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## **Efficient and strategy-proof multi-unit object allocation problems with money: decreasing incremental valuations with income effects**

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# Efficient and strategy-proof multi-unit object allocation problems with money: decreasing incremental valuations with income effects

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## Abstract

We consider the problem of allocating multiple units of an object and collecting payments. Each agent can receive multiple units, and his (consumption) bundle is a pair consisting of his quantity of the object and his payment. An agent's preference over bundles is not necessarily quasi-linear. A class of preferences is called a domain. We show that if a sufficiently rich domain includes at least one preference exhibiting decreasing incremental valuations and either positive or negative income effects, then no rule satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. Our result demonstrates the necessity of certain assumptions for the existence of a rule satisfying the four properties in various environments—for instance, the quasi-linearity assumption is necessary for the classical existence result under quasi-linear preferences with non-increasing incremental valuations (Vickrey 1961).

**Keywords** Efficiency · Strategy-proofness · Non-quasi-linear preferences · Decreasing incremental valuations · The inverse Vickrey rule · Multi-unit auctions

**JEL Classification** D44 · D47 · D71 · D82

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# 1 Introduction

## 1.1 Purpose

A primary objective of many government auctions is the efficient allocation of public resources. These resources often comprise identical objects. Notable examples include spectrum license auctions, such as the French 5G auction and the German 3G auction, as well as procurement auctions in specific industries, such as electricity procurement. Our paper focuses on multi-unit object allocation problems with money, where the objects are identical.

Each agent can receive multiple units of the object, and his (*consumption*) *bundle* consists of a pair specifying the number of units received and the payment. Each agent has a (possibly) non-quasi-linear preference over bundles that accommodates income effects or reflects non-linear borrowing costs.<sup>1</sup> An *allocation* specifies a bundle for each agent. An (*allocation*) *rule* is a function from a set of preference profiles to the set of allocations. A class of preferences is called a *domain*.<sup>2</sup>

An allocation is *efficient* for a given preference profile if no other allocation makes any agent better off without making someone else worse off, or decreasing the owner's revenue. A rule satisfies *efficiency* if it selects an efficient allocation for each preference profile. It satisfies *individual rationality* if no agent is worse off than receiving nothing and making no payment. It satisfies *no subsidy for losers* if an agent who receives no object makes a non-negative payment. This condition prevents the presence of "fake" agents whose sole interest is the participation subsidy. A rule satisfies *strategy-proofness* if no agent ever benefits from misrepresenting his preferences. Our goal is to identify domains on which a rule satisfying these properties exists.

## 1.2 Main results

We consider preferences that exhibit both decreasing incremental valuations and income effects. A preference exhibits *decreasing* (resp. *non-increasing*) *incremental valuations* if, at each bundle, the incremental willingness to sell exceeds (resp. is not smaller than) the incremental willingness to buy. If a firm's production technology exhibits decreasing returns to scale—a standard assumption in economic theory—then the corresponding preference exhibits decreasing incremental valuations. Moreover, most research on multi-unit object allocation problems with money assumes that incremental valuations are decreasing or, at the very least, non-increasing.

A preference exhibits *positive* (resp. *negative*) *income effects* if, as the payment decreases, the incremental willingness to buy increases (resp. decreases). Positive (resp. negative) income effects indicate that the object is a normal (resp. inferior) good. The sign of income effects depends on the context. For example, borrowing costs typically rise progressively as borrowing increases, leading to preferences that

<sup>1</sup> We consider a private values setting.

<sup>2</sup> More precisely, a set of preference profiles constitutes the domain of a rule. However, we assume that the domain of a rule is the Cartesian product of a common class of preferences. Thus, with slight abuse of terminology, we also refer to a class of preferences as a domain.

exhibit positive income effects. The presence of complementary goods can induce positive income effects: lower payments leave more cash available for purchasing complements, thereby increasing demand for the object. In contrast, the presence of substitute goods can give rise to negative income effects.<sup>3</sup> Therefore, we analyze both positive and negative income effects.

A simple and intuitive case of non-increasing incremental valuations is *constant incremental valuations*, where the incremental willingness to sell and buy at each bundle remains constant. Given a positive number  $\varepsilon > 0$ , a preference exhibits  $\varepsilon$ -*non-increasing incremental valuations* if it exhibits non-increasing incremental valuations and the difference between the incremental willingness to sell and that to buy at each bundle is less than  $\varepsilon$ . The  $\text{NI}(\varepsilon)$ -*minimal domain* is the class of quasi-linear preferences exhibiting  $\varepsilon$ -non-increasing incremental valuations. For sufficiently small  $\varepsilon > 0$ , the  $\text{NI}(\varepsilon)$ -minimal domain consists only of quasi-linear preferences with almost constant incremental valuations. Moreover, the Vickrey rule (Vickrey 1961) satisfies *efficiency, individual rationality, no subsidy for losers, and strategy-proofness* on this domain (Vickrey 1961; Holmström 1979). A domain is  $\text{NI}(\varepsilon)$ -*rich* if it contains the  $\text{NI}(\varepsilon)$ -minimal domain.

We establish the following result: *For each  $\varepsilon > 0$ , if an  $\text{NI}(\varepsilon)$ -rich domain includes at least one preference exhibiting both decreasing incremental valuations and either positive or negative income effects, then no rule satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness (Theorem 1 (i)).*

For each  $\varepsilon > 0$ , the quasi-linear, non-increasing incremental valuations domain is an  $\text{NI}(\varepsilon)$ -rich domain. Moreover, the Vickrey rule satisfies the four properties on the domain (Vickrey 1961; Holmström 1979). Thus, Theorem 1 (i) highlights the necessity of the quasi-linearity assumption for guaranteeing the existence of a rule satisfying the four properties on the domain. In fact, it demonstrates that relaxing the quasi-linearity assumption-by introducing even a single preference that exhibits either positive or negative *income effects*-leads to the non-existence of a rule satisfying the four properties.

Although the primary focus of this paper is on preferences exhibiting decreasing incremental valuations, we can also consider preferences with non-decreasing incremental valuations. The notions of *non-decreasing incremental valuations* and, for a given  $\varepsilon > 0$ ,  $\varepsilon$ -*non-decreasing incremental valuations* are defined analogously. The  $\text{ND}(\varepsilon)$ -*minimal domain* is the class of quasi-linear preferences exhibiting  $\varepsilon$ -non-decreasing incremental valuations. A domain is  $\text{ND}(\varepsilon)$ -*rich* if it contains the  $\text{ND}(\varepsilon)$ -minimal domain.

Analogous to the case of  $\text{NI}(\varepsilon)$ -rich domains (Theorem 1 (i)), we establish the following result: *For each  $\varepsilon > 0$ , if an  $\text{ND}(\varepsilon)$ -rich domain includes at least one preference exhibiting both decreasing incremental valuations and either positive or negative income effects, then no rule satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness (Theorem 1 (ii)).*

<sup>3</sup> For example, licenses for low- and high-frequency spectrums serve as substitutes for mobile carriers. Because low-frequency spectrums travel longer distances, a low-frequency spectrum license requires fewer base stations for mobile phone networks. As a result, such a license enables carriers to enter the market at lower financial costs, making it more attractive to carriers with lower cash balances. This factor causes low-frequency spectrum licenses to exhibit negative income effects.

Overall, our findings suggest that when a domain includes at least one non-quasi-linear preference with decreasing incremental valuations, it becomes difficult to ensure the existence of a rule satisfying the four properties. The reasoning behind this negative conclusion is as follows. In a typical efficient allocation, multiple agents receive the object. Consequently, an agent may benefit from engaging in a mutually beneficial exchange with another recipient or from misrepresenting his preference to obtain a more favorable quantity of the object. This flexibility, combined with income effects, makes it challenging for a rule to satisfy the four properties—especially *efficiency* and *strategy-proofness*.

This observation contrasts with the case of non-decreasing incremental valuations. In such a case, a single agent receives all the units in a typical efficient allocation, leaving no opportunity for an agent to acquire a more desirable quantity either by engaging in a mutually beneficial exchange with another recipient or by misrepresenting preferences. Thus, a rule satisfying the four properties exists even for non-quasi-linear preferences with non-decreasing incremental valuations (Saitoh and Serizawa 2008; Sakai 2008; Shinozaki et al. 2022).

Theorem 1 (ii) implies that the assumption of non-decreasing incremental valuations is necessary for this positive result. In fact, relaxing this assumption by introducing even a single non-quasi-linear preference that exhibits *decreasing incremental valuations* into the non-quasi-linear, non-decreasing incremental valuations domain leads to the non-existence of a rule satisfying the four properties.

### 1.3 Organization

The rest of the paper is structured as follows. Section 2 sets up the model. Section 3 introduces a new extension of the Vickrey rule to non-quasi-linear preferences, which we refer to as the inverse Vickrey rule. Section 4 presents the main result. Section 5 discusses the contribution of this paper relative to the previous literature. Section 6 outlines the proof of the main result. Most proofs are deferred to the Appendix, while the remaining ones are provided in the supplementary material.

## 2 Model

There are  $n \geq 2$  agents. The owner owns  $m \geq 2$  units of an object. Let  $N \equiv \{1, \dots, n\}$  denote the set of agents, and  $M \equiv \{0, \dots, m\}$  denote the set of quantities of the object.

Each agent  $i \in N$  receives  $x_i \in M$  units of the object. The amount of money paid by agent  $i$  is denoted by  $t_i \in \mathbb{R}$ . For each agent  $i \in N$ , his **consumption set** is  $M \times \mathbb{R}$ , and his **(consumption) bundle** is a pair  $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$ . Let  $\mathbf{0} \equiv (0, 0)$ .

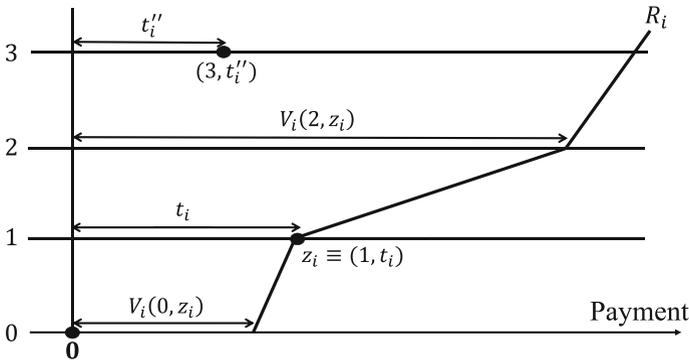


Fig. 1 The consumption set and an indifference curve.

### 2.1 Preferences

Each agent  $i \in N$  has a complete and transitive preference  $R_i$  over  $M \times \mathbb{R}$ . Let  $P_i$  and  $I_i$  be the strict and indifference relations associated with  $R_i$ , respectively. Preferences are privately known. Throughout the paper, we assume the following properties.

**Money monotonicity.** For each  $x_i \in M$  and each pair  $t_i, t'_i \in \mathbb{R}$  with  $t_i < t'_i$ , we have  $(x_i, t_i) P_i (x_i, t'_i)$ .

**Object monotonicity.** For each pair  $x_i, x'_i \in M$  with  $x_i > x'_i$  and each  $t_i \in \mathbb{R}$ , we have  $(x_i, t_i) P_i (x'_i, t_i)$ .

**Possibility of compensation.** For each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M$ , there exists a pair  $t_i, t'_i \in \mathbb{R}$  such that  $(x_i, t_i) R_i z_i$  and  $z_i R_i (x_i, t'_i)$ .

**Continuity.** For each  $z_i \in M \times \mathbb{R}$ , the upper contour set at  $z_i$ ,  $\{z'_i \in M \times \mathbb{R} : z'_i R_i z_i\}$ , and the lower contour set at  $z_i$ ,  $\{z'_i \in M \times \mathbb{R} : z_i R_i z'_i\}$ , are both closed.

A class of preferences satisfying the above four properties is called a **domain**. Our generic notation for a domain is  $\mathcal{R}$ . For each  $i \in N$ , each  $R_i \in \mathcal{R}$ , each  $z_i \in M \times \mathbb{R}$ , and each  $x_i \in M$ , possibility of compensation and continuity together imply that there exists a payment  $V_i(x_i, z_i) \in \mathbb{R}$  such that  $(x_i, V_i(x_i, z_i)) I_i z_i$ .<sup>4</sup> By money monotonicity, such a payment  $V_i(x_i, z_i)$  is unique. We call  $V_i(x_i, z_i)$  the **valuation** of  $x_i$  at  $z_i$  for  $R_i$ .

**Remark 1** Let  $R_i \in \mathcal{R}$ . For each  $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$ ,  $t_i = V_i(x_i, z_i)$ .

Figure 1 illustrates the consumption set of an agent  $i \in N$  and an indifference curve of his preference  $R_i \in \mathcal{R}$ . Each horizontal line corresponds to a quantity of the object. The intersections of the horizontal lines and the vertical line are the points at which payments are zero. Each point on a horizontal line indicates the amount of money that he pays. A solid line is an indifference curve of his preference  $R_i$ . By money monotonicity, a bundle is more preferable as it goes to the left on a horizontal line. Thus, the upper contour set at  $z_i$  corresponds to the left area of the indifference curve through  $z_i$ , so for example,  $(3, t_i'') P_i z_i \equiv (1, t_i)$ .

<sup>4</sup> For the formal proof of the existence of such a payment, see Lemma 1 of Kazumura and Serizawa (2016).

A preference  $R_i$  is **quasi-linear** if for each pair  $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$  and each  $\delta \in \mathbb{R}$ ,  $(x_i, t_i) I_i (x'_i, t'_i)$  implies  $(x_i, t_i - \delta) I_i (x'_i, t'_i - \delta)$ . Let  $\mathcal{R}^Q$  denote the class of quasi-linear preferences. We call  $\mathcal{R}^Q$  the **quasi-linear domain**.

**Remark 2** Let  $R_i \in \mathcal{R}^Q$ .

- (i) There is a **(quasi-linear) valuation function**  $v_i : M \rightarrow \mathbb{R}_+$  such that (i-i)  $v_i(0) = 0$ , (i-ii) for each  $x_i \in M \setminus \{0\}$ ,  $v_i(x_i) > v_i(x_i - 1)$ , and (i-iii) for each pair  $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$ ,  $(x_i, t_i) R_i (x'_i, t'_i)$  if and only if  $v_i(x_i) - t_i \geq v_i(x'_i) - t'_i$ .
- (ii) For each  $(x_i, t_i) \in M \times \mathbb{R}$  and each  $x'_i \in M$ ,  $V_i(x'_i, (x_i, t_i)) - t_i = v_i(x'_i) - v_i(x_i)$ .

### 2.2 Incremental valuations

Given  $R_i \in \mathcal{R}$ ,  $z_i \in M \times \mathbb{R}$ , and  $x_i \in M \setminus \{0\}$ , the **incremental valuation** of  $x_i$  at  $z_i$  for  $R_i$  is  $V_i(x_i, z_i) - V_i(x_i - 1, z_i)$ . If  $R_i \in \mathcal{R}^Q$ , then the incremental valuation of  $x_i$  at  $z_i$  for  $R_i$  is  $v_i(x_i) - v_i(x_i - 1)$ .

A preference  $R_i$  exhibits **non-increasing** (resp. **decreasing**) **incremental valuations** if for each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M \setminus \{0, m\}$ ,

$$V_i(x_i + 1, z_i) - V_i(x_i, z_i) \leq (\text{resp. } <) V_i(x_i, z_i) - V_i(x_i - 1, z_i).$$

It exhibits **constant incremental valuations** if for each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M \setminus \{0, m\}$ ,

$$V_i(x_i + 1, z_i) - V_i(x_i, z_i) = V_i(x_i, z_i) - V_i(x_i - 1, z_i).$$

It exhibits **non-decreasing incremental valuations** if for each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M \setminus \{0, m\}$ ,

$$V_i(x_i + 1, z_i) - V_i(x_i, z_i) \geq V_i(x_i, z_i) - V_i(x_i - 1, z_i).$$

Our definitions of the properties of incremental valuations naturally generalize the corresponding definitions for quasi-linear preferences.

Let  $\mathcal{R}^{NI}$ ,  $\mathcal{R}^D$ ,  $\mathcal{R}^C$ , and  $\mathcal{R}^{ND}$  denote the classes of preferences that exhibit non-increasing, decreasing, constant, and non-decreasing incremental valuations, respectively. We call  $\mathcal{R}^{NI}$ ,  $\mathcal{R}^C$ , and  $\mathcal{R}^{ND}$  the **non-increasing incremental valuations domain**, the **constant incremental valuations domain**, and the **non-decreasing incremental valuations domain**, respectively. Clearly,  $\mathcal{R}^{NI} \cap \mathcal{R}^{ND} = \mathcal{R}^C$ ,  $\mathcal{R}^D \subseteq \mathcal{R}^{NI}$ , and  $\mathcal{R}^D \cap \mathcal{R}^{ND} = \emptyset$ .

Convexity of a preference is a standard assumption in a classical divisible goods model. A preference is *convex* if the upper contour set at each bundle is a convex set (Mas-Colell et al. 1995). In our model with an indivisible object, a convex set can be defined by focusing only on feasible bundles. Specifically, a set  $L \subseteq M \times \mathbb{R}$  is *convex* if for each pair  $z_i, z'_i \in L$  and each  $\delta \in [0, 1]$ ,  $\delta z_i + (1 - \delta)z'_i \in M \times \mathbb{R}$  implies that  $\delta z_i + (1 - \delta)z'_i \in L$ . A set  $L \subseteq M \times \mathbb{R}$  is *strictly convex* if for each distinct pair

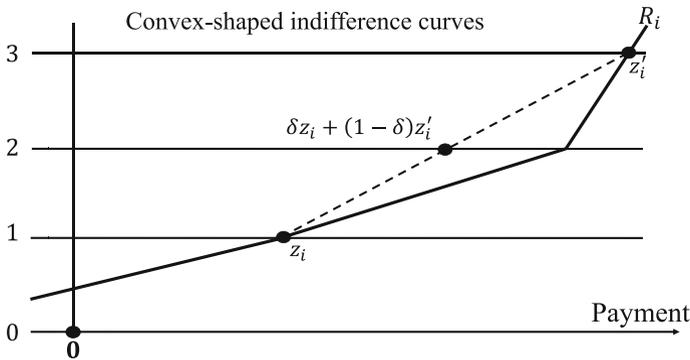


Fig. 2 An indifference curve of a preference with non-increasing incremental valuations.

$z_i, z'_i \in L$  and each  $\delta \in (0, 1)$ ,  $\delta z_i + (1 - \delta)z'_i \in M \times \mathbb{R}$  implies that  $\delta z_i + (1 - \delta)z'_i$  is in the interior of  $L$ .<sup>5</sup> A preference  $R_i$  is *convex* (resp. *strictly convex*) if for each  $z_i \in M \times \mathbb{R}$ , the upper contour set at  $z_i$ ,  $\{z'_i \in M \times \mathbb{R} : z'_i R_i z_i\}$ , is convex (resp. strictly convex).

**Remark 3** For each  $R_i \in \mathcal{R}$ ,  $R_i \in \mathcal{R}^{NI}$  (resp.  $R_i \in \mathcal{R}^D$ ) if and only if  $R_i$  is convex (resp. strictly convex).

As is well known, a convex preference has convex-shaped indifference curves (Mas-Colell et al. 1995). Figure 2 illustrates a convex-shaped indifference curve, where the upper contour set is a convex set.<sup>6</sup> Since convexity is equivalent to non-increasing incremental valuations (see Remark 3), a preference with non-increasing (resp. decreasing) incremental valuations has (resp. strictly) convex-shaped indifference curves.

