equiv \{\ell \in M \mid \max\{t^i_\ell, q^i_\ell\} < x^i_\ell\}$, $M'_0 \equiv \{\ell \in M \mid \min\{t^i_\ell, q^i_\ell\} \leq x^i_\ell \leq \max\{t^i_\ell, q^i_\ell\}\}$ and $M'_- \equiv \{\ell \in M \mid x^i_\ell < \min\{t^i_\ell, q^i_\ell\}\}$.

For each $\ell \in M'_+$, let λ_ℓ be such that $W_\ell - x^i_\ell = \sum_{j\neq i} \min\{q^j_\ell, \lambda_\ell\}$,[23] and for each $\ell \in M'_-$, let λ_ℓ be such that $W_\ell - x^i_\ell = \sum_{j\neq i} \max\{q^j_\ell, \lambda_\ell\}$.[24] For each $j \neq i$, let $R^j \in \mathcal{D}^j$ be such that

$$\begin{aligned} t_\ell(R^j) \equiv t^j_\ell &= \min\{q^j_\ell, \lambda_\ell\} \quad \text{for each } \ell \in M'_+ \\ t_\ell(R^j) \equiv t^j_\ell &= q^j_\ell \quad \text{for each } \ell \in M'_0 \\ t_\ell(R^j) \equiv t^j_\ell &= \max\{q^j_\ell, \lambda_\ell\} \quad \text{for each } \ell \in M'_-. \end{aligned}$$

By their definitions,

$$\sum_{j\neq i} t^j = W - x^i, \tag{8}$$

and

$$\left.\begin{aligned} &\text{for each } j \neq i, \\ &q^j_\ell \geq t^j_\ell \quad \text{for each } \ell \in M'_+ \\ &q^j_\ell = t^j_\ell \quad \text{for each } \ell \in M'_0 \\ &q^j_\ell \leq t^j_\ell \quad \text{for each } \ell \in M'_- \end{aligned}\right\}. \tag{9}$$

By the definition of $\widetilde{x}^i$,

$$\left.\begin{aligned} \widetilde{x}^i_\ell &= \max\{t^i_\ell, q^i_\ell\} \quad \text{for each } \ell \in M'_+ \\ \widetilde{x}^i_\ell &= x^i_\ell \quad \text{for each } \ell \in M'_0 \\ \widetilde{x}^i_\ell &= \min\{t^i_\ell, q^i_\ell\} \quad \text{for each } \ell \in M'_- \end{aligned}\right\}. \tag{10}$$

We show that for each $\ell \in M$, $f^i_\ell(t^i, t^{-i}) = \widetilde{x}^i_\ell$. Let $\ell \in M$.

Case 1: $\ell \in M'_+$.

Subcase 1.1: $t^i_\ell \geq q^i_\ell$. By (10), $\widetilde{x}^i_\ell = t^i_\ell$. By $\max\{t^i_\ell, q^i_\ell\} < x^i_\ell$ and (8), $\sum_{j\in N} t^j_\ell = W_\ell - x^i_\ell + t^i_\ell < W_\ell$. Thus by same-sidedness, $f^i_\ell(t^i, t^{-i}) \geq t^i_\ell$.

Suppose $f^i_\ell(t^i, t^{-i}) > t^i_\ell$. Then, by $q^i_\ell \leq t^i_\ell$ and richness relative to q^i, there is $\widehat{R}^i \in \mathcal{D}^i$ such that $t(\widehat{R}^i) = t^i$ and $q^i\ \widehat{P}^i\ f^i(t^i, t^{-i})$. By tops-onlyness, $q^i\ \widehat{P}^i\ f^i(t(\widehat{R}^i), t^{-i})$, contradicting individual rationality with respect to q^i. Thus, $f^i_\ell(t^i, t^{-i}) = t^i_\ell = \widetilde{x}^i_\ell$.

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[23] To see that for each $\ell \in M'_+$, such λ_ℓ exists, pick $\ell \in M'_+$ and let $h_\ell(s) \equiv \sum_{j\neq i} \min\{q^j_\ell, s\}$. Note that $h_\ell(\cdot)$ is continuous, $h_\ell(0) = 0$, and $h_\ell(\max\{q^j_\ell \mid j \neq i\}) = \sum_{j\neq i} q^j_\ell$. Also note that by $\ell \in M'_+$, $0 \leq W_\ell - x^i_\ell < W_\ell - q^i_\ell = \sum_{j\neq i} q^j_\ell$. Thus, by the intermediate value theorem, there is $\lambda_\ell \in [0, \max\{q^j_\ell \mid j \neq i\}]$ such that $h_\ell(\lambda_\ell) = W_\ell - x^i_\ell$.

[24] To see that for each $\ell \in M'_-$, such λ_ℓ exists, pick $\ell \in M'_-$ and let $h_\ell(s) \equiv \sum_{j\neq i} \max\{q^j_\ell, s\}$. Note that $h_\ell(\cdot)$ is continuous, $h_\ell(\min\{q^j_\ell \mid j \neq i\}) = \sum_{j\neq i} q^j_\ell$, and $h_\ell(W_\ell) = (n-1) \cdot W_\ell$. Moreover by $\ell \in M'_-$, $\sum_{j\neq i} q^j_\ell = W_\ell - q^i_\ell \leq W_\ell - x^i_\ell \leq (n-1) \cdot W_\ell$. Thus, by the intermediate value theorem, there is $\lambda_\ell \in [\min\{q^j_\ell \mid j \neq i\}, W_\ell]$ such that $h_\ell(\lambda_\ell) = W_\ell - x^i_\ell$.

Subcase 1.2: $t^i_\ell < q^i_\ell$. By (10), $\widetilde{x}^i_\ell = q^i_\ell$. Suppose $f^i_\ell(t^i,t^{-i}) > q^i_\ell$. Then, by $q^i_\ell > t^i_\ell$ and richness relative to q^i, there is $\widehat{R}^i \in \mathcal{D}^i$ such that $t(\widehat{R}^i) = t^i$ and $q^i \ \widehat{P}^i \ f^i(t^i,t^{-i})$. By tops-onlyness, $q^i \ \widehat{P}^i \ f^i(t(\widehat{R}^i),t^{-i})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) \le q^i_\ell = \widetilde{x}^i_\ell$.

Suppose $f^i_\ell(t^i,t^{-i}) < q^i_\ell$. Then by feasibility, for some $j \ne i$, $f^j_\ell(t^i,t^{-i}) > q^j_\ell$. By (9) and richness relative to q^j, there is $\widehat{R}^j \in \mathcal{D}^j$ such that $t(\widehat{R}^j) = t^j$ and $q^j \ \widehat{P}^j \ f^j(t^i,t^{-i})$. By tops-onlyness, $q^j \ \widehat{P}^j \ f^j(t^i,t(\widehat{R}^j),t^{-\{i,j\}})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) = q^i_\ell = \widetilde{x}^i_\ell$.

Case 2: $\ell \in M'_-$.

Subcase 2.1: $t^i_\ell \le q^i_\ell$. By (10), $\widetilde{x}^i_\ell = t^i_\ell$. By $\min\{t^i_\ell, q^i_\ell\} > x^i_\ell$ and (8), $\sum_{j\in N} t^j_\ell = W_\ell - x^i_\ell + t^i_\ell > W_\ell$. Thus by same-sidedness, $f^i_\ell(t^i,t^{-i}) \le t^i_\ell$.

Suppose $f^i_\ell(t^i,t^{-i}) < t^i_\ell$. Then, by $q^i_\ell \ge t^i_\ell$ and richness relative to q^i, there is $\widehat{R}^i \in \mathcal{D}^i$ such that $t(\widehat{R}^i) = t^i$ and $q^i \ \widehat{P}^i \ f^i(t^i,t^{-i})$. By tops-onlyness, $q^i \ \widehat{P}^i \ f^i(t(\widehat{R}^i),t^{-i})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) = t^i_\ell = \widetilde{x}^i_\ell$.

Subcase 2.2: $t^i_\ell > q^i_\ell$. By (10), $\widetilde{x}^i_\ell = q^i_\ell$. Suppose $f^i_\ell(t^i,t^{-i}) < q^i_\ell$. Then, by $q^i_\ell < t^i_\ell$ and richness relative to q^i, there is $\widehat{R}^i \in \mathcal{D}^i$ such that $t(\widehat{R}^i) = t^i$ and $q^i \ \widehat{P}^i \ f^i(t^i,t^{-i})$. By tops-onlyness, $q^i \ \widehat{P}^i \ f^i(t(\widehat{R}^i),t^{-i})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) \ge q^i_\ell = \widetilde{x}^i_\ell$.

Suppose $f^i_\ell(t^i,t^{-i}) > q^i_\ell$. Then by feasibility, for some $j \ne i$, $f^j_\ell(t^i,t^{-i}) < q^j_\ell$. Thus, by (9), richness relative to q^j, there is $\widehat{R}^j \in \mathcal{D}^j$ such that $t(\widehat{R}^j) = t^j$ and $q^j \ \widehat{P}^j \ f^j(t^i,t^{-i})$. By tops-onlyness, $q^j \ \widehat{P}^j \ f^j(t^i,t(\widehat{R}^j),t^{-\{i,j\}})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) = q^i_\ell = \widetilde{x}^i_\ell$.

Case 3: $\ell \in M'_0$. By (10), $\widetilde{x}^i_\ell = x^i_\ell$.

Subcase 3.1: $t^i_\ell \le q^i_\ell$. By $t^i_\ell \le x^i_\ell \le q^i_\ell$ and (8), $\sum_{j\in N} t^j_\ell = W_\ell - x^i_\ell + t^i_\ell \le W_\ell$. Thus by same-sidedness, for each $j \in N$, $f^j_\ell(t^i,t^{-i}) \ge t^j_\ell$. Thus, by (8), $\sum_{j\ne i} f^j_\ell(t^i,t^{-i}) \ge \sum_{j\ne i} t^j_\ell = W_\ell - x^i_\ell$. Thus by feasibility, $f^i_\ell(t^i,t^{-i}) = W_\ell - \sum_{j\ne i} f^j_\ell(t^i,t^{-i}) \le x^i_\ell$.

