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A general characterization of the minimum price Walrasian rule with reserve prices ☆,☆☆

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ABSTRACT

We consider economies consisting of arbitrary numbers of agents and objects, and study the multi-object allocation problem with monetary transfers. Each agent obtains at most one object (unit-demand), and has non-quasi-linear preferences, which accommodate income effects or nonlinear borrowing costs. The seller may derive benefit from objects. We show that on the non-quasi-linear domain, the minimum price Walrasian rule in which reserve prices are equal to the benefit the seller derives is the only rule satisfying four desirable properties; efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Moreover, we characterize the minimum price Walrasian rule by efficiency, overall individual rationality, and strategy-proofness.

1. Introduction

Auctions are popular methods to allocate public assets efficiently. Recent examples include spectrum license auctions, vehicle ownership auctions, land auctions, etc. An important feature of those auctions is that several objects are sold simultaneously, which promotes the efficiency of allocations. Another feature is that winning prices are often quite high.[1] This causes nonnegligible income effects for bidders or faces them with nonlinear borrowing costs. As a result, their preferences are not quasi-linear. Also, reserve prices have an important role to prevent small revenues for the seller and mitigate collusion among the buyers since they limits the maximum gain from the collusion (Ausubel and Cramton, 2004). It often happens that the number of bidders is smaller than the

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☆☆ Independently, Wakabayashi and Serizawa (2021) have studied the rules with reserve prices and Sakai and Serizawa (2021) have studied the rules for an arbitrary number of agents and objects. Our paper combines their works because there are many things in common.

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[1] For example, in the 3G Spectrum license auction in U.K. (2000), the total revenue for five licenses amounted to £22.5 billion, which is approximately 2.5% of the GDP of U.K. in 2000. See Binmore and Klemperer (2002) for details.

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number of objects because a large competitor excludes small bidders, which makes reserve prices important. In this article, by paying a special attention to reserve prices, we investigate the existence of efficient multi-object rules, and the possibility of characterizing them in non-quasi-linear environments.

In addition to efficiency, we impose other desirable properties; *individual rationality for the buyers* ensures that a buyer is not worse off from participation; *no-subsidy* ensures that the payment of each agent is always nonnegative; *strategy-proofness* ensures that they have dominant strategies of reporting their preferences truthfully.

In non-quasi-linear environments, a prominent rule is "the minimum price Walrasian" (MPW) rule (Demange and Gale, 1985). When bidders have unit-demand preferences, the MPW rule satisfies not only efficiency, but also individual rationality for the buyers, no-subsidy, and strategy-proofness. More importantly, under the assumptions 1) the number of agents is larger than the number of objects and 2) the seller derives no benefit from the objects, the MPW rule is the unique rule satisfying the four properties (Morimoto and Serizawa, 2015). This result implies the distinguished theoretical merit of the MPW rule.

However, since the MPW rule excludes reserve prices, it may generate small revenue and collusion. Objects to be auctioned are owned by the public or private sectors. Land and spectrum frequency licenses are examples. Those sectors derive benefit from the objects, and the benefit should be taken into account in order to allocate the objects efficiently. These factors are in violation of the assumptions 1) and 2) of Morimoto and Serizawa (2015). We extend their analysis by incorporating seller's benefit for arbitrary numbers of agents and objects.

In our model, there are n bidders (hereafter "agents") and m objects. Each agent obtains at most one object (unit-demand) and pays the seller. The seller derives benefit $v^x \geq 0$ from object x. He has a quasi-linear preference over the set of allocations, and so his *net revenue* of an allocation is the sum of the agents' payments minus the sum of the benefit of the objects he sells. An allocation is *efficient for a preference profile and the seller's benefit* if no allocation can increase the seller's net revenue without worsening any buyer's welfare.

An (*allocation*) *rule* determines, for each preference profile, the object each agent receives and how much each agent pays. We mainly focus on the above four properties of desirability of rules, but we also consider a condition we call *overall individual rationality*; it adds to individual rationality for the seller that each agent pays at least the benefit the seller enjoys from the object the agent receives.

For each preference profile, *Walrasian equilibria with reserve prices* exist (Alkan and Gale, 1990). Also, the set of Walrasian prices has a lattice structure. Hence, there are *minimum Walarasian equilibrium prices with reserve prices* (Demange and Gale, 1985). The "MPW rule with reserve prices" is the rule which assigns to each preference profile the MPW equilibrium with reserve prices. When reserve prices are set equal to $v = (v^1, \ldots, v^m)$, the MPW rule with the reserve prices satisfies efficiency (Proposition 1).

Extending Morimoto and Serizawa's (2015) result, we show that *the minimum price Walrasian rule with reserve prices v is the only rule satisfying efficiency for v, individual rationality for the buyers, no-subsidy, and strategy-proofness* (Theorem). We also show that *the minimum price Walrasian rule with reserve prices v is the only rule satisfying efficiency for v, overall individual rationality, and strategy-proofness* (Corollary 2).

We emphasize that reserve prices equal to v in Theorem does not directly follow from efficiency for v. Only when combined with individual rationality for the buyers, no subsidy, and strategy-proofness, does efficiency with respect to v imply that reserve prices are equal to v.[2] Although, the seller's benefit should be taken into account for practical applications, the consequence of doing so is not straightforward. This article analyzes the practically important applications and establishes results that can be applied to more general environments than considered in the previous literature.

We also emphasize that although our results are extensions of Morimoto and Serizawa's (2015), there are several points in which their proof fails in our model.[3] These points necessitates that we develop a novel proof technique, and makes our extensions far from trivial.

This article is organized as follows. Section 2 introduces the model and basic concepts and checks the properties of MPW rules with reserve prices. Our results are in Section 3. Section 4 provides sketches of our proofs in the intuitive way. Section 5 provides a discussion. Section 6 discusses related literature, and Section 7 concludes. Proofs are relegated to Appendix.

2. The model

Let $N \equiv \{1, \ldots, n\}$ be a set of agents (or buyers) and $M \equiv \{1, \ldots, m\}$ be a set of objects. Not consuming an object in M is called consuming the "null object." Let $L \equiv M \cup \{0\} = \{0, 1, \ldots, m\}$, where 0 denotes the null object. Call $a \in M$ a "real object." Numbers n and m are arbitrary. Each agent consumes at most one object (unit-demand). A typical **(consumption) bundle** for agent $i \in N$ is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$: agent i receives object x_i and pays t_i.

Each agent has a complete and transitive preference relation R_i over $L \times \mathbb{R}$. Let I_i and P_i be the indifference preference relation and strict preference relation associated with R_i. A typical class of preference relations is denoted by $\mathcal{R}$. We call $\mathcal{R}^n$ a **domain**. We introduce some properties of preference relations.

Continuity: For each $z_i \in L \times \mathbb{R}$, $\{z_i' \in L \times \mathbb{R} : z_i' \, R_i \, z_i\}$ and $\{z_i' \in L \times \mathbb{R} : z_i \, R_i \, z_i'\}$ are both closed.
Finite compensation: For each $(a, t) \in L \times \mathbb{R}$ and each $b \in L$, there exists $t' \in \mathbb{R}$ such that $(a, t) \; I_i \; (b, t')$.

[2] We demonstrate this point by an example in Section 3. See Example 1 on p. 5.
[3] See Appendix C for the detailed explanation.

Money monotonicity: For each $a \in L$ and each $t, t' \in \mathbb{R}$, if $t < t'$, then $(a,t)\ P_i\ \left(a, t'\right)$.
Object desirability: For each $a \in M$ and each $t \in \mathbb{R}$, $(a,t)\ P_i\ (0,t)$.

A preference relation $R_i \in \mathcal{R}$ is **classical** if it satisfies the above four properties. Let $\mathcal{R}^C$ be the set of all classical preference relations. A classical preference is such that all objects are "goods" for agent i, where goods are the objects which are preferred to the null object.

A preference relation $R_i \in \mathcal{R}$ is **extended** if it satisfies the first three properties. Let $\mathcal{R}^E$ be the set of all extended preference relations. Note that $\mathcal{R}^C \subsetneq \mathcal{R}^E$. A extended preference allows for "bads", where bads are the objects which are not preferred to the null object. We assume that $\mathcal{R} \subseteq \mathcal{R}^E$.

A **preference profile** is a list of preference relations $R \equiv (R_1, \ldots, R_n)$. Given $i \in N$ and $N' \subseteq N$, let $R_{-i} \equiv (R_j)_{j \neq i}$ and $R_{-N'} \equiv (R_j)_{j \in N \setminus N'}$.

An **(feasible) object allocation** is an n-tuple $x \equiv \left(x_1, \ldots, x_n\right) \in L^n$ such that for each $i, j \in N$ with $i \neq j$, if $x_i = x_j$, then $x_i = x_j = 0$. Hence, each real object is assigned to at most one agent. Let X be the set of all feasible object allocations. A **(feasible) allocation** is an n-tuple $z \equiv \left(z_1, \ldots, z_n\right) = ((x_1, t_1), \ldots, (x_n, t_n)) \in (L \times \mathbb{R})^n$ with $(x_1, \ldots, x_n) \in X$. Let Z be the set of all feasible allocations. We also write an allocation as a pair $z = (x, t)$. We denote the object allocation and payments at $z' \in Z$ by $x' = (x'_1, \ldots, x'_n)$ and $t' = (t'_1, \ldots, t'_n)$.

For each object $a \in M$, the seller derives benefit $v^a \in \mathbb{R}_+$. Let $v \equiv (v^1, \ldots, v^m) \in \mathbb{R}^m_+$ and $v^0 = 0$. Then, given $z \in Z$, the seller's net revenue is equal to $\sum_{i \in N} \left(t_i - v^{x_i}\right)$. The seller's benefit v is common knowledge. Let $\mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^m_+$.

An allocation $z' \in Z$ **(Pareto-)dominates** $z \in Z$ for $R \in \mathcal{R}$ (and for v) if (i) for each $i \in N$, $z'_i\ R_i\ z_i$ and (ii) $\sum_{i \in N} \left(t'_i - v^{x'_i}\right) > \sum_{i \in N} \left(t_i - v^{x_i}\right)$. An allocation $z \in Z$ is **(Pareto-)efficient** for $R \in \mathcal{R}^n$ (and for v) if there is no other allocation that dominates z.[4]

An **(allocation) rule** is a mapping $f = (x, t) : \mathcal{R}^n \to Z$. Given a rule $f = (x, t)$ and a preference profile $R \in \mathcal{R}^n$, we denote agent i's assigned object and payment by $f_i(R) = (x_i(R), t_i(R)) \in L \times \mathbb{R}$. Also, we write $f(R) = \left(f_1(R), \cdots, f_n(R)\right) \in Z$.

We introduce some desirable properties of rules.

(Pareto-)Efficiency (for v): For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R and v.
Individual rationality for the buyers: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ (0,0)$.
No-subsidy: For each $R \in \mathcal{R}$ and each $i \in N$, $t_i(R) \geq 0$.
Individual rationality for the seller: For each $R \in \mathcal{R}^n$ and each $i \in N$, $t_i(R) \geq v^{x_i(R)}$.
Overall individual rationality: For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R)\ R_i\ (0,0)$ and $t_i(R) \geq v^{x_i(R)}$.
Strategy-proofness: For each $R \in \mathcal{R}^n$, each $i \in N$ and each $R'_i \in \mathcal{R}$,

$$f_i(\overset{\text{true}}{R_i}, R_{-i})\ \overset{\text{true}}{R_i}\ f_i(\overset{\text{false}}{R'_i}, R_{-i}).$$

We must say "strategy-proofness for the buyers" precisely, not "strategy-proofness," to be parallel with individual rationality. However, since the seller's benefit is common knowledge, he cannot take a strategical action. Hence, we do not need to consider "strategy-proofness for the seller," and so we just use "strategy-proofness" as "strategy-proofness for the buyers" in this paper.

A **price (vector)** is an m-tuple $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$. Given $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$, we abuse notation and let p denote the $(m+1)$-tuple $(p^0, p^1, \ldots, p^m)$, where $p^0 = 0$ is the price of the null object, when this causes no confusion. Given $i \in N$, $R_i \in \mathcal{R}$ and $p \in \mathbb{R}^m_+$, agent i's **demand set** $D(R_i, p)$ is the set of his most preferred objects among $\{(0,0), (1, p^1), \ldots, (m, p^m)\}$, that is,

$$D(R_i, p) \equiv \left\{a \in L : \forall b \in L, (a, p^a)\ R_i\ (b, p^b)\right\}.$$

Next, we define the concept of a Walrasian equilibrium with reserve prices $r \equiv (r^1, \ldots, r^m) \in \mathbb{R}^m_+$, where $r^0 = 0$. It is a pair of an allocation and a price vector such that (i) each agent receives an object he demands and pays its price, (ii) the price of each object is no less than its reserve price, and (iii) the price of each unassigned object is equal to the reserve price. Given $r \equiv (r^1, \ldots, r^m) \in \mathbb{R}^m_+$, the set of all price vectors such that the price of each object is no less than its reserve price is denoted by $\mathbb{R}^m_{r+}$, that is,

$$\mathbb{R}^m_{r+} \equiv \{p \in \mathbb{R}^m_+ : \forall a \in M, p^a \geq r^a\}.$$

Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, a pair $(z, p) = ((x_i, t_i)_{i \in N}, (p^a)_{a \in M}) \in Z \times \mathbb{R}^m_{r+}$ is a **Walrasian equilibrium with reserve prices r for R** if

WE-i: for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$,
WE-ii: for each $a \in M \setminus \{x_i\}_{i \in N}$, $p^a = r^a$.

Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, let $W(R, r)$ be the set of Walrasian equilibria with reserve prices r for R, and define

$$Z(R, r) \equiv \{z \in Z : \exists p \in \mathbb{R}^m_{r+}, (z, p) \in W(R, r)\}$$

[4] This condition is equivalent to the following: there is no $z' \in Z$ such that (i) for each $i \in N$, $z'_i\ R_i\ z_i$, (ii) for some $j \in N$, $z'_j\ P_j\ z_j$ and (iii) $\sum_{i \in N}(t'_i - v^{x'_i}) \geq \sum_{i \in N}(t_i - v^{x_i})$.

and

$$P(R,r) \equiv \left\{ p \in \mathbb{R}^m_{r+} : \exists z \in Z, (z,p) \in W(R,r) \right\}.$$

Fact 1 *(Alkan and Gale, 1990).* For each $R \in \mathcal{R}^n$ and each $r \in \mathbb{R}^m_+$, there is a Walrasian equilibrium, that is, $W(R,r) \neq \emptyset$.

By Fact 1, for each $R \in \mathcal{R}^n$ and each $r \in \mathbb{R}^m_+$, $Z(R,r) \neq \emptyset$ and $P(R,r) \neq \emptyset$.

Fact 2 *(Demange and Gale, 1985).* For each $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, there is $p \in P(R,r)$ such that for each $p' \in P(R,r)$, $p \leq p'$.[5]

Fact 2 shows the existence of a minimum Walrasian equilibrium price. Given $R \in \mathcal{R}^n$ and $r \in \mathbb{R}^m_+$, we denote the **minimum Walrasian equilibrium price with reserve prices *r* for *R*** by $p_{\min}(R,r)$, and we define the set of the minimum price Walrasian allocations with reserve prices r for R by

$$Z_{\min}(R,r) \equiv \{z \in Z : (z, p_{\min}(R,r)) \in W(R,r)\}.$$

Given $r \in \mathbb{R}^m_+$, an allocation rule f is a **minimum price Walrasian rule with reserve prices *r*** if for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R,r)$.

We discuss the properties of the minimum price Walrasian rule with reserve prices.

Fact 3. Let $n, m \in \mathbb{N}$, $\upsilon \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, for each $r \in \mathbb{R}^m_+$, the minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies (i) individual rationality for the buyers, (ii) no-subsidy, and (iii) strategy-proofness (Demange and Gale, 1985).[6]

Note that reserve prices with which the minimum price Walrasian rule is associated are not necessarily equal to seller's benefit. Hence, the minimum price Walrasian rule with reserve prices is not necessarily efficient or individual rational for the seller. Proposition 1 shows the necessary and sufficient condition for these properties to be satisfied. The proof is relegated to Appendix B.1.

Proposition 1. *Let $n, m \in \mathbb{N}$, $\upsilon \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Let $r \in \mathbb{R}^m_+$.*

(i) *The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies efficiency if and only if $r = \upsilon$.*
(ii) *The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies individual rationality for the seller if and only if $r \geq \upsilon$.*

Note that in a competitive market, the seller sells each object if and only if its price is no less than the benefit he derives from the object. Hence, the "if" part of Proposition 1 (i) is essentially The First Welfare Theorem. On the other hand, the "only if" part of Proposition 1 (i) is not straightforward. The discrepancies between the reserve prices of objects and the benefit derived by the seller, if they exist, might distort allocations and cause inefficiency. However, since objects are indivisible, small discrepancies may not distort object allocations but only change payments, maintaining efficiency. Hence, the "only if" part of Proposition 1 (i) holds for the minimum price Walrasian rules with reserve prices, but not for a fixed preference profile.

The "if" part of Proposition 1 (ii) is straightforward, but the "only if" part of Proposition 1 (ii) also holds only for the minimum price Walrasian rules with reserve prices, but not for a fixed preference profile.

As a corollary of Fact 3 and Proposition 1, we get the following result.

Corollary 1. *Let $n, m \in \mathbb{N}$, $\upsilon \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, the minimum price Walrasian rule with reserve prices $r = \upsilon$ on $\mathcal{R}^n$ satisfies efficiency, overall individual rationality, no-subsidy, and strategy-proofness.*

3. Characterizations

In this section, we give two characterizations.

Theorem. *Let $n, m \in \mathbb{N}$, $\upsilon \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, a rule f on $\mathcal{R}^n$ satisfies efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness if and only if it is a minimum price Walrasian rule with the reserve prices $r = \upsilon$.*

Morimoto and Serizawa (2015) assume that $m < n$ and $r = \upsilon = \mathbf{0}$, and show that on the classical domain $\mathcal{R}^n$, a rule f satisfies the properties of Theorem if and only if it is a minimum price Walrasian rule. Our result generalizes their result for any $n, m \in \mathbb{N}$ and $\upsilon \in \mathbb{R}^m_+$. The proof of Theorem is provided in Section 4 (p. 6).

[5] $p \leq p'$ means that $p^a \leq p'^a$ for each $a \in M$.

[6] Precisely, they show that the minimum price Walrasian rule f is *group strategy-proof*: that is, for each $R \in \mathcal{R}^n$ and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N'$, $f_i(R'_{N'}, R_{-N'})\, P_i\, f_i(R)$.