Finally, we introduce preferences that exhibit “almost” constant incremental valuations. Given  $\varepsilon > 0$ , a preference  $R_i$  exhibits  **$\varepsilon$ -non-increasing incremental valuations** if for each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M \setminus \{0, m\}$ ,

$$0 \leq (V_i(x_i, z_i) - V_i(x_i - 1, z_i)) - (V_i(x_i + 1, z_i) - V_i(x_i, z_i)) < \varepsilon.$$

It exhibits  **$\varepsilon$ -non-decreasing incremental valuations** if for each  $z_i \in M \times \mathbb{R}$  and each  $x_i \in M \setminus \{0, m\}$ ,

$$0 \leq (V_i(x_i + 1, z_i) - V_i(x_i, z_i)) - (V_i(x_i, z_i) - V_i(x_i - 1, z_i)) < \varepsilon.$$

Given  $\varepsilon > 0$ , let  $\mathcal{R}^{NI}(\varepsilon)$  and  $\mathcal{R}^{ND}(\varepsilon)$  denote the classes of  $\varepsilon$ -non-increasing and  $\varepsilon$ -non-decreasing incremental valuations, respectively.

<sup>5</sup> We endow  $M \times \mathbb{R}$  with a distance function  $d : (M \times \mathbb{R})^2 \rightarrow \mathbb{R}_+$  such that for each pair  $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$ ,  $d((x_i, t_i), (x'_i, t'_i)) = |x_i - x'_i| + |t_i - t'_i|$ .

<sup>6</sup> Recall that the upper contour set at a bundle corresponds to the left area of the indifference curve through the bundle in the figure.

For each pair  $\varepsilon, \varepsilon'$  with  $0 < \varepsilon' < \varepsilon$ ,  $\mathcal{R}^{NI}(\varepsilon') \subsetneq \mathcal{R}^{NI}(\varepsilon) \subsetneq \mathcal{R}^{NI}$  and  $\mathcal{R}^{ND}(\varepsilon') \subsetneq \mathcal{R}^{ND}(\varepsilon) \subsetneq \mathcal{R}^{ND}$ . For each  $\varepsilon > 0$ ,  $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^{ND}(\varepsilon) = \mathcal{R}^C$ . As  $\varepsilon$  goes to 0, both  $\mathcal{R}^{NI}(\varepsilon)$  and  $\mathcal{R}^{ND}(\varepsilon)$  converge to  $\mathcal{R}^C$ , that is,  $\bigcap_{\varepsilon \in \mathbb{R}_{++}} \mathcal{R}^{NI}(\varepsilon) = \bigcap_{\varepsilon \in \mathbb{R}_{++}} \mathcal{R}^{ND}(\varepsilon) = \mathcal{R}^C$ .

### 2.3 Income effects

Although we do not explicitly account for income in our model, the payment of zero can be interpreted as the initial income. An increase in (relative) income by  $\delta > 0$  corresponds to a decrease in payment by  $\delta$ . Positive (resp. non-negative) income effects require that an increase in income (or, equivalently, a decrease in payment) by  $\delta$  leads to an increase (resp. no decrease) in the incremental valuation.

Formally, a preference  $R_i$  exhibits **positive** (resp. **non-negative**) **income effects** if for each pair  $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$  with  $x_i > x'_i$  and  $t_i > t'_i$ , and each  $\delta \in \mathbb{R}_{++}$ ,  $(x_i, t_i) I_i (x'_i, t'_i)$  implies  $(x_i, t_i - \delta) P_i (x'_i, t'_i - \delta)$  (resp.  $(x_i, t_i - \delta) R_i (x'_i, t'_i - \delta)$ ).

Let  $\mathcal{R}^{++}$  and  $\mathcal{R}^+$  denote the classes of preferences that exhibit positive and non-negative income effects, respectively. Note that  $\mathcal{R}^{++} \subsetneq \mathcal{R}^+$ ,  $\mathcal{R}^{++} \cap \mathcal{R}^Q = \emptyset$ , and  $\mathcal{R}^Q \subsetneq \mathcal{R}^+$ .

**Remark 4** Let  $R_i \in \mathcal{R}^{++}$ .

- (i) For each  $x_i \in M \setminus \{m\}$ ,  $V_i(x_i + 1, (x_i, t_i)) - t_i$  is strictly decreasing in  $t_i$ .
- (ii) For each  $x_i \in M \setminus \{0\}$ ,  $t_i - V_i(x_i - 1, (x_i, t_i))$  is strictly decreasing in  $t_i$ .

In contrast, negative (resp. non-positive) income effects require that an increase in income by  $\delta$  leads to a decrease (resp. no increase) in the incremental valuation. A preference  $R_i$  exhibits **negative** (resp. **non-positive**) **income effects** if for each pair  $(x_i, t_i), (x'_i, t'_i) \in M \times \mathbb{R}$  with  $x_i > x'_i$  and  $t_i > t'_i$ , and each  $\delta \in \mathbb{R}_{++}$ ,  $(x_i, t_i) I_i (x'_i, t'_i)$  implies  $(x'_i, t'_i - \delta) P_i (x_i, t_i - \delta)$  (resp.  $(x'_i, t'_i - \delta) R_i (x_i, t_i - \delta)$ ).

Let  $\mathcal{R}^{--}$  and  $\mathcal{R}^-$  denote the classes of preferences that exhibit negative and non-positive income effects, respectively. Note that  $\mathcal{R}^{--} \subsetneq \mathcal{R}^-$ ,  $\mathcal{R}^{--} \cap \mathcal{R}^Q = \emptyset$ , and  $\mathcal{R}^Q \subsetneq \mathcal{R}^-$ . Note also that  $\mathcal{R}^+ \cap \mathcal{R}^{--} = \emptyset$ ,  $\mathcal{R}^{++} \cap \mathcal{R}^- = \emptyset$ , and  $\mathcal{R}^+ \cap \mathcal{R}^- = \mathcal{R}^Q$ .

**Remark 5** Let  $R_i \in \mathcal{R}^{--}$ .

- (i) For each  $x_i \in M \setminus \{m\}$ ,  $V_i(x_i + 1, (x_i, t_i)) - t_i$  is strictly increasing in  $t_i$ .
- (ii) For each  $x_i \in M \setminus \{0\}$ ,  $t_i - V_i(x_i - 1, (x_i, t_i))$  is strictly increasing in  $t_i$ .

### 2.4 Allocations and rules

Let  $X \equiv \{(x_1, \dots, x_n) \in M^n : 0 \leq \sum_{i \in N} x_i \leq m\}$ . A **(feasible) allocation** is an  $n$ -tuple  $z \equiv (z_1, \dots, z_n) \equiv ((x_1, t_1), \dots, (x_n, t_n)) \in (M \times \mathbb{R})^n$  such that  $(x_1, \dots, x_n) \in X$ . Let  $Z$  denote the set of allocations. We denote the object allocation and the payments at  $z \in Z$  by  $x \equiv (x_1, \dots, x_n)$  and  $t \equiv (t_1, \dots, t_n)$ , respectively. We may write  $z \equiv (x, t) \in Z$ .

A **preference profile** is an  $n$ -tuple  $R \equiv (R_1, \dots, R_n) \in \mathcal{R}^n$ . Given  $R \in \mathcal{R}^n$  and  $i \in N$ , let  $R_{-i} \equiv (R_j)_{j \in N \setminus \{i\}}$ .

An **allocation rule**, or simply a **rule**, on  $\mathcal{R}$  is a function  $f : \mathcal{R}^n \rightarrow Z$ . With a slight abuse of notation, we may write  $f \equiv (x, t)$ , where  $x : \mathcal{R}^n \rightarrow X$  and  $t : \mathcal{R}^n \rightarrow \mathbb{R}^n$  are the object allocation and the payment rules associated with  $f$ , respectively. Agent  $i$ 's outcome bundle for a preference profile  $R$  under a rule  $f$  is denoted by  $f_i(R) = (x_i(R), t_i(R))$ , where  $x_i(R)$  and  $t_i(R)$  are his quantity of the object and his payment, respectively.

We introduce the properties of rules. The efficiency condition considers the preference of the owner, who is assumed to be interested only in his revenue.<sup>7</sup> An allocation  $z \equiv (x, t) \in Z$  (**Pareto**) **improves** another allocation  $z' \equiv (x', t') \in Z$  for  $R \in \mathcal{R}^n$  if (i) for each  $i \in N$ ,  $z'_i R_i z_i$ , (ii)  $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$ , and (iii) some agent has a strict relation in (i) or the inequality in (ii) is strict. An allocation  $z \in Z$  is (**Pareto**) **efficient** for  $R \in \mathcal{R}^n$  if there is no other allocation  $z' \in Z$  that improves  $z$  for  $R$ .

**Remark 6** Let  $R \in \mathcal{R}^n$  and  $z \equiv (x, t) \in Z$ . Then,  $z$  is efficient for  $R$  if and only if  $\sum_{i \in N} V_i(x_i, z_i) = \max_{x' \in X} \sum_{i \in N} V_i(x'_i, z_i)$ .

We are interested in the class of rules satisfying the following four basic properties.

**Efficiency.** For each  $R \in \mathcal{R}^n$ ,  $f(R)$  is efficient for  $R$ .

**Individual rationality.** For each  $R \in \mathcal{R}^n$  and each  $i \in N$ ,  $f_i(R) R_i \mathbf{0}$ .

**No subsidy for losers.** For each  $R \in \mathcal{R}^n$  and each  $i \in N$ , if  $x_i(R) = 0$ , then  $t_i(R) \geq 0$ .

**Strategy-proofness.** For each  $R \in \mathcal{R}^n$ , each  $i \in N$ , and each  $R'_i \in \mathcal{R}$ ,  $f_i(R) R_i f_i(R'_i, R_{-i})$ .

### 3 Inverse Vickrey rule

Given  $i \in N$ ,  $R_{-i} \in \mathcal{R}^{n-1}$ , and  $x_i \in M$ , let

$$\sigma_i(R_{-i}; x_i) \equiv \max_{x' \in X: x'_i = x_i} \sum_{j \in N \setminus \{i\}} V_j(x'_j, \mathbf{0}).$$

If  $R_{-i} \in (\mathcal{R}^Q)^{n-1}$ , then  $\sigma_i(R_{-i}; x_i) = \max_{x' \in X: x'_i = x_i} \sum_{j \in N \setminus \{i\}} v_j(x'_j)$ .

**Definition 1** (Vickrey 1961) Given  $\mathcal{R} \subseteq \mathcal{R}^Q$ , a rule  $f \equiv (x, t)$  on  $\mathcal{R}$  is a **Vickrey rule** if for each  $R \in \mathcal{R}^n$ , the following two conditions hold.

- (i)  $x(R) \in \arg \max_{x \in X} \sum_{i \in N} v_i(x_i)$ .
- (ii) For each  $i \in N$ ,  $t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R))$ .

Under the Vickrey rule, each agent pays for the impact he has on the other agents.

Research on object allocation problems with money for non-quasi-linear preferences has primarily focused on the generalized Vickrey rule, one of the possible extensions of the Vickrey rule to non-quasi-linear preferences (Saitoh and Serizawa

<sup>7</sup> Thus, the owner has a preference over  $Z$  such that an allocation  $z \equiv (x, t) \in Z$  is at least as good as another allocation  $z' \equiv (x', t') \in Z$  if and only if  $\sum_{i \in N} t_i \geq \sum_{i \in N} t'_i$ .

2008; Sakai 2008; Malik and Mishra 2021).<sup>8</sup> In this section, we propose an alternative extension of the Vickrey rule to non-quasi-linear preferences with non-increasing incremental valuations.

A **price** is  $p \in \mathbb{R}_+$ . Given  $R_i \in \mathcal{R}$  and  $x_i \in M$ , the **inverse-demand set** of  $x_i$  for  $R_i$  is defined as:

$$P(R_i, x_i) \equiv \{p \in \mathbb{R}_+ : \forall x'_i \in M, (x_i, px_i) R_i (x'_i, px'_i)\}.$$

When the preference  $R_i$  is clear from the context, we will abbreviate it by writing  $P(x_i)$  instead of  $P(R_i, x_i)$ .

**Remark 7** Let  $R_i \in \mathcal{R}^{NI}$  and  $x_i \in M$ .

- (i)  $P(x_i) \neq \emptyset$ .
- (ii)  $P(x_i)$  is closed.
- (iii) If  $x_i > 0$ , then  $\max P(x_i)$  exists.
- (iv)  $\sup P(0) = \infty$ .

The proof of Remark 7 can be found in the supplementary material.

An **inverse-demand function** for  $R_i \in \mathcal{R}^{NI}$  is a function  $p(R_i, \cdot) : M \rightarrow \mathbb{R}_+ \cup \{\infty\}$  such that for each  $x_i \in M \setminus \{0\}$ ,  $p(R_i, x_i) \in P(R_i, x_i)$ , and  $p(R_i, 0) \in P(R_i, 0) \cup \{\infty\}$ . If the underlying preference  $R_i$  is obvious from the context, we may omit it from  $p(R_i, \cdot)$ , simply writing  $p(\cdot)$ . Since  $P(x_i)$  is always non-empty (see Remark 7 (i)), an inverse-demand function is well-defined. Note that  $P(0)$  is unbounded above (see Remark 7 (iv)), so we allow  $p(0)$  to equal  $\infty$ .

**Definition 2** The **maximum inverse-demand function** for  $R_i \in \mathcal{R}^{NI}$  is an inverse-demand function  $\bar{p}(\cdot; R_i) : M \rightarrow \mathbb{R}_+ \cup \{\infty\}$  such that (i)  $\bar{p}(0; R_i) = \infty$ , and (ii) for each  $x_i \in M \setminus \{0\}$ ,  $\bar{p}(x_i; R_i) = \max P(x; R_i)$ .

For notational convenience, we may omit  $R_i$  from  $\bar{p}(R_i, x_i)$  when no confusion arises. Since  $\max P(x_i)$  exists for each  $x_i \in M \setminus \{0\}$  (see Remark 7 (iii)),  $\bar{p}$  is well-defined. Moreover, for convenience, let  $\bar{p}(m + 1) \equiv 0$ .

**Remark 8** Let  $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ . For each  $x_i \in M \setminus \{0\}$ ,  $\bar{p}(x_i) = v_i(x_i) - v_i(x_i - 1)$ .

Given a preference  $R_i \in \mathcal{R}^{NI}$ , let  $\bar{R}_i \in \mathcal{R}^Q$  be a quasi-linear preference such that for each  $x_i \in M \setminus \{0\}$ ,  $\bar{v}_i(x_i) - \bar{v}_i(x_i - 1) = \bar{p}(x_i)$ . If  $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ , then  $\bar{R}_i = R_i$  (see Remark 8).

For each preference profile  $R \in (\mathcal{R}^{NI})^n$ , the inverse Vickrey rule applies the transformed preference profile  $\bar{R}$  derived from  $R$  to the Vickrey rule.

**Definition 3** Given  $\mathcal{R} \subseteq \mathcal{R}^{NI}$ , a rule  $f \equiv (x, t)$  on  $\mathcal{R}$  is an **inverse(-demand-based generalized) Vickrey rule** if for each  $R \in \mathcal{R}^n$ , the following two conditions hold.

- (i)  $x(R) \in \arg \max_{x \in X} \sum_{i \in N} \bar{v}_i(x_i)$ .

<sup>8</sup> A rule  $f \equiv (x, t)$  on  $\mathcal{R}$  is a *generalized Vickrey rule* if for each  $R \in \mathcal{R}^n$ , the following two conditions hold. (i)  $x(R) \in \arg \max_{x \in X} \sum_{i \in N} V_i(x_i, \mathbf{0})$ . (ii) For each  $i \in N$ ,  $t_i(R) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i(R))$ .

(ii) For each  $i \in N$ ,  $t_i(R) = \sigma_i(\bar{R}_{-i}; 0) - \sigma_i(\bar{R}_{-i}; x_i(R))$ .

Since  $R_i \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$  implies  $\bar{R}_i = R_i$  (see Remark 8), on any  $\mathcal{R} \subseteq \mathcal{R}^{NI} \cap \mathcal{R}^Q$ , the inverse Vickrey rule coincides with the Vickrey rule.<sup>9</sup> However, the inverse Vickrey rule is different from the generalized Vickrey rule, and the two rules typically produce different outcomes.

## 4 Main result

In this section, we present the main result of the paper and compare it with the results of Baisa (2020), which is the paper most closely related to this paper.

### 4.1 Main result

First, we introduce a domain richness condition that we employ in our result. Given  $\varepsilon > 0$ , a domain  $\mathcal{R}$  is **NI( $\varepsilon$ )-rich** (resp. **ND( $\varepsilon$ )-rich**) if  $\mathcal{R} \supseteq \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  (resp.  $\mathcal{R} \supseteq \mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$ ). We call the domain  $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  (resp.  $\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$ ) the **NI( $\varepsilon$ )-minimal domain** (resp. the **ND( $\varepsilon$ )-minimal domain**). Note that both NI( $\varepsilon$ )-richness and ND( $\varepsilon$ )-richness depend on  $\varepsilon > 0$ . The smaller  $\varepsilon$  becomes, the weaker these conditions are. Specifically, when  $\varepsilon$  is sufficiently close to zero, NI( $\varepsilon$ )-richness (resp. ND( $\varepsilon$ )-richness) requires that a domain should include all quasi-linear preferences with *almost constant* and non-increasing (resp. non-decreasing) incremental valuations.

The following is the main result of the paper.

**Theorem 1** *Let  $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ . Let  $\varepsilon \in \mathbb{R}_{++}$ .*

- (i) *Let  $\mathcal{R}$  be NI( $\varepsilon$ )-rich and  $R_0 \in \mathcal{R}$ . No rule on  $\mathcal{R}$  satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*
- (ii) *Let  $\mathcal{R}$  be ND( $\varepsilon$ )-rich and  $R_0 \in \mathcal{R}$ . No rule on  $\mathcal{R}$  satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness.*

For each  $\varepsilon > 0$ , on both the NI( $\varepsilon$ )-minimal domain  $\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  and the ND( $\varepsilon$ )-minimal domain  $\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q$ , the Vickrey rule satisfies *efficiency, individual rationality, no subsidy for losers, and strategy-proofness* (Vickrey 1961; Holmström 1979). Theorem 1 shows that adding at least one non-quasi-linear preference with decreasing incremental valuations to these domains leads to an impossibility theorem. Thus, Theorem 1 highlights the strict tension between decreasing incremental valuations with income effects and the existence of a rule satisfying the four properties.

**Remark 9** In Theorem 1, we assume that  $\varepsilon > 0$ . When  $\varepsilon = 0$ , NI(0)-richness and ND(0)-richness are equivalent, and they require that a domain should contain all the quasi-linear preferences with *constant* incremental valuations. When  $\varepsilon = 0$ , the

<sup>9</sup> We can extend the definition of the inverse Vickrey to any  $\mathcal{R}$ , but if  $R_i$  does not exhibit non-increasing incremental valuations for some  $i \in N$ , then it does not necessarily coincide with the Vickrey rule, even when  $R \in (\mathcal{R}^Q)^n$ . In such a case, the inverse Vickrey rule can no longer be regarded as an extension of the Vickrey rule. Thus, we choose to define the inverse Vickrey rule only on  $\mathcal{R} \subseteq \mathcal{R}^{NI}$ .

existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* depends on  $m$ , the number of units of the object owned by the owner. When  $m$  is odd, no rule satisfies the four properties on  $\mathcal{R} \cup \{R_0\}$ , where  $\mathcal{R}$  is NI(0)-rich (or equivalently, ND(0)-rich) and  $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ . When  $m$  is even, however, the impossibility result does not necessarily hold. Indeed, we find the examples of  $R_0$  where the inverse Vickrey rule satisfies the four properties on  $(\mathcal{R}^C \cap \mathcal{R}^Q) \cup \{R_0\}$ . The formal results for the case of  $\varepsilon = 0$  can be found in the previous version of this paper (Shinozaki et al. 2022).