Suppose $f^i_\ell(t^i,t^{-i}) < x^i_\ell$. Then by (8), for some $j \ne i$, $f^j_\ell(t^i,t^{-i}) > t^j_\ell$. By (9), $t^j_\ell = q^j_\ell$. Thus, $q^j \ P^j \ f^j(t^i,t^{-i})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) = x^i_\ell = \widetilde{x}^i_\ell$.

Subcase 3.2: $t^i_\ell > q^i_\ell$. By $q^i_\ell \le x^i_\ell \le t^i_\ell$ and (8), $\sum_{j\in N} t^j_\ell = W_\ell - x^i_\ell + t^i_\ell \ge W_\ell$. Thus by same-sidedness, for each $j \in N$, $f^j_\ell(t^i,t^{-i}) \le t^j_\ell$. Thus, by (8), $\sum_{j\ne i} f^j_\ell(t^i,t^{-i}) \le \sum_{j\ne i} t^j_\ell = W_\ell - x^i_\ell$. Thus by feasibility, $f^i_\ell(t^i,t^{-i}) = W_\ell - \sum_{j\ne i} f^j_\ell(t^i,t^{-i}) \ge x^i_\ell$.

Suppose $f^i_\ell(t^i,t^{-i}) > x^i_\ell$. Then, by (8), for some $j \ne i$, $f^j_\ell(t^i,t^{-i}) < t^j_\ell$. By (9), $t^j_\ell = q^j_\ell$. Thus, $q^j \ P^j \ f^j(t^i,t^{-i})$, contradicting individual rationality with respect to q. Thus, $f^i_\ell(t^i,t^{-i}) = x^i_\ell = \widetilde{x}^i_\ell$.

Hence, $f^i(t^i,t^{-i}) = \widetilde{x}^i$. Next, let $\widehat{R}^i \in \mathcal{D}^i$ be such that $t(\widehat{R}^i) = x^i$. Then, by (8), $\sum_{j\ne i} t^j + t(\widehat{R}^i) = W$. Thus, by same-sidedness, $f^i(t(\widehat{R}^i),t^{-i}) = t(\widehat{R}^i) = x^i$. Thus, by $f^i(t^i,t^{-i}) = \widetilde{x}^i$, strategy-proofness implies that $\widetilde{x}^i \ R^i \ x^i$. Thus, (MSP.2) holds. □

3.2. Multi-dimensional single-peaked domains relative to the entitlements admit strategy-proof rules

In this subsection we show that, for each $q \in Z$, the domain $\mathcal{MSP}^*(q)$ admits a strategy-proof and tops-only rule that satisfies same-sidedness and individual rationality with respect to q. The rule that we will exhibit is the m-dimensional sequential allotment rule $g^q : \mathcal{MSP}^*(q) \to Z$ where, for each $R \in \mathcal{MSP}^*(q)$, $g^q(R) = (g^{q_\ell}_\ell(t_\ell(R_1),\dots,t_\ell(R_n)))_{\ell\in M}$ and, for each $\ell \in M$, (i) $g^{q_\ell}_\ell : \mathcal{MSP}^*(q_\ell) \to [0,W_\ell]$ is the one-dimensional sequential allotment rule that satisfies individual rationality with respect to q_ℓ (as defined in Barberà et al. (1997) on the domain of single-peaked preferences on $[0,W_\ell]$) and (ii) its sequential adjustment function is uniform (up to feasibility).[25] Sequential allotment rules are tops-only. Since the domain of single-peaked preferences on $[0,W_\ell]$ is a subset of $\mathcal{MSP}^*(q_\ell)$, any sequential allotment rule $f^{q_\ell}_\ell : \mathcal{MSP}^*(q_\ell) \to [0,W_\ell]$ can be identified with $f^{q_\ell}_\ell : X^N_\ell \to [0,W_\ell]$. Consequently, it can be extended to operate on the larger domain $\mathcal{MSP}^*(q_\ell)$. We now describe the family of one-dimensional sequential allotment rules.[26]

For each $q \in Z$ and $\ell \in M$, the (sequential) definition of $f^{q_\ell}_\ell : X^N_\ell \to [0,W_\ell]$ is as follows. Let $t_\ell = (t^1_\ell,\dots,t^n_\ell) \in X^N_\ell$ be arbitrary.

Suppose $\sum_{j\in N} t^j_\ell = W_\ell$. Then, $f^{q_\ell}_\ell(t_\ell) = t_\ell$.

Suppose $\sum_{j\in N} t^j_\ell > W_\ell$. If $t^i_\ell \ge q^i_\ell$ for all $i \in N$, then $f^{q_\ell}_\ell(t_\ell) = q_\ell$. Otherwise, each i with $t^i_\ell \le q^i_\ell$ receives t^i_ℓ and leaves the process with t^i_ℓ, while the other agents remain. The guaranteed entitlements of the remaining agents are weakly increased by distributing among them the not yet allotted amount uniformly. Agents with a top smaller than or equal to the new guaranteed entitlement receive the top and leave the process, while the others remain. The process proceeds this way until there is no agent with a top smaller than the current guaranteed entitlement, and the rule assigns to those remaining agents their last guaranteed entitlement.

Suppose $\sum_{j\in N} t^j_\ell < W_\ell$. If $t^i_\ell \le q^i_\ell$ for all $i \in N$, then $f^{q_\ell}_\ell(t_\ell) = q_\ell$. Otherwise, each i with $t^i_\ell \ge q^i_\ell$ receives t^i_ℓ and leaves the process with t^i_ℓ, while the other agents remain. The guaranteed entitlements of the remaining agents are weakly decreased uniformly, keeping them feasible and non-negative. Agents with a top larger than or equal to the new guaranteed entitlement receive the top and leave the process, while the others remain. The process proceeds this way until there is no agent with a top larger than the current guaranteed entitlement, and the rule assigns to those remaining agents their last guaranteed entitlement.

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[25] The rule $g^{q_\ell}_\ell$ coincides with the "uniform reallocation rule" defined and studied in Klaus et al. (1998).

[26] See Barberà et al. (1997) for a formal definition of all sequential allotment rules by means of sequential adjustment functions.

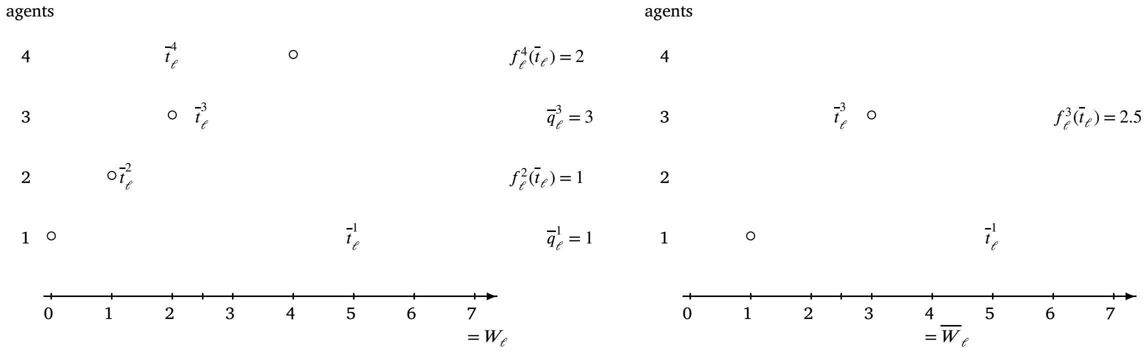

Fig. 10.a Fig. 10.b

Fig. 10. Initial and updated entitlements.

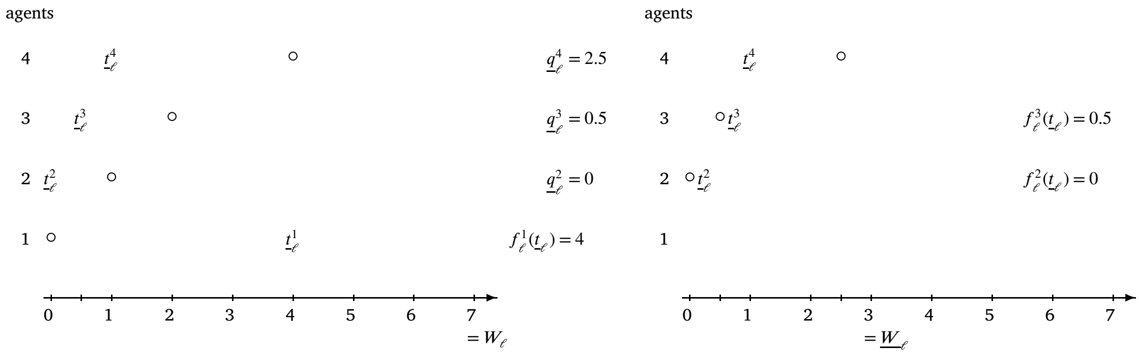

Fig. 11.a Fig. 11.b

Fig. 11. Initial and updated entitlements.

Barberà et al. (1997) establish that (i) $f_\ell^{q_\ell}(0,\dots,0)=f_\ell^{q_\ell}(W_\ell,\dots,W_\ell)=q_\ell$ and (ii) q_ℓ can be seen as a vector of guaranteed entitlements since, for each $i\in N$ and each $t_\ell^{-i}\in X_\ell^{N\setminus\{i\}}$, $f_\ell^{q_\ell,i}(q_\ell^i,t_\ell^{-i})=q_\ell^i$. For further reference, we state as Remark 4 three properties of any of these sequential allotment rules $f_\ell^{q_\ell}:X_\ell^N\to[0,W_\ell]$.