We emphasize that $r = v$ is not straightforward from the four properties of Theorem. Proposition 1 says that, by individual rationality for the seller, $r \geq v$ (Proposition 1-ii), but Theorem does not assume individual rationality for the seller. Proposition 1 also says that $r = v$ is necessary for a minimum price Walrasian rule with reserve prices r to be efficient (Proposition 1-i), but it is not true for non-Walrasian rules. Example 1 below demonstrates this point.

Example 1 *(Efficiency and no-subsidy).* Let $N = \{1,2\}$, $M = \{a,b\}$ and $v \in \mathbb{R}^2_+$ with $v^a = v^b = 2\varepsilon > 0$. Let $R = (R_1, R_2) \in \mathcal{R}^2$ be such that for each $i \in N$, $(0, -\varepsilon)\, I_i\, (a, \varepsilon)\, I_i\, (b, \varepsilon)$.

Let f be such that $f(R) = ((a,\varepsilon),(b,\varepsilon))$ and for each $R' \in \mathcal{R}^2 \setminus \{R\}$, $f(R') \in Z(R', v)$. Then, f satisfies efficiency and no-subsidy, but it is not the case that $r = v$ because $t_1(R) = \varepsilon < 2\varepsilon = v^a$.

In Theorem, $r = v$ does not follow from efficiency only, but it follows from the four properties together. Therefore, it is not trivial to obtain $r = v$ from the properties of Theorem.

The proposition below says that no-subsidy is equivalent to individual rationality for the seller if the rule satisfies efficiency and strategy-proofness. The proof is relegated to Appendix B.2.

Proposition 2. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. If a rule f on $\mathcal{R}^n$ satisfies efficiency, no-subsidy, and strategy-proofness, then f satisfies individual rationality for the seller.*

Note that in Example 1, by $t_1(R) = \varepsilon < v^{x_1(R)}$ and $t_2(R) = \varepsilon < v^{x_2(R)}$, f violates individual rationality for the seller. Therefore, this example also demonstrates that if strategy-proofness is dropped, then Proposition 2 does not hold.

The following example says that if efficiency is dropped, then Proposition 2 does not hold.[7]

Example 2 *(No-subsidy and Strategy-proofness).* Let $n = m \in \mathbb{N}$ and $v \in \mathbb{R}^m_+$ be such that for each $a \in M$, $v^a > 0$.

Let f be such that for each $R \in \mathcal{R}^n$ and $i \in N$, $f_i(R) = (i, 0)$. Then, f satisfies no-subsidy and strategy-proofness, but violates individual rationality for the seller.

As a corollary of Theorem and Proposition 2, we get the following result.

Corollary 2. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Then, a rule f on $\mathcal{R}^n$ satisfies efficiency, overall individual rationality, and strategy-proofness if and only if it is a minimum price Walrasian rule with reserve prices $r = v$.*

The following examples show the independence of each properties.

Example 3 *(Dropping efficiency).* Let $v \in \mathbb{R}^m_+$ and $r \in \mathbb{R}^m_+$ be such that $r \geq v$ and $r \neq v$. Then, the minimum price Walrasian rule with reserve prices r satisfies individual rationality for the buyers, no-subsidy (or individual rationality for the seller) and strategy-proofness, but violates efficiency (by Proposition 1).

Example 4 *(Dropping strategy-proofness).* Let $v \in \mathbb{R}^m_+$. Let $r = v$. Then, the maximum price Walrasian rule with reserve prices r satisfies efficiency, individual rationality for the buyers and no-subsidy (or individual rationality for the seller), but violates strategy-proofness.

Given $R \in \mathcal{R}^n$, $r \in \mathbb{R}^m_+$, and $e \in \mathbb{R}$, $(z, p) \in Z \times \mathbb{R}^m_{r+}$ is a **Walrasian equilibrium with reserve prices *r* and an entry fee *e*** if (WE-i*) for each $i \in N$, $x_i \in \{a \in L : \forall b \in L, (a, p^a + e)\, R_i\, (b, p^b + e)\}$ and $t_i = p^{x_i} + e$, and (WE-ii) for each $a \in M \setminus \{x_i\}_{i \in N}$, $p^a = r^a$ (Morimoto and Serizawa, 2015). Note that if $r = v$, for each $e \in \mathbb{R}$, the minimum price Walrasian rule with reserve prices r and an entry fee e satisfies efficiency and strategy-proofness.[8]

Example 5 *(Dropping individual rationality for the buyers).* Let $v \in \mathbb{R}^m_+$. Let $r = v$ and $e > 0$. Then, the minimum price Walrasian rule with reserve prices r and an entry fee e satisfies efficiency, no-subsidy (or individual rationality for the seller), and strategy-proofness, but violates individual rationality for the buyers.[9]

Example 6 *(Dropping no-subsidy).* Let $v \in \mathbb{R}^m_+$. Let $r = v$ and $e < 0$. Then, the minimum price Walrasian rule with reserve prices r and an entry fee e satisfies efficiency, individual rationality for the buyers, and strategy-proofness, but violates no-subsidy (or individual rationality for the seller).

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[7] We omit the counterexample dropping no-subsidy, since it is obvious that if the rule does not satisfy no-subsidy, then it also does not satisfy individual rationality for the seller.

[8] We can prove the result in the same way as in the case without an entry fee.

[9] The incentive not to participate is not covered by strategy-proofness. This is because each agent must pay at least a positive entry fee regardless of his outcome; especially even if he loses.

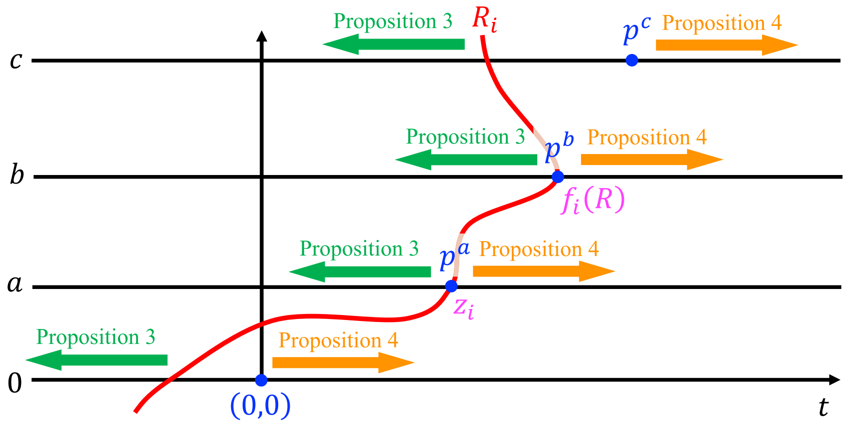

Fig. 1. Illustration of Propositions 3 and 4.

A rule f satisfies **no-subsidy for losers** if for each $R \in \mathcal{R}$ and $i \in N$ with $x_i(R) = 0$, $t_i(R) \geq 0$. Morimoto and Serizawa (2015) show that if $m < n$ and $v = \mathbf{0}$, then efficiency, individual rationality for the buyers, no-subsidy for losers and strategy-proofness imply no-subsidy. Hence, their result holds even if they weaken no-subsidy to no-subsidy for losers. However, in our setting, i.e., unless $m < n$ and $v = \mathbf{0}$, the above four properties do not imply no-subsidy (Zhou and Serizawa, 2018, Remark 8). Such a fact demonstrates the possibility that removing the assumptions that $m < n$ and $v = \mathbf{0}$ may bring results different from Morimoto and Serizawa (2015).

Example 7 *(Weakening no-subsidy).* [10] Let $m \geq n$ and $v = \mathbf{0}$. Let $r = v$ and $e < 0$. Then, the minimum price Walrasian rule with reserve prices r and an entry fee e satisfies efficiency, individual rationality for the buyers, no-subsidy for losers and strategy-proofness, but violates no-subsidy. This is because there is no agent $i \in N$ such that $x_i(R) = 0$, and no-subsidy for losers holds vacuously.

4. Sketch of proofs

In this section, we sketch the proof of Theorem. By Corollary 1, in order to prove Theorem, it suffices to show that if an allocation rule on $\mathcal{R}^n$ satisfies efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, then it is the minimum price Walrasian rule with reserve prices v. To prove this, we use the following propositions.

Proposition 3. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, and let $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R, v)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.*

In the single-object case, if an agent who obtains an object is charged more than the minimum Walrasian price, then the agent can misreport his preference downward, so that by efficiency, he obtains the object but by individual rationality for the buyers, his payment is cheaper, contradicting strategy-proofness. Proposition 3 generalizes such an idea to the multi-object case.

Proposition 4. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies Pareto-efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, and let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. Then, for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$.*

In the single-object case, if an agent can obtain an object with a price cheaper than minimum Walrasian price level, then when his valuation on the object is not high enough to obtain the object, he can get better by misreporting his preference upward to obtain the object, contradicting strategy-proofness. Proposition 4 generalizes such an idea to the multi-object case.

Fig. 1 illustrates Propositions 3 and 4. In Fig. 1, each horizontal line corresponds to an object, and the horizontal-axis represents transfers. A point on the horizontal lines is a consumption bundle of an agent. The distance of a bundle from the vertical line represents an agent's payment when the bundle is to the right hand side of the vertical line, while it represents the transfer to an agent from the seller when the bundle is to the left hand side of the vertical line. For example, the point z_i means that agent i receives object a and pays the amount equal to the distance of z_i from the vertical line. The red curve in this figure is the indifference curve of R_i. By money monotonicity, bundles in the left hand side of the indifference curve are preferred to those in the right hand side of it. Note that in Fig. 1, $D(R_i, p) = \{a, b\}$.

In Fig. 1, Proposition 3 is expressed by green arrows, which says that the outcome of the rule is at least as good as the minimum price Walrasian allocation for the buyers. Proposition 4 is expressed by orange arrows, which says that the payment is no less than the minimum Walrasian price. Then the intersection of the propositions in Fig. 1 is $\{(a, p^a), (b, p^b)\}$. That is, $f(R)$ is either (a, p^a) or

[10] This example is from Zhou and Serizawa (2018).

(b, p^b). This means that $(f(R), p)$ satisfies WE-i for the agent. Since it follows from efficiency that $(f(R), p)$ satisfies WE-ii, $(f(R), p)$ is the minimum price Walrasian equilibrium. This is what we want to show. Formally, we can show Theorem as follows.

Proof of Theorem. Let $R \in \mathcal{R}^n$ and $(z, p) \in W_{\min}(R, v)$. First we show $(f(R), p)$ satisfies WE-i. Let $i \in N$. Then,

$$f_i(R) \underset{\text{Prop.3}}{R_i} z_i \underset{z \in Z(R,v)}{R_i} (x_i(R), p^{x_i(R)}) \underset{\text{Prop.4}}{R_i} f_i(R),$$

which implies $f_i(R)\, I_i\, z_i\, I_i\, (x_i(R), p^{x_i(R)})$. By $z_i\, I_i\, (x_i(R), p^{x_i(R)})$, $x_i(R) \in D(R_i, p)$, and by $f_i(R)\, I_i\, (x_i(R), p^{x_i(R)})$, $t_i(R) = p^{x_i(R)}$. Hence, $(f(R), p)$ satisfies WE-i for agent i.

Next we show $(f(R), p)$ satisfies WE-ii. Suppose not. Let $M^+ = \{a \in M : p^a > v^a\}$. Since some $a \in M^+$ is assigned to no agent at $f(R)$, $\{x_i(R)\}_{i\in N} \cap M^+ \subsetneq M^+$. Since (z, p) satisfies WE-ii, $M^+ = \{x_i\}_{i\in N} \cap M^+$. Hence, $(*)$ $\{x_i(R)\}_{i\in N} \cap M^+ \subsetneq \{x_i\}_{i\in N} \cap M^+$, and so

$$\sum_{i\in N} (p^{x_i(R)} - v^{x_i(R)}) \underset{p\geq v}{=} \sum_{a\in\{x_i(R)\}_{i\in N}\cap M^+} (p^a - v^a) \underset{(*)}{<} \sum_{a\in\{x_i\}_{i\in N}\cap M^+} (p^a - v^a) \underset{p\geq v}{=} \sum_{i\in N} (p^{x_i} - v^{x_i}).$$

Since for each $i \in N$, $f_i(R)\, I_i\, z_i$, $t_i(R) = p^{x_i(R)}$, and $t_i = p^{x_i}$ (WE-i of $f(R)$ and z), z Pareto-dominates $f(R)$, contradicting efficiency. Hence, $(f(R), p)$ satisfies WE-ii. □

As we can see from the above proof, Propositions 3 and 4 are necessary to prove Theorem. Hence, in the following subsections, we sketch the proofs of the propositions. In each subsection, we first summarize the basic structure of the proof, next introduce some preliminaries and explain how to use the preliminaries, and finally sketch the proof. The formal proofs of Propositions 3 and 4 are relegated to Appendices B.3 and B.4.

Morimoto and Serizawa (2015) also use the results which are parallel to Propositions 3 and 4 to prove that a rule satisfying our four properties is a minimum price Walrasian rule. Although we owe them this basic structure of proof, there are several points that make their proof inapplicable to our model. In the last paragraph of each subsection, we briefly explain such a point. See Appendix C for more detailed discussions in this point.

4.1. Proof Sketch of Proposition 3

We show Proposition 3 in the following way: Let a rule f satisfy efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Suppose that for some preference profile R and some agent $i \in N$, $z_i\, P_i\, f_i(R)$, where z is a minimum price Walrasian allocation for R. Without loss of generality, let $i = 1$. First, by using $z_1\, P_1\, f_1(R)$, we construct a special preference R'_1 such that $(*)$ R'_1 evaluates objects other than x_1 so little, but (i) $x_1(R'_1, R_{-1}) \neq x_1$, and for some $j \in N$, (ii) $x_j(R'_1, R_{-1}) = x_1$ and (iii) $t_j(R'_1, R_{-1}) > t_1$. By (i) and (ii), $j \neq 1$. Without loss of generality, let $j = 2$. Then, since z is a Walrasian allocation for R, by (ii) and (iii), $z_2\, R_2\, z_1\, P_2\, f_2(R'_1, R_{-1})$. Second, by using $z_2\, P_2\, f_2(R'_1, R_{-1})$, we similarly construct a special preference R'_2 such that $x_2(R'_1, R'_2, R_{-1,2}) \neq x_2$. If $x_2(R'_1, R'_2, R_{-\{1,2\}}) = x_1$, then $(*)$ and (iii) violate individual rationality for agent 2. Hence, $x_2(R'_1, R'_2, R_{-\{1,2\}}) \neq x_1$, and so $x_2(R'_1, R'_2, R_{-\{1,2\}}) \notin \{x_1, x_2\}$. Therefore, by (ii), there is $k \notin \{1,2\}$ such that $x_k(R'_1, R'_2, R_{-\{1,2\}}) \in \{x_1, x_2\}$. Without loss of generality, let $k = 3$. By (ii), $z_3\, P_3\, f_3(R'_1, R'_2, R_{-\{1,2\}})$. Inductively and similarly, we construct the preferences of other agents, and replace their preferences with the constructed preferences one by one to obtain $z_n\, P_n\, f_n(R'_1, \ldots, R'_{n-1}, R_n)$. Finally, by using $z_n\, P_n\, f_n(R'_1, \ldots, R'_{n-1}, R_n)$, we construct R'_n such that $x_n(R') \neq x_n$. By (ii), $x_n(R') \in \{x_1, \ldots, x_{n-1}\}$. Hence, $(*)$ and (iii) imply that $f(R')$ fails individual rationality for agent n. This is a contradiction, and so $f_i(R)\, R_i\, z_i$ for each R and each i.

In the following, we provide detailed explanations about the construction of R' and the proofs of facts (i) to (iii).

The first class of preferences is a class of preferences that favor a certain bundle. Given a bundle $(a, t) \in M \times \mathbb{R}_+$, a preference relation $R_i \in \mathcal{R}$ is **(a,t)-favoring** if R_i prefers a priced at t to any other object even if offered for free; that is, if for each $b \in L \setminus \{a\}$, $(a, t)\, P_i\, (b, 0)$.

The second class is a special type of favoring preferences, which are parsimonious. Given $(a, t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, a (a, t)-favoring preference relation $R_i \in \mathcal{R}$ is **ε-greedy for (a,t)** if (1) R_i is not willing to pay more than $t + 2\varepsilon$ for a, that is, $(a, t + 2\varepsilon)\, I_i\, (0, 0)$, and moreover (2) R_i is not willing to pay more than ε for any other object, that is, for each $b \in M \setminus \{a\}$, $(b, \varepsilon)\, I_i\, (0, 0)$. Note that the smaller $\varepsilon \in \mathbb{R}_{++}$ is, the more parsimonious this preference is.

The third class is the class of **quasi-linear** preferences, whose indifference curves are parallel, that is, for each $(a, t), (b, t') \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}$,

$$(a, t)\, I_i\, (b, t') \iff (a, t - \delta)\, I_i\, (b, t' - \delta).$$

Note that given $\varepsilon \in \mathbb{R}_{++}$ and $(a, t) \in M \times \mathbb{R}_+$, a quasi-linear preference R_i that is ε-greedy for (a, t) is unique. Fig. 2 illustrates the quasi-linear preferences R_i that is ε-greedy for (a, t).

Next, we explain how to use quasi-linear and ε_i-greedy preferences in the proof. Let $R \in \mathcal{R}^n$ be a given preference profile and $z \in Z_{\min}(R, v)$ be a minimum price Walrasian allocation. Let R'_i be the quasi-linear and ε_i-greedy preference relation for $z_i = (x_i, t_i)$ with small ε_i.

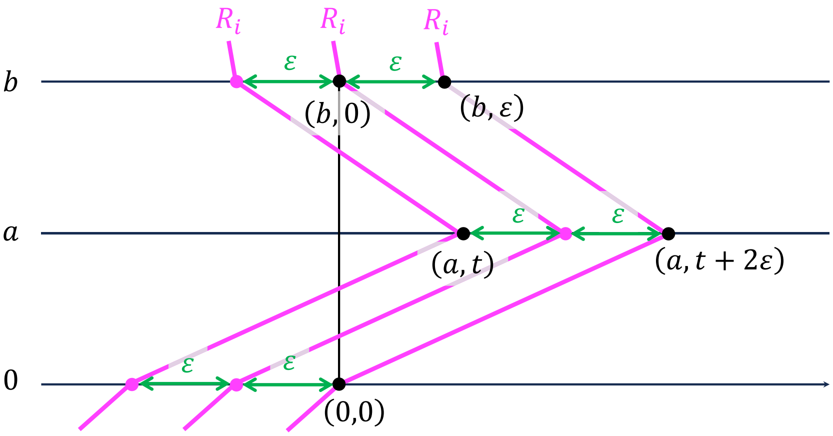

Fig. 2. Illustration of a ε-greedy and quasi-linear preference.