**Remark 10** It is worth noting that Theorem 1 (ii) is not a direct consequence of the existence of preferences with non-decreasing incremental valuations and those with decreasing incremental valuations. In fact, the Vickrey rule satisfies the four properties on the domain  $\mathcal{R}^Q$ , which includes both types of preferences (Vickrey 1961; Holmström 1979). Thus, the inclusion of preferences with both non-decreasing and decreasing incremental valuations does not necessarily imply an impossibility result. Instead, Theorem 1 (ii) shows that the impossibility arises not from the mere presence of these preferences, but from the interplay between decreasing incremental valuations and income effects.

## 4.2 Comparison with Baisa (2020)

Baisa (2020) is the paper most closely related to this paper. This subsection compares his findings with ours and highlights the key differences.

Baisa (2020) studies the multi-unit object allocation problem with money. He assumes that preferences exhibit (i) positive income effects, (ii) declining demand, and (iii) single-crossing, and derives results depending on a dimensionality of the domain.<sup>10</sup>

**Fact 1** (Baisa 2020) *Let  $\mathcal{R}$  be a domain consisting solely of preferences that exhibit positive income effects and declining demand, and that satisfies the single-crossing property.*

- (i) *Suppose that  $\mathcal{R}$  is single-dimensional. If  $n = 2$  or  $m = 2$ , then there exists a rule on  $\mathcal{R}$  satisfying efficiency, individual rationality, no subsidy, and strategy-proofness.<sup>11</sup>*
- (ii) *Suppose that  $\mathcal{R}$  is multi-dimensional. Then, no rule on  $\mathcal{R}$  satisfies efficiency, individual rationality, no deficit, and strategy-proofness.<sup>12</sup>*

First, it is clear that Theorem 1 is fundamentally different from Fact 1 (i), as Theorem 1 establishes an impossibility result.<sup>13</sup>

<sup>10</sup> See Baisa (2020) for the formal definitions of these notions.

<sup>11</sup> *No subsidy* requires that the payment of each agent should be non-negative. Note that *no subsidy* implies *no subsidy for losers*.

<sup>12</sup> *No deficit* requires that the sum of payments should be nonnegative. In general, there is no direct logical relationship between *no deficit* and *no subsidy for losers*. However, in the setting of Baisa (2020), *no deficit* implies *no subsidy for losers under efficiency, individual rationality, and strategy-proofness* (see Lemma 5 of Baisa (2020)).

<sup>13</sup> The definitions of the rules in Baisa (2020), as presented in Fact 1 (i), crucially rely on the single-dimensionality of a domain and therefore cannot be applied to the multi-dimensional domains in Theorem 1.

Second, although  $\text{NI}(\varepsilon)$ -richness or  $\text{ND}(\varepsilon)$ -richness makes the domains in Theorem 1 multi-dimensional, Theorem 1 does not follow from Fact 1 (ii). Note that Fact 1 (ii) assumes that a domain  $\mathcal{R}$  is multi-dimensional and includes only preferences exhibiting positive income effects (that is,  $\mathcal{R} \subseteq \mathcal{R}^{++}$ ), meaning that it contains multiple preferences with positive income effects. Therefore, to apply Fact 1 (ii) to a given domain, it must contain multiple preferences exhibiting positive income effects.<sup>14</sup> However, the domains in Theorem 1 may include only a single preference exhibiting (either positive or negative) income effects, as long as they are  $\text{NI}(\varepsilon)$ -rich or  $\text{ND}(\varepsilon)$ -rich. Consequently, Fact 1 (ii) cannot be applied to derive the conclusion of Theorem 1. In other words, Theorem 1 does not follow from Fact 1 (ii). On the other hand, any domain  $\mathcal{R}$  in Fact 1 (ii) satisfies neither  $\text{NI}(\varepsilon)$ -richness nor  $\text{ND}(\varepsilon)$ -richness, as it only includes preferences with positive income effects. Thus, Fact 1 (ii) does not follow from Theorem 1 either.

Furthermore, Theorem 1 covers preferences that may exhibit negative income effects or non-decreasing incremental valuations, whereas Fact 1 focuses on preferences with decreasing incremental valuations and positive income effects.<sup>15</sup>

In addition to the logical independence between the two results, they also have distinct implications. For instance, as discussed in Section 1.2, Theorem 1 (i) demonstrates the necessity of the quasi-linearity assumption for ensuring the existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* on  $\mathcal{R}^{NI} \cap \mathcal{R}^Q$ .<sup>16</sup> In contrast, Baisa (2020)'s impossibility result (Fact 1 (ii)) demonstrates the necessity of the single-dimensionality assumption for ensuring the existence of a rule satisfying *efficiency*, *individual rationality*, *no deficit*, and *strategy-proofness*. Therefore, the two results highlight the necessity of different assumptions in distinct existence results, complementing each other.

After outlining the proof of Theorem 1 from Section 6.1 to Section 6.3, we discuss the differences between our proof and Baisa (2020)'s proof in Section 6.4.

## 5 Related literature

The literature on object allocation problems with money is vast. We classify the literature into several strands based on assumptions on preferences and properties of rules, and discuss how our results differ from each strand.

Footnote 13 continued

Since Baisa (2020)'s rules are based on technical results, such as a fixed-point argument, we do not provide a formal definition here. For details, refer to Baisa (2020).

<sup>14</sup> If a domain  $\mathcal{R} \subseteq \mathcal{R}^{++}$  is a singleton, then any rule is *strategy-proof*, so the existence of a rule satisfying the four properties becomes trivial.

<sup>15</sup> As a minor point, when a preference exhibits positive income effects, declining demand implies decreasing incremental valuation, but the converse is not necessarily true. Thus, there are preferences with decreasing incremental valuations that violate declining demand. Since Fact 1 assumes that every preference in his domain satisfies declining demand, his result does not cover preferences with decreasing incremental valuations and positive income effects that violate declining demand, but our result covers such preferences.

<sup>16</sup> Note that  $\mathcal{R}^{NI} \cap \mathcal{R}^Q$  is  $\text{NI}(\varepsilon)$ -rich.

- (1) *Quasi-linearity and decreasing incremental valuations.* Most studies assume quasi-linear preferences with decreasing (or non-increasing) incremental valuations (Perry and Reny 2002, 2005; Ausubel 2004; Milgrom 2004; Krishna 2009; Ausubel et al. 2014, etc.). Our result (Theorem 1) differs from this strand by covering non-quasi-linear preferences and allowing for non-decreasing incremental valuations, as in Theorem 1 (ii).
- (2) *Heterogeneity and substitutability.* Many studies (Kelso and Crawford 1982; Gul and Stacchetti 1999, 2000; Milgrom 2004; Ausubel 2006, etc.) show that, in the case of heterogeneous objects, substitutability is essential for several positive results. For example, substitutability is sufficient for the existence of an equilibrium allocation (Kelso and Crawford 1982). Moreover, substitutability is both necessary and sufficient for several desirable properties of the Vickrey rule, such as *false-name-proofness*, *core allocation property*, and *shutting-out-proofness*, among others (Ausubel and Milgrom 2002; Shinozaki 2024). When preferences are quasi-linear and objects are identical, as in this paper, substitutability is equivalent to non-increasing incremental valuations (Kelso and Crawford 1982). Thus, our result (Theorem 1) contrasts sharply with this strand, suggesting that if preferences are not necessarily quasi-linear, then non-increasing incremental valuations lead to the non-existence of a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.
- (3) *Efficiency and strategy-proofness.* The class of rules satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* has been studied extensively in the literature. Thus, we classify this strand of the literature into substrands based on assumptions about preferences.
  - (3-1) *Quasi-linearity.* If a domain includes only quasi-linear preferences and is sufficiently rich, then the Vickrey rule is the only rule satisfying the four properties (Holmström 1979; Chew and Serizawa 2007). Our paper distinguishes itself from this strand by studying non-quasi-linear preferences.
  - (3-2) *Non-Quasi-linearity.* The literature on object allocation problems with money for non-quasi-linear preferences is small but expanding. This strand primarily examines rules that satisfy the four properties in various models. We further classify this strand based on the specific assumptions about preferences.<sup>17</sup>
  - (3-2-a) *Unit-demand.* The papers in this category have shown that the minimum price Walrasian rule (Demange and Gale 1985) is the only rule satisfying the four properties in various environments, as long as preferences are unit-demand (Saitoh and Serizawa 2008; Sakai 2008; Morimoto and Serizawa 2015; Zhou and Serizawa 2018; Wakabayashi et al. 2025). Because of the unit-demand assumption, the above results do not apply to multi-unit auc-

<sup>17</sup> More precisely, there is a third category of papers on object allocation problems with money for non-quasi-linear preferences: papers on hard budget constraints (Dobzinski et al. 2012; Lavi and May 2012, etc.). Papers in the first two categories, including this one, assume continuity of preferences, which excludes hard budget constraints. In contrast, papers in the third category assume quasi-linear preferences with hard budget constraints, excluding any income effect in the feasible consumption set. Thus, papers in the third category differ from those in the first two.

tions, which encompass many significant auctions.<sup>18</sup> Our paper contributes to the literature by addressing the multi-unit case, which is common in many important real-life auctions but is not covered by the above studies.

(3-2-b) *Multi-demand.* Other studies examine cases where multiple objects are demanded, as in this paper. In contrast to (3-2-a), the existence of a rule satisfying the four properties is not necessarily ensured. Studies in this category identify various conditions under which a rule satisfying the four properties exists or does not exist.

To the best of our knowledge, prior to this paper, Baisa (2020) was the only study to examine the multi-unit object allocation problem with money under non-quasi-linear preferences.<sup>19</sup> For a detailed comparison between Baisa (2020) and this paper, see Section 4.2.

For heterogeneous objects, Kazumura and Serizawa (2016) show that if a domain contains all unit-demand preferences and at least one multi-demand preference, no rule satisfies the four properties. Similarly, for heterogeneous objects, Malik and Mishra (2021) show that no rule satisfies the four properties on the dichotomous domain. However, on the dichotomous domain with non-negative income effects, the generalized Vickrey rule is the only rule satisfying these four properties. The domains considered in this paper do not include unit-demand or dichotomous preferences. Therefore, our result complements the findings of Kazumura and Serizawa (2016) and Malik and Mishra (2021), as their conclusions do not apply to our model.

## 6 Proof of Theorem 1: An outline

In this section, we outline the proof of Theorem 1. In the proof, the inverse-demand functions and the inverse Vickrey rule play crucial roles. The proof is divided into three parts.

*Part 1: Properties of inverse-demand.* In the first part, we examine the properties of inverse-demand functions for preferences with decreasing incremental valuations.

*Part 2: Identifying the rules satisfying the properties.* In the second part, using the properties of inverse-demand functions identified in Part 1, we determine the object allocation and payments under a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* when an agent has a preference with decreasing incremental valuations. Specifically, we show that when an agent's preference

<sup>18</sup> Examples of multi-unit auctions include car license auctions in Singapore, emission auctions, government bond auctions, electricity procurement auctions, and 3G spectrum auctions in Germany and Austria.

<sup>19</sup> Building on Baisa (2020) and this paper, Shinozaki (2023) examines the multi-unit object allocation problem with money under non-quasi-linear preferences and shows that, in the two-unit case, forgoing *no subsidy for losers* leads to the existence of a rule satisfying *efficiency*, *individual rationality*, and *strategy-proofness*, in contrast to the impossibility results of Baisa (2020) and this paper. Furthermore, Essig Aberg and Baisa (2024) assume an upper bound on non-negative income effects and identify the worst-case inefficiency of the (indirect) Vickrey auction mechanism over all undominated strategies, which depends on this upper bound.

exhibits decreasing incremental valuations and positive income effects, the outcome of the rule coincides with that of the inverse Vickrey rule.

*Part 3: Deriving a contradiction.* In the last part, we show that the rule identified in Part 2 fails to satisfy *efficiency*. This contradicts the assumption that it satisfies all four properties, including *efficiency*, thereby completing the proof. Specifically, when an agent's preference exhibits decreasing incremental valuations and positive income effects, the rule coincides with the inverse Vickrey rule, which we demonstrate does not satisfy *efficiency*.

In the following three subsections, we elaborate on each part. The formal proof can be found in the Appendix and the supplementary material.

## 6.1 Part 1: Properties of inverse-demand

In Part 1, we examine the properties of inverse-demand functions for preferences with decreasing incremental valuations.<sup>20</sup> The results presented in this subsection rely solely on the property of decreasing incremental valuations and thus apply not only to quasi-linear preferences but also to non-quasi-linear preferences. To the best of our knowledge, no previous paper has analyzed inverse-demand functions in a setting with non-quasi-linear preferences. We believe that the results in Part 1 are of independent interest, as they can serve as an analysis of Walrasian equilibrium for non-quasi-linear preferences with decreasing incremental valuations. Among other complications, non-quasi-linearity allows for indifference curves to “follow” a pivoting budget set. For example,  $\{x_i, x_i + 1\}$  may be demanded at a disconnected set of prices with non-empty interior.

First, we introduce the demand set, which is a counterpart of the inverse-demand set. Given a price  $p \in \mathbb{R}_+$ , the **budget set** at  $p$  is the set  $\{(x_i, px_i) : x_i \in M\}$ . Given a preference  $R_i \in \mathcal{R}$  and a price  $p \in \mathbb{R}_+$ , the **demand set** at  $p$  for  $R_i$  is defined as:

$$D(R_i, p) \equiv \{x_i \in M : \forall x'_i \in M, (x_i, px_i) R_i (x'_i, px'_i)\}.$$

When  $R_i$  is clear from the context, we will simply write  $D(p)$  instead of  $D(R_i, p)$ .

**Remark 11** Let  $R_i \in \mathcal{R}$ . For each  $x_i \in M$  and each  $p \in \mathbb{R}_+$ ,  $x_i \in D(p)$  if and only if  $p \in P(x_i)$ .

We say that a pair of distinct quantities  $x_i, x'_i \in M$  is **adjacent** if  $|x_i - x'_i| = 1$ .

**Lemma 1** Let  $R_i \in \mathcal{R}^D$ . Let  $p \in \mathbb{R}_+$ . For each pair  $x_i, x'_i \in D(p)$ ,  $|x_i - x'_i| \leq 1$ .

To understand the intuition behind Lemma 1, recall that a preference with decreasing incremental valuations has strictly convex-shaped indifference curves (see Remark 3 and Figure 2). Since the budget set  $\{(x_i, px_i) : x_i \in M\}$  forms a straight line, an indifference curve can be tangent to the budget line at most at two adjacent quantities.

<sup>20</sup> Note that any domain  $\mathcal{R}$  considered in Theorem 1 satisfies  $R_0 \in \mathcal{R}$ , where  $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ . Thus, the results in Part 1 concerning decreasing incremental valuations apply to a preference  $R_0$  not only in Theorem 1 (i) but also in Theorem 1 (ii).

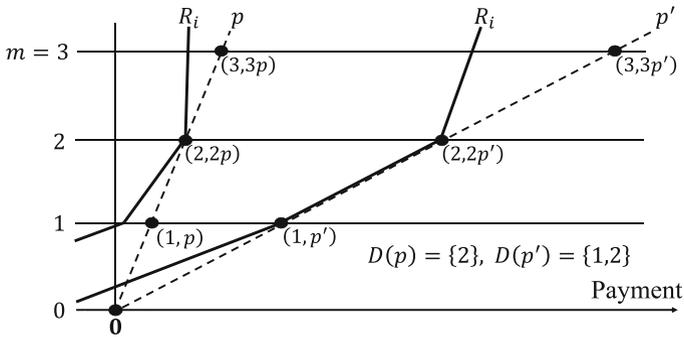


Fig. 3 An illustration of Lemma 1.

Figure 3 illustrate this point, with the dotted lines representing the budget lines at prices  $p$  and  $p'$ , respectively. Thus, an agent with a preference exhibiting decreasing incremental valuations demands at most two adjacent quantities.

Next, we examine the properties of the maximum inverse-demand function  $\bar{p}$ . Note that, by definition,  $\bar{p}(x_i) \in P(x_i)$ , so that  $x_i \in D(\bar{p}(x_i))$  (see Remark 11). Since decreasing incremental valuations imply that the demanded quantities are at most two (see Lemma 1), there are three possible cases: (i)  $D(\bar{p}(x_i)) = \{x_i\}$ , (ii)  $D(\bar{p}(x_i)) = \{x_i, x_i + 1\}$ , or (iii)  $D(\bar{p}(x_i)) = \{x_i, x_i - 1\}$ . The following lemma states that the third case must occur.

**Lemma 2** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \setminus \{0\}$ ,  $D(\bar{p}(x_i)) = \{x_i, x_i - 1\}$ .

The intuition behind the proof of Lemma 2 is as follows. Given that  $x_i \in D(\bar{p}(x_i))$  and that the demanded quantities are at most two (see Lemma 1), it suffices to show  $x_i - 1 \in D(\bar{p}(x_i))$ . If  $x_i - 1 \notin D(\bar{p}(x_i))$ , then by decreasing incremental valuations, an agent would prefer  $x_i$  units to any fewer quantity at the price  $\bar{p}(x_i)$ . Thus, even if the price increases slightly from  $\bar{p}(x_i)$ , he would still demand  $x_i$  units. However, this contradicts the definition of the maximum inverse-demand  $\bar{p}(x_i)$  for  $x_i$  units, where  $\bar{p}(x_i) = \max P(x_i)$ .

The following is a corollary from Lemma 2.

**Corollary 1** Let  $R_i \in \mathcal{R}^D$ .  $\bar{p}(x_i)$  is non-increasing in  $x_i$ .

To see why Corollary 1 follows from Lemma 2, note that for each  $x_i \in M \setminus \{m\}$ , we have  $D(\bar{p}(x_i + 1)) = \{x_i + 1, x_i\}$  (see Lemma 2). Thus,  $x_i \in D(\bar{p}(x_i + 1))$ , which implies  $\bar{p}(x_i + 1) \in P(x_i)$  (see Remark 11). Thus, by  $\bar{p}(x_i) = \max P(x_i)$ ,  $\bar{p}(x_i + 1) \leq \bar{p}(x_i)$ .

A notable feature of the maximum inverse-demand  $\bar{p}(x_i)$  of  $x_i$  units is that  $D(\bar{p}(x_i)) = \{x_i, x_i - 1\}$  (see Lemma 2). Given  $x_i \in M \setminus \{0\}$ , let

$$A(R_i, x_i) \equiv \{p \in \mathbb{R}_+ : D(p) = \{x_i, x_i - 1\}\}$$

denote the set of such prices. We call  $A(R_i, x_i)$  the **(lower) adjacent-demand price set** of  $x_i$  for  $R_i$ . When the underlying preference  $R_i$  is clear from context, we may

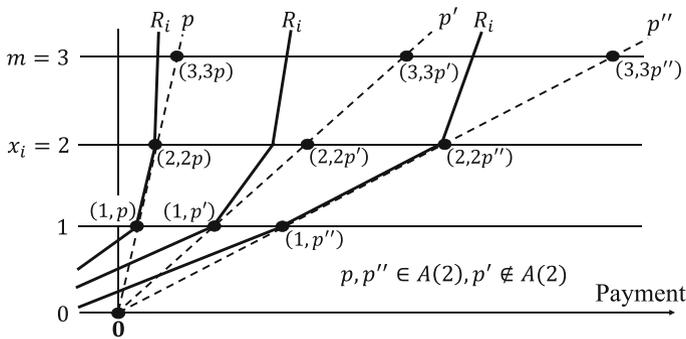


Fig. 4 An illustration of  $A(x_i)$ .

omit  $R_i$ , and simply write  $A(x_i)$ . Note that for each  $x_i \in M \setminus \{0\}$ ,  $\bar{p}(x_i) \in A(x_i)$  (see Lemma 2).

Figure 4 illustrates the adjacent-demand price set  $A(x_i)$  for  $x_i = 2$ . A key graphical feature of a price in  $A(x_i)$  is that an indifference curve is tangent to the budget line at the price only at the lower adjacent quantities,  $x_i$  and  $x_i - 1$ . Thus, in the figure,  $p, p'' \in A(2)$ . In contrast, since an indifference curve is tangent to the budget line at the price  $p'$  only at  $x'_i = 1$ , we have  $p' \notin A(2)$ .