Remark 4. Let q_ℓ be such that $\sum_{i\in N}q_\ell^i=W_\ell$ and $f_\ell^{q_\ell}:X_\ell^N\to[0,W_\ell]$ be the sequential allotment rule that satisfies individual rationality with respect to q_ℓ and its sequential adjustment function is uniform (up to feasibility). Then, for each $t_\ell=(t_\ell^1,\dots,t_\ell^n)\in X_\ell^N$, the following hold:
(i) At the end of the process, each agent i receives either t_ℓ^i or i's final guaranteed entitlement, which has been moving monotonically towards t_ℓ^i along the process.
(ii) *Uncompromisingness*: For each $i\in N$ and each $\hat{t}_\ell^i\in X_\ell$, if $f_\ell^{q_\ell,i}(t_\ell)<t_\ell^i$ and $f_\ell^{q_\ell,i}(t_\ell)\le\hat{t}_\ell^i$ ($f_\ell^{q_\ell,i}(t_\ell)>t_\ell^i$ and $f_\ell^{q_\ell,i}(t_\ell)\ge\hat{t}_\ell^i$), then $f_\ell^{q_\ell,i}(\hat{t}_\ell^i,t_\ell^{-i})=f_\ell^{q_\ell,i}(t_\ell)$.
(iii) *Top-monotonicity*: For each $i\in N$ and each $\hat{t}_\ell^i\in X_\ell$, if $\hat{t}_\ell^i\ge t_\ell^i$ ($\hat{t}_\ell^i\le t_\ell^i$), then $f_\ell^{q_\ell,i}(\hat{t}_\ell^i,t_\ell^{-i})\ge f_\ell^{q_\ell,i}(t_\ell)$ ($f_\ell^{q_\ell,i}(\hat{t}_\ell^i,t_\ell^{-i})\le f_\ell^{q_\ell,i}(t_\ell)$).

The next example illustrates one of these sequential allotment rules $f_\ell^{q_\ell}:X_\ell^N\to[0,W_\ell]$ by evaluating it at two different profiles of tops.

Example 1. Let $N=\{1,2,3,4\}$ and $W_\ell=7$. Assume $q_\ell=(0,1,2,4)$ is the initial vector of guaranteed entitlements, represented in Fig. 10.a by the four circles. In Figs. 10 and 11 the horizontal axes represent the assignments of the good while the vertical axes represent the agents. To simplify notation, we omit the reference to q_ℓ and write f_ℓ^i instead of $f_\ell^{q_\ell,i}$.

Consider the vector of tops $\bar{t}_\ell=(5,1,2.5,2)\in X_\ell^N$. Since $\sum_{j\in N}\bar{t}_\ell^j>7$, $\bar{t}_\ell^2=1=q_\ell^2$ and $\bar{t}_\ell^4=2<4=q_\ell^4$, $f_\ell^2(\bar{t}_\ell)=1$ and $f_\ell^4(\bar{t}_\ell)=2$, and agents 2 and 4 leave with their tops. The amount not allotted yet is $\overline{W}_\ell=4$. The new updated guaranteed entitlements for agents 1 and 3 that remain are $\bar{q}_\ell^1=q_\ell^1+x=x$ and $\bar{q}_\ell^3=q_\ell^3+x=2+x$, where x is such that $\bar{q}_\ell^1+\bar{q}_\ell^3=4$. Hence, $x=1$ and so $\bar{q}_\ell^1=1$ and $\bar{q}_\ell^3=3$, represented in Fig. 10.b by the two circles.

As $\bar{t}^3_\ell = 2.5 < 3 = \bar{q}^3_\ell$, $f^3_\ell(\bar{t}_\ell) = 2.5$. Since only agent 1 remains and one a half units have not been allotted yet, the new guaranteed entitlement for agent 1 is equal to $\bar{\bar{q}}^1_\ell = 1.5$, strictly smaller than $\bar{t}^1_\ell = 5$. Hence, $f^1_\ell(\bar{t}_\ell) = 1.5$. Therefore, $f^{q_\ell}_\ell(5,1,2.5,2) = (1.5,1,2.5,2)$.

Consider the vector of tops $\underline{t}_\ell = (4,0,0.5,1) \in X^N_\ell$. Since $\sum_{j\in N} \underline{t}^j_\ell < 7$ and $\underline{t}^1_\ell = 4 > 0 = q^1_\ell$ agent 1 leaves with its top. The amount not allotted yet is $\underline{W}_\ell = 3$, and the updated guaranteed entitlements for agents 2, 3 and 4 that remain are $\underline{q}^2_\ell = \max\{q^2_\ell - x, 0\} = \max\{1-x, 0\}$, $\underline{q}^3_\ell = \max\{q^3_\ell - x, 0\} = \max\{2-x, 0\}$ and $\underline{q}^4_\ell = \max\{q^4_\ell - x, 0\} = \max\{4-x, 0\}$, where x is such that $\underline{q}^2_\ell + \underline{q}^3_\ell + \underline{q}^4_\ell = 3$. Hence, $x = 1.5$ and so $\underline{q}^2_\ell = 0$, $\underline{q}^3_\ell = 0.5$ and $\underline{q}^4_\ell = 2.5$, represented in Fig. 11.b by three circles. Since $\underline{t}^2_\ell = \underline{q}^2_\ell = 0$ and $\underline{t}^3_\ell = \underline{q}^3_\ell = 0.5$ agents 2 and 3 leave with their tops and agent 4 receives its guaranteed entitlement 2.5. Therefore, $f^{q_\ell}_\ell(4,0,0.5,1) = (4,0,0.5,2.5)$. □

Proposition 1. *For each $q \in Z$, the rule $g^q : \mathcal{MSP}^*(q) \to Z$ whose sequential adjustment function is uniform (up to feasibility) satisfies strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q.*

Proof. Tops-onlyness and same-sidedness follow directly from the definition of g^q. It remains to verify strategy-proofness and individual rationality with respect to q.

Fix an arbitrary agent $i \in N$ and a profile $R \in \mathcal{MSP}^*(q_\ell)$. By Remark 4.(i), for all $\ell \in M$, $g^{q,i}_\ell(t_\ell(R^1),\dots,t_\ell(R^n))$ is either $t_\ell(R^i)$ or i's last updated guaranteed entitlement of good ℓ used in the sequential process to define $g^q_\ell(t_\ell(R^1),\dots,t_\ell(R^n))$. In both cases, $g^{q,i}_\ell(t_\ell(R^1),\dots,t_\ell(R^n)) \in MB(t^i_\ell, q^i_\ell)$. Hence,

$$g^{q,i}(R) \in MB(t^i, q^i).$$

By (MSP.1), $g^{q,i}(R)\ R^i\ q^i$ which means that g^q satisfies individual rationality with respect to q.

To show that g^q is strategy-proof, let $\widehat{R}^i \in \mathcal{MSP}^*(q^i)$ be arbitrary. We have to verify that

$$x^i \equiv g^{q,i}(R^i, R^{-i})\ R^i\ g^{q,i}(\widehat{R}^i, R^{-i}) \equiv \widehat{x}^i. \tag{11}$$

By Remark 4.(i), $x^i \in MB(t^i, q^i)$. Consider the case of $\widehat{x}^i \in MB(x^i, q^i)$. Then by $x^i \in MB(t^i, q^i)$, $\widehat{x}^i \in MB(t^i, q^i)$ and $x^i \in MB(t^i, \widehat{x}^i)$. Thus, by (MSP.1), (11) holds.

Next consider the case of $\widehat{x}^i \notin MB(x^i, q^i)$. By Remark 4.(ii) and 4.(iii),

$$\widehat{x}^i_\ell \in [0, x_\ell] \text{ if } x^i_\ell < t^i_\ell; \quad \widehat{x}^i_\ell \in [0, W_\ell] \text{ if } x^i_\ell = t^i_\ell; \quad \widehat{x}^i_\ell \in [x_\ell, W_\ell] \text{ if } x^i_\ell > t^i_\ell.$$

Thus, by $x^i \in MB(t^i, q^i)$, there are the following cases:

Case A:	$\widehat{x}^i_\ell \le q^i_\ell \le x^i_\ell < t^i_\ell$,	Case B:	$q^i_\ell \le \widehat{x}^i_\ell \le x^i_\ell < t^i_\ell$,
Case C:	$\widehat{x}^i_\ell \le q^i_\ell \le x^i_\ell = t^i_\ell$,	Case D:	$q^i_\ell \le \widehat{x}^i_\ell \le x^i_\ell = t^i_\ell$,
Case E:	$q^i_\ell \le x^i_\ell = t^i_\ell \le \widehat{x}^i_\ell$,	Case F:	$\widehat{x}^i_\ell \le x^i_\ell = t^i_\ell \le q^i_\ell$,
Case G:	$x^i_\ell = t^i_\ell \le q^i_\ell \le \widehat{x}^i_\ell$,	Case H:	$x^i_\ell = t^i_\ell \le \widehat{x}^i_\ell \le q^i_\ell$,
Case I:	$\widehat{x}^i_\ell \ge q^i_\ell \ge x^i_\ell > t^i_\ell$,	Case J:	$q^i_\ell \ge \widehat{x}^i_\ell \ge x^i_\ell > t^i_\ell$.

Thus by $\widehat{x}^i \notin MB(x^i, q^i)$, $\widehat{x}^i \notin MB(q^i, t^i)$. Let $\widetilde{x}^i$ be such that $\widetilde{x}^i_\ell = q^i_\ell$ for Cases A, C, G and I; $\widetilde{x}^i_\ell = \widehat{x}^i_\ell$ for Cases B, D, H and J; and $\widetilde{x}^i_\ell = x^i_\ell$ for Cases E and F. Then, $x^i \in MB(\widetilde{x}^i, t^i)$, $\widetilde{x}^i \in MB(x^i, q^i) \subseteq MB(q^i, t^i)$, and $\widetilde{x}^i$ is the closest point to $\widehat{x}^i$ in $MB(q^i, t^i)$. By (MSP.1), $\widetilde{x}^i \in MB(t^i, q^i)$ and $x^i \in MB(\widetilde{x}^i, t^i)$, $x^i\ R^i\ \widetilde{x}^i$. Since $\widetilde{x}^i$ is the closest point to $\widehat{x}^i$ in $MB(q^i, t^i)$, by (MSP.2) and $\widehat{x}^i \notin MB(q^i, t^i)$, $\widetilde{x}^i\ R^i\ \widehat{x}^i$. Thus, by transitivity, $x^i\ R^i\ \widehat{x}^i$ and (11) holds. □

Observe that, given $q \in Z$, there are many sequential allotment rules; in particular, those that use sequential adjustment functions that are not necessarily uniform (see Barberà et al., 1997).