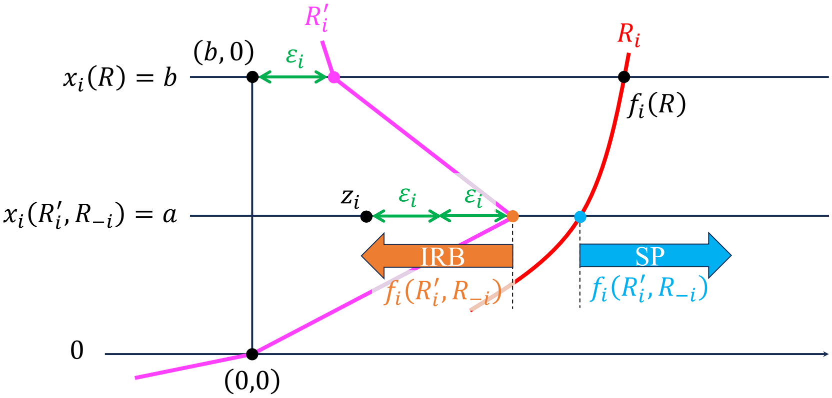

Fig. 3. Illustration of fact (i).

Suppose $z_i\ P_i\ f_i(R)$ for some $i \in N$. If $x_i(R_i', R_{-i}) = x_i$, then since ε_i-greediness and individual rationality for the buyers (IRB) require $t_i(R_i', R_{-i}) \leq t_i + 2\varepsilon_i$, for small ε_i, $z_i\ P_i\ f_i(R)$ implies $f_i(R_i', R_{-i})\ P_i\ f_i(R)$. See Fig. 3. This contradicts strategy-proofness (SP). Hence, (i) *if $z_i\ P_i\ f_i(R)$, then agent i cannot receive the object x_i at (R_i', R_{-i}).*[11]

Suppose no agent receives x_i at (R_i', R_{-i}). Since R_i' is quasi-linear and ε_i-greedy for z_i, he is indifferent between $f_i(R_i', R_{-i})$ and $(x_i, t_i + \varepsilon_i + t_i(R_i', R_{-i}))$. Hence, assigning $(x_i, t_i + \varepsilon_i + t_i(R_i', R_{-i}))$ to agent i makes no agent worse off, but increases the seller's net revenue. See Fig. 4. This contradicts efficiency. Hence, (ii) *the object x_i must be assigned to some agent at (R_i', R_{-i}).*[12]

Suppose that for other agent $j \neq i$, $x_j(R_i', R_{-i}) = x_i$ but $t_j(R_i', R_{-i}) < t_i + \varepsilon_i$. Let $\delta_j = t_i + \varepsilon_i - t_j(R_i', R_{-i})$ be the gap. We may assume that R_j is $f_j(R_i', R_{-i})$-favoring, but is almost indifferent between $f_j(R_i', R_{-i})$ and $(x_i(R_i', R_{-i}), 0)$.[13] Therefore, since $(x_i(R_i', R_{-i}), -\delta_j)\ P_j\ f_j(R_i', R_{-i})$ and $(x_j(R_i', R_{-i}), t_j(R_i', R_{-i}) + \delta_j + t_i(R_i', R_{-i}))\ I_i'\ f_i(R_i', R_{-i})$, by exchanging i's object and j's object at (R_i', R_{-i}), we can improve agents' welfare while keeping the seller's net revenue. See Fig. 5. This contradicts efficiency. Hence, (iii) *if other agent $j \neq i$ receives x_i at (R_i', R_{-i}), then the j's payment at (R_i', R_{-i}) is no less than $t_i + \varepsilon_i$.*[14]

Now, we demonstrate how to use the above facts (i), (ii), and (iii) to prove Proposition 3 for the case of $n = 3$. Suppose that for some agent, say agent 1, $z_1\ P_1\ f_1(R)$.

First, for a small ε_1, we construct the quasi-linear preference relation R_1' that is ε_1-greedy for z_1. Then, by (i) and (ii), some other agent, say agent 2, receives x_1 at (R_1', R_2, R_3), i.e., $x_2(R_1', R_2, R_3) = x_1$. By (iii), $t_1 + \varepsilon_1 \leq t_2(R_1', R_2, R_3)$, and so $z_1\ P_2\ f_2(R_1', R_2, R_3)$. Hence, by WE-i, $z_2\ R_2\ z_1\ P_2\ f_2(R_1', R_2, R_3)$.

Next, for small $\varepsilon_2 < \varepsilon_1$, we construct the quasi-linear preference relation R_2' that is ε_2-greedy for z_2. By $z_2\ P_2\ f_2(R_1', R_2, R_3)$ and (i), $x_2(R_1', R_2', R_3) \neq x_2$. If $x_2(R_1', R_2', R_3) = x_1$, then by ε_2-greediness and (iii), $(0,0)\ I_2'\ (x_1, \varepsilon_2)\ P_2'\ (x_1, t_1 + \varepsilon_1)\ R_2'\ f_2(R_1', R_2', R_3)$. This contradicts individual rationality for the buyers. Hence, $x_2(R_1', R_2', R_3) \neq x_1$, and so $x_2(R_1', R_2', R_3) \notin \{x_1, x_2\}$. By (ii), x_1 and x_2

[11] This statement corresponds to Lemma 1. See Appendix B.3 on p. 17.

[12] This statement corresponds to Lemma 2 (i). See Appendix B.3 on p. 17.

[13] It follows from Fact 6 in Appendix A on p. 14.

[14] This statement corresponds to Lemma 2 (ii). See Appendix B.3 on p. 17.

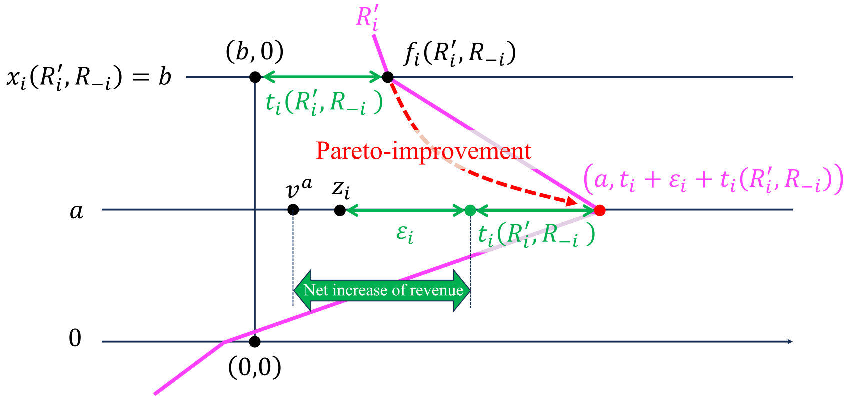

Fig. 4. Illustration of fact (ii).

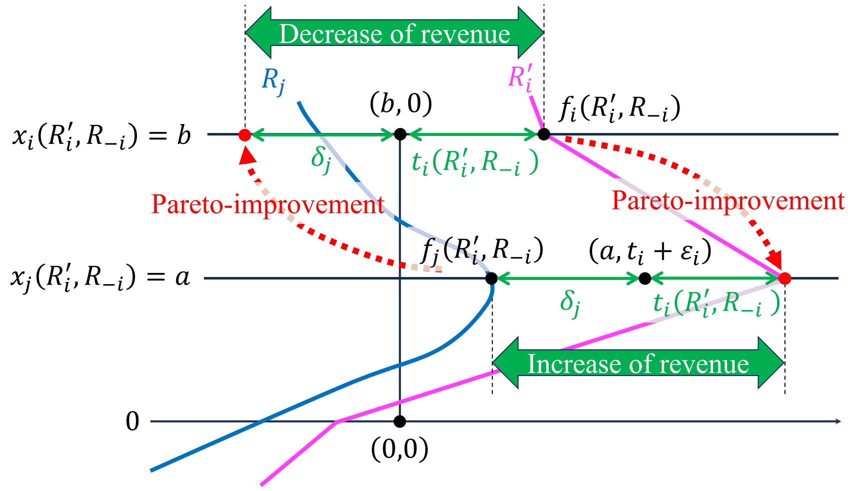

Fig. 5. Illustration of fact (iii).

are assigned to some agents, so that $x_3(R_1', R_2', R_3) \in \{x_1, x_2\}$. Without loss of generality, let $x_3(R_1', R_2', R_3) = x_1$. By WE-i and (iii), $z_3 \, R_3 \, z_1 \, P_3 \, f_3(R_1', R_2', R_3)$.

Finally, for small $\varepsilon_3 < \varepsilon_2 < \varepsilon_1$, we construct the quasi-linear preference relation R_3' that is ε_3-greedy for z_3. Then, by $z_3 \, P_3 \, f_3(R_1', R_2', R_3)$ and (i), $x_3(R_1', R_2', R_3') \neq x_3$. By (ii), x_1, x_2 and x_3 are assigned to some agents, and so $x_3(R_1', R_2', R_3') \in \{x_1, x_2\}$. Without loss of generality, let $x_3(R_1', R_2', R_3') = x_1$. Then, by ε_3-greediness and (iii), $(0,0) \, I_3' \, (x_1, \varepsilon_3) \, P_3' \, (x_1, t_1 + \varepsilon_1) \, P_3' \, f_3(R_1', R_2', R_3')$. However, this contradicts individual rationality for the buyers. This completes the proof.

We finally discuss how our proof of Proposition 3 is different from Morimoto and Serizawa's (2015) proof. For each $i \in N$ and $a \in M$, let V_i^a be such that $(a, V_i^a) \, I_i \, (0,0)$. Let $V^a(1)$ be the first highest among $\{V_1^a, \dots, V_n^a\}$, $V^a(2)$ be the second highest among $\{V_1^a, \dots, V_n^a\}$, and so on. In Morimoto and Serizawa's (2015) proof of the result that is parallel to our Proposition 3, $V^a(m)$, i.e., the m-th highest among $\{V_1^a, \dots, V_n^a\}$, plays an important role. However, when $m > n$, $V^a(m)$ cannot be specified. Hence, their proof technique cannot be applied to our model where n and m are arbitrary finite numbers. See more details in Appendix C.1.

4.2. *Proof Sketch of Proposition 4*

We show Proposition 4 in the following way: Let R be a given preference profile and p be a minimum Walrasian price for R. For agent i, let R_i' be the p-**indifferent** preference relation, i.e., all the objects are indifferent at price p. Moreover, we assume that each p-indifferent preference relation, R_i' satisfies a condition, which we call the **income effect condition**. Then, p-indifference implies that whenever some or all preferences of R are replaced with the p-indifferent preference, the minimum Walrasian equilibrium price remains the same. We start from R', and inductively replaces the preferences of some set N' of agents, $R'_{N'}$, with their original preferences, $R_{N'}$, and show that for each $N' \subseteq N$ and each $i \in N$, $t_i(R_{N'}, R'_{N'}) \geq p^{x_i(R_{N'}, R'_{N'})}$. When $N' = N$, the proof is completed.

Let $\tilde{R}$ be a preference profile in the induction process, that is, for some $N' \subseteq N$, $\tilde{R} = (R_{N'}, R'_{N'})$. As induction hypothesis, we assume that for each $N'' \subseteq N$ with $|N''| \leq |N'| - 1$ and each $i \in N$, $t_i(R_{N''}, R'_{-N''}) \geq p^{x_i(R_{N''}, R'_{-N''})}$. Proposition 3 implies: (i) for each

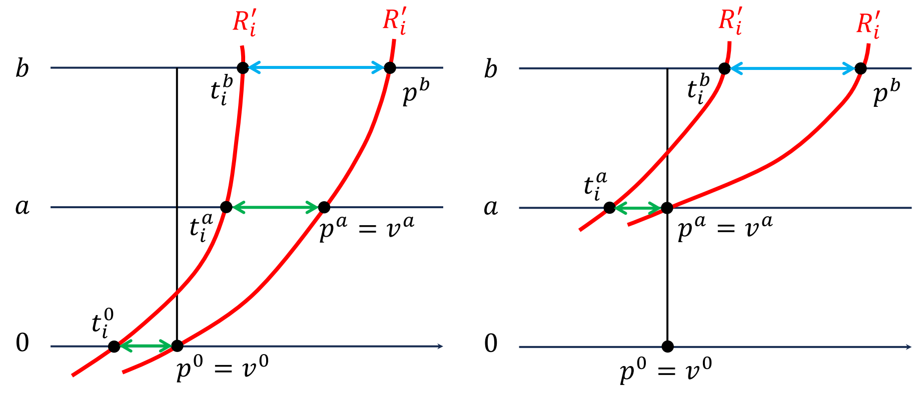

Fig. 6. Illustration of a $p_{\min}(R, v)$-indifferent preference satisfying the income effect condition.

$i \in N$, if $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$, then $x_i(\tilde{R}) \in D(\tilde{R}_i, p)$ and $t_i(\tilde{R}) = p^{x_i(\tilde{R})}$. Moreover, individual rationality for the seller implies: (ii) for each $i \in N$, if $p^{x_i(\tilde{R})} = v^{x_i(\tilde{R})}$, then $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$. Strategy-proofness and (i) imply: (iii) for each $i \in N'$, if $t_i(R_{N''}, R'_{-N''}) \geq p^{x_i(R_{N''}, R'_{-N''})}$, where $N'' = N' \setminus \{i\}$, then $t_i(R_{N'}, R'_{-N'}) \geq p^{x_i(R_{N'}, R'_{-N'})}$. Efficiency, the income effect condition, and (iii) imply: (iv) for each $i \in N \setminus N'$, if $p^{x_i(\tilde{R})} > v^{x_i(\tilde{R})}$, $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$. Hence, for each $i \in N$, if $i \in N'$, then by (iii) and induction hypothesis, $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$, if $i \in N \setminus N'$ and $p^{x_i(\tilde{R})} = v^{x_i(\tilde{R})}$, then by (ii), $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$, and if $i \in N \setminus N'$ and $p^{x_i(\tilde{R})} > v^{x_i(\tilde{R})}$, then by (iv), $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}$.

In the following, we provide detailed explanations about the construction of R' and the proofs of facts (i) to (iv).

To demonstrate the idea of the proof of Proposition 4, we prove the proposition in the simplest case where $M = \{a, b\}$ and $n = 2$. Although the proof of this case is much simpler than that of the general case of arbitrary numbers of agents and objects, it includes many important points of the idea of the proof in the general case. After the proof of the simplest case, we explain how to extend the proof to the general case.

Let $R = (R_1, R_2)$. If some $i \in N$ demands 0 at the price vector $v = (v^a, v^b)$, or if two agents demand different objects at v, then $p_{\min}(R, v) = v$ and so individual rationality for the seller (Proposition 2) implies Proposition 4 directly. Hence, assume that two agents demand the same real object, say object b only at v, that is, two agents compete for b at v. We refer object b as a **competitive object**, and objects 0 and a as **noncompetitive objects**. Moreover, we refer an agent who receives b at some allocation as a **competitive agent**, and an agent who receives 0 or a as a **noncompetitive agent**. For each $i \in N$, let v_i^0 and v_i^a be such that $(b, v_i^0)\, I_i\, (0, 0)$ and $(b, v_i^a)\, I_i\, (a, v^a)$, and let $v_i = \min\{v_i^0, v_i^a\}$. Since each $i \in N$ demands only b at v, $v_i > v^b$. Assume $v_2 \geq v_1$ since the case of $v_2 \leq v_1$ can be treated by a symmetric way. Then $v_1 = \min\{v_1^0, v_1^a, v_2^0, v_2^a\} > v^b$.

First, we show the structure of the minimum price Walrasian equilibrium. Given $p \in P(R, v)$, if $p^b < v_1 \leq v_2$, then by the definition of v_i and $p^a \geq v^a$, $D(R_1, p) = D(R_2, p) = \{b\}$. However, this is a contradiction, and so $p^b \geq v_1$. Hence, (v^a, v_1) is a lower bound of Walrasian equilibrium prices. Moreover, by $b \in D(R_2, (v^a, v_1))$ and $c \in D(R_1, (v^a, v_1))$ where $c \in \{0, a\}$ is such that $v_1 = v_1^c$, (v^a, v_1) is a Walrasian equilibrium price. Hence, $p_{\min}(R, v) = (v^a, v_1)$. Note that the price of a competitive object is strictly higher than its reserve price, and the price of a noncompetitive object is equal to its reserve price.

Next, we construct a preference which is used in the proof. Note that by $p^b_{\min}(R, v) = v_1$ and $v_1 > v^b$, $p^b_{\min}(R, v) > 0$. A preference relation $R'_i \in \mathcal{R}$ is $\boldsymbol{p_{\min}(R,v)}$**-indifferent** if (1) $(a, p^a_{\min}(R, v))\, I'_i\, (b, p^b_{\min}(R, v))\, I'_i\, (0, 0)$ whenever $p^a_{\min}(R, v) > 0$, and (2) $(a, p^a_{\min}(R, v))\, I'_i\, (b, p^b_{\min}(R, v))\, P'_i\, (0, 0)$ otherwise. Fig. 6 illustrates a $p_{\min}(R, v)$-indifferent preference relation. A $p_{\min}(R, v)$-indifferent preference R'_i satisfies the **income effect condition** if for each $t_i^b < p^b_{\min}(R, v)$, (3) $v^a - t_i^a < p^b_{\min}(R, v) - t_i^b$, and (4) if $p^a_{\min}(R, v) > 0$, then $v^0 - t_i^0 < p^b_{\min}(R, v) - t_i^b$, where t_i^c with $c \in \{0, a\}$ is such that $(c, t_i^c)\, I'_i\, (b, t_i^b)$. Fig. 6 also illustrates the income effect condition.

Let $\tilde{\mathcal{R}} = \{(R_1, R_2), (R'_1, R_2), (R_1, R'_2), (R'_1, R'_2)\}$ where R'_i for agent $i \in N$ is a $p_{\min}(R, v)$-indifferent preference relation satisfying the income effect condition. For each $\tilde{R} \in \tilde{\mathcal{R}}$, by $D(R'_i, (v^a, v_1)) \supseteq \{a, b\}$, $p_{\min}(\tilde{R}, v) = (v^a, v_1)$. Hence, the minimum Walrasian price does not change for any $\tilde{R} \in \tilde{\mathcal{R}}$,[15] and we denote it by $p_{\min}$. Given $\tilde{R} \in \tilde{\mathcal{R}}$, we call agent $i \in N$ with $\tilde{R}_i = R_i$ an **original preference agent**, and agent $i \in N$ with $\tilde{R}_i = R'_i$ a $\boldsymbol{p_{\min}}$**-indifferent agent**.