For each  $x_i \in M \setminus \{0\}$ , we have (i)  $\bar{p}(x_i) \in A(x_i)$  (see Lemma 2) (ii)  $\bar{p}(x_i) = \max P(x_i)$  (by definition), and (iii)  $A(x_i) \subseteq P(x_i)$  (by definition). Thus, we have the following.

**Remark 12** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \setminus \{0\}$ ,  $\bar{p}(x_i) = \max A(x_i)$ .

Note that for each  $x_i \in M$ , by  $\bar{p}(x_i + 1) \leq \bar{p}(x_i)$  (see Corollary 1), the closed interval  $[\bar{p}(x_i + 1), \bar{p}(x_i)]$  is well-defined.

**Lemma 3** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \setminus \{0\}$ ,  $\min \left( A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)] \right)$  exists.

Note that  $A(x_i)$  is bounded, and by continuity of  $R_i$ , it is also closed. Thus, since the closed interval  $[\bar{p}(x_i + 1), \bar{p}(x_i)]$  is also bounded and closed, Lemma 3 holds.

**Definition 4** The **minimum adjacent-demand price function** for  $R_i$  is a function  $\underline{p}(R_i \cdot) : M \rightarrow \mathbb{R}_+ \cup \{\infty\}$  such that (i)  $\underline{p}(R_i, 0) = \infty$ , and (ii) for each  $x_i \in M \setminus \{0\}$ ,  $\underline{p}(R_i, x_i) = \min \left( A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)] \right)$ .

Since for each  $x_i \in M \setminus \{0\}$ ,  $\min \left( A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)] \right)$  exists (see Lemma 3),  $\underline{p}$  is well-defined.<sup>21</sup> Moreover, since for each  $x_i \in M \setminus \{0\}$ ,  $\underline{p}(x_i) \in P(x_i)$  (which follows from  $A(x_i) \subseteq P(x_i)$ ),  $\underline{p}$  is an inverse-demand function. When the underlying

<sup>21</sup> Note that the minimum adjacent-demand price function is not defined as the function that selects the minimum element of  $A(x_i)$  for each  $x_i \in M$ . Indeed, such a function does not necessarily satisfy the properties of  $\underline{p}$  discussed below.

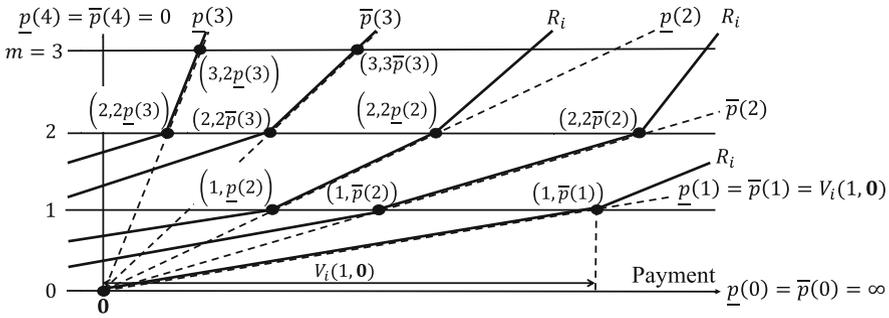


Fig. 5 An illustration of  $\bar{p}$  and  $\underline{p}$ .

preference  $R_i$  is clear from the context, we may simply write  $\underline{p}(\cdot)$  instead of  $\underline{p}(R_i, \cdot)$ . For notational convenience, let  $\underline{p}(m + 1) \equiv 0$ .

By definition,  $\bar{p}(0) = \underline{p}(0) = \infty$  and  $\bar{p}(m + 1) = \underline{p}(m + 1) = 0$ . Moreover, for each  $x_i \in M \setminus \{0\}$ , since  $\underline{p}(x_i) \in A(x_i)$  (by definition) and  $\bar{p}(x_i) = \max A(x_i)$  (see Remark 12), we have  $\bar{p}(x_i) \geq \underline{p}(x_i)$ . Thus, we have the following.

**Remark 13** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \cup \{m + 1\}$ ,  $\bar{p}(x_i) \geq \underline{p}(x_i)$ .

**Lemma 4** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M$ ,  $\underline{p}(x_i) > \bar{p}(x_i + 1)$ .

The sketch of the proof of Lemma 4 is as follows. First, note that  $\underline{p}(0) = \infty > \bar{p}(1)$ . Moreover, by object monotonicity,  $\underline{p}(m) = \min \left( A(m) \cap [\bar{p}(m + 1), \bar{p}(m)] \right) > 0 = \bar{p}(m + 1)$ . Now, let  $x_i \in M \setminus \{0, m\}$ . By definition,  $\underline{p}(x_i) \in A(x_i)$ , so an agent demands  $x_i$  and  $x_i - 1$  units at the price  $\underline{p}(x_i)$ . Similarly, by  $\bar{p}(x_i + 1) \in A(x_i + 1)$ , he demands  $x_i + 1$  and  $x_i$  units at the price  $\bar{p}(x_i + 1)$ . If  $\underline{p}(x_i) = \bar{p}(x_i + 1)$ , then he would demand  $x_i + 1$ ,  $x_i$ , and  $x_i - 1$  units at  $\bar{p}(x_i + 1)$ . However, by decreasing incremental valuations, the demanded quantity can be at most two (see Lemma 1), which leads to a contradiction. Thus,  $\underline{p}(x_i) \neq \bar{p}(x_i + 1)$ . Moreover, by definition,  $\underline{p}(x_i) \in [\bar{p}(x_i + 1), \bar{p}(x_i)]$ , so  $\underline{p}(x_i) \geq \bar{p}(x_i + 1)$ . Therefore,  $\underline{p}(x_i) > \bar{p}(x_i + 1)$ .

Combining Lemma 4 and Remark 13, we obtain the following.

**Corollary 2** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M$ ,  $\bar{p}(x_i) > \bar{p}(x_i + 1)$  and  $\underline{p}(x_i) > \underline{p}(x_i + 1)$ .

Figure 5 illustrates  $\bar{p}$  and  $\underline{p}$  for  $m = 3$ . Note that (i) for each  $x_i \in M \cup \{m + 1\}$ ,  $\bar{p}(x_i) \geq \underline{p}(x_i)$  (see Remark 13), (ii) for each  $x_i \in M$ ,  $\underline{p}(x_i) > \bar{p}(x_i + 1)$  (see Lemma 4), (iii) for each  $x_i \in M$ ,  $\underline{p}(x_i) > \underline{p}(x_i + 1)$  (see Corollary 2), and (iv)  $\underline{p}(1) = \bar{p}(1) = V_i(1, 0)$ .<sup>22</sup>

Note that for each  $x_i \in M \setminus \{m\}$ ,  $\bar{p}(x_i + 1) < \underline{p}(x_i)$  (see Lemma 4), so the open interval  $(\bar{p}(x_i + 1), \underline{p}(x_i))$  is well-defined.

**Lemma 5** Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \setminus \{m\}$  and each  $p \in (\bar{p}(x_i + 1), \underline{p}(x_i))$ ,  $D(p) = \{x_i\}$ .

<sup>22</sup> To verify (iv), let  $p \in A(1)$ . Then,  $D(p) = \{1, 0\}$ , so  $(1, p) I_i \mathbf{0}$ , which implies  $p = V_i(1, \mathbf{0})$ . Thus, by  $\underline{p}(1), \bar{p}(1) \in A(1)$  (see Lemma 2), we have  $\underline{p}(1) = \bar{p}(1) = V_i(1, \mathbf{0})$ .

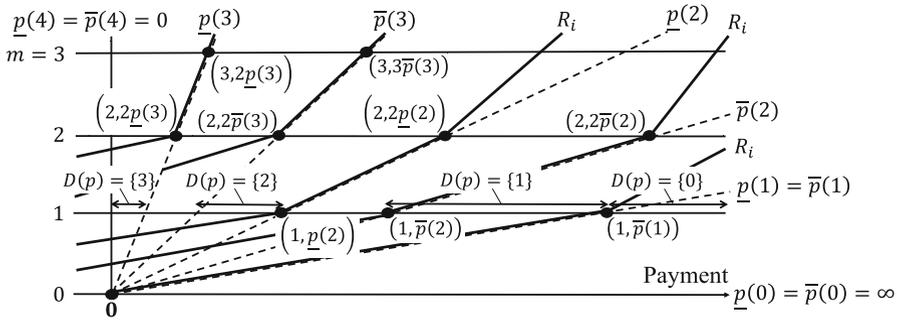


Fig. 6 An illustration of Lemma 5.

To see why Lemma 5 holds, suppose that an agent does not demand  $x_i$  units at some price  $p \in (\bar{p}(x_i + 1), \underline{p}(x_i))$ . If he demands more than  $x_i$  units at  $p$ , then by decreasing incremental valuations, he is willing to pay a higher price for  $x_i + 1$  units than  $p$ . Consequently, at some price  $p' > p > \bar{p}(x_i + 1)$ , he would demand  $x_i + 1$  units, contradicting the definition of the maximum inverse-demand  $\bar{p}(x_i + 1)$  of  $x_i + 1$  units. Similarly, if he demands fewer than  $x_i$  units at  $p$ , then he is willing to pay a lower price for  $x_i$  units at some price  $p' < p < \underline{p}(x_i)$ . However, such a price  $p'$  can be chosen within  $A(x_i) \cap [\bar{p}(x_i + 1), \underline{p}(x_i)]$ , contradicting that  $\underline{p}(x_i) = \min(A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$  (by definition). Thus, he demands only  $x_i$  units at  $p$ .

Figure 6 illustrates Lemma 5. As shown in the figure, an open interval  $(\bar{p}(x_i + 1), \underline{p}(x_i))$  (indicated by an arrow above the horizontal line corresponding to one unit) corresponds to the set of prices at which an agent demands only  $x_i$  units, that is,  $D(p) = \{x_i\}$ .

Lemma 5 provides no information about the demand set when a price lies in  $(\underline{p}(x_i), \bar{p}(x_i))$ . Decreasing incremental valuations only imply that the demanded quantities at prices in  $(\underline{p}(x_i), \bar{p}(x_i))$  are no greater than  $x_i$ .<sup>23</sup> Thus, in general, we cannot identify the demand set at prices in  $(\underline{p}(x_i), \bar{p}(x_i))$ . However, the following remark states that when  $R_i \in \mathcal{R}^+$ ,  $\underline{p}(x_i) = \bar{p}(x_i)$ , so the open interval  $(\underline{p}(x_i), \bar{p}(x_i))$  becomes an empty set.

**Remark 14** Let  $R_i \in \mathcal{R}^D$ . If  $R_i \in \mathcal{R}^+$ , then for each  $x_i \in M \setminus \{0\}$ ,  $|A(x_i)| = 1$ , so by  $\bar{p}(x_i), \underline{p}(x_i) \in A(x_i)$ ,  $\bar{p}(x_i) = \underline{p}(x_i)$ .

The proof of Remark 14 can be found in the supplementary material.

Since  $\mathcal{R}^Q \subseteq \mathcal{R}^+$  and  $\mathcal{R}^{++} \subseteq \mathcal{R}^+$ , Remark 14 holds for any quasi-linear preference with decreasing incremental valuations as well as for any preference with decreasing incremental valuations and positive income effects.

Figure 7 illustrates Lemma 5 for  $R_i \in \mathcal{R}^+$ . As shown in the figure, if  $R_i \in \mathcal{R}^+$ , then the set of prices  $\mathbb{R}_+$  (indicated by the horizontal line corresponding to one unit, to the right of the vertical line in the figure) is partitioned according to the demand.

<sup>23</sup> This follows from  $\bar{p}(x_i) > \bar{p}(x_i + 1)$  (see Corollary 2) and the definition of  $\bar{p}(x'_i)$  for  $x'_i > x_i$ .

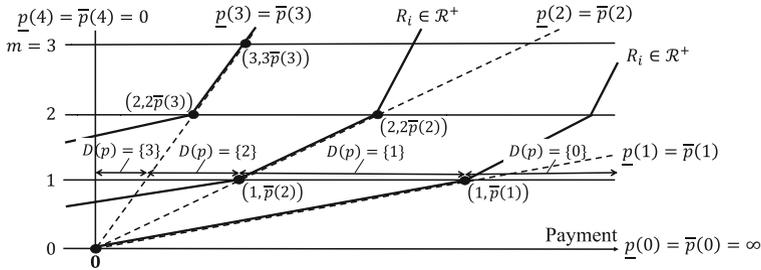


Fig. 7 An illustration of Lemma 5 for  $R_i \in \mathcal{R}^+$

Thus, when  $R_i \in \mathcal{R}^+$ , we can completely identify the demand set (or equivalently, the inverse-demand sets) using  $\bar{p}(x_i)$ .<sup>24</sup>

We illustrate the discussion in this subsection for a simple quasi-linear preference with decreasing incremental valuations.

**Example 1** Let  $m = 4$ . Let  $R_i \in \mathcal{R}^D \cap \mathcal{R}^Q$  be such that  $v_i(1) = 10, v_i(2) = 15, v_i(3) = 18,$  and  $v_i(4) = 19$ . We compute  $A(x_i), \bar{p}(x_i),$  and  $\underline{p}(x_i)$  for  $R_i$ . First, we identify the set  $A(x_i)$  for each  $x_i \in M \setminus \{0\}$ . If  $p \in A(x_i)$ , then  $\bar{D}(p) = \{x_i, x_i - 1\}$ , which implies  $(x_i, px_i) I_i (x_i - 1, p \cdot (x_i - 1))$ . Thus,  $v_i(x_i) - px_i = v_i(x_i - 1) - p \cdot (x_i - 1)$ , or equivalently,  $v_i(x_i) - v_i(x_i - 1) = p$ . For  $x_i = 1, v_i(1) = 10 = p$ , so  $A(1) = \{10\}$ . Similarly, we obtain  $A(2) = \{v_i(2) - v_i(1)\} = \{5\}, A(3) = \{v_i(3) - v_i(2)\} = \{3\}, A(4) = \{v_i(4) - v_i(3)\} = \{1\}$ . For each  $x_i \in M \setminus \{0\}, \bar{p}(x_i) = \max A(x_i)$  (see Remark 12). Thus,  $\bar{p}(1) = \max A(1) = 10, \bar{p}(2) = \max A(2) = 5, \bar{p}(3) = \max A(3) = 3,$  and  $\bar{p}(4) = \max A(4) = 1$ . By definition, for each  $x_i \in M \setminus \{0\}, \underline{p}(x_i) = \min (A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$ . Thus,  $\underline{p}(1) = \min (A(1) \cap [5, 10]) = 10, \underline{p}(2) = \min (A(2) \cap [3, 5]) = 5, \underline{p}(3) = \min (A(3) \cap [1, 3]) = 3,$  and  $\underline{p}(4) = \min (A(4) \cap [0, 1]) = 1$ .

We then verify that  $\bar{p}$  and  $\underline{p}$  identified above satisfy the properties discussed in this subsection.

- We verify that Lemma 2 holds for  $x_i = 2$ . We have  $v_i(1) - \bar{p}(2) = 10 - 5 = 5, v_i(2) - 2\bar{p}(2) = 15 - 10 = 5, v_i(3) - 3\bar{p}(2) = 18 - 15 = 3,$  and  $v_i(4) - 4\bar{p}(2) = 19 - 20 = -1$ . Thus,  $D(\bar{p}(2)) = \{2, 1\}$ .
- We verify Corollary 1. We have  $\bar{p}(1) = 10, \bar{p}(2) = 5, \bar{p}(3) = 3,$  and  $\bar{p}(4) = 1$ . Thus,  $\bar{p}(x_i)$  is non-increasing in  $x_i$ .
- We verify Lemma 4. We have  $\underline{p}(0) = \infty > 10 = \bar{p}(1), \underline{p}(1) = 10 > 5 = \bar{p}(2), \underline{p}(2) = 5 > 3 = \bar{p}(3), \underline{p}(3) = 3 > 1 = \bar{p}(4),$  and  $\underline{p}(4) = 1 > 0 = \bar{p}(5)$ .
- We verify Lemma 5 for  $x_i = 2$ . Let  $p = 4 \in (3, 5) = (\bar{p}(3), \underline{p}(2))$ . Then,  $v_i(1) - p = 10 - 4 = 6, v_i(2) - 2p = 15 - 8 = 7, v_i(3) - 3p = 18 - 12 = 6,$  and  $v_i(4) - 4p = 19 - 16 = 3$ . Thus,  $D(p) = \{2\}$ .

<sup>24</sup> Indeed, for each  $p \in \mathbb{R}_+, (i) D(p) = \{m\}$  if  $p = 0, (ii) D(p) = \{x_i, x_i - 1\}$  if  $p = \bar{p}(x_i)$  for some  $x_i \in M \setminus \{0\},$  and (iii)  $D(p) = \{x_i\}$  if  $p \in (\bar{p}(x_i + 1), \bar{p}(x_i))$  for some  $x_i \in M \setminus \{0\}$ . Note that this exhausts all the possible cases, so it completely identifies the demand set. Moreover, (i) for each  $x_i \in M \setminus \{0\}, P(x_i) = [\bar{p}(x_i + 1), \bar{p}(x_i)],$  and (ii)  $P(0) = [\bar{p}(1), \bar{p}(0)]$ .

- We verify Remark 14. We have  $A(1) = \{10\}$ ,  $A(2) = \{5\}$ ,  $A(3) = \{3\}$ , and  $A(4) = \{1\}$ . Thus, for each  $x_i \in M \setminus \{0\}$ ,  $|A(x_i)| = 1$ .  $\square$

**Remark 15** In Example 1, we focused on a quasi-linear preference with decreasing incremental valuations for simplicity. However, we can determine  $A(x_i)$ ,  $\bar{p}(x_i)$ , and  $\underline{p}(x_i)$  in the same way for a non-quasi-linear preference with decreasing incremental valuations. Indeed, for each  $R_i \in \mathcal{R}^D$  and each  $x_i \in M \setminus \{0\}$ , decreasing incremental valuations imply that  $D(p) = \{x_i, x_i - 1\}$  if and only if  $(x_i, px_i) I_i (x_i - 1, p(x_i - 1))$ , which is equivalent to  $V_i(x_i, (x_i - 1, p \cdot (x_i - 1))) - p \cdot (x_i - 1) = p$  (see Lemma 11 in Appendix A). Thus, for each  $x_i \in M \setminus \{0\}$ , the set  $A(x_i)$  can be identified by solving the following equation for  $p$ :

$$V_i(x_i, (x_i - 1, p \cdot (x_i - 1))) - p \cdot (x_i - 1) = p.$$

We then obtain  $\bar{p}(x_i)$  since  $\bar{p}(x_i) = \max A(x_i)$  for each  $x_i \in M \setminus \{0\}$  (see Remark 12). Finally, we determine  $\underline{p}(x_i)$  using  $\underline{p}(x_i) = \min (A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$  for each  $x_i \in M \setminus \{0\}$ .

## 6.2 Part 2: Identifying the rules satisfying the properties

In the second part, we identify the object allocation and the payments under a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*. The results of this subsection rely solely on decreasing incremental valuations and domain richness conditions, which are weaker than NI( $\varepsilon$ )-richness and ND( $\varepsilon$ )-richness and will be formally introduced in Sections 6.2.3 and 6.2.4. Thus, the results do not rely on income effects or multi-dimensionality of a domain (see Section 4.2 for the discussion of multi-dimensionality).

From here to the end of Section 6.3, we fix a domain  $\mathcal{R}$  and a rule  $f$  on  $\mathcal{R}$ . In the following results, we impose assumptions on  $\mathcal{R}$  or  $f$  as needed to clarify the assumptions required for each result.

We begin by discussing the basic properties of rules that form the foundation of our subsequent analysis.