4. Corollaries

4.1. The maximal domain property

A domain $\mathcal{D}$ is *maximal for a list of properties* if (i) there is a rule on $\mathcal{D}$ satisfying the properties and (ii) there is no domain $\mathcal{D}' \supsetneq \mathcal{D}$ such that there is a rule on $\mathcal{D}'$ satisfying the properties. Note that by Theorem 1 and Proposition 1, we have the following result.

Corollary 1. *The set of all multi-dimensional single-peaked preferences relative to q is the unique maximal domain for strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q that is top-separable and rich relative to $q \in Z$.*[27]

[27] One might see Corollary 1 as being more interesting than Theorem 1. However, we present the results this way because (i) Theorem 1 answers our initial research question and (ii) Corollary 1 follows from Theorem 1 (*i.e.*, Corollary 1 is a corollary while Theorem 1 is not).

Proof. By Proposition 1, there is a rule on $\mathcal{MSP}^*(q)$ satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q. Thus, $\mathcal{MSP}^*(q)$ satisfies (i) of domain maximality.

Let $\mathcal{D}$ be rich relative to $q \in Z$. Theorem 1 states $\mathcal{D} \subseteq \mathcal{MSP}^*(q)$.[28] Thus, $\mathcal{MSP}^*(q)$ also satisfies condition (ii) of domain maximality. Since $\mathcal{D}$ is an arbitrary top-separable and rich domain relative to $q \in Z$, $\mathcal{MSP}^*(q)$ is the unique maximal domain for these properties. □

Note that the definition of maximal domain does not imply that a maximal domain for a list of properties has to be unique. Thus, the uniqueness claim of Corollary 1 demonstrates that multi-dimensional single-peakedness relative to q is essential for our properties on rules (strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q).

4.2. The model with variable entitlements

We consider now a variant of our model where the entitlements are a variable as in earlier work; for instance, Ching and Serizawa (1998) and Mizobuchi and Serizawa (2006).

Let

$$Q = \{q = (q^1, \dots, q^n) \in \mathbb{R}^N_+ \mid \sum_{i \in N} q^i = W\}$$

be the set of entitlement profiles. A *rule* now is a mapping $f : \mathcal{D} \times Q \to Z$ that assigns to every profile $(R, q) \in \mathcal{D} \times Q$ of preferences and entitlements a feasible allotment $f(R, q) \in Z$.

The following result is a direct corollary of Theorem 1.

Corollary 2. *Let $\mathcal{D}$ be an admissible domain that is top-separable and rich relative to all $q \in Q$ and let $f : \mathcal{D} \times Q \to Z$ be a rule satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to $q \in Q$. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of multi-dimensional single-peaked preferences.*

Proof. Let $i \in N$, $R^i \in \mathcal{D}^i$, $x^i \in X$ and $y^i \in X$ be such that $y^i \in MB(x^i, t^i)$. We need to show $y^i\ R^i\ x^i$. Let $q^i = x^i$. Then, $x^i \in MB(q^i, t^i)$. By Theorem 1, R^i is multi-dimensional single-peaked relative to q^i. Thus, (MSP.1) in the definition of multi-dimensional single-peaked relative to q^i implies $y^i\ R^i\ x^i$. □

4.3. The case of one private good

We consider now the special case of our model where W units of one perfectly divisible private good have to be allotted among the set of agents N. This is a special case of our model with $m = 1$. We therefore set $W_1 = W$ and $X = [0, W]$.

In this case, the domain requirement of top-separability is vacuously satisfied. We only require that the domain be rich relative to q. Sprumont (1991) studies this problem assuming that preferences are continuous and single-peaked. He characterizes the uniform rule as the unique rule that is strategy-proof, efficient and anonymous on the single-peaked domain. The statement in Corollary 3 below corresponds to our Theorem 1 for this one-dimensional case.

Corollary 3. *Assume $m = 1$ and let $\mathcal{D} = \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$ be an admissible domain that is rich relative to $q \in Z$ and let $f : \mathcal{D} \to Z$ be a rule satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to q. Then, for each $i \in N$, $\mathcal{D}^i$ is a set of multi-dimensional single-peaked preferences relative to q^i.*

In the one-dimensional case with single-peaked preferences, same-sidedness is indeed equivalent to efficiency. However, Morimoto et al. (2013) show that in the multiple-dimensional case with continuous, strictly convex, and separable preferences, efficiency implies same-sidedness, with the converse not being true.

Massó and Neme (2001) identify *the* maximal domain of preferences that admits a strategy-proof, efficient and strong symmetric rule.[29] Massó and Neme (2004) identify *a* maximal domain of preferences that admits a strategy-proof, efficient, tops-only and continuous rule. These two domains are similar to the one described here in Fig. 1.a, with $q^i = \frac{W}{n}$. The main differences between the two domains and ours are that (i) the unique top condition is not imposed from the outset and (ii) preferences with some (and very specific) indifference intervals at the same side of the peak have to be excluded. The reason underlying the second difference is that Massó and Neme (2001, 2004) require the rule to be efficient but not individually rational (because agents do not have entitlements). For the case of a variable amount of the good, Ching and Serizawa (1998) show that the single-plateaued domain is *the* unique maximal domain containing the domain of single-peaked preferences while admitting a strategy-proof, efficient and symmetric rule.

[28] As said, Theorem 1 holds without explicitly requiring that $\mathcal{D}$ is top-separable.

[29] A rule $f : \mathcal{D} \to Z$ is *strong symmetric* if for all $R \in \mathcal{D}$ and all $i, j \in N$ such that $R_i = R_j$, $f^i(R) = f^j(R)$.

5. Discussion

5.1. Multi-dimensional single-peakedness relative to the entitlement for public and private goods

The notion of multi-dimensional single-peakedness relative to the entitlement used in this paper corresponds to the definition of semilattice single-peakedness (relative to a semilattice) over the set of social alternatives used in Chatterji and Massó (2018) in a public good context. We now show how to obtain in our private goods case a semilattice on X for which the set of semilattice single-peaked preferences and multi-dimensional single-peakedness relative to the entitlement coincide.

Let $q^i \in X$ be given. Define the binary relation $\succeq^{q^i}$ over X as follows. For each pair $x^i, y^i \in X$, set

$$x^i \succeq^{q^i} y^i \Leftrightarrow x^i \in MB(y^i, q^i).$$

It is immediate to check that the binary relation $\succeq^{q^i}$ is reflexive (for all $x \in X$, $x^i \succeq^{q^i} x^i$) and antisymmetric (for all $x^i, y^i \in X$, $x^i \succeq^{q^i} y^i$ and $y^i \succeq^{q^i} x^i$ imply $x^i = y^i$).

Statement 1 in the Appendix, Subsection A.1, says that $\succeq^{q^i}$ is a semilattice; namely, it is transitive and, for any pair $x^i, y^i \in X$, $\sup_{\succeq^{q^i}}\{x^i, y^i\}$ does exist. Moreover, our definition of multi-dimensional single-peakedness relative to q^i here corresponds to the notion of semilattice single-peakedness on $(X, \succeq^{q^i})$ given by Chatterji and Massó (2018) for the case of a public good (*i.e.*, if X were the set of social alternatives). Indeed, given any triple t^i, x^i, y^i of alternatives, a preference ordering that has t^i as its top must rank the supremum of the pair (t^i, y^i) above the supremum of the pair (x^i, y^i) (see Figs. 8.a and 8.b).

5.2. Other related literature: the case of many private goods

There is a rich literature studying rules in economies with more than one private good. In contrast to our approach, most of them assume a given domain of preferences and identify rules satisfying desirable properties on the domain.

Morimoto et al. (2013) study the multi-dimensional extension of Sprumont (1991).[30] They show that on the class of continuous, strictly convex, and separable preferences a rule satisfies strategy-proofness, unanimity, weak symmetry and non-bossiness if and only if it is the uniform rule.[31] This result extends to the class of continuous, strictly convex, and multi-dimensional single-peaked preferences. They observe that the uniform rule is not efficient on these domains. Hence, in multi-dimensional problems efficiency may be lost, even if the rules are restricted to operate only on subdomains of multi-dimensional single-peaked preferences. Adachi (2010) provides a similar characterization of the uniform rule using strategy-proofness, same-sidedness and envy-freeness.

Cho and Thomson (2017) study maximal domains for strategy-proofness, same-sidedness, and no-envy in the setting similar to Subsection 4.2, where only the total endowments are variables. They decompose multidimensional single-peakedness into two conditions; top-separability and commodity-wise single-peakedness, and examine them one by one. First they assume commodity-wise single-peakedness, and show top-separability, that is, the class of multidimensional single-peaked preferences is a maximal domain for the three axioms in the class of commodity-wise single-peaked preferences. Next, they assume a condition similar to, but weaker than, multidimensional single-peakedness, which they call "plateau-separability," and show commodity-wise weak single-plateauedness, which is also similar to, but weaker than, commodity-wise single-peakedness.

Mas-Colell (1992) considers an economy with private goods and production. Agents have continuous and convex preferences (*i.e.*, they might or might not be satiated). He defines the notion of Walrasian equilibrium with slacks (an extension of the notion of competitive equilibrium in this more general setting with potentially satiated agents). The main contribution of Mas-Colell (1992) is to identify sufficient conditions on the economy under which a Walrasian equilibrium with slacks does exist and is efficient; the issue of truthful revelation is not addressed.[32]

Barberà and Jackson (1995) consider an exchange economy with private goods. Agents have initial entitlements and continuous, strictly quasi-concave and increasing preferences, which are represented by utility functions. For the case of two agents they characterize fixed-proportion trading as the class of all strategy-proof and individually rational rules. For the case of an arbitrary number of agents, they characterize fixed-proportion anonymous trading as the class of all strategy-proof, non-bossy, anonymous and tie-free rules. In contrast with the rule that we exhibit in Proposition 1, those rules are not tops-only (although they are tops-only on the range). Observe that their assumption that agents have increasing preferences and our assumption that agents are satiated (*i.e.*, for each $x^i \in X$ there exists at least one $R^i \in \mathcal{D}^i$ such that $t(R^i) = x^i$) imply that the two domains are different and they reflect two very distant economic settings.