Next, we explain how to use $p_{\min}$-indifferent preferences satisfying the income effect condition. Let a rule f satisfy efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Let $\tilde{R} \in \tilde{\mathcal{R}}$.

If $t_i(\tilde{R}) \geq p^{x_i(\tilde{R})}_{\min}$, then by Proposition 3 and WE-i, $f_i(\tilde{R})\, \tilde{R}_i\, z_i\, \tilde{R}_i\, (x_i(\tilde{R}), p^{x_i(\tilde{R})}_{\min})\, \tilde{R}_i\, f_i(\tilde{R})$, where $z \in W_{\min}(R, v)$. Therefore, $f_i(\tilde{R}) \tilde{I}_i\, z_i\, \tilde{I}_i\, (x_i(\tilde{R}), p^{x_i(\tilde{R})}_{\min})$, which implies $x_i(\tilde{R}) \in D(\tilde{R}_i, p_{\min})$ and $t_i(\tilde{R}) = p^{x_i(\tilde{R})}_{\min}$. See Fig. 7. Hence, (i) *if the payment of an agent is no less than the minimum Walrasian price, then the allocation rule satisfies WE-i for the agent.*[16]

[15] Lemma 6 on p. 21 shows the same result for the general case.

[16] This statement corresponds to Lemma 3. See Appendix B.4 on p. 19.

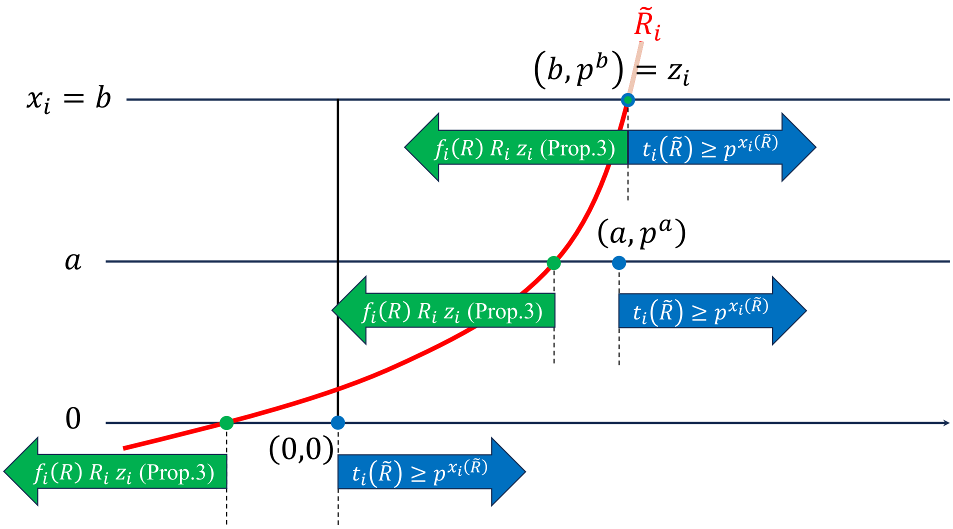

Fig. 7. Illustration of fact (i).

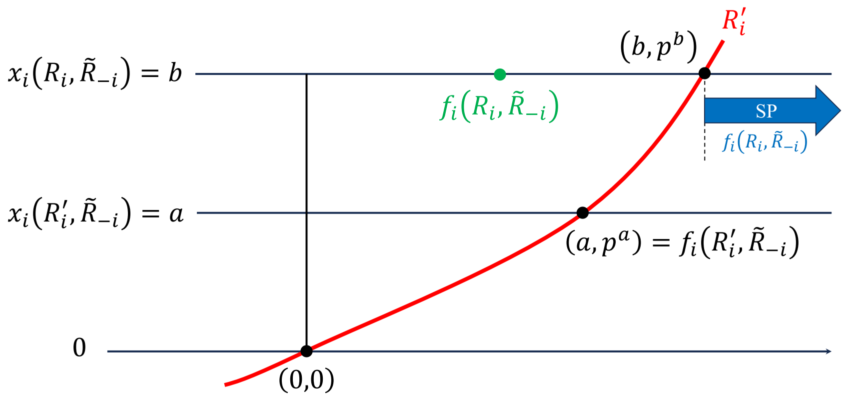

Fig. 8. Illustration of fact (iii).

If i is a noncompetitive agent at $f(\tilde{R})$, then by $p_{\min}^{x_i(\tilde{R})} = v^{x_i(\tilde{R})}$ and individual rationality for the seller (Proposition 2), $t_i(\tilde{R}) \geq p_{\min}^{x_i(\tilde{R})}$. Hence, (ii) *for a noncompetitive agent, his payment is no less than the minimum Walrasian price.*

Assume that i is an original preference agent at $\tilde{R}$ and $t_i(R_i', \tilde{R}_j) \geq p_{\min}^{x_i(R_i', \tilde{R}_j)}$, where $j \neq i$. Then, by $p_{\min}$-indifference and (i), $(x_i(\tilde{R}), p_{\min}^{x_i(\tilde{R})})\ I_i'\ f_i(R_i', \tilde{R}_j)$. Hence, if $t_i(\tilde{R}) < p_{\min}^{x_i(\tilde{R})}$, then $f_i(\tilde{R})\ P_i'\ f_i(R_i', \tilde{R}_j)$. See Fig. 8. However, this contradicts strategy-proofness. Hence, (iii) *for an original agent, if his payment when his preference is p_{min}-indifferent is no less than the minimum Walrasian price, then so is his payment when his preference is replaced with the original one.*[17]

Assume that agent i is competitive and $p_{\min}$-indifferent at $f(\tilde{R})$, and $b \in D(\tilde{R}_j, p_{\min})$, where $j \neq i$. Since i is competitive, j is noncompetitive. By (i) and (ii), $x_j(\tilde{R}) \in D(\tilde{R}_j, p_{\min})$ and $t_j(\tilde{R}) = p_{\min}^{x_j(\tilde{R})}$, so that by $b \in D(\tilde{R}_j, p_{\min})$, $(x_i(\tilde{R}), p_{\min}^{x_i(\tilde{R})}) = (b, p_{\min}^b)\ \tilde{I}_j\ f_j(\tilde{R})$. Hence, if $t_i(\tilde{R}) < p_{\min}^{x_i(\tilde{R})}$, then by the income effect condition, some reassignment improves agents' welfare while keeping the seller's revenue. See Fig. 9. However, this contradicts efficiency. Hence, (iv) *for a competitive and $p_{\min}$-indifferent agent, if another noncompetitive agent demands the competitive object, then the payment of the competitive $p_{\min}$-indifferent agent is no less than the minimum Walrasian price.*[18]

By using the above facts (i)-(iv), we sketch the proof of Proposition 4.

First, we show $t_i(R_1', R_2') \geq p_{\min}^{x_i(R_1', R_2')}$ for each i. If i is noncompetitive, then by (i), $t_i(R_1', R_2') \geq p_{\min}^{x_i(R_1', R_2')}$. If i is competitive, then j is noncompetitive and by $p_{\min}$-indifference, $b \in D(R_j', p_{\min})$. Hence, by (iv), $t_i(R_1', R_2') \geq p_{\min}^{x_i(R_1', R_2')}$.

[17] This statement is formally shown in the proof of Proposition 4. See Appendix B.4 on p. 19.

[18] This statement corresponds to Lemma 7 in Appendix B.4 on p. 19.

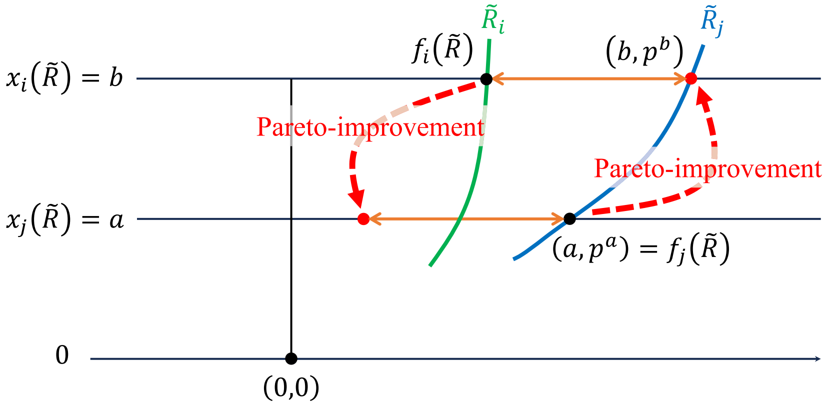

Fig. 9. Illustration of fact (iv).

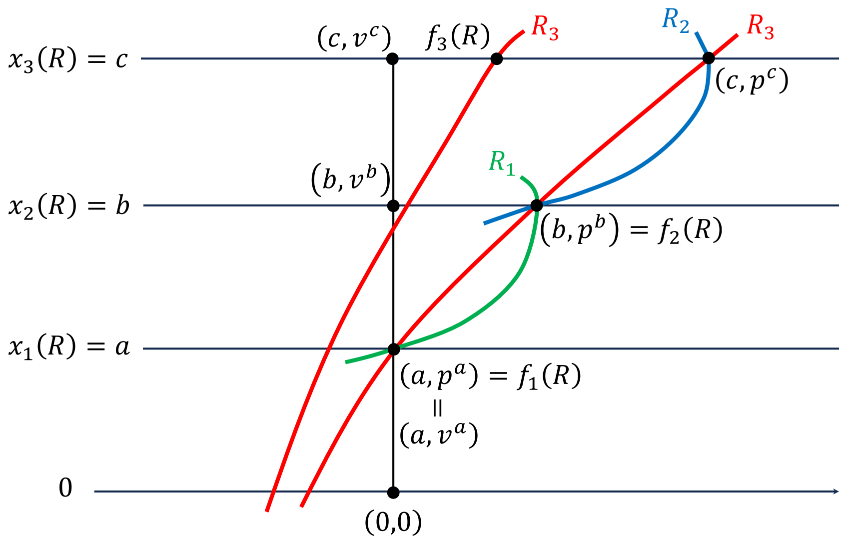

Fig. 10. Illustration of a construction of a sequence.

Next, we show $t_i(R_1, R_2') \geq p_{\min}^{x_i(R_1, R_2')}$ for each i. By $t_1(R_1', R_2') \geq p_{\min}^{x_1(R_1', R_2')}$ and (iii), $t_1(R_1, R_2') \geq p_{\min}^{x_1(R_1, R_2')}$. By $p_{\min}^b = v_1$, $b \in D(R_1, p_{\min})$. Hence, by (iv), $t_2(R_1, R_2') \geq p_{\min}^{x_2(R_1, R_2')}$. In the same way, we can show $t_i(R_1', R_2) \geq p_{\min}^{x_i(R_1', R_2)}$ for each i.

Finally, we show $t_i(R_1, R_2) \geq p_{\min}^{x_i(R_1, R_2)}$ for each i. By $t_1(R_1', R_2) \geq p_{\min}^{x_1(R_1', R_2)}$ and (iii), $t_1(R_1, R_2) \geq p_{\min}^{x_1(R_1, R_2)}$. By $t_2(R_1, R_2') \geq p_{\min}^{x_2(R_1, R_2')}$ and (iii), $t_2(R_1, R_2) \geq p_{\min}^{x_2(R_1, R_2)}$.

Now, we explain how to extend the proof of the simplest case to the general case. In order to prove Proposition 4, we extend the above facts (i)-(iv) in the simplest case to the general case. We can extend facts (i)-(iii) in straightforward ways. However, as shown in the paragraph before Fig. 9, the proof of fact (iv) highly depends on $m = 2$. Hence, the difficulty of extending the sketch to the general case arises from fact (iv). To overcome this difficulty, we show the existence of the sequence of agents such that: (a) it starts from a noncompetitive agent, (b) each agent demands his own object and his next agent's object, and (c) it is terminated by a competitive and $p_{\min}$-indifferent agent.[19] If the payment of the final agent is less than the minimum Walrasian price, then by the income effect condition, some reassignment improves agents' welfare while keeping the seller's revenue. See Fig. 10. Hence, by the sequence, we can extend fact (iv) to the general case, and in the same way as in the simplest case, we can show Proposition 4.

Finally, we discuss how our proof of Proposition 4 is different from Morimoto and Serizawa's (2015) proof. To show the result for a competitive and $p_{\min}$-indifferent agent, they construct a sequence of agents that starts from the agent who receives the null object, not a noncompetitive object. However, if $m \geq n$, agents do not necessarily receive the null object. Hence, as we explained in the previous paragraph, instead we show the existence of a sequence of agents that starts from a noncompetitive agent. See more details in Appendix C.2.

[19] See Lemma 4 in Appendix B.4 on p. 19.

5. Discussion: multi-seller model

In this paper, we assume that there is a single seller, who is endowed with all the objects. However in this section, we consider a multi-seller model, i.e., the situation where there are m sellers, and each seller is endowed with one object.

To set up the multi-seller model, let $S \equiv \{1, \dots, m\}$ be the set of sellers. Seller $i \in S$ is endowed with object $i \in M$. A consumption bundle of seller $i \in S$ is a pair $z_i \equiv (x_i, t_i) \in \{0,1\} \times \mathbb{R}$, where $x_i = 1$ means that i sells his object while $x_i = 0$ means that i keeps his object, and t_i is a transfer to seller i. Each seller $i \in S$ has a quasilinear preference relation R_i with valuation v^i, i.e., for each two bundles $z_i \equiv (x_i, t_i)$ and $z'_i \equiv (x'_i, t'_i)$, $z_i\ R_i\ z'_i$ if and only if $t_i - x_i \cdot v^i \geq t'_i - x'_i \cdot v^i$. A **(feasible) allocation of multi-seller model** is an $(n+m)$-tuple $z = (x, t) \in (L^n \times \{0,1\}^m) \times \mathbb{R}^{n+m}$ such that

(i) for each $i, j \in N$ with $i \neq j$, if $x_i = x_j$, then $x_i = x_j = 0$,

(ii) for each $a \in S$, $x_a = \begin{cases} 1 & \text{if } \exists i \in N,\ x_i = a \\ 0 & \text{otherwise} \end{cases}$.

(iii) $\sum_{i \in N} t_i = \sum_{a \in S} t_a$.

Given a feasible allocation $z = (z_i)_{i \in N \cup S}$ of the multi-seller model, we can construct a feasible allocation $z' = (z'_i)_{i \in N}$ of the single-seller model, i.e., the original model, as follows: for each $i \in N$, $z'_i = z_i$. Given a feasible allocation $z = (z_i)_{i \in N}$ of the single-seller model, we can also construct a feasible allocation $z' = (z'_i)_{i \in N \cup S}$ of the multi-seller model as follows: for each $i \in N$, $z'_i = z_i$; for each $a \in S$, if for some $i \in N$, $x_i = a$, then $x'_a = 1$, and otherwise $x'_a = 0$; and $(t'_a)_{a \in M}$ is such that $\sum_{a \in S} t'_a = \sum_{i \in N} t_i$. Therefore, the two feasible sets of the two models are essentially equivalent. Hence, we also use Z as the set of all feasible allocations of the multi-seller model.

An allocation $z' \in Z$ **multi-seller (Pareto-)dominates** $z \in Z$ for $R \in \mathcal{R}^n$ if (i) for each $i \in N \cup S$, $z'_i\ R_i\ z_i$, and (ii) for some $j \in N \cup S$, $z'_j\ P_j\ z_j$. An allocation $z \in Z$ is **multi-seller efficient** for $R \in \mathcal{R}^n$ if there is no $z' \in Z$ that multi-seller dominates z. To distinguish two conditions of efficiency, we call the original condition of efficiency of the single-seller model[20] as *single-seller efficiency*. However, as we show in Proposition 5, the two conditions of efficiency are equivalent. The proof is relegated to Appendix B.5.

Proposition 5. *Let $z \in Z$ and $R \in \mathcal{R}^n$. Then, z is multi-seller efficient for R if and only if z is single-seller efficient for R.*

By Proposition 5, our characterization results of the minimum price Walrasian rule in the single-seller model also hold in the multi-seller model. Hence, our characterization results can be applied to the situation where there are m sellers, and each seller is endowed with one object.

6. Related literature

In the seminal work of Myerson (1981), the reserve price has an important role in characterizing optimal auctions. In a symmetric environment where there is a single object and preferences are quasi-linear, the Vickrey auction rule with a suitably set reserve price maximizes seller revenues. Since his article, a vast number of articles analyzes optimal auctions in environments of single object and quasi-linear preferences. However, there are several strands of literature analyzing auction rules of multiple objects in environments where preferences are non-quasi-linear. We discuss such strands of literature.

The first strand of the literature we discuss analyzes efficient auction rules of homogeneous goods for non-quasi-linear but unit-demand preferences. In the cases where objects are homogeneous, Saitoh and Serizawa (2008), and Sakai (2008) characterize the generalized Vickrey rule by efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness.

The second strand of the literature extends the setting of the first strand to heterogeneous objects. Morimoto and Serizawa (2015) show that in cases where the number of agents is greater than objects, the MPW rule is the unique rule satisfying efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Zhou and Serizawa (2018) maintain the assumption of unit-demand, but focus on the special class of preferences, "the common-tiered domains." It says that objects are partitioned into several tiers, and if objects are equally priced, agents prefer an object in a higher tier to one in a lower tier. They show that when the tier including nth highest objects is singleton, the MPW rule is the only rule satisfying the above four properties on the common-tiered domains.

The third strand of the literature analyze efficient auction rules with non-quasi-linear preferences admitting multi-demand. Kazumura and Serizawa (2016) study classes of preferences that include unit-demand preferences and additionally includes at least one multi-demand preference, and show that no rule satisfies the four properties on such a domain. Malik and Mishra (2021) study the special classes of preferences, "dichotomous" domains. A preference is *dichotomous* if there is a set of objects such that the valuations of its supersets are constant and the valuations of other sets are zero. A *dichotomous domain* includes all such dichotomous preferences for a given set of objects. They show that no rule satisfies the four properties on a dichotomous domain, but that the generalized Vickrey rule is the only rule satisfying the four properties on a class of dichotomous preferences exhibiting positive income effects.

[20] See p.5 for the definition of the original condition of efficiency.

This strand also includes Baisa (2020). He assumes that objects are homogeneous, and preferences are non-quasi-linear and multi-demand. He shows that on the class of preferences exhibiting decreasing marginal valuations, positive income effect, and single-crossing property, if the preferences are parametrized by one dimensional types, there is a rule satisfying efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, but that if types of preferences are multi-dimensional, no rule satisfies these properties.

The above three strands of literature on efficient auction rules in non-quasi-linear environments takes no account of the seller's benefit from objects to be auctioned, and excludes reserve prices. Our article is different from these strands of literature in this point.