### 6.2.1 Efficiency

First, we introduce a basic lemma related to *efficient* rules.

**Lemma 6** (No remaining object) *Suppose that  $f$  satisfies efficiency. For each  $R \in \mathcal{R}^n$ ,  $\sum_{i \in N} x_i(R) = m$ .*

Since Lemma 6 directly follows from *efficiency* and object monotonicity, we omit the proof.

### 6.2.2 Strategy-proofness

We then discuss the properties of *strategy-proof* rules.

Given  $i \in N$  and  $R_{-i} \in \mathcal{R}$ , let

$$M_i(R_{-i}) \equiv \{x_i \in M : \exists R_i \in \mathcal{R} \text{ s.t. } x_i(R_i, R_{-i}) = x_i\}.$$

A well-known property of *strategy-proof* rules in this model is that an agent's payment depends solely on the quantity of the object and the preferences of the other agents. Thus, once the quantity is fixed, his own preferences do not affect the payment. Formally, if  $f$  is *strategy-proof*, then given  $i \in N$ ,  $R_{-i} \in \mathcal{R}^{n-1}$ , and  $x_i \in M_i(R_{-i})$ , there exists a (unique) payment  $t_i(R_{-i}; x_i) \in \mathbb{R}$  such that for some  $R_i \in \mathcal{R}$ ,  $f_i(R_i, R_{-i}) = (x_i, t_i(R_{-i}; x_i))$ .<sup>25</sup> Given  $i \in N$ ,  $R_{-i} \in \mathcal{R}^{n-1}$ , and  $x_i \in M_i(R_{-i})$ , let  $z_i(R_{-i}; x_i) \equiv (x_i, t_i(R_{-i}; x_i))$ .

With these notations in hand, *strategy-proofness* implies that for each  $R \in \mathcal{R}^n$  and each  $i \in N$ , agent  $i$ 's outcome bundle  $f_i(R)$  for  $R$  under  $f$  maximizes his welfare among the set  $\{z_i(R_{-i}; x_i) : x_i \in M_i(R_{-i})\}$ .<sup>26</sup> We formalize this observation in the following lemma.

**Lemma 7** (Strategy-proofness) *Suppose that  $f$  satisfies strategy-proofness. For each  $R \in \mathcal{R}^n$ , each  $i \in N$ , and each  $x_i \in M_i(R_{-i})$ ,  $f_i(R) R_i z_i(R_{-i}; x_i)$ .*

### 6.2.3 Characterization of the Vickrey rule

It is well known that if a domain includes only quasi-linear preferences and is convex, then the Vickrey rule is the only rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* (Holmström 1979).<sup>27</sup>

As discussed in Section 6.2.2, under a *strategy-proof* rule, an agent's payment for a certain quantity depends solely on the other agents' preferences. This, together with Holmström (1979)'s characterization result, implies that if the other agents have quasi-linear preferences, then an agent's payment under a rule satisfying the four properties is identical to the payment under the Vickrey rule. The following fact summarizes this observation.

**Fact 2** (Holmström 1979) *Suppose that  $f$  satisfies efficiency, individual rationality, no subsidy for losers, and strategy-proofness. Suppose also that  $\mathcal{R} \cap \mathcal{R}^Q$  is convex. For each  $i \in N$ , each  $R_{-i} \in (\mathcal{R} \cap \mathcal{R}^Q)^{n-1}$ , and each  $x_i \in M_i(R_{-i})$ ,  $t_i(R_{-i}; x_i) = \sigma_i(R_{-i}; 0) - \sigma_i(R_{-i}; x_i)$ .*

<sup>25</sup> To see this, suppose that there exist two distinct payments  $t_i, t'_i \in \mathbb{R}$  such that for some pair  $R_i, R'_i \in \mathcal{R}$ ,  $f_i(R_i, R_{-i}) = (x_i, t_i)$  and  $f_i(R'_i, R_{-i}) = (x_i, t'_i)$ . By  $t_i \neq t'_i$ , either  $t'_i < t_i$  or  $t_i < t'_i$ . Without loss of generality, assume that  $t'_i < t_i$ . Then, by money monotonicity,  $f_i(R'_i, R_{-i}) = (x_i, t'_i) P_i (x_i, t_i) = f_i(R_i, R_{-i})$ . However, this contradicts *strategy-proofness*.

<sup>26</sup> Indeed, if there exists some  $x_i \in M_i(R_{-i})$  such that  $z_i(R_{-i}; x_i) P_i f_i(R)$ , then by  $x_i \in M_i(R_{-i})$ , for some  $R'_i \in \mathcal{R}$ , we would have  $f_i(R'_i, R_{-i}) = z_i(R_{-i}; x_i) P_i f_i(R)$ . However, this contradicts *strategy-proofness*.

<sup>27</sup> A domain  $\mathcal{R} \subseteq \mathcal{R}^Q$  is *convex* if for each pair  $R_i, R'_i$  with valuations functions  $v_i(\cdot), v'_i(\cdot)$  and each  $\lambda \in [0, 1]$ , the quasi-linear preference  $R_i^\lambda$  with valuation function  $v_i^\lambda(\cdot) = \lambda v_i(\cdot) + (1 - \lambda)v'_i(\cdot)$  belongs to  $\mathcal{R}$ .

### 6.2.4 Identifying the rules satisfying the properties

Now, we identify the object allocation and payments under  $f$ .

In the following three lemmas in this subsection, we make the following assumptions:

- Let  $n = 2$ .
- Suppose that  $f$  satisfies *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.
- Suppose that  $\mathcal{R} \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$ .
- Suppose that  $\mathcal{R} \cap \mathcal{R}^Q$  is convex.
- Let  $R_1 \in \mathcal{R} \cap \mathcal{R}^D$ .

Note that, for simplicity, we restrict our attention to the case of two agents (that is,  $n = 2$ ). In Appendix D, we discuss how the proof for the two-agent case extends to the case of three or more agents. Note also that we only assume that agent 1 has a preference  $R_1$  with decreasing incremental valuations, so it may or may not exhibit income effects.

Given a preference  $R_i \in \mathcal{R}^D$ , let  $\underline{R}_i \in \mathcal{R}^D \cap \mathcal{R}^Q$  be a quasi-linear preference such that for each  $x_i \in M \setminus \{0\}$ ,  $\underline{v}_i(x_i) - \underline{v}_i(x_i - 1) = \underline{p}(x_i)$ .<sup>28</sup>

Note that if  $R_i \in \mathcal{R}^D \cap \mathcal{R}^+$ , then  $\underline{p}(x_i) = \bar{p}(x_i)$  for each  $x_i \in M$  (see Remark 14), so  $\underline{R}_i = \bar{R}_i$ . Note also that if  $R_i \in \mathcal{R}^D \cap \mathcal{R}^Q$ , then  $\bar{R}_i = R_i$  (Remark 8), so since  $\mathcal{R}^Q \subseteq \mathcal{R}^+$ ,  $\underline{R}_i = \bar{R}_i = R_i$ .

The next lemma identifies the object allocation under  $f$  when agent 2 has a quasi-linear preference.

**Lemma 8** (Object allocation (i)) *Let  $R_2 \in \mathcal{R} \cap \mathcal{R}^Q$ .*

- (i) *For each  $x \in X$  with  $x_2 \leq x_2(R)$ ,  $\underline{v}_1(x_1(R)) + v_2(x_2(R)) \geq \underline{v}_1(x_1) + v_2(x_2)$ .*
- (ii) *For each  $x \in X$  with  $x_2 \geq x_2(R)$ ,  $\bar{v}_1(x_1(R)) + v_2(x_2(R)) \geq \bar{v}_1(x_1) + v_2(x_2)$ .*
- (iii) *If  $R_1 \in \mathcal{R}^+$ , then  $x(R) \in \arg \max_{x \in X} (\bar{v}_1(x_1) + v_2(x_2))$ .*

To show Lemma 8, we prove the following two lemmas.

First, the following lemma identifies the object allocation under  $f$  when agent 2 has a quasi-linear preference with constant incremental valuations. Note that, in contrast with Lemma 8, we assume that agent 2 has a quasi-linear preference with *constant incremental valuations*.

**Lemma 9** (Object allocation (ii)) *Let  $x_1 \in M$ . Let  $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$  have a constant incremental valuation  $\Delta v_2 \in (\bar{p}(R_1, x_1 + 1), \underline{p}(R_1, x_1))$ . Then,  $x_1(R) = x_1$  and  $x_2(R) = m - x_1$ .*

The intuition behind the proof of Lemma 9 is as follows. Since agent 2 has a quasi-linear preference, agent 1’s payment is equal to that under the Vickrey rule (see Fact 2). Thus, for each  $x'_1 \in M_1(R_2)$ ,  $t_1(R_2; x'_1) = \sigma_1(R_2; 0) - \sigma_1(R_2; x'_1) = v_2(m) - v_2(m - x'_1) = \Delta v_2 \cdot x'_1$ , where the last equality follows from constant incremental valuations

<sup>28</sup> Note that for each  $x_i \in M$ ,  $\underline{p}(x_i + 1) < \underline{p}(x_i)$  (see Corollary 2). Thus,  $\underline{R}_i \in \mathcal{R}^D$ .

of  $R_2$ . Thus, the set  $\{z_1(R_2; x'_1) : x'_1 \in M_1(R_2)\} = \{(x'_1, \Delta v_2 \cdot x'_1) : x'_1 \in M_1(R_2)\}$  forms a straight line, which can be regarded as a (partial) budget set at a price  $p = \Delta v_2$ .<sup>29</sup> Since the price  $p = \Delta v_2$  lies in  $(\bar{p}(R_1, x_1 + 1), \underline{p}(R_1, x_1))$ , agent 1 demands only  $x_1$  units at the price (see Lemma 5). Consequently, *efficiency* implies that  $x_1(R) = x_1$ . Furthermore, since *efficiency* implies that all quantities are allocated to agents (see Lemma 6), we have  $x_2(R) = m - x_1(R) = m - x_1$ .

Lemma 9 implies that  $M_2(R_1) = M$ . Thus, the domain of the function  $t_2(R_1; \cdot)$  is  $M$ .

Second, the following lemma identifies the **incremental payments** of agent 2 under  $f$ , denoted as  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$ .

**Lemma 10** (Incremental payments)

(i) For each  $x_2 \in M \setminus \{0\}$ ,

$$t_2(R_1; x_2) - t_2(R_1; x_2 - 1) \in [\underline{p}(R_1, m - x_2 + 1), \bar{p}(R_1, m - x_2 + 1)].$$

(ii) If  $R_1 \in \mathcal{R}^+$ , then for each  $x_2 \in M \setminus \{0\}$ ,  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) = \bar{p}(R_1, m - x_2 + 1)$ .

To see why Lemma 10 (i) holds, suppose for some  $x_2 \in M \setminus \{0\}$ ,  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$  does not lie within  $[\underline{p}(m - x_2 + 1), \bar{p}(m - x_2 + 1)]$ . If  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) < \underline{p}(m - x_2 + 1)$ , then agent 2 receives  $x_2 - 1$  units at a relatively low incremental payment of  $x_2$  units, by declaring  $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$  whose constant incremental valuation  $\Delta v_2$  lies in  $(\bar{p}(R_1, m - x_2 + 2), \underline{p}(R_1, m - x_2 + 1))$  (see Lemma 9). Since the incremental payment of  $x_2$  units is low, agent 2 could benefit by misrepresenting his preference so as to win one additional unit, contradicting *strategy-proofness*. Analogously, if  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) > \bar{p}(m - x_2 + 1)$ , then agent 2 receives  $x_2$  units at a relatively high incremental payment of  $x_2$  units, and could benefit by misrepresenting his preference to receive one fewer unit, contradicting *strategy-proofness*. Lemma 10 (ii) follows from Lemma 10 (i), since if  $R_1 \in \mathcal{R}^+$ , then for each  $x_1 \in M$ ,  $\bar{p}(R_1, x_1) = \underline{p}(R_1, x_1)$  (see Remark 14).

Now, we move on to the proof of Lemma 8. Figure 8 illustrates the proof of Lemma 8 (i) for  $m = 4$ ,  $x_2(R) = 3$ , and  $x_2 = 1$ . By *strategy-proofness*,  $v_2(3) - v_2(1) \geq t_2(R_1; 3) - t_2(R_1; 1)$  (see Lemma 7). By the lower bounds of agent 2's incremental payments under  $f$  identified in Lemma 10 (i),  $t_2(R_1; 3) - t_2(R_1; 1) \geq \underline{p}(R_1, 2) + \underline{p}(R_1; 1) = \underline{v}_1(3) - \underline{v}_1(1)$ . Thus,  $v_2(3) - v_2(1) \geq \underline{v}_1(3) - \underline{v}_1(1)$ , so  $\underline{v}_1(1) + v_2(3) \geq \underline{v}_1(3) + v_2(1)$ . An analogous argument applies to Lemma 8 (ii), where the upper bounds of the incremental payments identified in Lemma 10 (i) are used. Lemma 8 (iii) follows from Lemma 8 (i) and (ii), since if  $R_1 \in \mathcal{R}^+$ ,  $\underline{R}_1 = \bar{R}_1$  (see Remark 14).

Lemmas 8 and 10 together imply that, when  $R_1 \in \mathcal{R}^+$  and agent 2 has a quasi-linear preference with non-increasing incremental valuations, the outcome allocation under  $f$  coincides with that under the inverse Vickrey rule. To see this, suppose that

<sup>29</sup> Note that the set may be a *partial* budget set, since not all quantities may be available to agent 1 due to  $M_1(R_2) \subseteq M$ .

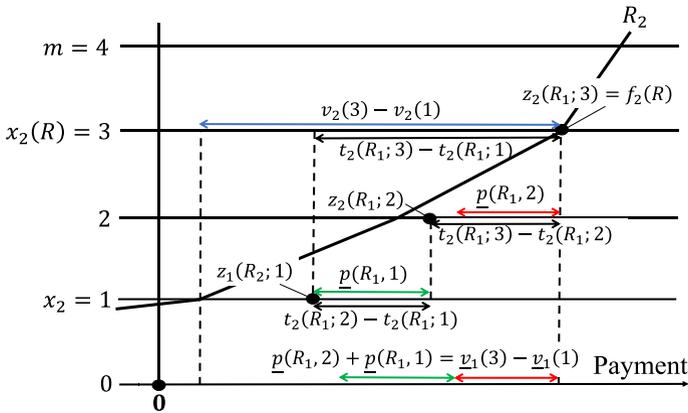


Fig. 8 An illustration of the proof of Lemma 8.

$R_1 \in \mathcal{R}^+$  and agent 2 has a quasi-linear preference with non-increasing incremental valuations. By Lemma 8 (iii), the object allocation under  $f$  coincides with that under the inverse Vickrey rule. Since agent 2 has a quasi-linear preference, agent 1's payment under  $f$  is equivalent to that under the (inverse) Vickrey rule (see Fact 2).<sup>30</sup> Moreover, Lemma 10 (ii) implies that agent 2's payment  $t_2(R_1; x_2)$  of  $x_2$  units under  $f$  coincides with that under the inverse Vickrey rule: (i)  $t_2(R_1; 0) = 0 = \sigma_2(\bar{R}_1; m) - \sigma_2(\bar{R}_1; m)$  (which follows from *individual rationality* and *no subsidy for losers*),<sup>31</sup> and (ii) for each  $x_2 \in M \setminus \{0\}$ ,

$$\begin{aligned} \sigma_2(\bar{R}_1; 0) - \sigma_2(\bar{R}_1; x_2) &= \bar{v}_1(m) - \bar{v}_1(m - x_2) \\ &= \sum_{x'_2=1}^{x_2} \bar{p}(R_1, m - x'_2 + 1) = \sum_{x'_2=1}^{x_2} (t_2(R_1; x'_2) - t_2(R_1; x'_2 - 1)) \\ &= t_2(R_1; x_2) - t_2(R_1; 0) = t_2(R_1; x_2), \end{aligned}$$

where the third equality follows from Lemma 10 (ii), and the last equality from  $t_2(R_1; 0) = 0$ . Thus, we have the following.

**Corollary 3** *Suppose that  $R_1 \in \mathcal{R}^+$ . Let  $R_2 \in \mathcal{R} \cap \mathcal{R}^{NI} \cap \mathcal{R}^Q$ . Then,  $f(R)$  is an outcome allocation under the inverse Vickrey rule for  $R$ .*

As already noted, the results presented in this subsection (Lemmas 8, 9, and 10, and Corollary 3) do not rely on income effects of  $R_1$ . Thus, in particular, they hold when  $R_1$  is quasi-linear. We verify in the following example that the outcome allocation

<sup>30</sup> Note that when  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ ,  $\bar{R}_2 = R_2$  (see Remark 8), so agent 1's payment under the Vickrey rule coincides with that under the inverse Vickrey rule.

<sup>31</sup> To see  $t_2(R_1; 0) = 0$ , note that there is  $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$  such that  $x_2(R) = 0$  (see Lemma 9). By *individual rationality*,  $t_2(R) \leq 0$ . By *no subsidy for losers*,  $t_2(R) \geq 0$ . Thus,  $t_2(R) = 0$ .

under  $f$  coincides with that under the (inverse) Vickrey rule when  $R_1 \in \mathcal{R}^D \cap \mathcal{R}^Q$  and  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$  (see Corollary 3).<sup>32</sup>

**Example 2** Let  $m = 4$ . Let  $R_1 \in \mathcal{R}^D \cap \mathcal{R}^Q$  be the same preference as in Example 1:  $v_1(1) = 10, v_1(2) = 15, v_1(3) = 18$ , and  $v_1(4) = 19$ . As shown in Example 1, we have  $\bar{v}_1(1) = \bar{p}(R_1, 1) = v_1(1) = 10, \bar{v}_1(2) - \bar{v}_1(1) = \bar{p}(R_1, 2) = v_1(2) - v_1(1) = 5, \bar{v}_1(3) - \bar{v}_1(2) = \bar{p}(R_1, 3) = v_1(3) - v_1(2) = 3$ , and  $\bar{v}_1(4) - \bar{v}_1(3) = \bar{p}(R_1, 4) = v_1(4) - v_1(3) = 1$ . Thus,  $\bar{R}_1 = R_1$ . Let  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$  be such that  $v_2(1) = 8, v_2(2) = 11, v_2(3) = 13$ , and  $v_2(4) = 15$ . Suppose that  $R_2 \in \mathcal{R}$ .

We identify the object allocation for  $R$  under  $f$ . Note that  $\bar{v}_1(4) + v_2(0) = 19, \bar{v}_1(3) + v_2(1) = 26, \bar{v}_1(2) + v_2(2) = 28, \bar{v}_1(1) + v_2(3) = 23$ , and  $\bar{v}_1(0) + v_2(4) = 15$ . Thus,  $x = (x_1, x_2) = (2, 2)$  uniquely maximizes  $\bar{v}_1(x'_1) + v_2(x'_2)$  among all  $x' \in X$ , so  $x(R) = x$  (see Lemma 8 (iii)). By  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q, \bar{R}_2 = R_2$  (see Remark 8). Thus,  $x(R) = (x_1(R), x_2(R)) = (2, 2)$  also uniquely maximizes  $\bar{v}_1(x_1) + \bar{v}_2(x_2)$  among all  $x \in X$ . Thus,  $x(R)$  coincides with the object allocation for  $R$  under the (inverse) Vickrey rule.