Moulin (2017) studies a family of collective decision problems where each agent i's preferences are single-peaked over the set of alternatives of interest to agent i, and where this set is one-dimensional. The model includes among others the voting model Moulin (1980), when the set is a common subset of real numbers, and Sprumont's (1991) division problem, when the set of alternatives has n private components, and each agent i cares only about its one-dimensional private component. Moulin (2017) shows the existence

[30] Amorós (2002) previously studied this extension. However, he only considered the case of two agents, which can easily be seen as being equivalent to a public good case: by feasibility, once the assignment of an agent is determined (the "public good"), the assignment of the other agent is determined as well.

[31] A rule $f : \mathcal{D} \to Z$ is *weak symmetric* if for all $R \in \mathcal{D}$ and all $i, j \in N$ such that $R^i = R^j$, $f^i(R)\, I^i\, f^j(R)$. A rule $f : \mathcal{D} \to Z$ is *non-bossy* if the change of one agent's preference does not alter assignments unless it alters its own assignment.

[32] Hurwicz (1972) and Zhou (1991) address this issue for two-agents pure exchange economies. Later, Serizawa (2002) and Serizawa and Weymark (2003) extend the analysis to many-agents pure exchange economies.

of strategy-proof rules satisfying additional desirable properties like efficiency and fairness. The existence result in Proposition 1 here can be recast as a multi-dimensional version of Moulin (2017) where we are able to weaken the requirement that the domain of preferences be single-peaked over a one-dimensional set to the requirement that the domain of preferences be multi-dimensional single-peaked over a multi-dimensional set.

5.3. Our axioms

We first deal with the indispensability of same-sidedness in Theorem 1. Example 2 shows that same-sidedness is indispensable in Theorem 1.

Example 2 *(No-trade rule)*. Let $\mathcal{D}$ be an admissible and top-separable domain that is rich relative to $q \in Z$, but includes preferences that are not multi-dimensional single-peaked relative to q. Let f be the rule such that for each $R \in \mathcal{D}$, $f(R) = q$. Then, f is strategy-proof, top-only, individually rational with respect to q, but not same-sided.

Two of the key ingredients in our analysis are the features that (i) the preference domain is rich relative to the entitlements and (ii) we postulate the existence of a tops-only rule on the domain.

We first consider tops-onlyness. Results establishing that the property of tops-onlyness follows from strategy-proofness are abundant in the literature in restricted domains; see for instance the earlier ones in Sprumont (1991 and 1995), Barberà et al. (1991) and Barberà et al. (1993) or more recent ones in Chatterji and Sen (2011) and Chatterji and Zeng (2018). However, our previous statement requires two types of qualifications. First, the rule must satisfy, in addition to strategy-proofness, some additional property like efficiency, complete range, unanimity, or similar. Second, these results are obtained for rules on particular domains such as continuous, single-peaked, separable, multi-dimensional single-peaked or alike, or domains that in public good environments satisfy well-structured richness. The most important difficulty in establishing that tops-onlyness follows from the other axioms is, in our private goods setting, that the information that we have about the domain is very weak and limited (except richness). Indeed, in our setting, tops-onlyness does not follow from strategy-proofness and our other axioms on rules on rich domains (see Remark 5 below). Furthermore, the problem is that only relatively arbitrary and non-compelling domains can be identified for which any strategy-proof, same-sided, individually rational with respect to q and non-bossy rule ought to be tops-only. Hence, these kind of identifications do not seem to be necessarily interesting *per se*. In Subsection A.2 in the Appendix we present top-compatible domains that constitute an illustration of this type of result for the simplest case where $m = 1$.

However, we have not been able to find an example showing that the statement of Theorem 1 does not hold in the absence of tops-onlyness.

We next consider richness and provide an example (Example 3a, adapted from Chatterji and Massó (2018)) that shows that one may construct a non-tops-only rule that satisfies our remaining axioms on a domain that violates multi-dimensional single-peakedness relative to the entitlements and richness relative to q. We conclude with two examples that show the indispensability of strategy-proofness and individual rationality with respect to q respectively for Theorem 1.

Example 3a. Let $m = 1$ and the total amount of the single good available W_1 be one. Let $n = 2$. In this simple case, the division problem can be reformulated as a pure public good problem as follows. Let $A = [0, 1]$ and $a \in A$ denote the division whereby agent 1 receives a and agent 2 receives $1 - a$ and where agents' preferences are directly formulated on A. That is, saying that agent 2 strictly prefers a to b means now that agent 2 strictly prefers $1 - a$ to $1 - b$. Assume that $q := q^1 = q^2 = 1/2$.

The preference domain will be assumed to be identical across agents. To describe the preferences of agents on A, we partition it into the following four intervals: $X \equiv [0, 0.25), Y \equiv [0.25, 0.5), q \equiv [0.5, 0.5], Z \equiv (0.5, 1]$. We will postulate that there are five categories of preferences R^i in the domain. Each preference in the category R^i_a will be assumed to have a common structure as shown in the table below, where for instance, R^i_x ranks the block X above the block Y, Y above q and finally block Z is ranked last. Analogous restrictions apply to R^i_y, etc. Furthermore, the ranking of alternatives *within* each of the blocks X, Y, Z will be assumed to be single-peaked. Given a preference R^i drawn from this domain, we will let $\tau_k(R^i)$ denote the peak of the preference R^i restricted to the block $k \in \{X, Y, Z\}$.

R^i_x	R^i_y	R^i_z	R'^i_z	R^i_q
X	Y	Z	Z	q
Y	X	Y	q	Z
q	q	q	X	X
Z	Z	X	Y	Y

.

To see that the domain does not satisfy richness, consider the assignments $z^i = 0.1$, $x^i = 0.2$ and $y^i = 0.3$ and note that the hypotheses of the property are satisfied but the domain does not contain a preference $R^i_{x^i}$ for which $y^i \, P^i_{x^i} \, z^i$. To define a two-agents rule on this domain, we proceed as follows. Given a profile $R = (R^1, R^2)$ of preferences, let $\sigma_k(R) = max(\tau_k(R^1), \tau_k(R^2))$ for $k \in \{X, Y, Z\}$.[33]

[33] Recall that a preference R^i induces a single-peaked preference on each $k \in \{X, Y, Z\}$: The alternative $\sigma_k(R)$ is the one chosen by a fixed voter rule applied to the interval $k \in \{X, Y, Z\}$ where the fixed vote is located at the upper end of the interval.

Finally, consider the non-tops-only, and unanimous rule $f : \mathcal{D}^1 \times \mathcal{D}^2 \to A$ defined by the following table where we suppress, for $k \in \{X, Y, Z\}$, the dependence of $\sigma_k(R)$ on R for notational convenience:

f	R^2_x	R^2_y	R^2_z	R'^2_z	R^2_q
R^1_x	σ_X	σ_Y	σ_Y	q	q
R^1_y	σ_Y	σ_Y	σ_Y	q	q
R^1_z	σ_Y	σ_Y	σ_Z	σ_Z	q
R'^1_z	q	q	σ_Z	σ_Z	q
R^1_q	q	q	q	q	q

.

By construction, f satisfies individual rationality with respect to q. To verify that f is strategy-proof, we proceed in two steps. The configuration of preferences in the table is used to argue that no agent can manipulate to a preferred block. Within a block, strategy-proofness is guaranteed by the assumption that preferences within the block are single-peaked and a particular fixed voter rule is used. Even though preferences within a block are single-peaked, the overall preferences are not multi-dimensional single-peaked relative to the entitlements.

It is straightforward to exhibit the indispensability of two of our axioms, respectively strategy-proofness and individual rationality with respect to q, for Theorem 1.

Example 3b. The setting is identical to the one in Example 3a. The preference domain is different (but identical across agents).

(i) Assume the universal domain of preferences. This domain is rich relative to q^i for each i, violates multi-dimensional single-peakedness relative to the entitlements, but admits a dictatorial rule which satisfies all our axioms other than individual rationality with respect to q^i for each i.

(ii) Consider now the case of the set of preferences $\mathcal{D}(q^i) = \hat{\mathcal{D}}(q^i) \cup \mathcal{D}^{SP}$ where $\hat{\mathcal{D}}(q^i)$ is the set of all complete, transitive preference that are represented by continuous utility functions that attain a unique maximum and attain a minimum at q^i whenever the top is different from q^i, and $\mathcal{D}^{SP}$ is the set of all single-peaked preferences on A with respect to the natural order. Since $\mathcal{D}^{SP}$ is rich relative to q^i for each i, so is $\mathcal{D}(q^i)$. The rule that selects the median of the two tops and $q^1 = 0.5$ on the domain $\mathcal{D}(q^1) \times \mathcal{D}(q^2)$ (which corresponds to the uniform rule in the original private good setting) satisfies all our axioms except strategy-proofness and the domain is evidently not multi-dimensional single-peakedness relative to the entitlements.

Finally, it is natural to ask whether a smaller domain could be identified by replacing the requirement of strategy-proofness in Theorem 1 with stronger properties like group strategy-proofness. However, the answer is negative, as strategy-proof rules on multidimensional extensions of single-peaked domains are not efficient on the range. Consequently, since same-sidedness implies that the rules have full range, they are not efficient and accordingly they are not group strategy-proof.

6. Conclusion

We study the problem of assigning multiple private goods to a set of agents with entitlements where preferences display satiation. We identify a preference domain, the multi-dimensional single-peaked domain relative to the entitlements, which is implied by the existence of a rule satisfying strategy-proofness, tops-onlyness, same-sidedness and individual rationality with respect to entitlements acting on a rich domain of preferences. This domain is shown to suffice for the design of a strategy-proof rule exhibiting the aforementioned properties. This domain turns out to coincide with one identified in earlier work in the context of public goods.

Declaration of competing interest

We listed all of financially supporting institutions to our paper in the acknowledgments of the manuscript. None of the supporting institutions inappropriately influenced (biased) our work.

Appendix A

A.1. Proof that $\succeq^{q^i}$ is a semilattice

We show that the binary relation $\succeq^{q^i}$ over X defined in Subsection 5.1 is a semilattice.