The fourth strand of literature works on optimal auctions for non-quasi-linear preferences. On the unit-demand setting, Kazumura et al. (2020) and Sakai and Serizawa (2023) show that the MPW rule maximizes ex-post revenue among the class of auction rules satisfying individual rationality for the buyers, no-subsidy, non-wastefulness, equal treatment of equals, and strategy-proofness, and such a revenue maximizing rule is unique. These works also exclude reserve prices by non-wastefulness, which means that no agent prefers his own bundle to unassigned object with no payment.

The fifth strand works on efficient auction rules with reserve prices. Sakai (2013) studies strategy-proof auction in the single object setting on the quasi-linear domain, and assumes "non-imposition," which requires that the payment of the agent with zero valuation be zero. He shows that the allocation rule satisfies weak efficiency, non-imposition, and strategy-proofness if and only if it is the Vickrey auction rule with a reserve price or no-trade rule, where no-trade rule is the rule such that for each preference profile, each agent gets no object and pays nothing.

Andersson and Svensson (2014) study a housing allocation model where preferences are non-quasi-linear and unit demand, and rents are bounded not only below by reserve prices but also above by price ceilings set by governments. They introduce "rationing price equilibrium (RPE)," which a hybrid of Walrasian equilibrium and a rationing mechanism with fixed prices for a given priority structure. A RPE is not Pareto-efficient, but constrained efficient for a given priority structure. They show that the minimum RPE price uniquely exists, and that the minimum RPE mechanism is group strategy-proof. Our result is different from this strand in that we derive reserve prices from the seller's benefit from objects.

The final strand is about auctions with agents' budget constraints. When there are (weak) budget constraints, agents have non-quasi-linear preferences. Che and Gale (1998) consider auctions with soft and hard budget constraints, and show that first-price auctions with reserve price achieve higher expected revenues and efficiency than second-price auctions with same reserve price. Recently, Herings and Zhou (2022) study a matching with contracts model where buyers have unit demanded and non-quasi-linear preferences, and face financial constraints. In their model, a competitive equilibrium may not exist. Hence they introduce a new equilibrium concept, *quantity-constrained competitive equilibrium (QCCE)*, and show the existence of the equilibrium. In a QCCE, the buyers have an expectation about the supply of contracts, which is called a *quantity constraint*, and they demand the best contract out of the contracts that they expect to be supplied. Moreover, they also show that the set of QCCE outcomes is equivalent to the set of *stable* outcomes, and forms the lattice structure.

7. Conclusion

We extended the result of Morimoto and Serizawa (2015) to the settings where there are arbitrary numbers of agents and objects, and the seller may derive benefit from objects to be auctioned. Then we showed 1) the minimum price Walrasian rule with reserve prices set equal to the benefit the seller derives is a unique rule satisfying efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, and 2) it is also a unique rule satisfying efficiency, overall individual rationality, and strategy-proofness. Our result demonstrates that the minimum price Walrasian rule has distinguished theoretical merits and applications in a variety of environments.

Declaration of competing interest

We listed all of financially supporting institutions to our paper (joint with Yuya Wakabayashi and Ryosuke Sakai) entitled "A general characterization of the minimum price Walrasian rule with reserve prices" in the acknowledge of the manuscript, which we submit to Games and Economic Behavior. However, none of the supporting institutions inappropriately influenced (biased) our work.

Appendix A. Preliminary definitions and results

Given $p \in \mathbb{R}^m_{r+}$ and $R \in \mathcal{R}^n$, $M' \subseteq M$ is **overdemanded at p for R** if

$$|\{i \in N : D(R_i, p) \subseteq M'\}| > |M'|,$$

and $M' \subseteq M$ is **weakly overdemanded at p for R** if

$$|\{i \in N : D(R_i, p) \subseteq M'\}| \geq |M'|.$$

Given $p \in \mathbb{R}^m_{r+}$ and $R \in \mathcal{R}^n$, $M' \subseteq M$ is **underdemanded at p for R** if

$$[\forall a \in M', p^a > r^a] \text{ and } |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| < |M'|,$$

and $M' \subseteq M$ is **weakly underdemanded at p for R** if

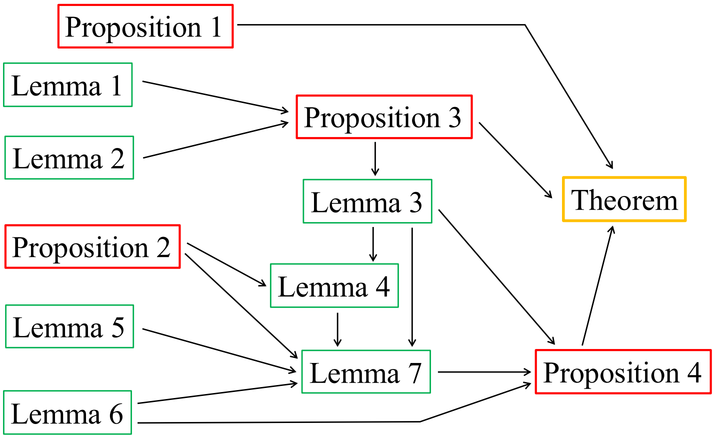

Fig. 11. Relationships among lemmas and propositions for Theorem.

$[\forall a \in M', p^a > r^a]$ and $|\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| \leq |M'|$.

The above concepts are the counterparts of excess demand and supply of classical exchange economics, according to which prices are adjusted. If objects are (weakly) underdemanded, by decreasing the prices of these objects, we can balance demand and supply. However, if the price of some object is equal to its reserve price, then we cannot decrease its price any more. Hence, we exclude such objects in the definition of a set of (weak) underdemanded objects.

The above two definitions are the necessary and sufficient conditions for obtaining the minimum Walrasian equilibrium price.

Fact 4 *(Theorem 1 in Morimoto and Serizawa, 2015).* [21] Let $n, m \in \mathbb{N}$ and $\mathcal{R} = \mathcal{R}^E$. Then, for each $R \in \mathcal{R}^n$, each $r \in \mathbb{R}^m_+$, and each $p \in \mathbb{R}^m_{r+}$, $p = p_{\min}(R, r)$ if and only if no set is overdemanded at p for R and no set is weakly underdemanded at p for R, that is, for each $M' \subseteq M$,

(i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \leq |M'|$,
(ii) $[\forall a \in M', p^a > r^a] \Longrightarrow |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|$.

Given $R_i \in \mathcal{R}$, $a \in L$ and $(b,t) \in L \times \mathbb{R}$, $V_i(a;(b,t))$ is the **compensation of a from (b,t) for R_i** which is defined by $(a, V_i(a;(b,t)))\, I_i\, (b,t)$. The compensation for R_i' is denoted by V_i'.

Fact 5 *(Lemma 5 in Morimoto and Serizawa, 2015).* Let $R \in \mathcal{R}^n$ and $z \in Z$. For each $i, j \in N$, if $t_i + t_j < V_i(x_j; z_i) + V_j(x_i; z_j)$, then there exists $z' \in Z$ that dominates z for R.

Given $(a,t) \in M \times \mathbb{R}_+$, a preference relation $R_i \in \mathcal{R}$ is **(a,t)-favoring** if for each $b \in L \setminus \{a\}$, $(a,t)\, P_i\, (b,0)$. Let $\mathcal{R}^F((a,t))$ be the set of all (a,t)-favoring preference relations. Note that $\mathcal{R}^F((a,t)) \subsetneq \mathcal{R}^C$.

Fact 6 *(Lemma 8 in Morimoto and Serizawa, 2015).* Let f satisfy no-subsidy and strategy-proofness. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $x_i(R) \in M$. Then, for each $R_i' \in \mathcal{R}^F(f_i(R))$, $f_i(R_i', R_{-i}) = f_i(R)$.

Finally, we provide a flowchart to reach Theorem. Fig. 11 describes the overall structure of the proof of Theorem. The arrows in the figure illustrate the relationships among lemmas and propositions for Theorem. For example, Lemmas 1 and 2 are applied to prove Proposition 3. Proposition 3 is applied to prove Lemma 3, which is in turn applied to prove Lemmas 4 and 7, and so on.

Appendix B. Proofs for propositions

B.1. Proof of Proposition 1

Proposition 1. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Let $r \in \mathbb{R}^m_+$. Then, the following statements hold.*

(i) *The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies efficiency if and only if $r = v$.*

[21] Precisely, Morimoto and Serizawa (2015) show the above statement for the only case of $r = \mathbf{0}$. However, we can show the same result for the case of $r \geq \mathbf{0}$ by using their technique. For the proof, see Wakabayashi et al. (2022) which is the previous version of this paper.

(ii) *The minimum price Walrasian rule with r on $\mathcal{R}^n$ satisfies individual rationality for the seller if and only if $r \geq v$.*

Proof. Let f be the minimum price Walrasian rule with r. First, we show (i).

If. Assume $r = v$. Let $R \in \mathcal{R}^n$, $z \equiv f(R)$ and $p = p_{\min}(R, r)$. Suppose to the contrary that there is some $z' \in Z$ such that (a) for each $i \in N$, $z'_i \, R_i \, z_i$, and (b) $\sum_{i \in N}(t'_i - v^{x'_i}) > \sum_{i \in N}(t_i - v^{x_i})$. Then, for each $i \in N$,

$$(x'_i, t'_i) = z'_i \underset{\text{(a)}}{R_i} z_i = (x_i, p^{x_i}) \underset{x_i \in D(R_i, p)}{R_i} (x'_i, p^{x'_i}),$$

which implies $t'_i \leq p^{x'_i}$. Hence,

$$\sum_{i \in N}(p^{x'_i} - v^{x'_i}) \underset{p^{x'_i} \geq t'_i}{\geq} \sum_{i \in N}(t'_i - v^{x'_i}) \underset{\text{(b)}}{>} \sum_{i \in N}(t_i - v^{x_i}) \underset{\text{(WE-i)}}{=} \sum_{i \in N}(p^{x_i} - v^{x_i}). \tag{1}$$

By WE-ii, $\{x_i\}_{i \in N} \supseteq \{a \in M : p^a > r^a\}$, and so $(*)$ $x \in \arg\max_{x'' \in X} \sum_{i \in N}(p^{x''_i} - r^{x''_i})$. Hence,

$$\sum_{i \in N}(p^{x_i} - v^{x_i}) \underset{r=v}{=} \sum_{i \in N}(p^{x_i} - r^{x_i}) \underset{(*)}{\geq} \sum_{i \in N}(p^{x'_i} - r^{x'_i}) \underset{r=v}{=} \sum_{i \in N}(p^{x'_i} - v^{x'_i}).$$

However, this inequality contradicts (1).

Only if. Assume that f satisfies efficiency. We show $r = v$ in the following two steps.

Step 1: $r \geq v$. Suppose to the contrary that there is some $a \in M$ with $r^a < v^a$. Let $\varepsilon \in (0, \frac{v^a - r^a}{2})$. Let $R \in (\mathcal{R}^Q)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, r^a + 2\varepsilon) \, I_i \, (b, r^b + \varepsilon) \, I_i \, (0, 0)$, and let $p = p_{\min}(R, r)$.

First, we show that there is some $j \in N$ with $x_j(R) = a$. Suppose to the contrary that for each $i \in N$, $x_i(R) \neq a$. Then, by (WE-ii), $p^a = r^a$. Let $i \in N$. Note that by $x_i(R) \neq a$, $D(R_i, p) \setminus \{a\} \neq \emptyset$. By the definition of $R_i \in \mathcal{R}^Q$ and $p \geq r$, for each $b \in M \setminus \{a\}$,

$$(a, p^a) = (a, r^a) \, P_i \begin{cases} (a, r^a + \varepsilon) \, I_i \, (b, r^b) \, R_i \, (b, p^b) \\ (a, r^a + 2\varepsilon) \, I_i \, (0, 0) \end{cases}.$$

Hence, $D(R_i, p) = \{a\}$, but this contradicts $D(R_i, p) \setminus \{a\} \neq \emptyset$. Therefore, there is some $j \in N$ with $x_j(R) = a$. By WE-i, $t_j(R) = p^a$.

Next, we derive a contradiction. By $R_j \in \mathcal{R}^Q$,

$$(a, r^a + 2\varepsilon) \, I_j \, (0, 0) \Longleftrightarrow (a, p^a) \, I_j \, (0, p^a - r^a - 2\varepsilon).$$

Also, by the definition of ε,

$$2\varepsilon < v^a - r^a \Longleftrightarrow -v^a < -r^a - 2\varepsilon \Longleftrightarrow p^a - v^a < p^a - r^a - 2\varepsilon.$$

Let $z' \in Z$ be such that $z'_j = (0, p^a - r^a - 2\varepsilon)$ and for each $i \in N \setminus \{j\}$, $z'_i = f_i(R)$. Then, by $f_j(R) = (a, p^a) \, I_j \, (0, p^a - r^a - 2\varepsilon) = z'_j$ and $t_j(R) - v^{x_j(R)} = p^a - v^a < p^a - r^a - 2\varepsilon = t'_j - v^{x'_j}$, z' dominates $f(R)$ for R. However, this is a contradiction. **(End of Step 1)**

Step 2: $r = v$. By Step 1, $v \leq r$. Suppose to the contrary that there is some $a \in M$ with $v^a < r^a$. Let $M^0 \equiv \{b \in M : r^b = 0\}$, $\underline{r} \equiv \min\{r^b\}_{b \in M \setminus M^0}$ and $\varepsilon \in (0, \min\{\underline{r}, \frac{r^a - v^a}{2}\})$. Let $R \in (R^Q)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, v^a + 2\varepsilon) \, I_i \, (b, \varepsilon) \, I_i \, (0, 0)$, and let $p = p_{\min}(R, r)$. By $0 \leq v^a < r^a$, $a \notin M^0$. Since for each $i \in N$ and each $b \in M \setminus M^0$, by $\varepsilon < r^b$, $(0, 0) \, I_i \, (b, \varepsilon) \, P_i \, (b, r^b)$, $\{x_i(R)\}_{i \in N} \subseteq M^0 \cup \{0\}$.

First, we show that for each $i \in N$, $x_i(R) \in M$. Suppose to the contrary that there is some $j \in N$ with $x_j(R) = 0$. Then, by $f_j(R) = (0, 0) \, I_j \, (a, v^a + 2\varepsilon)$ and $t_j(R) - v^{x_j(R)} = 0 < 2\varepsilon = (v^a + 2\varepsilon) - v^a$, $((a, v^a + 2\varepsilon), f_{-j}(R))$ dominates $f(R)$ for R. However, this is a contradiction. Hence, for each $i \in N$, $x_i(R) \in M$, and so $\{x_i(R)\}_{i \in N} \subseteq M^0$.

Next, we derive a contradiction. Let $j \in N$ and $b \in L$ be such that $x_j(R) = b$. By $\{x_i(R)\}_{i \in N} \subseteq M^0$, $b \in M^0$. By $R_j \in \mathcal{R}^Q$,

$$(b, \varepsilon) \, I_j \, (a, v^a + 2\varepsilon) \Longleftrightarrow (b, p^b) \, I_j \, (a, v^a + \varepsilon + p^b).$$

By $r^b \geq v^b \geq 0$ (Step 1) and $b \in M^0$, $v^b = r^b = 0$. Hence,

$$p^b - v^b \underset{v^b = r^b = 0}{=} p^b \underset{0 < \varepsilon}{<} p^b + \varepsilon = (v^a + \varepsilon + p^b) - v^a.$$

Let $z' \in Z$ be such that $z'_j = (a, v^a + \varepsilon + p^b)$ and for each $i \in N \setminus \{j\}$, $z'_i = f_i(R)$. Then, by $f_j(R) = (b, p^b) \, I_j \, (a, v^a + \varepsilon + p^b) = z'_j$ and $t_j(R) - v^{x_j(R)} = p^b - v^b < (v^a + \varepsilon + p^b) - v^a = t'_j - v^{x'_j}$, z' dominates $f(R)$ for R. However, this is a contradiction. Hence, $r = v$ holds.

Next, we show (ii).

If. Assume $r \geq v$. Let $R \in \mathcal{R}$ and $i \in N$. By $f(R) \in Z_{\min}(R, r)$, there is some $p \in \mathbb{R}^m_{r+}$ with $(f(R), p) \in W_{\min}(R, r)$. By $t_i(R) = p^{x_i(R)}$ and $p \geq r \geq v$, $t_i(R) \geq r^{x_i(R)} \geq v^{x_i(R)}$.

Only if. Assume that f satisfies individual rationality for the seller. Suppose to the contrary that there is some $a \in M$ with $r^a < v^a$. Let $\varepsilon \in (0, \frac{v^a - r^a}{2})$. Let $R \in (\mathcal{R}^Q)^n$ be such that for each $i \in N$ and each $b \in M \setminus \{a\}$, $(a, r^a + 2\varepsilon)\, I_i\, (b, r^b + \varepsilon)\, I_i\, (0,0)$, and let $p = p_{\min}(R, r)$.

First, we show that there is some $j \in N$ with $x_j(R) = a$. Suppose to the contrary that for each $i \in N$, $x_i(R) \neq a$. Then, by (WE-ii), $p^a = r^a$. Let $i \in N$. Note that by $x_i(R) \neq a$, $D(R_i, p) \setminus \{a\} \neq \emptyset$. By $R_i \in \mathcal{R}^Q$ and $p \geq r$, for each $b \in M \setminus \{a\}$,

$$(a, p^a) = (a, r^a)\, P_i \begin{cases} (a, r^a + \varepsilon)\, I_i\, (b, r^a)\, R_i\, (b, p^b) \\ (a, r^a + 2\varepsilon)\, I_i\, (0,0) \end{cases}.$$

Hence, $D(R_i, p) = \{a\}$, but this contradicts $D(R_i, p) \setminus \{a\} \neq \emptyset$. Hence, there is some $j \in N$ with $x_j(R) = a$.