Next, we identify the payments under  $f$ . Since agent 2 has a quasi-linear preference, agent 1's payment is equivalent to that under the (inverse) Vickrey rule (see Fact 2). Thus,  $t_1(R) = \sigma_1(R_2; 0) - \sigma_1(R_2; 2) = v_2(4) - v_2(2) = 4 = \bar{v}_2(4) - \bar{v}_2(2) = \sigma_1(\bar{R}_2; 0) - \sigma_1(\bar{R}_2; 2)$ . Furthermore,  $t_2(R) = (t_2(R_1; 1) - t_2(R_1; 0)) + (t_2(R_1; 2) - t_2(R_1; 1)) = \bar{p}(R_1, 4) + \bar{p}(R_1, 3) = 4 = \bar{v}_1(4) - \bar{v}_1(2) = \sigma_2(\bar{R}_1; 0) - \sigma_1(\bar{R}_1; 2)$ , where the first equality follows from  $t_2(R_1; 0) = 0$  (by *individual rationality and no subsidy for losers*), and the second one from Lemma 10 (ii). Thus, the payments for  $R$  under  $f$  coincide with those under the (inverse) Vickrey rule.  $\square$

### 6.3 Part 3: Deriving a contradiction

In the last part, we complete the proof.

Let  $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$  and  $\varepsilon \in \mathbb{R}_{++}$ . Suppose that the domain  $\mathcal{R}$  is NI( $\varepsilon$ )-rich or ND( $\varepsilon$ )-rich, and  $R_0 \in \mathcal{R}$ . For contradiction, suppose that  $f$  satisfies *efficiency, individual rationality, no subsidy for losers, and strategy-proofness*.

Let  $R_1 = R_0$ . As in Section 6.2.4, we focus on the two-agent case, so let  $n = 2$ .

If no rule satisfies the four properties on a domain  $\mathcal{R}$ , then no rule satisfies them on any superdomain  $\mathcal{R}' \supseteq \mathcal{R}$ . Thus, to prove Theorem 1 (i) (resp. Theorem 1 (ii)), it suffices to consider the smallest domain,  $(\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$  (resp.  $(\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ ). Since these domains satisfy the assumptions made in Section 6.2.4 (that is,  $\mathcal{R} \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$  and convexity of  $\mathcal{R} \cap \mathcal{R}^Q$ ), we can apply the results from Section 6.2.4 to these domains.

We now provide a sketch of the proof, aiming to derive a contradiction.

First, consider Theorem 1 (i). Let  $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ . Since  $R_1$  exhibits either positive or negative income effects, there are two cases. We outline the proof for the case of positive income effects. Thus, suppose  $R_1 \in \mathcal{R}^{++}$ . Let  $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  be such that  $v_2(1) = \bar{p}(R_1, m) + \varepsilon'$ , and for each  $x_2 \in M \setminus \{0, 1\}, v_2(x_2) - v_2(x_2 -$

<sup>32</sup> Note that when agents have quasi-linear preferences with non-increasing incremental valuations, the inverse Vickrey rule coincides with the Vickrey rule (see Remark 8).

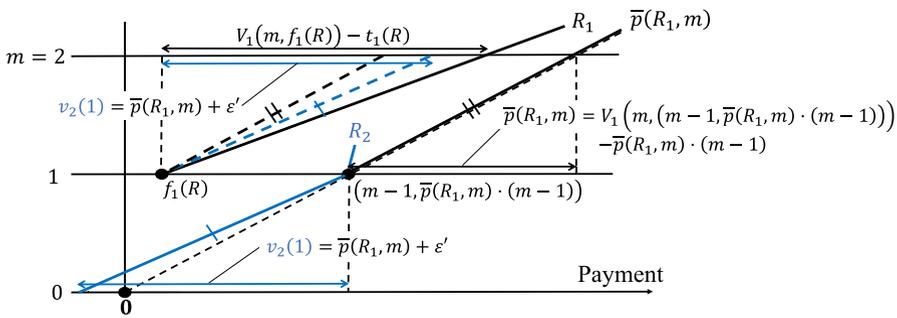


Fig. 9 An illustration of  $t_1(R) < \bar{p}(R_1, m) \cdot (m - 1)$  and  $V_1(m, f_1(R)) - t_1 > v_2(1)$

1) =  $\bar{p}(R_1, m) - \varepsilon''$ , where  $\varepsilon', \varepsilon''' > 0$  are sufficiently small positive numbers. Then,  $v_2(1) > \bar{p}(R_1, m) = \bar{v}_1(m) - \bar{v}_1(m - 1) > v_2(x_2) - v_2(x_2 - 1)$  for each  $x_2 \in M \setminus \{0, 1\}$ . Thus, the object allocation  $x = (x_1, x_2) = (m - 1, 1)$  uniquely maximizes  $\bar{v}(x_1) + v_2(x_2)$  among all  $x \in X$ , so by positive income effects of  $R_1$ ,  $x(R) = (m - 1, 1)$  (see Lemma 8 (iii)). Since agent 2 has a quasi-linear preference, agent 1’s payment under  $f$  coincides with that under the Vickrey rule (see Fact 2). Thus, by  $x_1(R) = m - 1$ ,

$$\begin{aligned} t_1(R) &= t_1(R_2, m - 1) = \sigma_1(R_2, 0) - \sigma_1(R_2, m - 1) = v_2(m) - v_2(1) \\ &= (\bar{p}(R_1, m) - \varepsilon'') \cdot (m - 1) \\ &< \bar{p}(R_1, m) \cdot (m - 1). \end{aligned}$$

Thus, positive income effects of  $R_1$  imply that the incremental valuation of  $m$  units at  $f_1(R)$  for  $R_1$ ,  $V_1(m, f_1(R)) - t_1(R)$ , is greater than that at  $(m - 1, \bar{p}(R_1, m) \cdot (m - 1))$ ,  $V_1(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1))) - \bar{p}(R_1, m) \cdot (m - 1)$ , which is equivalent to  $\bar{p}(R_1, m)$  (by  $\bar{p}(R_1, m) \in A(R_1, m)$ ; see Lemma 2). Since  $\varepsilon' > 0$  is sufficiently small,  $V_1(m, f_1(R)) - t_1(R)$  still exceeds  $\bar{p}(R_1, m) + \varepsilon' = v_2(1)$ . Figure 9 illustrates this point for  $m = 2$ , where the solid line starting from  $f_1(R)$  and extending upwards corresponds to  $V_1(m, f_1(R)) - t_1(R)$  and the blue dotted line from  $f_1(R)$  represents  $v_2(1) = \bar{p}(R_1, m) + \varepsilon'$ . Therefore, reallocating one unit from agent 2 to agent 1 Pareto improves  $f(R)$  for  $R$ , contradicting efficiency. In the formal proof, the choice of  $\varepsilon', \varepsilon'' > 0$  depends on  $R_1$  and  $\varepsilon$ , making the analysis more intricate.

The key points of the above discussion are summarized as follows:

- The discussion begins by considering agent 2’s preference  $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  such that  $v_2(1) > \bar{p}(R_1, m) > v_2(x_2) - v_2(x_2 - 1)$  for each  $x_2 \in M \setminus \{0, 1\}$ . We can choose such a preference from a domain only when the domain is multi-dimensional, that is, it is NI( $\varepsilon$ )-rich for  $\varepsilon > 0$ . Then,  $\varepsilon$ -non-increasing incremental valuations of  $R_2$  imply that  $x(R) = (m - 1, 1)$ , and  $t_1(R) < \bar{p}(R_1, m) \cdot (m - 1)$ , that is, agent 1’s payment is relatively small.
- Since agent 1’s payment is small, positive income effects of  $R_1$  make the incremental valuation of  $m$  units at  $f_1(R)$  large, enabling us to Pareto improve the outcome allocation under  $f$  for  $R$ .

The proof for the case of negative income effects is similar:  $\varepsilon$ -non-increasing incremental valuations make agent 1’s payment under  $f$  small, and negative income effects of  $R_1$  imply that reallocating one unit from agent 1 to agent 2 Pareto improves  $f(R)$  for  $R$ , contradicting *efficiency*. Moreover, the proof of Theorem 1 (ii) follows a similar argument, where  $\varepsilon$ -non-decreasing incremental valuations make agent 1’s payment under  $f$  relatively large, and (either positive or negative) income effects of  $R_1$  allow us to Pareto improve the outcome allocation under  $f$  for  $R$ . Note that the construction of agent 2’s quasi-linear preference  $R_2$  depends on whether  $R_1$  exhibits positive or negative income effects and whether  $\mathcal{R}$  is NI( $\varepsilon$ )-rich or ND( $\varepsilon$ )-rich.

Recall that the results in Steps 1 and 2 depend solely on decreasing incremental valuations; hence, neither income effects nor NI( $\varepsilon$ )-richness (or ND( $\varepsilon$ )-richness) are required. These assumptions only come into play in Part 3. As explained above, NI( $\varepsilon$ )-richness (resp. ND( $\varepsilon$ )-richness) ensures that agent 1’s payment is small (resp. large), and income effects allow for a Pareto improvement of the outcome allocation under  $f$ .

When  $R_1 \in \mathcal{R}^+$  and  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ , the outcome allocation under the rule  $f$  coincides with that under the inverse Vickrey rule (see Corollary 3). Thus, in this case, the discussion above reduces to showing the failure of *efficiency* of the inverse Vickrey rule. To illustrate the above discussion, we consider the following example, which demonstrates that the inverse Vickrey rule violates *efficiency*.

**Example 3** Let  $n = 2$  and  $m = 2$ . Let  $R_0 \in \mathcal{R}^D \cap \mathcal{R}^{++}$  be such that (i)  $V_0(1, \mathbf{0}) = 4$ , and (ii) for each  $t_0 \in [0, 4]$ ,  $V_0(2, (1, t_0)) - t_0 = -\frac{1}{2}t_0 + 3$ . Let  $\varepsilon = 0.05$ . Let  $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ . Let  $f$  be an inverse Vickrey rule on  $\mathcal{R}$ . Let  $R_1 = R_0$ .

First, we compute  $\bar{R}_1$ . As discussed in Remark 15, we can identify  $A(R_1, x_1)$  by solving  $V_1(x_1, (x_1 - 1, p \cdot (x_1 - 1))) - p \cdot (x_1 - 1) = p$  for  $p$ . We have  $p = V_1(1, \mathbf{0}) = 4$ . Thus,  $A(R_1, 1) = \{4\}$ , so  $\bar{p}(R_1, 1) = \max A(R_1, 1) = 4$ . We also have  $V_1(2, (1, p)) - p = -\frac{1}{2}p + 3 = p$  if and only if  $p = 2$ . Thus,  $A(R_1, 2) = \{2\}$ , so  $\bar{p}(R_1, 2) = 2$ . Thus,  $\bar{v}_1(1) = \bar{p}(R_1, 1) = 4$  and  $\bar{v}_1(2) - \bar{v}_1(1) = \bar{p}(R_1, 2) = 2$ .

Let  $R_2 \in \mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q$  be such that  $v_2(1) = 2.01 = \bar{p}(R_1, 2) + 0.01$  and  $v_2(2) - v_2(1) = 1.97 = \bar{p}(R_1, 2) - 0.03$ . By  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ ,  $\bar{R}_2 = R_2$  (see Remark 8). Thus,  $x = (x_1, x_2) = (1, 1)$  uniquely maximizes  $\bar{v}_1(x'_1) + \bar{v}_2(x'_2)$  among all  $x' \in X$ , so  $x(R) = (1, 1)$ . Thus,  $t_1(R) = \sigma_1(\bar{R}_2; 0) - \sigma_1(\bar{R}_2; 1) = \bar{v}_2(2) - \bar{v}_2(1) = v_2(2) - v_2(1) = 1.97$ . Then,

$$\left( V_1(2, f_1(R)) - t_1(R) \right) - v_2(1) = (-0.985 + 3) - 2.01 = 2.015 - 2.01 > 0.$$

Thus, reallocating one unit from agent 2 to agent 1 Pareto improves  $f(R)$  for  $R$  (see Remark 6), so  $f(R)$  is not efficient for  $R$ . □

### 6.4 Comparison with Baisa (2020)’s proof

We explain how our proof of Theorem 1 differs from the impossibility result in Baisa (2020) (Fact 1 (ii)).

We first briefly outline his proof. A key observation in his proof is the *interdependence* between  $V_2(1, \mathbf{0})$  and  $V_1(2, z_1(R_2; 1)) - t_1(R_2; 1)$ . Specifically, he shows that if a rule  $f$  satisfies *efficiency*, *individual rationality*, *no deficit* (or *no subsidy for losers*), and *strategy-proofness*, then for each pair  $R_1, R_2 \in \mathcal{R}$ , if  $V_2(1, \mathbf{0}) = t_2(R_1; 1)$ , then<sup>33</sup>

$$V_2(1, \mathbf{0}) = V_1(2, z_1(R_2; 1)) - t_1(R_2; 1).$$

He then establishes that this interdependence is incompatible with the combination of positive income effects and multi-dimensionality of a domain.<sup>34</sup>

In Baisa (2020)’s proof, the combination of positive income effects and multi-dimensionality of a domain plays a crucial role in establishing the failure of the interdependence. In contrast, our proof shows that the combination of (either positive or negative) income effects and NI( $\epsilon$ )-richness (or ND( $\epsilon$ )-richness, which corresponds to multi-dimensionality of a domain) allows us to establish the failure of *efficiency*. Thus, while both proofs involve income effects and multi-dimensionality, these elements play different roles.

Moreover, recall that a key observation in our proof is that the outcome allocation under a rule satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness* coincides with that under the inverse Vickrey rule (when  $R_1 \in \mathcal{R}^+$  and  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ ; see Corollary 3). As demonstrated in the following example, under the inverse Vickrey rule, the interdependence discussed in Baisa (2020) does not necessarily hold. Therefore, our proof strategy differs fundamentally from his.

**Example 4** Let  $n = 2$  and  $m = 2$ . Consider the same setting as Example 3. As shown in Example 3,  $\bar{v}_1(1) = \bar{p}(R_1, 1) = 4$ ,  $\bar{v}_1(2) - \bar{v}_1(1) = \bar{p}(R_1, 2) = 2$ , and agent 1’s outcome bundle for  $R$  under the inverse Vickrey rule  $f$  is  $f_1(R) = (1, 1.97)$ . Let  $R'_2 \in \mathcal{R}^{NI}(\epsilon) \cap \mathcal{R}^Q$  be such that  $v'_2(1) = 2.01$  and  $v'_2(2) - v'_2(1) = 1.98$ . By  $R'_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$ ,  $\bar{R}'_2 = R'_2$  (see Remark 8). Thus,  $x = (x_1, x_2) = (1, 1)$  uniquely maximizes  $\bar{v}_1(x'_1) + \bar{v}'_2(x'_2)$  among all  $x' \in X$ , so  $x(R_1, R'_2) = (x_1(R_1, R'_2), x_2(R_1, R'_2)) = (1, 1)$ . Thus,  $t_1(R_1, R'_2) = \sigma_1(\bar{R}'_2; 0) - \sigma_1(\bar{R}'_2; 1) = \bar{v}'_2(2) - \bar{v}'_2(1) = v'_2(2) - v'_2(1) = 1.98$ . Then,  $v_2(1) = v'_2(1) = 2.01$ , but

$$\begin{aligned} V_1(2, f_1(R)) - t_1(R) &= -0.985 + 3 = 2.015 > 2.01 = -0.99 + 3 \\ &= V_1(2, f_1(R_1, R'_2)) - t_1(R_1, R'_2). \end{aligned}$$

<sup>33</sup> See Corollary 2 of Baisa (2020).

<sup>34</sup> Specifically, let  $R_1 \in \mathcal{R}^{++}$ . Then, multi-dimensionality of a domain implies that there exists a pair  $R_2, R'_2 \in \mathcal{R}$  such that  $V_2(1, \mathbf{0}) = V'_2(1, \mathbf{0}) = t_2(R_1; 1)$ , and for each  $t_2 \in \mathbb{R}$ ,  $V'_2(2, (1, t_2)) - t_2 > V_2(2, (1, t_2)) - t_1$ . Then, Baisa (2020) shows that  $t_1(R'_2; 1) > t_1(R_2; 1)$  (see Proposition 2 (4) of Baisa (2020)). Thus, positive income effects of  $R_1$  imply  $V_1(2, z_1(R'_2; 1)) - t_1(R'_2; 1) < V_1(2, z_1(R_2; 1)) - t_1(R_2; 1)$ . However, this contradicts the interdependence discussed above, as  $V_2(1, \mathbf{0}) = V'_2(1, \mathbf{0}) = t_2(R_1; 1)$  implies

$$V_1(2, z_1(R_2; 1)) - t_1(R_2; 1) = V_2(1, \mathbf{0}) = V'_2(1, \mathbf{0}) = V_1(2, z_1(R'_2; 1)) - t_1(R'_2; 1).$$

Thus, the interdependence of Baisa (2020) does not hold under the inverse Vickrey rule.  $\square$

## Appendix

### A Proof of Part 1: Properties of inverse-demand

In this section, we present the proofs of the results in Part 1 of the proof of Theorem 1, as discussed in Section 6.1.

First, we prove the following two lemmas, which will be used in the proofs of the lemmas in Part 1.

**Lemma 11** *Let  $R_i \in \mathcal{R}^D$ . For each  $x_i \in M \setminus \{0\}$  and each  $p \in \mathbb{R}_+$ , the following statements are mutually equivalent:*

- (i)  $D(p) = \{x_i, x_i - 1\}$ .
- (ii)  $(x_i, px_i) I_i (x_i - 1, p \cdot (x_i - 1))$ .
- (iii)  $p \cdot (x_i - 1) = V_i(x_i - 1, (x_i, px_i))$ .
- (iv)  $px_i = V(x_i, (x_i - 1, p \cdot (x_i - 1)))$

**Proof** Let  $x_i \in M \setminus \{0\}$  and  $p \in \mathbb{R}_+$ . The equivalence between (ii), (iii), and (iv) follows from the definition of the valuation. It is also straightforward that (i) implies (ii). We now proceed to prove that (iii) implies (i). Suppose  $p \cdot (x_i - 1) = V_i(x_i - 1, (x_i, px_i))$ . Then,

$$p = px_i - V_i(x_i - 1, (x_i, px_i)). \tag{1}$$

Let  $x'_i \in M \setminus \{x_i, x_i - 1\}$ . We show  $(x_i, px_i) P_i (x'_i, px'_i)$ , which, together with  $(x_i, px_i) I_i (x_i, p \cdot (x_i - 1))$  (recall (ii) is equivalent to (iii)), establishes  $D(p) = \{x_i, x_i - 1\}$ . If  $x'_i > x_i$ , then

$$V_i(x'_i, (x_i, px_i)) - px_i < (x'_i - x_i) \cdot (px_i - V_i(x_i - 1, (x_i, px_i))) = (x'_i - x_i) \cdot p,$$

where the inequality follows from decreasing incremental valuations and the equality from (1). Thus,  $V_i(x'_i, (x_i, px_i)) < px'_i$ , which implies  $(x_i, px_i) P_i (x'_i, px'_i)$ . If  $x'_i < x_i - 1$ , then

$$px_i - V_i(x'_i, (x_i, px_i)) > (x_i - x'_i) \cdot (px_i - V_i(x_i - 1, (x_i, px_i))) = (x_i - x'_i) \cdot p,$$

where the inequality follows from decreasing incremental valuations and the equality from (1). Thus,  $V_i(x'_i, (x_i, px_i)) < px'_i$ , which implies  $(x_i, px_i) P_i (x'_i, px'_i)$ , as required.  $\square$

**Lemma 12** *Let  $R_i \in \mathcal{R}^D$ . Let  $p \in \mathbb{R}_+$  be such that  $D(p) = \{0\}$ . For each  $x_i \in M \setminus \{0\}$ ,  $(x_i - 1, p \cdot (x_i - 1)) P_i (x_i, px_i)$ .*

**Proof** Let  $x_i \in M \setminus \{0\}$ . Suppose for contradiction that  $(x_i, px_i) R_i (x_i - 1, p \cdot (x_i - 1))$ . Then,  $p \cdot (x_i - 1) \geq V_i(x_i - 1, (x_i, px_i))$ , or equivalently,  $px_i - V_i(x_i - 1, (x_i, px_i)) \geq p$ . Thus,

$$px_i - V_i(0, (x_i, px_i)) \geq (px_i - V_i(x_i - 1, (x_i, px_i))) \cdot x_i \geq px_i,$$

where the first inequality follows from decreasing incremental valuations. Thus,  $V_i(0, (x_i, px_i)) \leq 0$ , which implies  $(x_i, px_i) R_i \mathbf{0}$ . However, by  $x_i > 0$ , this contradicts  $D(p) = \{0\}$ .  $\square$

Next, we present the proofs of the lemmas introduced in Section 6.1.