Statement 1. *The binary relation $\succeq^{q^i}$ over X is a semilattice.*

Proof. We have already argued that $\succeq^{q^i}$ is reflexive and antisymmetric.

We first establish that $\succeq^{q^i}$ is a partial order by showing that it is transitive. Let $x^i, y^i, z^i \in X$ be such that $x^i \succeq^{q^i} y^i$ and $y^i \succeq^{q^i} z^i$. Equivalently, assume that $x^i \in MB(y^i, q^i)$ and $y^i \in MB(z^i, q^i)$ hold. Fix $\ell \in M$. The fact that $y^i \in MB(z^i, q^i)$ means that $\min\{z^i_\ell, q^i_\ell\} \leq$

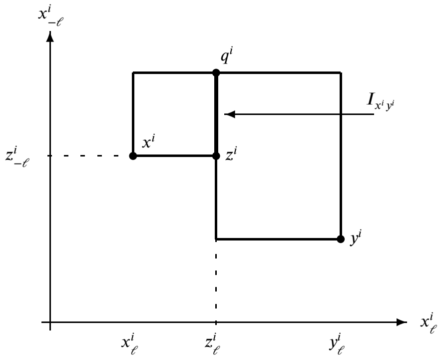

Fig. 12. Identification of z^i.

$y^i_\ell \le \max\{z^i_\ell, q^i_\ell\}$. We distinguish between two cases. First, $z^i_\ell \le q^i_\ell$, and so $z^i_\ell \le y^i_\ell \le q^i_\ell$. The fact that $x^i \in MB(y^i, q^i)$ means that $\min\{y^i_\ell, q^i_\ell\} = y^i_\ell \le x^i_\ell \le q^i_\ell = \max\{y^i_\ell, q^i_\ell\}$, which implies that $\min\{z^i_\ell, q^i_\ell\} \le x^i_\ell \le \max\{z^i_\ell, q^i_\ell\}$. Second, $q^i_\ell \le z^i_\ell$, and so $q^i_\ell \le y^i_\ell \le z^i_\ell$. The fact that $x^i \in MB(y^i, q^i)$ means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell \le x^i_\ell \le y^i_\ell = \max\{y^i_\ell, q^i_\ell\}$, which implies that $\min\{z^i_\ell, q^i_\ell\} \le x^i_\ell \le \max\{z^i_\ell, q^i_\ell\}$. Hence, in both cases, $x^i_\ell \in [\min\{z^i_\ell, q^i_\ell\}, \max\{z^i_\ell, q^i_\ell\}]$, and so $x^i_\ell \in MB(z^i_\ell, q^i_\ell)$. Since this holds for all $\ell \in M$, we have that $x^i \in MB(z^i, q^i)$. Thus, $x^i \succeq^{q^i} z^i$, and $\succeq^{q^i}$ is transitive.

To complete the proof that $\succeq^{q^i}$ is a semilattice, we have to show that, for any arbitrary pair $x^i, y^i \in X$, there exists $\sup_{\succeq^{q^i}} \{x^i, y^i\}$. This will be done in the proof of Claim 1 below. But before stating it, note that the set of upper bounds of x^i and y^i can be written as

$$I_{x^iy^i} = MB(x^i, q^i) \cap MB(y^i, q^i).$$

Note that this set is non-empty since $q^i \in I_{x^iy^i}$. Define z^i as follows; for each $\ell \in M$,

(a) if $\min\{x^i_\ell, y^i_\ell\} \le q^i_\ell \le \max\{x^i_\ell, y^i_\ell\}$, set $z^i_\ell = q^i_\ell$,

(b) if $q^i_\ell \le \min\{x^i_\ell, y^i_\ell\}$, set $z^i_\ell = \min\{x^i_\ell, y^i_\ell\}$,

(c) if $\max\{x^i_\ell, y^i_\ell\} \le q^i_\ell$, set $z^i_\ell = \max\{x^i_\ell, y^i_\ell\}$.

Fig. 12 illustrates the definition of z^i as the projection of q^i to $MB(x^i, y^i)$; that is, z^i is the closest assignment to q^i in $MB(x^i, y^i)$ according to the L_1-norm.

Claim 1. $\sup_{\succeq^{q^i}} \{x^i, y^i\} = z^i$.

Proof of Claim 1. We verify Claim 1 in two steps.

Step 1. The assignment z^i is an upper bound of x^i and y^i. Namely, $z^i \succeq^{q^i} x^i$ and $z^i \succeq^{q^i} y^i$ or, equivalently, $z^i \in MB(x^i, q^i) \cap MB(y^i, q^i)$. We show that $z^i \in MB(y^i, q^i)$. To show that $z^i \in MB(x^i, q^i)$ is similar, and therefore omitted.

To obtain a contradiction, assume $z^i \notin MB(y^i, q^i)$. By (1), there exists $\ell \in M$ such that either $z^i_\ell < \min\{y^i_\ell, q^i_\ell\}$ or $\max\{y^i_\ell, q^i_\ell\} < z^i_\ell$.

Assume $z^i_\ell < \min\{y^i_\ell, q^i_\ell\}$ holds. We proceed by distinguishing between two cases.

Case 1: $z^i_\ell < q^i_\ell \le y^i_\ell$. Since $z^i_\ell \ne q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \le z^i_\ell$ which is a contradiction.

Case 2: $z^i_\ell < y^i_\ell \le q^i_\ell$. Since $z^i_\ell \ne q^i_\ell$ Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \le z^i_\ell$ which is a contradiction.

Assume $\max\{y^i_\ell, q^i_\ell\} < z^i_\ell$ holds. We proceed by distinguishing between two cases.

Case 1: $q^i_\ell \le y^i_\ell < z^i_\ell$. Since $z^i_\ell \ne q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $q^i_\ell = z^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell = z^i_\ell$ which is a contradiction.

Case 2: $z^i_\ell < y^i_\ell \le q^i_\ell$. Since $z^i_\ell \ne q^i_\ell$, Case (a) does not hold. Case (b) does not hold either, because otherwise $z^i_\ell = y^i_\ell$. In order that Case (c) holds, $\max\{y^i_\ell, q^i_\ell\} = z^i_\ell$, but then, $y^i_\ell \le z^i_\ell$ which is a contradiction.

Hence, $z^i \in MB(y^i, q^i)$ as well as $z^i \in MB(x^i, q^i)$. Accordingly, z^i is an upper bound of x^i and y^i.

Step 2. The assignment z^i is the smallest upper bound of x^i and y^i. Namely, if $\hat{z}^i \in I_{x^iy^i}$, then $\hat{z}^i \succeq^{q^i} z^i$ (or equivalently, $\hat{z}^i \in MB(z^i, q^i)$), which is the same that, for all $\ell \in M$, $\min\{z^i_\ell, q^i_\ell\} \le \hat{z}^i_\ell \le \max\{z^i_\ell, q^i_\ell\}$. Fix $\ell \in M$. We distinguish among the three cases (a), (b) or (c), depending under which one of them z^i_ℓ is defined.

Assume it is case (a). Then, $\min\{z^i_\ell, q^i_\ell\} \le q^i_\ell = z^i_\ell \le \max\{z^i_\ell, q^i_\ell\}$. First, suppose $x^i_\ell \le y^i_\ell$. Then, $x^i_\ell \le q^i_\ell = z^i_\ell \le y^i_\ell$. Because, $\hat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \le \hat{z}^i_\ell \le \max\{x^i_\ell, q^i_\ell\}$. which means that $\max\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \ge \hat{z}^i_\ell$ and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{12}$$

Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$, and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{13}$$

By (12) and (13), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $y^i_\ell \leq q^i_\ell = z^i_\ell \leq x^i_\ell$. Because, $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{14}$$

Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = q^i_\ell = z^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{q^i_\ell, z^i_\ell\}. \tag{15}$$

By (14) and (15), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$.

Assume it is case (b). Then, $q^i_\ell \leq z^i_\ell = \min\{x^i_\ell, y^i_\ell\}$. First, suppose $x^i_\ell \leq y^i_\ell$. Then, $q^i_\ell \leq z^i_\ell = x^i_\ell \leq y^i_\ell$. Because, $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $q^i_\ell \leq \widehat{z}^i_\ell \leq z^i_\ell = x^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{16}$$

Because $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = q^i_\ell \leq \widehat{z}^i_\ell \leq z^i_\ell = x^i_\ell$ and so

$$\widehat{z}^i_\ell \leq \max\{q^i_\ell, z^i_\ell\}. \tag{17}$$

By (16) and (17), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$. Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $q^i_\ell \leq z^i_\ell = y^i_\ell \leq x^i_\ell$. Because, $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = q^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell. \tag{18}$$

Moreover, $q^i_\ell \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = y^i_\ell = z^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{19}$$

By (18) and (19), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$.

Assume it is case (c). Then, $\max\{x^i_\ell, y^i_\ell\} = z^i_\ell \leq q^i_\ell$. First, suppose $x^i_\ell \leq y^i_\ell$. Then, $x^i_\ell \leq y^i_\ell = z^i_\ell \leq q^i_\ell$. Because $\widehat{z}^i \in MB(y^i, q^i)$, $\min\{y^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\}$, which means that $\min\{y^i_\ell, q^i_\ell\} = y^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{q^i_\ell, z^i_\ell\} \leq \widehat{z}^i_\ell. \tag{20}$$

Moreover, $\widehat{z}^i_\ell \leq \max\{y^i_\ell, q^i_\ell\} = q^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}. \tag{21}$$

By (20) and (21), we have $\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}$.

Suppose now that $y^i_\ell \leq x^i_\ell$. Then, $y^i_\ell \leq x^i_\ell = z^i_\ell \leq q^i_\ell$. Because $\widehat{z}^i \in MB(x^i, q^i)$, $\min\{x^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\}$, which means that $\min\{x^i_\ell, q^i_\ell\} = x^i_\ell = z^i_\ell \leq \widehat{z}^i_\ell$ and so

$$\min\{z^i_\ell, q^i_\ell\} \leq \widehat{z}^i_\ell.$$

Moreover, $\widehat{z}^i_\ell \leq \max\{x^i_\ell, q^i_\ell\} = q^i_\ell$, and so

$$\widehat{z}^i_\ell \leq \max\{z^i_\ell, q^i_\ell\}.$$

This shows that $\widehat{z}^i \succeq^{q^i} z^i$, and that z^i is the smallest upper bound of x^i and y^i. Hence $\sup_{\succeq^{q^i}} \{x^i, y^i\} = z^i$.[34] □

A.2. Top-compatible domains and tops-onlyness

Assume $m = 1$. Given a preference R^i and an entitlement q^i, let $U(R^i, q^i)$ denote the weak upper contour set of R^i at q^i.