Next, we derive a contradiction. By $x_j(R) = a$, $a \in D(R_j, p)$. Then,

$$(a, p^a) \underset{a \in D(R_j, p)}{R_j} (0,0) \underset{\text{def. of } R}{I_j} (a, r^a + 2\varepsilon) \underset{r^a + 2\varepsilon < v^a}{P_j} (a, v^a),$$

which implies $p^a < v^a$. Therefore, $t_j(R) = p^a < v^a = v^{x_j(R)}$, but this contradicts individual rationality for the seller. □

B.2. Proof of Proposition 2

Proposition 2. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. If a rule f on $\mathcal{R}^n$ satisfies efficiency, no-subsidy, and strategy-proofness, then f satisfies individual rationality for the seller.*

Proof. Let f satisfy efficiency, no-subsidy, and strategy-proofness. Let $R \in \mathcal{R}^n$ and $i \in N$. We show $t_i(R) \geq v^{x_i(R)}$. Suppose to the contrary that $t_i(R) < v^{x_i(R)}$. By $t_i(R) < v^{x_i(R)}$ and no-subsidy, $f_i(R) \in M \times \mathbb{R}_+$. Let $R'_i \in \mathcal{R}^F(f_i(R))$ be such that $-V'_i(0; f_i(R)) < v^{x_i(R)} - t_i(R)$. Then, by Fact 6, $f_i(R'_i, R_{-i}) = f_i(R)$. Let $z' \in Z$ be such that $z'_i \equiv (0, V'_i(0; f_i(R'_i, R_{-i})))$ and for each $j \in N \setminus \{i\}$, $z'_j \equiv f_j(R'_i, R_{-i})$. Then, $z'_i\, I'_i\, f_i(R'_i, R_{-i})$ and for each $j \in N \setminus \{i\}$, $z'_j\, I_j\, f_j(R'_i, R_{-i})$. Also,

$$\begin{aligned}\sum_{j \in N} \left(t'_j - v^{x'_j}\right) &= V'_i(0; f_i(R'_i, R_{-i})) + \sum_{j \in N \setminus \{i\}} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})}\right) \\ &\underset{\text{def. of } R'_i}{>} \left(t_i(R'_i, R_{-i}) - v^{x_i(R'_i, R_{-i})}\right) + \sum_{j \in N \setminus \{i\}} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})}\right) \\ &= \sum_{j \in N} \left(t_j(R'_i, R_{-i}) - v^{x_j(R'_i, R_{-i})}\right).\end{aligned}$$

However, these equations contradict that $f(R'_i, R_{-i})$ is efficient for (R'_i, R_{-i}). Therefore, we have $t_i(R) \geq v^{x_i(R)}$. □

B.3. Proofs of Proposition 3

In order to prove Proposition 3, we use two lemmas.

Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, a preference relation $R_i \in \mathcal{R}$ is **ε-greedy for (a,t)** if (i) R_i is (a,t)-favoring, (ii) $(a, t+2\varepsilon)\, I_i\, (0,0)$, (iii) for each $b \in M \setminus \{a\}$, $(b, \varepsilon)\, I_i\, (0,0)$. Let $\mathcal{R}^F((a,t); \varepsilon)$ be the set of all ε-greedy preference relations for (a,t). Note that $\mathcal{R}^F((a,t); \varepsilon) \subsetneq \mathcal{R}^F((a,t)) \subsetneq \mathcal{R}^C$. A preference relation $R_i \in \mathcal{R}$ is **quasi-linear** if for each $(a,t), (b,t') \in L \times \mathbb{R}$ and each $\delta \in \mathbb{R}$,

$$(a,t)\, I_i\, (b,t') \iff (a, t-\delta)\, I_i\, (b, t'-\delta).$$

Let $\mathcal{R}^Q$ be the set of all quasi-linear preference relations. Note that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$. Given $(a,t) \in M \times \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_{++}$, the unique quasi-linear preference relation that is ε-greedy for (a,t) is denoted by $R^Q((a,t); \varepsilon)$.

Lemma 1 says that if an agent misreports $R'_i = R^Q(z_i; \varepsilon_i)$ where ε_i is small enough, then he does not get x_i under the rule.

Lemma 1. *Let f satisfy individual rationality for the buyers, no-subsidy, and strategy-proofness. Let $R \in \mathcal{R}^n$, $i \in N$ and $z_i \in M \times \mathbb{R}_+$ be such that $z_i\, P_i\, f_i(R)$. Let $\varepsilon_i \in (0, \frac{1}{2}(V_i(x_i; f_i(R)) - t_i))$ and $R'_i = R^Q(z_i; \varepsilon_i)$. Then, $x_i(R'_i, R_{-i}) \neq x_i$.*

Proof. Note that by $z_i\, P_i\, f_i(R)$, $t_i < V_i(x_i; f_i(R))$, and so we can pick $\varepsilon_i \in (0, \frac{1}{2}(V_i(x_i; f_i(R)) - t_i))$. Suppose to the contrary that $x_i(R'_i, R_{-i}) = x_i$. Then,

$$f_i(R'_i, R_{-i}) \underset{\text{Ind. Rat. for Buyers.}}{R'_i} (0,0) \underset{\varepsilon_i\text{-greediness}}{I'_i} (x_i, t_i + 2\varepsilon_i) \underset{\text{def. of } \varepsilon_i}{P'_i} (x_i, V_i(x_i; f_i(R))).$$

By $x_i(R'_i, R_{-i}) = x_i$, $t_i(R'_i, R_{-i}) < V_i(x_i; f_i(R))$, which implies $f_i(R'_i, R_{-i})\, P_i\, f_i(R)$. However, this relation contradicts strategy-proofness. Hence, $x_i(R'_i, R_{-i}) \neq x_i$. □

Lemma 2 says that if there is a group of agents N' such that the preference relation of each agent $i \in N'$ is $R^Q(z_i; \varepsilon_i)$, then for each $i \in N'$, (i) there is an agent j assigned x_i, and (ii) if j differs from i, then j's payment is strictly larger than t_i.

Lemma 2. *Let f satisfy efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Let $N' \subseteq N$, $z \in Z$, $R \in \mathcal{R}^n$ and $(\varepsilon_i)_{i\in N'} \in \mathbb{R}^{|N'|}_{++}$ be such that for each $i \in N'$, $x_i \in M$, $t_i \geq v^{x_i}$, and $R_i = R^Q(z_i;\varepsilon_i)$. Then, for each $i \in N'$, there exists some $j \in N$ such that*

(i) $x_j(R) = x_i$,
(ii) *if $j \neq i$, then $t_j(R) \geq t_i + \varepsilon_i$.*

Proof. Let $i \in N'$.

(i) Suppose to the contrary that for each $j \in N$, $x_j(R) \neq x_i$. Let

$$\delta_i \equiv \begin{cases} \varepsilon_i & \text{if } x_i(R) \in M \\ 2\varepsilon_i & \text{if } x_i(R) = 0 \end{cases}.$$

Then, by $x_i(R) \neq x_i$ and $R_i = R^Q(z_i;\varepsilon_i)$, $(x_i, t_i + \delta_i)\, I_i\, (x_i(R), 0)$. Moreover, by $R_i \in R^Q$,

$$(x_i, t_i + \delta_i + t_i(R))\, I_i\, (x_i(R), t_i(R)).$$

Note

$$(t_i + \delta_i + t_i(R)) - v^{x_i} \underset{t_i \geq v^{x_i}}{\geq} \delta_i + t_i(R) \underset{\delta_i > 0}{>} t_i(R) \underset{v^{x_i(R)} \geq 0}{\geq} t_i(R) - v^{x_i(R)}.$$

Therefore, since for each $j \in N$, $x_j(R) \neq x_i$, $((x_i, t_i + \delta_i + t_i(R)), f_{-i}(R))$ dominates $f(R)$ for R. However, this is a contradiction. Hence, there exists some $j \in N$ such that $x_j(R) = x_i$.

(ii) Let $j \in N$ be such that $x_j(R) = x_i$ and $j \neq i$. Suppose to the contrary that $t_j(R) < t_i + \varepsilon_i$. Let $R'_j \in \mathcal{R}^F(f_j(R))$ be such that for each $a \in L \setminus \{x_i\}$, $-V'_j(a; f_j(R)) < t_i + \varepsilon_i - t_j(R)$. Let $R' \equiv (R'_j, R_{-j})$. Then, by Fact 6, $f_j(R') = f_j(R)$, which implies $x_i(R') \neq x_i$. Hence, we get $-V'_j(x_i(R'); f_j(R')) < t_i + \varepsilon_i - t_j(R')$. Let

$$\delta'_i \equiv \begin{cases} \varepsilon_i & \text{if } x_i(R') \in M \\ 2\varepsilon_i & \text{if } x_i(R') = 0 \end{cases}.$$

By $x_i(R') \neq x_i$ and $R_i = R^Q(z_i;\varepsilon_i)$, $(x_i, t_i + \delta'_i)\, I_i\, (x_i(R'), 0)$. By $R_i \in \mathcal{R}^Q$, $(x_i, t_i + \delta'_i + t_i(R'))\, I_i\, (x_i(R'), t_i(R'))$, which implies $V_i(x_j(R'); f_i(R')) = t_i + \delta'_i + t_i(R')$. Hence, by $-V'_j(x_i(R'); f_j(R')) < t_i + \varepsilon_i - t_j(R')$ and $V_i(x_j(R'); f_i(R')) = t_i + \delta'_i + t_i(R') \geq t_i + \varepsilon_i + t_i(R')$,

$$\begin{aligned} &V'_j(x_i(R'); f_j(R')) + V_i(x_j(R'); f_i(R')) \\ &> -(t_i + \varepsilon_i) + t_j(R') + t_i + \varepsilon_i + t_i(R') \\ &= t_j(R') + t_i(R'). \end{aligned}$$

Hence, by Fact 5, there exists some $z \in Z$ that dominates $f(R')$ for R'. However, this is a contradiction. Hence, $t_j(R) \geq t_i + \varepsilon_i$. □

Finally, we show Proposition 3.

Proposition 3. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, and let $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R, v)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.*

Proof. Suppose to the contrary that for some $i \in N$, $z_i\, P_i\, f_i(R)$. Without loss of generality, let $i = 1$. Let $\varepsilon_0 \equiv \frac{1}{2}(V_1(x_1; f_1(R)) - t_1)$ and $R^{(0)} \equiv R$. Then, we show the following claim by induction. The last condition (iii) derives a contradiction.

Claim 1. *For each $k \in \{1, \dots, n\}$, there exist $N(k) \equiv \{1, \dots, k\} \subseteq N$, $(\varepsilon_i)_{i\in N(k)} \in \mathbb{R}^k_{++}$ and $R^{(k)} \equiv (R'_{N(k)}, R_{-N(k)}) \in \mathcal{R}^n$ such that*

(i) *for each $i \in N(k)$, $x_i \in M$,*
(ii) *for each $i \in N(k)$, $0 < \varepsilon_i < \min\{\varepsilon_{i-1}, \frac{1}{2}(V_i(x_i; f_i(R^{(i-1)})) - t_i)\}$ and $R'_i = R^Q_i(z_i;\varepsilon_i)$,*
(iii) $x_k(R^{(k)}) \notin \{x_i\}_{i\in N(k)}$.

Induction Base. Let $k = 1$.

(i-1) By $z_1\, P_1\, f_1(R)$, if $x_1 = 0$, $(0,0)\, P_i\, f_1(R)$. However, this contradicts individual rationality for the buyers. Hence, $x_1 \in M$.
(ii-1) By $z_1\, P_1\, f_1(R)$, we can pick $\varepsilon_1 \in (0, \frac{1}{2}(V_1(x_1; f_1(R)) - t_1))$, and let $R'_1 = R^Q_1(z_1;\varepsilon_1)$.
(iii-1) By (i-1), (ii-1) and Lemma 1, $x_1(R^{(1)}) \neq x_1$.

Induction Hypothesis. Let $s \in \{1,\ldots,n-1\}$. Assume that there exist $N(s) \equiv \{1,\ldots,s\} \subseteq N$, $(\varepsilon_i)_{i\in N(s)} \in \mathbb{R}^s_{++}$ and $R^{(s)} \equiv (R'_{N(s)}, R_{-N(s)}) \in \mathcal{R}^n$ such that

(i-s) for each $i \in N(s)$, $x_i \in M$,
(ii-s) for each $i \in N(s)$, $0 < \varepsilon_i < \min\{\varepsilon_{i-1}, \frac{1}{2}(V_i(x_i; f_i(R^{(i-1)})) - t_i)\}$ and $R'_i = R^Q_i(z_i;\varepsilon_i)$,
(iii-s) $x_s(R^{(s)}) \notin \{x_i\}_{i\in N(s)}$.

Induction Argument. We consider the case $s+1$. By (i-s), $(t_i)_{i\in N(s)} \geq (v^{x_i})_{i\in N(s)}$, (ii-$s$) and Lemma 2 (i), for each $i \in N(s)$, there exists some $j \in N$ such that $x_j(R^{(s)}) = x_i$. In particular, by (iii-s), there exists some $k \in N \setminus N(s)$ such that $x_k(R^{(s)}) \in \{x_i\}_{i\in N(s)}$. Without loss of generality, let $k \equiv s+1$. Moreover, let $l \in N(s)$ be such that $x_{s+1}(R^{(s)}) = x_l$. By $s+1 \neq l$ and Lemma 2 (ii), $t_{s+1}(R^{(s)}) \geq t_l + \varepsilon_l > t_l$. Therefore,

$$z_{s+1} \underset{\text{WE-i}}{R_{s+1}} z_l \underset{t_l < t_{s+1}(R^{(s)})}{P_{s+1}} (x_l, t_{s+1}(R^s)) = f_{s+1}(R^{(s)}).$$

(i-$(s+1)$) By individual rationality for the buyers and $z_{s+1}\, P_{s+1}\, f_{s+1}(R^{(s)})$, $x_{s+1} \in M$.
(ii-$(s+1)$) By (ii-s) and $z_{s+1}\, P_{s+1}\, f_{s+1}(R^{(s)})$, we can pick ε_{s+1} such that

$$0 < \varepsilon_{s+1} < \min\left\{\varepsilon_s, \frac{1}{2}(V_{s+1}(x_{s+1}; f_{s+1}(R^{(s)})) - t_{s+1})\right\},$$

and let $R'_{s+1} = R^Q_{s+1}(z_{s+1};\varepsilon_{s+1})$.

(iii-$(s+1)$) By $z_{s+1}\, P_{s+1}\, f_{s+1}(R^{(s)})$, (i-$(s+1)$), (ii-$(s+1)$) and Lemma 1, $x_{s+1}(R^{(s+1)}) \neq x_{s+1}$. Suppose to the contrary that $x_{s+1}(R^{(s+1)}) \in \{x_i\}_{i\in N(s)}$. Let $l' \in N(s)$ be such that $x_{s+1}(R^{(s+1)}) = x_{l'}$. Then, by $s+1 \in N \setminus N(s)$ and $l' \in N(s)$, $s+1 \neq l'$. By $s+1 \neq l'$, $x_{l'} \in M$, $t_{l'} \geq v^{x_{l'}}$, $R_{l'} = R^Q(z_{l'};\varepsilon_{l'})$, and Lemma 2 (ii), $t_{s+1}(R^{(s+1)}) \geq t_{l'} + \varepsilon_{l'}$. By $t_{l'} \geq 0$ and $\varepsilon_{s+1} < \varepsilon_s < \cdots < \varepsilon_{l'+1} < \varepsilon_{l'}$, $\varepsilon_{s+1} + 0 < \varepsilon_{l'} + t_{l'} \leq t_{s+1}(R^{(s+1)})$. Therefore,

$$(0,0) \underset{x_{l'} \neq x_{s+1},\, \varepsilon_{s+1}\text{-greediness}}{I'_{s+1}} (x_{l'}, \varepsilon_{s+1}) \underset{\varepsilon_{s+1} < t_{s+1}(R^{(s+1)})}{P'_{s+1}} (x_{l'}, t_{s+1}(R^{(s+1)})) \underset{x_{l'} = x_{s+1}(R^{(s+1)})}{=} f_{s+1}(R^{(s+1)}).$$

This contradicts individual rationality for the buyers. Hence, $x_{s+1}(R^{(s+1)}) \notin \{x_i\}_{i\in N(s)}$, and so $x_{s+1}(R^{(s+1)}) \notin \{x_i\}_{i\in N(s+1)}$. **(End of Claim 1)**

Let $k = n$. Then by Claim 1 (iii), $x_n(R^{(n)}) \notin \{x_i\}_{i\in N(n)} = \{x_i\}_{i\in N}$. By Claim 1 (i), $|\{x_i\}_{i\in N}| = n$. By Claim 1 (i) & (ii), $(t_i)_{i\in N} \geq (v^{x_i})_{i\in N}$, and Lemma 2 (i), for each $i \in N$, there exists some $j \in N$ such that $x_j(R^{(n)}) = x_i$, which implies $\{x_i\}_{i\in N} \subseteq \{x_i(R^{(n)})\}_{i\in N}$. Hence, by $|\{x_i\}_{i\in N}| = n$ and $|\{x_i(R^{(n)})\}_{i\in N}| \leq n$, $\{x_i\}_{i\in N} = \{x_i(R^{(n)})\}_{i\in N}$. Therefore, there exists some $j \in N$ such that $x_n(R^{(n)}) = x_j$, contradicting $x_n(R^{(n)}) \notin \{x_i\}_{i\in N}$. □

B.4. Proofs of Proposition 4

In order to prove Proposition 4, we use five lemmas.

Lemma 3 says that if an agent's payment is larger than or equal to the minimum Walrasian price with v of the object he receives, then $f_i(R)$ satisfies (WE-i) for him.

Lemma 3. *Let f satisfy efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. For each $i \in N$, if $t_i(R) \geq p^{x_i(R)}$, then $x_i(R) \in D(R_i, p)$ and $t_i(R) = p^{x_i(R)}$.*

Proof. Let $i \in N$ be such that $t_i(R) \geq p^{x_i(R)}$. By $p = p_{\min}(R, v)$, there exists some $z \in Z$ such that $(z, p) \in W_{\min}(R, v)$. Then,

$$f_i(R) \underset{\text{Prop. 3}}{R_i} z_i \underset{\text{WE-i}}{R_i} (x_i(R), p^{x_i(R)}) \underset{p^{x_i(R)} \leq t_i(R)}{R_i} f_i(R),$$

which implies $f_i(R)\, I_i\, z_i\, I_i\, (x_i(R), p^{x_i(R)})$. By $z_i\, I_i\, (x_i(R), p^{x_i(R)})$, $x_i(R) \in D(R_i, p)$, and by $f_i(R)\, I_i\, (x_i(R), p^{x_i(R)})$, $t_i(R) = p^{x_i(R)}$. □

Given $r \in \mathbb{R}^m_+$ and $p \in \mathbb{R}^m_{r+}$, we say that an object $a \in M$ is **competitive** if $p^a > r^a$. Moreover, we say that an object $a \in M$ is **noncompetitive** if $p^a = r^a$. We denote the sets of competitive objects, noncompetitive real objects, and noncompetitive objects, respectively, by

$$M^+_r(p) \equiv \{a \in M : p^a > r^a\},$$
$$M^0_r(p) \equiv \{a \in M : p^a = r^a\},$$
$$L^0_r(p) \equiv \{a \in L : p^a = r^a\}.$$

If an agent receives a competitive object $a \in M_r^+(p)$ at an allocation z, then we call him a **competitive agent**. Similarly, we call an agent a **noncompetitive agent** if he receives a noncompetitive object $a \in L_r^0(p)$.