**Proof of Lemma 1.** Let  $x_i \in M$  and  $p \in \mathbb{R}_+$  be such that  $x_i \in D(p)$ . We show that if  $D(p) \neq \{x_i\}$ , then  $D(p) = \{x_i, x_i + 1\}$  or  $\{x_i, x_i - 1\}$ . Suppose that  $D(p) \neq \{x_i\}$ . Then, there exists  $x'_i \in M \setminus \{x_i\}$  such that  $x'_i \in D(p)$ . By  $x'_i \neq x_i$ , either  $x'_i > x_i$  or  $x'_i < x_i$ .

Suppose that  $x'_i > x_i$ . We show  $D(p) = \{x_i, x_i + 1\}$ . To establish this, we show that, for each  $x''_i \in M$  with  $x''_i \geq x_i + 2$ ,  $x''_i \notin D(p)$ . Let  $x'' \in M$  with  $x''_i \geq x_i + 2$ . By  $x_i \in D(p)$ ,  $(x_i, px_i) R_i (x_i + 1, p \cdot (x_i + 1))$ . Thus,  $V_i(x_i + 1, (x_i, px_i)) \leq p \cdot (x_i + 1)$ , or equivalently,  $V_i(x_i + 1, (x_i, px_i)) - px_i \leq p$ . Thus, by  $x''_i \geq x_i + 2$  and decreasing incremental valuations,

$$V_i(x''_i, (x_i, px_i)) - px_i < (x''_i - x_i) \cdot (V_i(x_i + 1, (x_i, px_i)) - px_i) \leq (x''_i - x_i) \cdot p,$$

or equivalently,  $V_i(x''_i, (x_i, px_i)) < px''_i$ . Thus,  $(x_i, px_i) P_i (x''_i, px''_i)$ , so that  $x''_i \notin D(p)$ . Thus,  $D(p) \cap \{x_i + 2, \dots, m\} = \emptyset$ . By  $x'_i > x_i$  and  $x'_i \in D(p)$ ,  $x'_i = x_i + 1$ . Thus, by  $x_i + 1, x_i \in D(p)$ ,  $(x_i + 1, p \cdot (x_i + 1)) I_i (x_i, px_i)$ . Thus, by decreasing incremental valuations,  $D(p) = \{x_i, x_i + 1\}$  (see Lemma 11).

If  $x'_i < x_i$ , then a parallel argument implies that  $D(p) = \{x_i, x_i - 1\}$ .  $\square$

**Proof of Lemma 2.** By  $\bar{p}(x_i) \in P(x_i)$ ,  $x_i \in D(\bar{p}(x_i))$  (see Remark 11). By decreasing incremental valuations, one of the following cases hold:  $D(\bar{p}(x_i)) = \{x_i\}$ ,  $D(\bar{p}(x_i)) = \{x_i, x_i - 1\}$ , or  $D(\bar{p}(x_i)) = \{x_i, x_i + 1\}$  (see Lemma 1). To show  $D(\bar{p}(x_i)) = \{x_i, x_i - 1\}$ , suppose for contradiction that  $D(\bar{p}(x_i)) = \{x_i\}$  or  $D(\bar{p}(x_i)) = \{x_i, x_i + 1\}$ . In either case,  $x_i - 1 \notin D(\bar{p}(x_i))$ . Thus, by  $x_i \in D(\bar{p}(x_i))$ ,  $(x_i, \bar{p}(x_i) \cdot x_i) P_i (x_i - 1, \bar{p}(x_i) \cdot (x_i - 1))$ . This implies  $\bar{p}(x_i) \cdot (x_i - 1) > V_i(x_i - 1, (x_i, \bar{p}(x_i) \cdot x_i))$ , or equivalently,

$$\bar{p}(x_i) \cdot x_i - V_i(x_i - 1, (x_i, \bar{p}(x_i) \cdot x_i)) > \bar{p}(x_i). \tag{2}$$

Let  $p \in \mathbb{R}_+$  be such that  $p > \max \{V_i(1, \mathbf{0}), \bar{p}(x_i)\}$ . By  $p > V_i(1, \mathbf{0})$ ,  $\mathbf{0} P_i (1, p)$ . Thus, decreasing incremental valuations imply  $D(p) = \{0\}$ . Thus, by decreasing incremental valuations,  $(x_i - 1, p \cdot (x_i - 1)) P_i (x_i, px_i)$  (see Lemma 12). Thus,  $p \cdot (x_i - 1) < V_i(x_i - 1, (x_i, px_i))$ , or equivalently,

$$px_i - V_i(x_i - 1, (x_i, px_i)) < p. \tag{3}$$

By continuity of  $R_i$ , the function  $p'x_i - V_i(x_i - 1, (x_i, p'x_i)) - p'$  is continuous in  $p'$ . Thus, by (2), (3), and  $p > \bar{p}(x_i)$ , the intermediate value theorem implies that there exists  $p' \in (\bar{p}(x_i), p)$  such that  $p'x_i - V_i(x_i - 1, (x_i, p'x_i)) = p'$ . Thus,  $p' \cdot (x_i - 1) = V_i(x_i - 1, (x_i, p'x_i))$ . Thus, by decreasing incremental valuations,  $D(p') = \{x_i, x_i - 1\}$  (see Lemma 11), so  $p' \in P(x_i)$ . However, by  $p' > \bar{p}(x_i)$ , this contradicts the definition of  $\bar{p}(x_i)$ , where  $\bar{p}(x_i) = \max P(x_i)$ .  $\square$

**Proof of Lemma 3.** Let  $x_i \in M \setminus \{0\}$ . First, we show  $A(x_i)$  is a closed set. Let  $(p_k)_{k \in \mathbb{N}}$  be a sequence in  $A(x_i)$ , and  $p \in \mathbb{R}_+$  be such that  $\lim_{k \rightarrow \infty} p_k = p$ . For each  $k \in \mathbb{N}$ , by  $p_k \in A(x_i)$ ,  $D(p_k) = \{x_i, x_i - 1\}$ . Thus,  $(x_i, p_k x_i) I_i(x_i - 1, p_k \cdot (x_i - 1))$ . By continuity of  $R_i$  and  $\lim_{k \rightarrow \infty} p_k = p$ ,  $(x_i, p x_i) I_i(x_i - 1, p \cdot (x_i - 1))$ . Thus, by decreasing incremental valuations,  $D(p) = \{x_i, x_i - 1\}$  (see Lemma 11). Thus,  $p \in A(x_i)$ .

Since the closed interval  $[\bar{p}(x_i + 1), \bar{p}(x_i)]$  is a closed set, the intersection  $A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$  is also a closed set. Clearly,  $A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$  is bounded since  $\bar{p}(x_i + 1)$  is its lower bound and  $\bar{p}(x_i)$  is its upper bound. By  $\bar{p}(x_i) \in A(x_i)$  (see Lemma 2),  $\bar{p}(x_i) \in A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$ . Thus,  $A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$  is non-empty. Thus,  $A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$  has the minimum element.  $\square$

**Proof of Lemma 4.** Since  $\max P(1)$  exists (see Remark 7 (iii)),  $\underline{p}(0) = \infty > \max P(1) = \bar{p}(1)$ .

Next, we show  $\underline{p}(m) > 0 = \bar{p}(m + 1)$ . By definition,  $\underline{p}(m) = \min(A(m) \cap [\bar{p}(m + 1), \bar{p}(m)])$ . Thus,  $\underline{p}(m) \in A(m) \cap [\bar{p}(m + 1), \bar{p}(m)]$ , so  $\underline{p}(m) \in A(m)$ . Thus,

$$\underline{p}(m) = V_i\left(m, (m - 1, \underline{p}(m) \cdot (m - 1))\right) - \underline{p}(m) \cdot (m - 1) > 0 = \bar{p}(m + 1),$$

where the first equality follows from  $\underline{p}(m) \in A(m)$  (see Lemma 11), and the inequality from object monotonicity.

Let  $x_i \in M \setminus \{0, m\}$ . Finally, we show that  $\underline{p}(x_i) > \bar{p}(x_i + 1)$ . By definition,  $\underline{p}(x_i) = \min(A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$ , so  $\underline{p}(x_i) \in A(x_i)$  and  $\underline{p}(x_i) \in [\bar{p}(x_i + 1), \bar{p}(x_i)]$ . By  $\underline{p}(x_i) \in A(x_i)$ ,  $D(\underline{p}(x_i)) = \{x_i, x_i - 1\}$ . By  $\bar{p}(x_i + 1) \in A(x_i + 1)$  (see Lemma 2),  $D(\bar{p}(x_i + 1)) = \{x_i + 1, x_i\}$ . Thus,  $D(\underline{p}(x_i)) \neq D(\bar{p}(x_i + 1))$ , so  $\underline{p}(x_i) \neq \bar{p}(x_i + 1)$ . By  $\underline{p}(x_i) \in [\bar{p}(x_i + 1), \bar{p}(x_i)]$ ,  $\underline{p}(x_i) \geq \bar{p}(x_i + 1)$ . Thus,  $\underline{p}(x_i) > \bar{p}(x_i + 1)$ .  $\square$

**Proof of Lemma 5.** Let  $x_i \in M \setminus \{m\}$  and  $p \in (\bar{p}(x_i + 1), \underline{p}(x_i))$ . The proof consists of two steps.

STEP 1. First, we show  $x_i \in D(p)$ . Suppose for contradiction that  $x_i \notin D(p)$ . Then, by decreasing incremental valuations, we have either  $(x_i + 1, p \cdot (x_i + 1)) P_i(x_i, p x_i)$  or  $(x_i - 1, p \cdot (x_i - 1)) P_i(x_i, p x_i)$ . There are two cases.

CASE 1.  $(x_i + 1, p \cdot (x_i + 1)) P_i(x_i, p x_i)$ .

By  $(x_i + 1, p \cdot (x_i + 1)) P_i(x_i, p x_i)$ ,  $V_i(x_i, (x_i + 1, p \cdot (x_i + 1))) < p x_i$ , or equivalently,

$$p \cdot (x_i + 1) - V_i(x_i, (x_i + 1, p \cdot (x_i + 1))) > p. \tag{4}$$

Let  $p' \in \mathbb{R}_+$  be such that  $p' > \max \{V_i(1, \mathbf{0}), p\}$ . By  $p' > V_i(1, \mathbf{0})$ ,  $\mathbf{0} P_i(1, p')$ . Thus, by decreasing incremental valuations,  $D(p') = \{0\}$ . Thus, by decreasing incremental valuations,  $(x_i, p'x_i) P_i(x_i + 1, p' \cdot (x_i + 1))$  (see Lemma 12), which implies

$$p' \cdot (x_i + 1) - V_i(x_i, (x_i + 1, p' \cdot (x_i + 1))) < p'. \tag{5}$$

By continuity of  $R_i$ , the function  $p'' \cdot (x_i + 1) - V_i(x_i, (x_i + 1, p'' \cdot (x_i + 1))) - p''$  is continuous in  $p''$ . Thus, by (4), (5), and  $p' > p$ , the intermediate value theorem implies that there exists  $p'' \in (p, p')$  such that  $p''(x_i + 1) - V_i(x_i, (x_i + 1, p'' \cdot (x_i + 1))) = p''$ . Thus, by decreasing incremental valuations,  $p'' \in P(x_i + 1)$  (see Lemma 11). However, by  $p'' > p > \bar{p}(x_i + 1)$ , this contradicts the definition of  $\bar{p}(x_i + 1)$ , where  $\bar{p}(x_i + 1) = \max P(x_i + 1)$ .

CASE 2.  $(x_i - 1, p \cdot (x_i - 1)) P_i(x_i, px_i)$ .

By  $(x_i - 1, p \cdot (x_i - 1)) P_i(x_i, px_i)$ ,  $V_i(x_i - 1, (x_i, px_i)) > p \cdot (x_i - 1)$ , or equivalently,

$$px_i - V_i(x_i - 1, (x_i, px_i)) < p. \tag{6}$$

By  $\bar{p}(x_i + 1) \in A(x_i + 1)$  (see Lemma 2),  $D(\bar{p}(x_i + 1)) = \{x_i + 1, x_i\}$ . Thus,  $(x_i, \bar{p}(x_i + 1) \cdot x_i) P_i(x_i - 1, \bar{p}(x_i + 1) \cdot (x_i - 1))$ . Thus,  $V_i(x_i - 1, (x_i, \bar{p}(x_i + 1) \cdot x_i)) < \bar{p}(x_i + 1) \cdot (x_i - 1)$ , or equivalently,

$$\bar{p}(x_i + 1) \cdot x_i - V_i(x_i - 1, (x_i, \bar{p}(x_i + 1) \cdot x_i)) > \bar{p}(x_i + 1). \tag{7}$$

By continuity of  $R_i$ , the function  $p'x_i - V_i(x_i - 1, (x_i, p'x_i)) - p'$  is continuous in  $p'$ . Thus, by (6), (7), and  $p > \bar{p}(x_i + 1)$ , the intermediate value theorem implies that there exists  $p' \in (\bar{p}(x_i + 1), p)$  such that  $p'x_i - V_i(x_i - 1, (x_i, p'x_i)) = p'$ . Thus, by decreasing incremental valuations,  $p' \in A(x_i)$  (see Lemma 11). By  $p' \in (\bar{p}(x_i + 1), p)$ ,  $p \in (\bar{p}(x_i + 1), \underline{p}(x_i))$ , and  $\underline{p}(x_i) \leq \bar{p}(x_i)$  (see Remark 13),  $p' \in [\bar{p}(x_i + 1), \bar{p}(x_i)]$ . Thus,  $p' \in A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$ . However, by  $p' < p < \underline{p}(x_i)$ , this contradicts the definition of  $\underline{p}(x_i)$ , where  $\underline{p}(x_i) = \min(A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$ .

STEP 2. Now, we complete the proof. By Step 1,  $x_i \in D(p)$ . By decreasing incremental valuations,  $D(p) = \{x_i\}$ ,  $\{x_i, x_i + 1\}$ , or  $\{x_i, x_i - 1\}$  (see Lemma 1). If  $D(p) = \{x_i, x_i + 1\}$ , then  $p \in P(x_i + 1)$ . However, by  $p > \bar{p}(x_i + 1)$ , this contradicts  $\bar{p}(x_i + 1) = \max P(x_i + 1)$ . Instead, if  $D(p) = \{x_i, x_i - 1\}$ , then  $p \in A(x_i)$ . By the same argument as in Case 2 of Step 1,  $p \in [\bar{p}(x_i + 1), \bar{p}(x_i)]$ , so  $p \in A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)]$ . However, by  $p < \underline{p}(x_i)$ , this contradicts  $\underline{p}(x_i) = \min(A(x_i) \cap [\bar{p}(x_i + 1), \bar{p}(x_i)])$ . Thus,  $D(p) = \{x_i\}$ . □

## B Proof of Part 2: Identifying the rules satisfying the properties

In this section, we present the proofs of the results in Part 2 of the proof of Theorem 1, as discussed in Section 6.2. Note that we focus only on the two-agent case, that is, we assume  $n = 2$  throughout the section.

First, we present the proofs of Lemmas 9 and 10.

**Proof of Lemma 9.** Since efficiency implies  $x_1(R) + x_2(R) = m$  (see Lemma 6), we only have to show  $x_1(R) = x_1$ . Suppose for contradiction that  $x_1(R) \neq x_1$ . Since  $R_2$  is quasi-linear,  $t_1(R) = t_1(R_2; x_1(R)) = \sigma_1(R_{-1}; 0) - \sigma_1(R_{-1}; x_1(R)) = v_2(m) - v_2(m - x_1(R))$  (see Fact 2). By constant incremental valuations of  $R_2$ ,  $v_2(m) - v_2(m - x_1(R)) = \Delta v_2 \cdot (m - (m - x_1(R))) = \Delta v_2 \cdot x_1(R)$ , where  $\Delta v_2$  is a constant incremental valuation associated with  $R_2$ . Thus,  $t_1(R) = \Delta v_2 \cdot x_1(R)$ . By  $\Delta v_2 \in (\bar{p}(R_1, x_1 + 1), \underline{p}(R_1, x_1))$ ,  $D(R_1, \Delta v_2) = \{x_1\}$  (see Lemma 5). Thus, by  $x_1(R) \neq x_1$ ,  $(x_1, \Delta v_2 \cdot x_1) P_1 (x_1(R), \Delta v_2 \cdot x_1(R)) = f_1(R)$ , which implies  $\Delta v_2 \cdot x_1 < V_1(x_1, f_1(R))$ . By rearranging this, we get

$$\begin{aligned} v_2(x_1) - v_2(x_1(R)) &= \Delta v_2 \cdot (x_1 - x_1(R)) < V_1(x_1, f_1(R)) - \Delta v_2 \cdot x_1(R) \\ &= V_1(x_1, f_1(R)) - t_1(R), \end{aligned}$$

where the first equality follows from constant incremental valuations of  $R_2$ . Thus, if  $x_1 > x_1(R)$ , then reallocating  $x_1 - x_1(R)$  units from agent 2 to agent 1 Pareto improves  $f(R)$  for  $R$  (see Remark 6). Instead, if  $x_1 < x_1(R)$ , then reallocating  $x_1(R) - x_1$  units from agent 1 to agent 2 Pareto improves  $f(R)$  for  $R$  (see Remark 6). In either case, we derive a contradiction to efficiency.  $\square$

**Proof of Lemma 10.** Since  $R_1 \in \mathcal{R}^+$  implies  $\bar{p}(R_1, x_1) = \underline{p}(R_1, x_1)$  for each  $x_1 \in M$  (see Remark 14), Lemma 10 (ii) follows from Lemma 10 (i). Thus, we only show Lemma 10 (i). Let  $x_2 \in M \setminus \{0\}$ . Suppose for contradiction that

$$t_2(R_1; x_2) - t_2(R_1; x_2 - 1) \notin [\underline{p}(R_1, m - x_2 + 1), \bar{p}(R_1, m - x_2 + 1)].$$

There are two cases.

CASE 1.  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) < \underline{p}(R_1, m - x_2 + 1)$ .

By  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) < \underline{p}(R_1, m - x_2 + 1)$ , we can choose  $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$  with a constant incremental valuation  $\Delta v_2 \in (\bar{p}(R_1, m - x_2 + 2), \underline{p}(R_1, m - x_2 + 1))$ , such that  $\Delta v_2 > t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$ . Since  $\Delta v_2 \in (\bar{p}(R_1, m - x_2 + 2), \underline{p}(R_1, m - x_2 + 1))$ ,  $x_2(R) = x_2 - 1$  (see Lemma 9). By  $v_2(x_2) - v_2(x_2 - 1) = \Delta v_2 > t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$ ,  $v_2(x_2) - t_2(R_1, x_2) > v_2(x_2 - 1) - t_2(R_1, x_2 - 1)$ . Thus, by  $x_2(R) = x_2 - 1$ ,  $z_2(R_1; x_2) P_2 z_2(R_1; x_2 - 1) = f_2(R)$ , so agent 2 benefits by misrepresenting his preference to receive  $x_2$  units. However, this contradicts strategy-proofness (see Lemma 7).

CASE 2.  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) > \bar{p}(R_1, m - x_2 + 1)$ .