TOP-COMPATIBILITY: A domain $\mathcal{D}^i$ is *top-compatible with* q^i if
(i) it satisfies the strict preference property with respect to q^i; namely, each $R^i \in \mathcal{D}^i$ is antisymmetric restricted to pairs $x^i, y^i \in U(R^i, q^i)$ for which either $[x^i < t^i$ and $y^i < t^i]$ or $[x^i > t^i$ and $y^i > t^i]$ hold;

[34] It is easy to see that $z^i = \arg\max_{r^i \in I_{x^i y^i}} \|r^i - q^i\|_{L_1}$. Moreover, $\sup_{\succeq^{q^i}} X = q^i$.

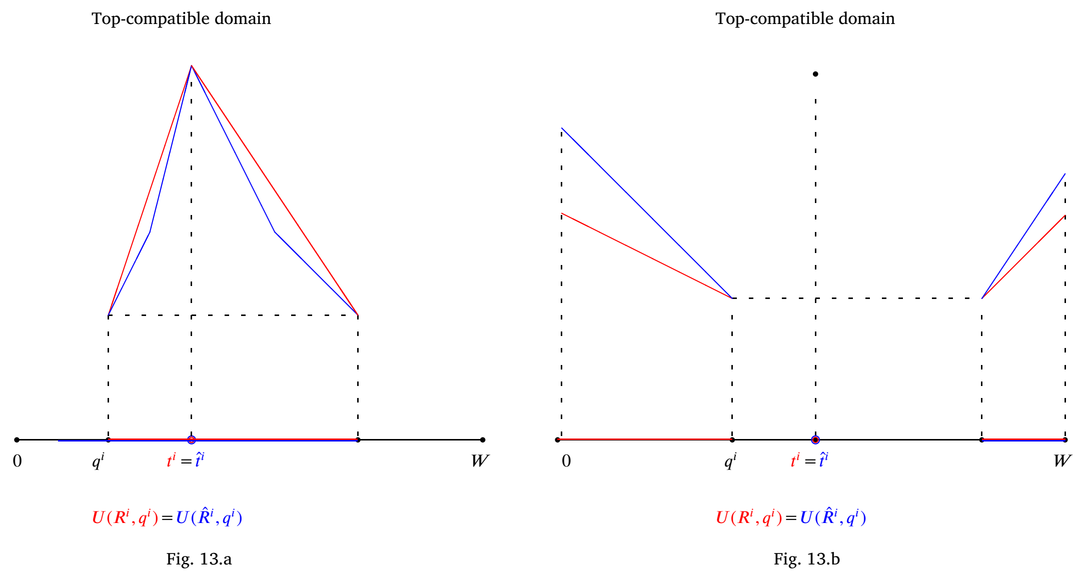

Fig. 13. Top-compatibility.

(ii) for any pair $R^i, \hat{R}^i \in \mathcal{D}^i$ such that $t(R^i) = t(\hat{R}^i)$, we have that $U(R^i, q^i) = U(\hat{R}^i, q^i)$, and furthermore, for any pair $x^i, y^i \in U(R^i, q^i)$ for which either $[x^i < t^i$ and $y^i < t^i]$ or $[x^i > t^i$ and $y^i > t^i]$ hold, it is the case that $x^i\, P^i\, y^i \Leftrightarrow x^i\, \hat{P}^i\, y^i$.

Figs. 13.a and 13.b illustrate the definition of top-compatibility with q^i by partially exhibiting two preferences on two different domains that are top-compatible with q^i. The preferences on assignments outside the upper contour sets are not depicted because top-compatibility does not impose any condition on them.

Proposition 2. *Suppose $\mathcal{D}^i$ is top-compatible with q^i for each $i \in N$. Then, every rule $f : \mathcal{D}^1 \times \cdots \times \mathcal{D}^n \to Z$ that is strategy-proof, same-sided, individually rational with respect to q and non-bossy is tops-only.*[35]

Proof of Proposition 2. We first prove in Lemma 1 below that the statement of Proposition 2 holds for $n = 2$. Then, we extend the argument to the general case for any n and non-bossy rule.

Lemma 1. *Suppose $\mathcal{D}^i$ is top-compatible with q^i for each $i \in \{1, 2\}$. Then, every rule $f : \mathcal{D}^1 \times \mathcal{D}^2 \to Z$ that is strategy-proof, same-sided and individually rational with respect to q is tops-only.*

Proof of Lemma 1. Suppose $(x^1, x^2) = f(R^1, R^2) \neq f(\hat{R}^1, R^2) = (\hat{x}^1, \hat{x}^2)$ where $t(R^1) = t(\hat{R}^1)$. By same-sidedness, it must be that $t(R^1) + t(R^2) \neq W$. Since (x^1, x^2) and $(\hat{x}^1, \hat{x}^2)$ are two different allotments, it must be that $x^1 \neq \hat{x}^1$. Assume $t(R^1) + t(R^2) > W$ so that by same-sidedness, $x^1 \leq t(R^1)$ and $\hat{x}^1 \leq t(\hat{R}^1)$; an analogous argument applies if $t(R^1) + t(R^2) < W$. By individual rationality with respect to q, $x^1, \hat{x}^1 \in U(R^1, q^1) = U(\hat{R}^1, q^1)$. By condition (i) of top compatibility with q^1 either $x^1\, P^1\, \hat{x}^1$ or $\hat{x}^1\, P^1\, x^1$ hold. Suppose $x^1\, P^1\, \hat{x}^1$ holds. Then, by (ii) of top-compatibility with q^1, it must be that $x^1\, \hat{P}^1\, \hat{x}^1$ also holds, wherein, agent 1 manipulates at $f(\hat{R}^1, R^2)$ to R^1. An identical argument applies if $\hat{x}^1\, P^1\, x^1$ holds. Therefore, it must be that $f(R^1, R^2) = f(\hat{R}^1, R^2)$. A similar argument applies to agent 2 switching from R^2 to $\hat{R}^2$ with the same top. Hence, $f(R^1, R^2) = f(\hat{R}^1, \hat{R}^2)$ whenever $t(R^i) = t(\hat{R}^i)$ for each $i \in \{1, 2\}$. ☐

To prove Proposition 2, let $n > 2$ and suppose that $(x^1, \ldots, x^n) = f(R^1, \ldots, R^n) \neq f(\hat{R}^1, \ldots, R^n) = (\hat{x}^1, \ldots, \hat{x}^n)$ where $t(R^1) = t(\hat{R}^1)$. By non-bossiness, it must be that $x^1 \neq \hat{x}^1$. The remainder of the argument is identical to the one above for Lemma 1, successively applied to agents 2 to n. ☐

Example 4 exhibits a domain and an entitlement q such that the domain is top-compatible with q and admits a rule that is strategy-proof, same-sided and individually rational with respect to q but which is neither non-bossy nor tops-only. Hence, non-bossiness is indispensable in the statement of Proposition 2. The domain in the example can also be specified so as to satisfy our richness condition.[36]

[35] A rule $f : \mathcal{D}^1 \times \cdots \times \mathcal{D}^n \to Z$ is *non-bossy* if for all $R \in \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$, $i \in N$ and $\hat{R}^i \in \mathcal{D}^i$, $f^i(R^i, R^{-i}) = f^i(\hat{R}^i, R^{-i})$ implies $f(R^i, R^{-i}) = f(\hat{R}^i, R^{-i})$. Note that when $n = 2$, non-bossiness is vacuous.

[36] Since $m = 1$, the domain satisfies top-separability vacuously.

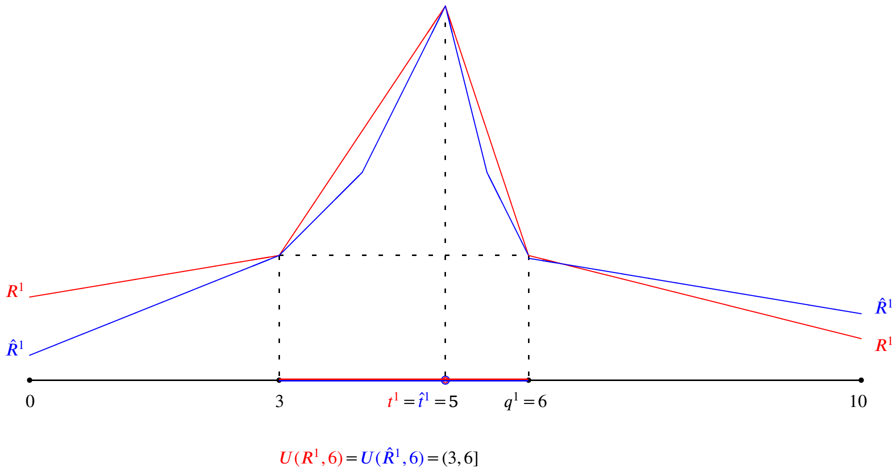

Fig. 14. A bossy, non-tops only rule.

Example 4. Let $N = \{1, 2, 3\}$, $m = 1$ and $W = 10$. To ease the notation in the example we omit the subscript 1. For each $i \in \{1, 2, 3\}$ and entitlement $q \in Z$, consider the domain $\mathcal{SPTC}(q^i)$ of single-peaked preferences on $[0, 10]$ that in addition satisfies conditions (i) and (ii) of top-compatibility with q^i, and so $\mathcal{SPTC}(q^i)$ is top-compatible with q^i.