Given $N' \subseteq N$, Lemma 4 describes a sufficient condition for constructing a sequence of agents such that (a) it starts from a noncompetitive agent, (b) each agent demands his object and his next agent's object, and (c) the final agent must be in N'.

Lemma 4. *Let f satisfy efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness. Let $R \in \mathcal{R}^n$, $p = p_{\min}(R, v)$ and $N' \subseteq N$. If*

(i) *for some $j \in N'$, $p^{x_j(R)} > v^{x_j(R)}$,*
(ii) *for each $i \in N \setminus N'$, $t_i(R) \geq p^{x_i(R)}$,*

then there exists $\{i_k\}_{k=1}^K \subseteq N$ with $K \geq 2$ such that

(a) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,
(b) *for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ and $t_{i_k}(R) = p^{x_{i_k}(R)}$,*
(c) $i_K \in N'$.

Proof. Assume that (i) and (ii) hold. First, we construct $\{j_k\}_{k=1}^{K'} \subseteq N$ with $K' \geq 2$ as follows.

Step 1: By (i), we can pick $j_1 \in N'$ such that $p^{x_{j_1}(R)} > v^{x_{j_1}(R)}$ and go to the next step.

Step $s \geq 2$: Since for each $k \in \{1, \ldots, s-1\}$, $p^{x_{j_k}(R)} > v^{x_{j_k}(R)}$, $\{x_{j_1}(R), \ldots, x_{j_{s-1}}(R)\}$ is not weakly underdemanded at p for R (Fact 4-ii). Hence, there exists some $j_s \in N \setminus \{j_k\}_{k=1}^{s-1}$ such that $D(R_{j_s}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{s-1}}(R)\} \neq \emptyset$. If $p^{x_{j_s}(R)} = v^{x_{j_s}(R)}$, we terminate this procedure. Otherwise, we go to the next step.

Since $\{a \in M : p^a > v^a\}$ is not weakly underdemanded at p for R (Fact 4-ii),

$$|N| \geq |\{i \in N : D(R_i, p) \cap \{a \in M : p^a > v^a\} \neq \emptyset\}| > |\{a \in M : p^a > v^a\}|.$$

Hence, there exists some $l \in N$ such that $p^{x_l(R)} = v^{x_l(R)}$. Therefore, the above process terminates in a finite step. As the result, we get $\{j_k\}_{k=1}^{K'} \subseteq N$ with $K' \geq 2$ such that

(K'-i) $j_1 \in N'$,
(K'-ii) for each $k \in \{2, \ldots, K'\}$, $D(R_{j_k}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{k-1}}(R)\} \neq \emptyset$,
(K'-iii) $p^{x_{j_{K'}}(R)} = v^{x_{j_{K'}}(R)}$.

Next, we construct $\{i_k\}_{k=1}^K \subseteq \{j_k\}_{k=1}^{K'}$ with $K \geq 2$ as follows.

Step 1: Let $i_1 \equiv j_{K'}$. By (K'-ii), $D(R_{i_1}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{K'-1}}(R)\} \neq \emptyset$. Hence, there exists some $i_2 \in \{j_1, \ldots, j_{K'-1}\}$ such that $x_{i_2}(R) \in D(R_{i_1}, p)$. If $i_2 \in N'$, we terminate this procedure. Otherwise, we go to the next step.

Step $s \geq 2$: $i_s \in \{j_k\}_{k=1}^{K'}$ is determined by the previous step. By (K'-ii), $D(R_{i_s}, p) \cap \{x_{j_1}(R), \ldots, x_{j_{K''-1}}(R)\} \neq \emptyset$, where $j_{K''} = i_s$. Hence, there exists some $i_{s+1} \in \{j_1, \ldots, j_{K''-1}\}$ such that $x_{i_{s+1}}(R) \in D(R_{i_s}, p)$. If $i_{s+1} \in N'$, we terminate this procedure. Otherwise, we go to the next step.

By $j_1 \in N'$, $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$. Since the number of agents is finite and the number of the left agents in $\{j_k\}_{k=1}^{K'}$ strictly decreases in each step, the above procedure terminates in a finite step.[22]

Hence, we get $\{i_k\}_{k=1}^K \subseteq \{j_k\}_{k=1}^{K'}$ with $K \geq 2$ such that

(K-i) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,
(K-ii) for each $k \in \{1, \ldots, K-1\}$, $x_{i_{k+1}} \in D(R_{i_k}, p)$,
(K-iii) for each $k \in \{2, \ldots, K-1\}$, $i_k \in N \setminus N'$,
(K-iv) $i_K \in N'$.

(a) By (K-i), $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$.

(b) By Proposition 2 and (K-i), $t_{i_1}(R) \geq v^{x_{i_1}(R)} = p^{x_{i_1}(R)}$. Therefore, by (ii) and (K-iii), for each $k \in \{1, \ldots, K-1\}$, $t_{i_k}(R) \geq p^{x_{i_k}(R)}$. Hence, by Lemma 3 and (K-ii), for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ and $t_{i_k}(R) = p^{x_{i_k}(R)}$.

(c) By (K-iv), $i_K \in N'$. □

[22] More precisely, the proof is as follows: By $j_1 \in N'$, $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$. Let $I(l) \equiv \{j_k : k < k' \text{ with } j_{k'} = i_l\}$. By the construction of the sequence, $I(l+1) \subsetneq I(l) \subsetneq \cdots \subsetneq I(1) \subsetneq \{j_k\}_{k=1}^{K'}$. By $\{j_k\}_{k=1}^{K'} \cap N' \neq \emptyset$, $I(l+1) \subsetneq I(l)$ and finiteness of agents, the above procedure terminate in a finite step.

An object has positive income effect if the object is more preferred against other objects as income increases, or equivalently as payments decreases.[23] If $(a, p^a)\, I_i\, (b, p^b)$ and for $\delta > 0$, $(a, p^a - \delta)\, P_i\, (b, p^b - \delta)$, then we say that the preference exhibits **positive income effect** for a against b.[24] Moreover, we can rewrite the positive income effect condition as $p^a - V_i(a; (b, t^b)) < p^b - t^b$, where $t^b = p^b - \delta$. We use the latter expression because it is more useful for proofs.

Lemma 5 says that if there exists a sequence of agents satisfying (i) it starts from a noncompetitive agent, (ii) each agent demands his object and his next agent's object, and (iii) the final agent exhibits positive income effect for the noncompetitive object in (i) against his own object, then we can construct an allocation that dominates the allocation rule outcome.

Lemma 5. *Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. If there exists $\{i_k\}_{k=1}^K \subseteq N$ with $K \geq 2$ such that*

(i) $p^{x_{i_1}(R)} = v^{x_{i_1}(R)}$,
(ii) *for each* $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}(R), x_{i_{k+1}}(R)\} \subseteq D(R_{i_k}, p)$ *and* $t_{i_k}(R) = p^{x_{i_k}(R)}$,
(iii) $v^{x_{i_1}(R)} - V_{i_K}(x_{i_1}(R); f_{i_K}(R)) < p^{x_{i_K}(R)} - t_{i_K}(R)$,

then there exists some $z' \in Z$ that dominates $f(R)$ for R.

Proof. Assume that there exists $\{i_k\}_{k=1}^K \subseteq N$ with $K \geq 2$ which satisfies (i), (ii) and (iii). Let $N^- \equiv N \setminus \{i_k\}_{k=1}^K$. Let $z' \in Z$ be such that

(a) for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} \equiv \left(x_{i_{k+1}}(R), p^{x_{i_{k+1}}(R)}\right)$,
(b) $z'_{i_K} \equiv (x_{i_1}(R), V_{i_K}(x_{i_1}(R); f_{i_K}(R)))$,
(c) for each $j \in N^-$, $z'_j \equiv f_j(R)$.

Then, by (ii) and (a), for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k}\, I_{i_k}\, f_{i_k}(R)$. Hence, by (b) and (c), for each $j \in N$, $z'_j\, I_j\, f_j(R)$. Also,

$$
\begin{aligned}
&\sum_{j \in N} (t'_j - v^{x'_j}) \\
\underset{\text{(a,b,c)}}{=}\; &\sum_{k=1}^{K-1} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(V_{i_K}(x_{i_1}(R); f_{i_K}(R)) - v^{x_{i_1}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
\underset{\text{(iii)}}{>}\; &\sum_{k=1}^{K-1} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(t_{i_K}(R) - p^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
=\; &\sum_{k=1}^{K-2} \left(p^{x_{i_{k+1}}(R)} - v^{x_{i_{k+1}}(R)}\right) + \left(t_{i_K}(R) - v^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
\underset{\text{(i)}}{=}\; &\left(p^{x_{i_1}(R)} - v^{x_{i_1}(R)}\right) + \sum_{k=2}^{K-1} \left(p^{x_{i_k}(R)} - v^{x_{i_k}(R)}\right) + \left(t_{i_K}(R) - v^{x_{i_K}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
\underset{\text{(ii)}}{=}\; &\sum_{k=1}^{K} \left(t_{i_k}(R) - v^{x_{i_k}(R)}\right) + \sum_{j \in N^-} \left(t_j(R) - v^{x_j(R)}\right) \\
=\; &\sum_{j \in N} \left(t_j(R) - v^{x_j(R)}\right).
\end{aligned}
$$

Hence, z' dominates $f(R)$ for R. □

Given $p \in \mathbb{R}^m_+$, a classical preference relation $R_i \in \mathcal{R}^C$ is ***p*-indifferent** if

(i) $[\forall a \in M, p^a > 0] \Longrightarrow [\forall a, b \in L, (a, p^a)\, I_i\, (b, p^b)]$,
(ii) $[\exists a \in M, p^a = 0] \Longrightarrow [\forall a, b \in M, (a, p^a)\, I_i\, (b, p^b)]$.

Note that to satisfy object desirability, we have to consider two cases: the prices of all objects are strictly larger than zero; and there is an object whose price is zero. Given $p \in \mathbb{R}^m_+$, let $\mathcal{R}^I(p)$ be the set of all p-indifferent preference relations. Note that $\mathcal{R}^I(p) \subsetneq \mathcal{R}^C$.

Lemma 6 says that even if agents' preferences are replaced by p-indifferent preferences, the minimum price is unchanged.

[23] Although income is not modeled explicitly, the zero payment corresponds to the endowed income. When an agent's income increases by $\delta > 0$, then his payment for each object decreases by δ.

[24] Our positive income effect concept corresponds to that of a normal good from standard consumer theory (e.g., Baisa, 2020).

Lemma 6. *Let $R \in \mathcal{R}^n$, $r \in \mathbb{R}^m_+$ and $p = p_{\min}(R, r)$. Let $N' \subseteq N$, $R'_{N'} \in \mathcal{R}^I(p)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. Then, $p = p_{\min}(R', r)$.*

Proof. It suffices to show (i) and (ii) of Fact 4. Let $M' \subseteq M$.

(i) We consider the following three cases.

Case 1: $M' \subsetneq M$. By $R'_{N'} \in \mathcal{R}^I(p)^{|N'|}$, for each $i \in N'$, $M \subseteq D(R'_i, p)$. Hence, by $M' \subsetneq M$, for each $i \in N'$, $D(R'_i, p) \nsubseteq M'$. Since M' is not overdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \subseteq M'\}| \le |\{i \in N : D(R_i, p) \subseteq M'\}| \le |M'|.$$

Case 2: $M' = M$ and $\forall a \in M$, $p^a > 0$. For each $i \in N'$, by $D(R'_i, p) = L$, $D(R'_i, p) \nsubseteq M$. Hence, since M is not overdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \subseteq M\}| \le |\{i \in N : D(R_i, p) \subseteq M\}| \le |M|.$$

Case 3: $M' = M$ and $\exists a \in M$, $p^a = 0$. By object desirability, for each $i \in N$, $(a, p^a) = (a, 0)\ P_i\ (0, 0)$, so that $D(R_i, p) \subseteq M$. Since M is not overdemanded at p for R, $|N| = |\{i \in N : D(R_i, p) \subseteq M\}| \le |M|$. Therefore, we have

$$|\{i \in N : D(R'_i, p) \subseteq M\}| \le |N| = |\{i \in N : D(R_i, p) \subseteq M\}| \le |M|.$$

(ii) Assume that for each $a \in M'$, $p^a > r^a$. By $R'_{N'} \in \mathcal{R}^I(p)^{|N'|}$, for each $i \in N'$, $M \subseteq D(R'_i, p)$. Hence, for each $i \in N'$, $D(R'_i, p) \cap M' \neq \emptyset$. Since M' is not weakly underdemanded at p for R,

$$|\{i \in N : D(R'_i, p) \cap M' \neq \emptyset\}| \ge |\{i \in N : D(R_i, p) \cap M' \neq \emptyset\}| > |M'|.$$

Therefore, $p = p_{\min}(R', r)$. □

Given $r \in \mathbb{R}_+$ and $p \in \mathbb{R}^m_{r+}$, a p-indifferent preference relation $R_i \in \mathcal{R}^I(p)$ exhibits **positive income effect for noncompetitive objects** if

$$\text{(i)}\ [\forall a \in M, p^a > 0] \Longrightarrow \begin{bmatrix} \forall a \in L^0_r(p) \text{ and } \forall (b, t) \in M^+_r(p) \times \mathbb{R}_+ \text{ with } t < p^b, \\ r^a - V_i(a; (b, t)) < p^b - t \end{bmatrix},$$
$$\text{(ii)}\ [\exists a \in M, p^a = 0] \Longrightarrow \begin{bmatrix} \forall a \in M^0_r(p) \text{ and } \forall (b, t) \in M^+_r(p) \times \mathbb{R}_+ \text{ with } t < p^b, \\ r^a - V_i(a; (b, t)) < p^b - t \end{bmatrix}.$$

Given $r \in \mathbb{R}_+$ and $p \in \mathbb{R}^m_{r+}$, let $\mathcal{R}^{I+}_r(p)$ be the set of all p-indifferent preference relations exhibiting positive income effect for noncompetitive objects. Note that $\mathcal{R}^{I+}_r(p) \subsetneq \mathcal{R}^I(p) \subsetneq \mathcal{R}^C$.

Given $R \in \mathcal{R}^n$, we consider an profile $\tilde{R}$ where some agents have the original preferences, i.e., $\tilde{R}_i = R_i$, and some agents have the $p_{\min}(R, \upsilon)$-indifferent preferences exhibiting positive income effect for noncompetitive objects, i.e., $\tilde{R}_i = R'_i \in \mathcal{R}^{I+}_\upsilon(p_{\min}(R, \upsilon))$. We call an agent i such that $\tilde{R}_i = R_i$ an **original preference agent**, and an agent i such that $\tilde{R}_i = R'_i$ a **$p_{\min}(R,\upsilon)$-indifferent agent**.

Lemma 7 says that if the conclusion of Proposition 4 holds for any original preference agents at a preference profile, then the result of Proposition 4 also holds for any $p_{\min}(R, \upsilon)$-indifferent agents.

Lemma 7. *Let f satisfy efficiency, individual rationality for the buyers, no-subsidy and strategy-proofness. Let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, \upsilon)$. Let $N' \subseteq N$, $R'_{N'} \in \mathcal{R}^{I+}_\upsilon(p)^{|N'|}$ and $R' \equiv (R'_{N'}, R_{-N'})$. If for each $i \in N \setminus N'$, $t_i(R') \ge p^{x_i(R')}$, then for each $i \in N'$, $t_i(R') \ge p^{x_i(R')}$.*

Proof. Assume that for each $i \in N \setminus N'$, $t_i(R') \ge p^{x_i(R')}$. Note that by $R'_{N'} \in \mathcal{R}^{I+}_\upsilon(p)^{|N'|} \subseteq \mathcal{R}^I(p)^{|N'|}$ and Lemma 6, $p = p_{\min}(R', \upsilon)$.

Suppose to the contrary that for some $j \in N'$, $t_j(R') < p^{x_j(R')}$. By Proposition 2, $\upsilon^{x_j(R')} \le t_j(R') < p^{x_j(R')}$. Hence, since $p^{x_j(R')} > \upsilon^{x_j(R')}$ and for each $i \in N \setminus N'$, $t_i(R') \ge p^{x_i(R')}$, Lemma 4 implies that there exists $\{i_k\}^{K'}_{k=1} \subseteq N$ with $K' \ge 2$ which satisfies the following conditions:

(a) $p^{x_{i_1}(R')} = \upsilon^{x_{i_1}(R')}$,
(b) for each $k \in \{1, \dots, K'-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R'_{i_k}, p)$ and $t_{i_k}(R') = p^{x_{i_k}(R')}$,
(c) $i_{K'} \in N'$.

We construct $\{i_k\}^K_{k=1}$ from $\{i_k\}^{K'}_{k=1}$ as follows: If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, let $K \equiv K'$ and $\{i_k\}^K_{k=1} \equiv \{i_k\}^{K'}_{k=1}$. If $t_{i_{K'}}(R') \ge p^{x_{i_{K'}}(R')}$, let $K \equiv K' + 1$, $i_K = j$ and $\{i_k\}^K_{k=1} \equiv \{i_k\}^{K'}_{k=1} \cup \{i_K\}$. Note that if $t_{i_{K'}}(R') \ge p^{x_{i_{K'}}(R')}$, then by $t_j(R') < p^{x_{i_K}(R')}$, $i_{K'} \neq j = i_K$.

We show that $\{i_k\}^K_{k=1}$ satisfies the following conditions:

(i) $p^{x_{i_1}(R')} = \upsilon^{x_{i_1}(R')}$,

(ii) for each $k \in \{1, \dots, K-1\}$, $\{x_{i_k}(R'), x_{i_{k+1}}(R')\} \subseteq D(R'_{i_k}, p)$ and $t_{i_k}(R') = p^{x_{i_k}(R')}$,
(iii) $i_K \in N'$ and $t_{i_K}(R') < p^{x_{i_K}(R')}$.

(i): It follows from (a).

(ii): If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, then (ii) follows from (b). Hence, let $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$. Then by $i_{K'} = i_{K-1}$ and Lemma 3, $x_{i_{K-1}}(R') \in D(R'_{i_{K-1}}, p)$ and $t_{i_{K-1}}(R') = p^{x_{i_{K-1}}(R')}$. Therefore, we have only to show $x_{i_K}(R') \in D(R'_{i_{K-1}}, p)$. By $v^{x_j(R')} < p^{x_j(R')}$ and $i_K = j$, $x_{i_K}(R') = x_j(R') \in M$. By (c) and $i_{K'} = i_{K-1}$, $R'_{i_{K-1}} \in \mathcal{R}^{I+}_v(p) \subseteq \mathcal{R}^I(p)$. Hence, by $x_{i_K}(R') \in M$, $x_{i_K}(R') \in D(R'_{i_{K-1}}, p)$, and so (ii) holds.

(iii): If $t_{i_{K'}}(R') < p^{x_{i_{K'}}(R')}$, then (iii) follows from (c). Hence, let $t_{i_{K'}}(R') \geq p^{x_{i_{K'}}(R')}$. By $t_j(R') < p^{x_j(R')}$ and $i_K = j$, $t_{i_K}(R') < p^{x_{i_K}(R')}$. Therefore, by $i_K = j \in N'$, (iii) holds.

Finally, in order to derive a contradiction, we apply Lemma 5 to $\{i_k\}_{k=1}^K$, which concludes that there is an allocation Pareto-dominating $f(R')$ for R'. Note that by (i) and (ii), to apply Lemma 5, we only need to show

$$v^{x_{i_1}(R')} - V'_{i_K}(x_{i_1}(R'); f_{i_K}(R')) < p^{x_{i_K}(R')} - t_{i_K}(R'). \tag{2}$$

By (i), $x_{i_1}(R') \in L^0_v(p)$. By (iii), $t_{i_K}(R') < p^{x_{i_K}(R')}$. Hence, by Proposition 2, $v^{x_{i_K}(R')} \leq t_{i_K}(R') < p^{x_{i_K}(R')}$, and so $x_{i_K}(R') \in M^+_v(p)$.

If for each $a \in M$, $p^a > 0$, then by $R'_{i_K} \in \mathcal{R}^{I+}_v(p)$, $x_{i_1}(R') \in L^0_v(p)$, $x_{i_K}(R') \in M^+_v(p)$, and $t_{i_K}(R') < p^{x_{i_K}(R')}$, Condition (i) of $\mathcal{R}^{I+}_v(p)$ implies (2). Hence, we assume that for some $a \in M$, $p^a = 0$. Then,

$$f_{i_1}(R') \underset{\text{(ii)}}{=} \left(x_{i_1}(R'), p^{x_{i_1}(R')}\right) \underset{x_{i_1}(R') \in D(R'_{i_1}, p)}{R'_{i_1}} (a, p^a) = (a, 0) \underset{\text{object desirability}}{P'_{i_1}} (0, 0).$$

Therefore, if $x_{i_1}(R') = 0$, then by money desirability, $t_{i_1}(R') < 0$, contradicting no-subsidy. Thus, $x_{i_1}(R') \in M$. By $x_{i_1}(R') \in L^0_v(p)$, this implies $x_{i_1}(R') \in M^0_v(p)$. Hence, by $x_{i_K}(R') \in M^+_v(p)$, $t_{i_K}(R') < p^{x_{i_K}(R')}$, and $R'_{i_K} \in \mathcal{R}^{I+}_v(p)$, Condition (ii) of $\mathcal{R}^{I+}_v(p)$ implies (2). □

Finally, we show Proposition 4

Proposition 4. *Let $n, m \in \mathbb{N}$, $v \in \mathbb{R}^m_+$ and $\mathcal{R} = \mathcal{R}^C$. Assume that f satisfies efficiency, individual rationality for the buyers, no-subsidy, and strategy-proofness, and let $R \in \mathcal{R}^n$ and $p = p_{\min}(R, v)$. Then, for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$.*

Proof. Let $R' \in \mathcal{R}^{I+}_v(p)^n$. We prove the following claim by induction.

Claim 2. *For each $S \subseteq N$ and each $i \in N$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$, where $R'_{-S} = (R'_i)_{i \in N \setminus S}$.*

Induction Base. Let $j \in N$ and $S = \{j\}$. By Lemma 7, it suffices to show that $t_j(R_j, R'_{-j}) \geq p^{x_j(R_j, R'_{-j})}$. Suppose to the contrary that $t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})}$.

If $x_j(R_j, R'_{-j}) = 0$, then $t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})} = 0$. However, this contradicts no-subsidy. Hence, $x_j(R_j, R'_{-j}) \in M$. By $x_j(R_j, R'_{-j}) \in M$ and $R'_j \in \mathcal{R}^{I+}_v(p) \subseteq \mathcal{R}^I(p)$, $x_j(R_j, R'_{-j}) \in D(R'_j, p)$.

By applying Lemma 7 to the case of $N' = N$, we have: for each $i \in N$, $t_i(R') \geq p^{x_i(R')}$. In particular, $t_j(R') \geq p^{x_j(R')}$. By Lemma 6, $p = p_{\min}(R', v)$. Hence, by Lemma 3, $x_j(R') \in D(R'_j, p)$ and $t_j(R') = p^{x_j(R')}$. Therefore,

$$\begin{aligned}
f_j(R_j, R'_{-j})\, P'_j &\left(x_j(R_j, R'_{-j}), p^{x_j(R_j, R'_{-j})}\right) && \text{by } t_j(R_j, R'_{-j}) < p^{x_j(R_j, R'_{-j})} \\
I'_j &\left(x_j(R'), p^{x_j(R')}\right) && \text{by } x_j(R_j, R'_{-j}), x_j(R') \in D(R'_j, p) \\
&= f_j(R'). && \text{by } t_j(R') = p^{x_j(R')}
\end{aligned}$$

Hence, $f_j(R_j, R'_{-j})\, P'_j\, f_j(R')$, but this contradicts strategy-proofness. Therefore, $t_j(R_j, R'_{-j}) \geq p^{x_j(R_j, R'_{-j})}$.

Induction Hypothesis. Let $n' \leq n$. Assume that for each $S' \subseteq N$ with $|S'| \leq n' - 1$ and each $i \in N$, $t_i(R_{S'}, R'_{-S'}) \geq p^{x_i(R_{S'}, R'_{-S'})}$.

Induction Argument. Let $S \subseteq N$ be such that $|S| = n'$. By Lemma 7, it suffices to show that for each $i \in S$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$. Suppose to the contrary that for some $k \in S$, $t_k(R_S, R'_{-S}) < p^{x_k(R_S, R'_{-S})}$.

By no-subsidy, $x_k(R_S, R'_{-S}) \in M$. By $x_k(R_S, R'_{-S}) \in M$ and $R'_k \in \mathcal{R}^I(p) \subseteq \mathcal{R}^{I+}_v(p)$, $x_k(R_S, R'_{-S}) \in D(R'_k, p)$.

Let $S' \equiv S \setminus \{k\}$. By $|S'| = n' - 1$ and the Induction Hypothesis, for each $i \in N$, $t_i(R_{S'}, R'_{-S'}) \geq p^{x_i(R_{S'}, R'_{-S'})}$. In particular, $t_k(R_{S'}, R'_{-S'}) \geq p^{x_k(R_{S'}, R'_{-S'})}$. By Lemma 6, $p = p_{\min}((R_{S'}, R'_{-S'}), v)$. Hence, by Lemma 3, $x_k(R_{S'}, R'_{-S'}) \in D(R'_k, p)$ and $t_k(R_{S'}, R'_{-S'}) = p^{x_k(R_{S'}, R'_{-S'})}$. Therefore,

$$\begin{aligned} f_k(R_S, R'_{-S})\, P'_k &\left(x_k(R_S, R'_{-S}), p^{x_k(R_S, R'_{-S})}\right) && \text{by } t_k(R_S, R'_{-S}) < p^{x_k(R_S, R'_{-S})} \\ I'_k &\left(x_k(R_{S'}, R'_{-S'}), p^{x_k(R_{S'}, R'_{-S'})}\right) && \text{by } x_k(R_S, R'_{-S}), x_k(R_{S'}, R'_{-S'}) \in D(R'_k, p) \\ &= f_k(R_{S'}, R'_{-S'}) && \text{by } t_k(R_{S'}, R'_{-S'}) = p^{x_k(R_{S'}, R'_{-S'})} \end{aligned}$$

which implies $f_k(R_{S'}, R_k, R'_{-S})\, P'_k\, f_k(R_{S'}, R'_k, R'_{-S})$, but this contradicts strategy-proofness. Hence, for each $i \in S$, $t_i(R_S, R'_{-S}) \geq p^{x_i(R_S, R'_{-S})}$. **(End of Claim 2)**

Let $S = N$, then for each $i \in N$, $t_i(R) \geq p^{x_i(R)}$. □

B.5. Proof of Proposition 5

Proposition 5. *Let $z \in Z$ and $R \in \mathcal{R}^n$. Then, z is multi-seller efficient for R if and only if z is single-seller efficient for R.*

Proof. Only if. Suppose that z is multi-seller efficient for R but is not single-seller efficient for R. Then, there is some $z' \in Z$ such that (i) for each $i \in N$, $z'_i\, R_i\, z_i$ and (ii) $\sum_{i\in N}(t'_i - v^{x'_i}) > \sum_{i\in N}(t_i - v^{x_i})$. Let

$$S_1 = \left\{a \in S : \exists i \in N, x_i = a \text{ and } \forall j \in N, x'_i \neq a\right\},$$
$$S_2 = \left\{a \in S : \exists i \in N, x_i = a \text{ and } \exists j \in N, x'_i = a\right\},$$
$$S_3 = \left\{a \in S : \forall i \in N, x_i \neq a \text{ and } \exists j \in N, x'_i = a\right\}.$$

Note that $S_1 \cup S_2 = \{x_i\}_{i\in N} \setminus \{0\}$ and $S_2 \cup S_3 = \{x'_i\}_{i\in N} \setminus \{0\}$. By (ii),

$$\begin{aligned} &\sum_{i\in N}(t'_i - v^{x'_i}) > \sum_{i\in N}(t_i - v^{x_i}) \\ \iff &\sum_{a\in S} t'_a - \sum_{i\in N} v^{x'_i} > \sum_{a\in S} t_a - \sum_{i\in N} v^{x_i} && \text{By feasibility (iii)} \\ \iff &\sum_{a\in S} t'_a - \sum_{a\in S_2\cup S_3} v^a > \sum_{a\in S} t_a - \sum_{a\in S_1\cup S_2} v^a && \text{By } \begin{cases} S_1 \cup S_2 = \{x_i\}_{i\in N} \setminus \{0\} \\ S_2 \cup S_3 = \{x'_i\}_{i\in N} \setminus \{0\} \end{cases} \\ \iff &\sum_{a\in S} t'_a - \sum_{a\in S_3} v^a > \sum_{a\in S} t_a - \sum_{a\in S_1} v^a \\ \iff &\sum_{a\in S} t'_a > \sum_{a\in S_1}(t_a - v^a) + \sum_{a\in S_2} t_a + \sum_{a\in S_3}(t_a + v^a). \end{aligned} \tag{3}$$

Let $z'' \in Z$ be such that for each $i \in N$, $z''_i = z'_i$, for each $a \in S$, $x''_a = x'_a$, and that

$$t''_a > \begin{cases} t_a - v^a & \text{if } a \in S_1 \\ t_a & \text{if } a \in S_2 \\ t_a + v^a & \text{if } a \in S_3 \end{cases}$$

and $\sum_{a\in S} t''_a = \sum_{a\in S} t'_a$. By (3), such z'' exists. Then, for each $i \in N$, $z''_i\, R_i\, z_i$ and for each $a \in S$, $t''_a - x''_a \cdot v^a > t_a - x_a \cdot v^a$. However, this contradicts multi-seller efficiency of z for R.

If. Suppose that z is single-efficient for R but is not multi-seller efficient for R. Then, there is some $z' \in Z$ such that (i) for each $i \in N$, $z'_i\, R_i\, z_i$, (ii) for each $a \in S$, $t'_a - x'_a \cdot v^a \geq t_a - x_a \cdot v^a$, and (iii) for some $b \in S$, $t'_b - x'_b \cdot v^b > t_b - x_b \cdot v^b$. By (ii) and (iii),

$$\sum_{a\in S}(t'_a - x'_a \cdot v^a) > \sum_{a\in S}(t_a - x_a \cdot v^a).$$

By $\sum_{i\in N} t_i = \sum_{a\in S} t_a$ and $\sum_{i\in N} t'_i = \sum_{a\in S} t'_a$,

$$\sum_{i\in N} t'_i - \sum_{a\in S} x'_a \cdot v^a > \sum_{i\in N} t_i - \sum_{a\in S} x_a \cdot v^a.$$

By $\{v^{x_i}\}_{i\in N} \setminus \{0\} = \{x_a \cdot v^a\}_{a\in S} \setminus \{0\}$ and by $\{v^{x'_i}\}_{i\in N} \setminus \{0\} = \{x'_a \cdot v^a\}_{a\in S} \setminus \{0\}$,

$$\sum_{i\in N} t'_i - \sum_{i\in N} v^{x'_i} > \sum_{i\in N} t_i - \sum_{i\in N} v^{x_i}.$$

However, this contradicts single-seller efficiency of z for R. □

Appendix C. Relationship to Morimoto and Serizawa (2015)

In this section, we explain the relationship between Morimoto and Serizawa (2015) and our works, which might be helpful to clarify the contribution of this paper.

C.1. Challenging point for Proposition 3

In the proof of the result which is parallel to our Proposition 3, Morimoto and Serizawa (2015) show a sufficient condition for agent i to receive an object a as Lemma 10 (hereafter M & S's Lemma 10), which is stated below as Fact 7. M & S's Lemma 10 is an indispensable part of their proof of Proposition 3 in the sense that it is repeatedly applied in their proof.

Fact 7 *(M & S's Lemma 10).* Let $n > m$ and $v = \mathbf{0}$. Let f satisfy efficiency, individual rationality for the buyers, no-subsidy and strategy-proofness. Let $R \in \mathcal{R}^n$, $a \in M$, $i \in N$ and $z \in Z$ be such that for each $j \in N$, $z_j \, R_j \, (0,0)$. Assume

(a) for each $b \in M \setminus \{a\}, V_i(b;(0,0)) < V^m(b;(0,0))$,[25]
(b) for each $j \in N \setminus \{i\}$, $f_j(R) \, R_j \, z_j$, and
(c) $V_i(a;(0,0)) > \max\left\{V_j(a;z_j) : j \in N \setminus \{i\}\right\}$.

Then, $x_i(R) = a$.

However, we do not assume $n > m$. When $n \leq m$, there is no $R \in \mathcal{R}^n$ that satisfies condition (a) of M & S's Lemma 10. Hence, M & S's Lemma 10 cannot be applied to obtain $x_i(R) = a$ for any $R \in \mathcal{R}^n$. Moreover, even if $n > m$, when $v \neq \mathbf{0}$, it may be the case that an object a is assigned to no agent. Owing to these problems, we cannot adopt the proof strategy of Morimoto and Serizawa (2015) for Proposition 3. Therefore, we use Lemmas 1 and 2 in Appendix B.3 for the proof of Proposition 3. Lemma 1 provides a sufficient condition for an agent i *not* to receive a given object a. Lemma 2 provides a sufficient condition for an object a to be assigned to some agent j, and the lower bound of agent j's payment. These two lemmas enable us to take a path to prove Proposition 3, which is different from Morimoto and Serizawa (2015).

C.2. Challenging points for Proposition 4

There are two difficulties to apply Morimoto and Serizawa's (2015) proof to our model.

The first difficulty is the applicability of the preferences which they use. Given $z \in Z$, $R_0 \in \mathcal{R}$ is **z-indifferent** if for each $j,k \in N$, $z_j \, I_0 \, z_k$. Let $R' = (R_0, \dots, R_0)$. Then, Morimoto and Serizawa's (2015) Lemma 11 (hereafter M & S's Lemma 11) states that for any $N' \subseteq N$, $p_{\min}(R'_{N'}, R_{-N'}, \mathbf{0}) = p_{\min}(R, \mathbf{0})$. M & S's Lemma 11 is a counterpart of Lemma 6 in Appendix B.4. However, as demonstrated by Examples 8 and 9 below, unless $n > m$ and $v = \mathbf{0}$, M & S's Lemma 11 does not hold.

Example 8. Let $N = \{1,2\}$, $M = \{a,b,c\}$. Let $v = r = \mathbf{0}$ and $R \in \mathcal{R}^2$. Let $(z,p) \in W_{\min}(R,r)$. By $r = \mathbf{0}$, $x_1 \in M$ and $x_2 \in M$. Without loss of generality, let $x_1 = a$ and $x_2 = b$. Let $R' \in \mathcal{R}^2$ be such that for each $i \in N$, $(c,0) \, P'_i \, z_1 \, I'_i \, z_2$. Then, R'_1 and R'_2 are z-indifferent preferences, but $p \neq p_{\min}(R',r)$. Therefore, M & S's Lemma 11 does not hold.

Example 9. Let $N = \{1,2,3\}$ and $M = \{a,b\}$. Let $v = r \in \mathbb{R}^2_{++}$ and $R \in \mathcal{R}^3$ be such that for each $i \in N$, $V_i(a;(0,0)) < r^a$. Let $(z,p) \in W_{\min}(R,r)$. Then, $a \in M \setminus \{x_i\}_{i\in N}$ and for some $j \in N$, $z_j = (0,0)$. Let $R' \in \mathcal{R}^3$ be such that for each $i \in N$, $(a, r^a) \, P'_i \, z_1 \, I'_i \, z_2 \, I'_i \, z_3$. Then, R'_1, R'_3 and R'_2 are z-indifferent preferences, but $p \neq p_{\min}(R',r)$. Therefore, M & S's Lemma 11 does not hold.

The second difficulty is in constructing a sequence of agents, which we explained in Subsection 4.2. To show that the payment for an object a is no less than its minimum Walrasian price, Morimoto and Serizawa (2015) construct a sequence of agents such that (a) it starts from an agent assigned the null object, (b) each agent demands his object and his next agent's object, and (c) it is terminated by the agent assigned a. However, unless $n > m$ and $v = r = 0$, since no agent may receive the null object in minimum Walrasian equilibria, there may be no sequence staring from an agent assigned the null object. In such a case, their proof cannot be applied.

We overcome the above problems to prove Proposition 4 as follows. First, instead of z-indifferent preferences, we use p-indifferent preferences defined in Appendix B.4. The p-indifferent preferences are more complicated, but with respect for these preferences, we can show that our Lemma 6 in Appendix B.4 holds, which is the counterpart of M & S's Lemma 11. As for the second difficulty, we consider a sequence of agents starting from an agent assigned a noncompetitive object instead of the null object. If the preference of the last agent of the sequence exhibits positive income effect for noncompetitive objects, then it is possible to construct an allocation that Pareto-dominates the outcome of f from the sequence. Such a sequence shows Proposition 4.

[25] Given k, $V^k(b;(0,0))$ is the kth highest compensation in $\{V_j(b;(0,0))\}_{j\in N}$.

Data availability

No data was used for the research described in the article.

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