Let $\widehat{f}^q : \mathcal{SPTC}(q^1) \times \mathcal{SPTC}(q^2) \times \mathcal{SPTC}(q^3) \to Z$ be a rule that is defined similarly to a sequential allotment rule with entitlements q, as defined in Subsection 3.2, but its sequential adjustment function is modified as follows. At those profiles $(R^1, R^2, R^3) \in \mathcal{SPTC}(q^1) \times \mathcal{SPTC}(q^2) \times \mathcal{SPTC}(q^3)$ where only one agent leaves the process at the first step, say agent j, the guaranteed endowments of the other two remaining agents are modified by giving (or subtracting) the remainder (up to its top) to agent $(j + 1)(mod\ 3)$ if $0\, R^j\, 10$ or to $(j + 2)(mod\ 3)$ if $10\, P^j\, 0$. By its definition, $\widehat{f}^q$ is same-sided. Moreover, it is easy to check that $\widehat{f}^q$ is strategy-proof by a simple adaptation of the proof used in Barberà et al. (1997) to show that any sequential allotment rule is strategy-proof. However, due to this modification of the sequential adjustment function, $\widehat{f}^q$ does not belong to the class of sequential allotment rules, as defined in Barberà et al. (1997), because the sequential adjustment function used to define $\widehat{f}^q$ is not tops-only while those used by Barberà et al. (1997) are tops-only. This is also the reason why $\widehat{f}^q$ is not replacement monotonic (the property that together with strategy-proofness and efficiency characterize the class of sequential allotment rules).[37] However, $\widehat{f}^q$ is individually rational with respect to q for the same reason that any sequential allotment rule satisfies this property: Each agent $i \in N$ receives either its top-assignment t^i or else an assignment $x^i \neq t^i$ that belongs to the interval whose extreme points are t^i and q^i and, by single-peakedness on this interval, $x^i\, R^i\, q^i$.

Finally, $\widehat{f}^q$ is neither non-bossy nor tops-only. To see that, assume $q = (6, 2, 2)$ and consider the profile $(R^1, R^2, R^3) \in \mathcal{SPTC}(6) \times \mathcal{SPTC}(2) \times \mathcal{SPTC}(2)$ and $\hat{P}^1 \in \mathcal{SPTC}(6)$, where $(t^1, t^2, t^3) = (5, 6, 7)$, $\hat{t}^1 = 5$, $U(R^1, 6) = U(\hat{R}^1, 6) = [3, 6]$, $0\, P^1\, 10$, and $10\, \hat{P}^1\, 0$. Fig. 14 depicts a pair of preferences R^1 and $\hat{R}^1$ of agent 1 with these features.
Agent 1 leaves the process at the first step of the sequential definition of $\widehat{f}^q$ at the two profiles (R^1, R^2, R^3) and $(\hat{R}^1, R^2, R^3)$, while agents 2 and 3 remain. The adjusted entitlements at profile (R^1, R^2, R^3) for agents 2 and 3 are $\overline{q}^2 = 3$ and $\overline{q}^3 = 2$ and, accordingly, $\widehat{f}^q(R^1, R^2, R^3) = (5, 3, 2)$. At the same time, the adjusted entitlements at profile $(\hat{R}^1, R^2, R^3)$ for agents 2 and 3 are $\hat{q}^2 = 2$ and $\hat{q}^3 = 3$ and, accordingly, $\widehat{f}^q(\hat{R}^1, R^2, R^3) = (5, 2, 3)$. Hence, $\widehat{f}^q$ is neither non-bossy nor tops-only. □

Remark 5. The non-tops-only rule $\widehat{f}^q$ of Example 4, if applied to the domain of one-dimensional single-peaked preferences relative to the entitlements $q = (q^1, q^2, q^3) \in Z$, denoted by $\mathcal{SP}(q) = \mathcal{SP}(q^1) \times \mathcal{SP}(q^2) \times \mathcal{SP}(q^3)$, is strategy-proof, same-sided and individually rational with respect to q but it is not tops-only.[38]

Thus, Remark 5 demonstrates that tops-onlyness is not a consequence of strategy-proofness and our axioms on rich domains. However, since the domain $\mathcal{SP}(q)$ is multi-dimensional single-peaked with respect to the entitlements, the rule of Example 4 can not be considered as showing that tops-onlyness is indispensable in the statement of Theorem 1.

Note that Sprumont (1991) and Barberà et al. (1997) show that, in the one-dimensional case under single-peaked preferences, tops-onlyness follows from strategy-proofness and efficiency if anonymity or replacement monotonicity are additionally required, respectively. In contrast, Remark 5 shows that in the larger domain of one-dimensional single-peaked preferences relative to the entitlements, this is not the case if efficiency together with either anonymity or replacement monotonicity are replaced by same-sidedness and individual rationality with respect to the entitlements.

[37] A rule $f : \mathcal{D}^1 \times \cdots \times \mathcal{D}^n \to Z$ is *replacement monotonic* if for all $R \in \mathcal{D}^1 \times \cdots \times \mathcal{D}^n$, $i \in N$ and $\hat{R}^i \in \mathcal{D}^i$, $f^i(R^i, R^{-i}) \geq f^i(\hat{R}^i, R^{-i})$ implies $f^j(R^i, R^{-i}) \leq f^j(\hat{R}^i, R^{-i})$ for all $j \neq i$. Note that replacement monotonicity implies non-bossiness.

[38] Observe that the domain $\mathcal{SP}(q)$ is rich relative to q.

We summarize the above discussion by arguing that in our private goods setting, from the perspective of deriving domains that are interesting for mechanism design, it is more fruitful to postulate the existence of *one* tops-only rule rather than attempt to rule out all non-tops-only rules endogenously.

Data availability

No data was used for the research described in the article.

References

Adachi, T., 2010. The uniform rule with several commodities: a generalization of Sprumont's characterization. J. Math. Econ. 46, 952–964.
Amorós, A., 2002. Single-peaked preferences with several commodities. Soc. Choice Welf. 19, 57–67.
Barberà, S., Jackson, M., 1995. Strategy-proof exchange. Econometrica 63, 51–87.
Barberà, S., Sonnenschein, H., Zhou, L., 1991. Voting by committees. Econometrica 59, 595–609.
Barberà, S., Gul, F., Stacchetti, E., 1993. Generalized median voter schemes and committees. J. Econ. Theory 61, 262–289.
Barberà, S., Jackson, M., Neme, A., 1997. Strategy-proof allotment rules. Games Econ. Behav. 18, 1–21.
Barberà, S., Berga, D., Moreno, B., 2020. Arrow on domain conditions: a fruitful road to travel. Soc. Choice Welf. 54, 237–258.
Benassy, J.-P., 1982. The Economics of Market Disequilibrium. Academic Press.
Berga, D., 1998. Strategy-proofness and single-plateaued preferences. Math. Soc. Sci. 35, 105–120.
Bonifacio, A., Massó, J., Neme, P., 2023. Preference restrictions for strategy-proof and simple rules: local and weakly single-peaked domains. J. Math. Econ. 106, 102845.
Chatterji, S., Massó, J., 2018. On strategy-proofness and the salience of single-peakedness. Int. Econ. Rev. 59, 163–189.
Chatterji, S., Sen, A., 2011. Tops-only domains. Econ. Theory 46, 255–282.
Chatterji, S., Zeng, H., 2018. On random social choice functions with the tops-only property. Games Econ. Behav. 109, 413–435.
Chatterji, S., Zeng, H., 2019. Random mechanism design on multidimensional domains. J. Econ. Theory 182, 25–105.
Chatterji, S., Sanver, R., Sen, A., 2013. On domains that admit well-behaved strategy-proof social choice functions. J. Econ. Theory 148, 1050–1073.
Ching, S., Serizawa, S., 1998. A maximal domain for the existence of strategy-proof rules. J. Econ. Theory 78, 157–166.
Cho, W.J., Thomson, W., 2017. On the extension of the uniform rule for more than one commodity: existence and maximality results. Mimeo.
Hurwicz, L., 1972. On informationally decentralized systems. In: McGuire, C.B., Radner, R. (Eds.), Decision and Organization. North-Holland, Amsterdam, pp. 297–336.
Klaus, B., Peters, H., Storcken, T., 1998. Strategy-proof division with single-peaked preferences and individual endowments. Soc. Choice Welf. 15, 297–311.
Le Breton, M., Weymark, J., 1999. Strategy-proof social choice with continuous separable preferences. J. Math. Econ. 32, 47–85.
Mas-Colell, A., 1992. Equilibrium theory with possibly satiated preferences. In: Majumdar, M. (Ed.), Equilibrium and Dynamics: Essays in Honor of David Gale. St. Martin's Press, New York, pp. 201–213.
Massó, J., Neme, A., 2001. A maximal domain of preferences for the division problem. Games Econ. Behav. 37, 367–387.
Massó, J., Neme, A., 2004. A maximal domain of preferences for strategy-proof, efficient, and simple rules in the division problem. Soc. Choice Welf. 23, 187–206.
Mizobuchi, H., Serizawa, S., 2006. Maximal domain for strategy-proof rules in allotment economies. Soc. Choice Welf. 27, 195–210.
Morimoto, S., Serizawa, S., Ching, S., 2013. A characterization of the uniform rule with several commodities and agents. Soc. Choice Welf. 40, 871–911.
Moulin, H., 1980. On strategy-proofness and single-peakedness. Public Choice 35, 437–455.
Moulin, H., 1984. Generalized Condorcet-winners for single peaked and single Plateau preferences. Soc. Choice Welf. 1, 127–147.
Moulin, H., 2017. One-dimensional mechanism design. Theor. Econ. 12, 587–619.
Serizawa, S., 2002. Inefficiency of strategy-proof rules for pure exchange economies. J. Econ. Theory 106, 219–241.
Serizawa, S., Weymark, J.A., 2003. Efficient strategy-proof exchange and minimum consumption guarantees. J. Econ. Theory 109, 246–263.
Sprumont, Y., 1991. The division problem with single-peaked preferences: a characterization of the uniform allocation rule. Econometrica 59, 509–519.
Sprumont, Y., 1995. Strategyproof collective choice in economic and political environments. Can. J. Econ. 28, 68–107.
Zhou, L., 1991. Inefficiency of strategy-proof allocation mechanisms in pure exchange economies. Soc. Choice Welf. 8, 247–257.