By  $t_2(R_1; x_2) - t_2(R_1; x_2 - 1) > \bar{p}(R_1, m - x_2 + 1)$ , we can choose  $R_2 \in \mathcal{R}^C \cap \mathcal{R}^Q$  with a constant incremental valuation  $\Delta v_2 \in (\bar{p}(R_1, m - x_2 + 1), \underline{p}(R_1, m - x_2))$ , such that  $\Delta v_2 < t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$ . Since  $\Delta v_2 \in (\bar{p}(R_1, m - x_2 + 1), \underline{p}(R_1, m - x_2))$ ,  $x_2(R) = x_2$  (see Lemma 9). By  $v_2(x_2) - v_2(x_2 - 1) = \Delta v_2 < t_2(R_1; x_2) - t_2(R_1; x_2 - 1)$ ,  $v_2(x_2 - 1) - t_2(R_1, x_2 - 1) > v_2(x_2) - t_2(R_1, x_2)$ . Thus, by  $x_2(R) = x_2$ ,  $z_2(R_1; x_2 - 1) P_2 z_2(R_1; x_2) = f_2(R)$ , so agent 2 benefits by misrepresenting his preference to receive  $x_2 - 1$  units. However, this contradicts *strategy-proofness* (see Lemma 7).  $\square$

Then, we present the proof of Lemma 8.

**Proof of Lemma 8.** We show Lemma 8 (i). Let  $x \in X$  be such that  $x_2 \leq x_2(R)$ . By *strategy-proofness*,  $f_2(R) = z_2(R_1; x_2(R)) R_2 z_2(R_1; x_2)$  (see Lemma 7). Since  $R_2$  is quasi-linear,  $v_2(x_2(R)) - t_2(R_1; x_2(R)) \geq v_2(x_2) - t_2(R_1; x_2)$ , or equivalently,

$$v_2(x_2(R)) - v_2(x_2) \geq t_2(R_1; x_2(R)) - t_2(R_1; x_2). \tag{8}$$

We have<sup>35</sup>

$$\begin{aligned} t_2(R_1; x_2(R)) - t_2(R_1; x_2) &= \sum_{x'_2=x_2+1}^{x_2(R)} (t_2(R_1; x'_2) - t_2(R_1; x'_2 - 1)) \\ &\geq \sum_{x'_2=x_2+1}^{x_2(R)} \underline{p}(R_1, m - x'_2 + 1) \\ &= \underline{v}_1(m - x_2) - \underline{v}_1(m - x_2(R)), \end{aligned} \tag{9}$$

where the inequality follows from Lemma 10 (i). Combining (8) and (9), we obtain

$$v_2(x_2(R)) - v_2(x_2) \geq \underline{v}_1(m - x_2) - \underline{v}_1(m - x_2(R)). \tag{10}$$

Then,

$$\begin{aligned} \underline{v}_1(x_1(R)) + v_2(x_2(R)) &= \underline{v}_1(m - x_2(R)) + v_2(x_2(R)) \\ &\geq \underline{v}_1(m - x_2) + v_2(x_2) \\ &\geq \underline{v}_1(x_1) + v_2(x_2), \end{aligned}$$

where the equality holds since *efficiency* implies  $x_1(R) + x_2(R) = m$  (see Lemma 6), and the first inequality follows from (10).

The proof of Lemma 8 (ii) follows similarly to Lemma 8 (i), where we use the upper bounds of the incremental payments under the rule  $f$  identified in Lemma 10 (i) instead of the lower bounds. Lemma 8 (iii) follows from Lemma 8 (i) and (ii), since  $R_1 \in \mathcal{R}^+$  implies  $\underline{R}_1 = \bar{R}_1$  (see Remark 14).  $\square$

<sup>35</sup> For notational convenience, let  $\sum_{x'_2=x_2+1}^{x_2} (t_2(R_1; x'_2) - t_2(R_1; x'_2 - 1)) \equiv 0$  and  $\sum_{x'_2=x_2+1}^{x_2} \underline{p}(m - x'_2 - 1) \equiv 0$ .

### C Part 3: Deriving a contradiction

In this section, we provide the formal proofs for the arguments presented in Section 6.3, where we discuss how the contradiction is derived. Throughout this section, we assume  $n = 2$ . Suppose for contradiction that there exists a rule  $f$  on  $\mathcal{R}$  satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*, where  $R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ ,  $\varepsilon \in \mathbb{R}_{++}$ , and  $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$  or  $\mathcal{R} = (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ . Let  $R_1 = R_0$ .

#### C.1 Proof of Theorem 1 (i)

Let  $\mathcal{R} = (\mathcal{R}^{NI}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ . By  $R_1 = R_0 \in \mathcal{R}^{++} \cup \mathcal{R}^{--}$ , either  $R_1 \in \mathcal{R}^{++}$  or  $R_1 \in \mathcal{R}^{--}$ . We consider the case where  $R_1$  exhibits positive income effects. The argument for the case of negative income effects is similar and is discussed in the supplementary material.

Suppose  $R_1 \in \mathcal{R}^{++}$ . Let  $\varepsilon_1 \in \mathbb{R}_{++}$  be a sufficiently small positive number compared to  $R_1$  and  $\varepsilon$ , such that  $\varepsilon_1 < \min \{ \bar{p}(R_1, m) \cdot (m - 1), (m - 1) \cdot \varepsilon \}$ . By  $\bar{p}(R_1, m) > \bar{p}(R_1, m + 1) = 0$  (see Corollary 2) and  $\varepsilon > 0$ , we can choose such  $\varepsilon_1 > 0$ . Then,

$$\begin{aligned} \bar{p}(R_1, m) &= V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1))\right) - \bar{p}(R_1, m) \cdot (m - 1) \\ &< V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1), \end{aligned}$$

where the equality follows from  $\bar{p}(R_1, m) \in A(R_1, m)$  (see Lemma 11), and the inequality from positive income effects of  $R_1$  (see Remark 4 (i)). Thus,

$$\varepsilon_2 \equiv V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1) - \bar{p}(R_1, m) > 0. \tag{11}$$

Let  $\varepsilon_3 \in \mathbb{R}_{++}$  be a positive number sufficiently small compared to  $\varepsilon_2$  and  $\varepsilon$ , such that  $\varepsilon_3 < \varepsilon_2$  and  $\varepsilon_1 + (m - 1) \cdot \varepsilon_3 < (m - 1) \cdot \varepsilon$ . We can choose such  $\varepsilon_3 > 0$ , since  $\varepsilon_2 > 0$  and  $\varepsilon_1 < (m - 1) \cdot \varepsilon$ .

Let  $R_2 \in \mathcal{R}^{NI} \cap \mathcal{R}^Q$  be such that  $v_2(1) = \bar{p}(R_1, m) + \varepsilon_3$ , and for each  $x_2 \in M \setminus \{0, 1\}$ ,  $v_2(x_2) - v_2(x_2 - 1) = \bar{p}(R_1, m) - \frac{\varepsilon_1}{m - 1}$ . We can verify object monotonicity of  $R_2$  by using  $\varepsilon_1 < \bar{p}(R_1, m) \cdot (m - 1)$ , and  $R_2 \in \mathcal{R}^{NI}(\varepsilon)$  by using  $\varepsilon_1 + (m - 1) \cdot \varepsilon_3 < (m - 1) \cdot \varepsilon$ .

Since  $v_2(1) > \bar{p}(R_1, m) = \bar{v}_1(m) - \bar{v}_1(m - 1) > v_2(x_2) - v_2(x_2 - 1)$  for each  $x_2 \in M \setminus \{0, 1\}$ , the object allocation  $x = (x_1, x_2) = (m - 1, 1)$  uniquely maximizes  $\bar{v}_1(x'_1) + v_2(x'_2)$  among all  $x' \in X$ . Thus, by positive income effects of  $R_1$ ,  $x(R) = (x_1(R), x_2(R)) = (m - 1, 1)$  (see Lemma 8 (iii)).

Since  $R_2$  is quasi-linear, agent 1's payment under the rule  $f$  is equivalent to that under the Vickrey rule (see Fact 2). Thus, by  $x_1(R) = m - 1$ ,

$$\begin{aligned} t_1(R) &= t_1(R_2; m - 1) = \sigma_1(R_2; 0) - \sigma_1(R_2, m - 1) \\ &= v_2(m) - v_2(1) = \bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1. \end{aligned} \tag{12}$$

Therefore,

$$\begin{aligned} & \left( V_1(x_1(R) + 1, f_1(R)) - t_1(R) \right) - \left( v_2(x_2(R)) - v_2(x_2(R) - 1) \right) \\ &= V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1) \\ &\quad - (\bar{p}(R_1, m) + \varepsilon_3) \\ &= \varepsilon_2 - \varepsilon_3 > 0, \end{aligned}$$

where the first equality follows from  $f_1(R) = (m - 1, \bar{p}(R_1, m) \cdot (m - 1) - \varepsilon_1)$  (see (12)) and  $x_2(R) = 1$ , the second equality from the definition of  $\varepsilon_2$  (see (11)), and the inequality from  $\varepsilon_3 < \varepsilon_2$ . Thus, reallocating one unit from agent 2 to agent 1 Pareto improves  $f(R)$  for  $R$  (see Remark 6). However, this contradicts efficiency.  $\square$

### C.2 Proof of Theorem 1 (ii)

Let  $\mathcal{R} = (\mathcal{R}^{ND}(\varepsilon) \cap \mathcal{R}^Q) \cup \{R_0\}$ . Since  $R_1 = R_0 \in \mathcal{R}^{++} \cup \mathcal{R}^{--}$ , either  $R_1 \in \mathcal{R}^{++}$  or  $R_1 \in \mathcal{R}^{--}$ . We consider the case where  $R_1$  exhibits negative income effects. The argument for the case of positive income effects is similar and is discussed in the supplementary material.

Suppose  $R_1 \in \mathcal{R}^{--}$ . Let  $\varepsilon_1 \in \mathbb{R}_{++}$  be a sufficiently small positive number compared to  $\varepsilon$  and  $R_1$ , such that  $\varepsilon_1 < (m - 1) \cdot \varepsilon$ , and for each  $x_1 \in M \setminus \{0, m\}$ ,  $\varepsilon_1 < (m - 1) \cdot \min \{ \bar{p}(R_1, x_1) - \bar{p}(R_1, m), \underline{p}(R_1, x_1) - \bar{p}(R_1, m) \}$ . We can choose such  $\varepsilon_1 > 0$ , since  $\varepsilon > 0$ , and for each  $x_1 \in M \setminus \{0, m\}$ ,  $\bar{p}(R_1, x_1) \geq \underline{p}(R_1, x_1) > \bar{p}(R_1, m)$  (see Lemma 4 and Remark 13). We have

$$\begin{aligned} \bar{p}(R_1, m) &= V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1))\right) - \bar{p}(R_1, m) \cdot (m - 1) \\ &< V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1), \end{aligned}$$

where the equality follows from  $\bar{p}(R_1, m) \in A(R_1, m)$  (see Lemma 11), and the inequality from negative income effects of  $R_1$  (see Remark 5 (i)). Thus,

$$\varepsilon_2 \equiv V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1) - \bar{p}(R_1, m) > 0. \tag{13}$$

Let  $\varepsilon_3 \in \mathbb{R}_{++}$  be a positive number sufficiently small compared to  $\varepsilon_1$  and  $\varepsilon_2$ , such that  $(m - 1) \cdot \varepsilon_3 \leq \varepsilon_1$  and  $\varepsilon_3 < \varepsilon_2$ . Since  $\varepsilon_1, \varepsilon_2 > 0$ , we can choose such  $\varepsilon_3 > 0$ .

Let  $R_2 \in \mathcal{R}^Q$  be such that  $v_2(1) = \bar{p}(R_1, m) + \varepsilon_3$ , and for each  $x_2 \in M \setminus \{0, 1\}$ ,  $v_2(x_2) - v_2(x_2 - 1) = \bar{p}(R_1, m) + \frac{\varepsilon_1}{m - 1}$ . We can verify  $R_2 \in \mathcal{R}^{ND}(\varepsilon)$  by using  $\varepsilon_1 < (m - 1) \cdot \varepsilon$  (the “ $\varepsilon$ -” part) and  $(m - 1) \cdot \varepsilon_3 \leq \varepsilon_1$  (the “non-decreasing incremental valuations” part).

For each  $x \in X$  with  $x_1 < m - 1$ ,<sup>36</sup>

$$\begin{aligned} \bar{v}_1(m - 1) + v_2(1) &= \sum_{x'_1=1}^{m-1} \bar{p}(R_1, x'_1) + (\bar{p}(R_1, m) + \varepsilon_3) \\ &> \sum_{x'_1=1}^{x_1} \bar{p}(R_1, x'_1) + (m - 1 - x_1) \cdot \left( \bar{p}(R_1, m) + \frac{\varepsilon_1}{m - 1} \right) \\ &\quad + (\bar{p}(R_1, m) + \varepsilon_3) \\ &= \bar{v}_1(x_1) + v_2(m - x_1) \\ &\geq \bar{v}_1(x_1) + v_2(x_2), \end{aligned}$$

where the first inequality follows from  $x_1 < m - 1$ , and  $\varepsilon_1 < (m - 1) \cdot (\bar{p}(R_1, x'_1) - \bar{p}(R_1, m))$  for each  $x'_1 \in M \setminus \{0, m\}$ . Moreover, by  $\varepsilon_3 > 0$ ,

$$\bar{v}_1(m - 1) + v_2(1) = \sum_{x'_1=1}^{m-1} \bar{p}(R_1, x'_1) + (\bar{p}(R_1, m) + \varepsilon_3) > \sum_{x'_1=1}^m \bar{p}(R_1, x'_1) = \bar{v}_1(m).$$

Thus, the object allocation  $x = (x_1, x_2) = (m - 1, 1)$  uniquely maximizes  $\bar{v}_1(x'_1) + v_2(x'_2)$  among all  $x' \in X$ . Also, for each  $x \in X$  with  $x_1 < m - 1$ ,<sup>37</sup>

$$\begin{aligned} \underline{v}_1(m - 1) + v_2(1) &= \sum_{x'_1=1}^{m-1} \underline{p}(R_1, x'_1) + (\bar{p}(R_1, m) + \varepsilon_3) \\ &> \sum_{x'_1=1}^{x_1} \underline{p}(R_1, x'_1) + (m - 1 - x_1) \cdot \left( \bar{p}(R_1, m) + \frac{\varepsilon_1}{m - 1} \right) \\ &\quad + (\bar{p}(R_1, m) + \varepsilon_3) \\ &= \underline{v}_1(x_1) + v_2(m - x_1) \\ &\geq \underline{v}_1(x_1) + v_2(x_2), \end{aligned}$$

where the first inequality follows from  $x_1 < m - 1$ , and  $\varepsilon_1 < (m - 1) \cdot (\underline{p}(R_1, x'_1) - \bar{p}(R_1, m))$  for each  $x'_1 \in M \setminus \{0, m\}$ . Moreover,

$$\underline{v}_1(m - 1) + v_2(1) = \sum_{x'_1=1}^{m-1} \underline{p}(R_1, x'_1) + (\bar{p}(R_1, m) + \varepsilon_3) > \sum_{x'_1=1}^m \underline{p}(R_1, x'_1) = \underline{v}_1(m),$$

where the inequality follows from  $\bar{p}(R_1, m) \geq \underline{p}(R_1, m)$  (see Remark 13) and  $\varepsilon_3 > 0$ . Thus, the object allocation  $x = (x_1, x_2) = (m - 1, 1)$  also uniquely maximizes

<sup>36</sup> For notational convenience, let  $\sum_{x'_1=1}^0 \bar{p}(R_1, x'_1) \equiv 0$ .

<sup>37</sup> For notational convenience, let  $\sum_{x'_1=1}^0 \underline{p}(R_1, x'_1) \equiv 0$ .

$v_1(x'_1) + v_2(x'_2)$  among all  $x' \in X$ . Thus,  $x(R) = (x_1(R), x_2(R)) = (m - 1, 1)$  (see Lemma 8 (i) and (ii)).

Since  $R_2$  is quasi-linear, agent 1's payment under the rule  $f$  is equivalent to that under the Vickrey rule (see Fact 2). Thus, by  $x_1(R) = m - 1$ ,

$$\begin{aligned} t_1(R) &= t_1(R_2; m - 1) = \sigma_1(R_2; 0) - \sigma_1(R_2, m - 1) \\ &= v_2(m) - v_2(1) = \bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1. \end{aligned} \tag{14}$$

Therefore,

$$\begin{aligned} & \left( V_1(x_1(R) + 1, f_1(R)) - t_1(R) \right) - \left( v_2(x_2(R)) - v_2(x_2(R) - 1) \right) \\ &= V_1\left(m, (m - 1, \bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1)\right) - (\bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1) \\ & \quad - (\bar{p}(R_1, m) + \varepsilon_3) \\ &= \varepsilon_2 - \varepsilon_3 > 0, \end{aligned}$$

where the first equality follows from  $f_1(R) = (m - 1, \bar{p}(R_1, m) \cdot (m - 1) + \varepsilon_1)$  (see (14)) and  $x_2(R) = 1$ , the second equality from the definition of  $\varepsilon_2$  (see (13)), and the inequality from  $\varepsilon_3 < \varepsilon_2$ . Thus, reallocating one unit from agent 2 to agent 1 Pareto improves  $f(R)$  for  $R$  (see Remark 6). However, this contradicts *efficiency*.  $\square$

### D Extension to three or more agents

In Parts 2 and 3 of the proof of Theorem 1, we focus on the two-agent case. In this section, we explain how the arguments presented there are extended to the case of three or more agents.

Intuitively, if agents 3, . . . ,  $n$  have preferences with sufficiently small incremental valuations, then by *efficiency*, we can ignore them. Therefore, the arguments for the two-agent case apply to the case of three or more agents.

Formally, as in Parts 2 and 3, let (i)  $R_1 = R_0 \in \mathcal{R}^D \cap (\mathcal{R}^{++} \cup \mathcal{R}^{--})$ , (ii)  $\mathcal{R}$  be a domain such that  $R_0 \in \mathcal{R}$ ,  $\mathcal{R} \supseteq \mathcal{R}^C \cap \mathcal{R}^Q$ , and  $\mathcal{R} \cap \mathcal{R}^Q$  is convex, and (iii)  $f$  be a rule on  $\mathcal{R}$  satisfying *efficiency*, *individual rationality*, *no subsidy for losers*, and *strategy-proofness*.

Let  $\delta \in \mathbb{R}_{++}$  be a positive number sufficiently small compared to  $R_1$ .<sup>38</sup> For each  $i \in N \setminus \{1, 2\}$ , let  $R_i \in \mathcal{R}^C \cap \mathcal{R}^Q$  have a constant incremental valuation  $\Delta v_i = \delta$ . Then, *efficiency* implies that for each  $R_2 \in \mathcal{R}$  and each  $i \in N \setminus \{1, 2\}$ ,  $x_i(R_2, R_{-2}) = 0$ . Thus, at the outcome object allocation under  $f$ , we can ignore agents 3, . . . ,  $n$ , effectively reducing the situation to the two-agent case for the object allocation.

Moreover, by imposing the lower bound  $\delta$  on agent 2's (quasi-linear) incremental valuations, agent 1's payment under the Vickrey rule (and thus under  $f$ ) for the case of three or more agents becomes equivalent to that in the two-agent case. Specifically,

<sup>38</sup> For example, let  $0 < \delta < \min \left\{ V_1(m, \mathbf{0}) - V_1(m - 1, \mathbf{0}), V_1(m, (m - 1, 0)) \right\}$ . For such a  $\delta > 0$ , the subsequent discussion holds.

for each  $R_2 \in \mathcal{R} \cap \mathcal{R}^Q$  with  $v_2(x_2) - v_2(x_2 - 1) > \delta$  for each  $x_2 \in M \setminus \{0\}$ , and each  $x_1 \in M_1(R_{-1})$ , since agents  $2, \dots, n$  have quasi-linear preferences, agent 1's payment under  $f$  is equivalent to that under the Vickrey rule (see Fact 2):

$$t_1(R_{-1}; x_1) = \sigma_1(R_{-1}; 0) - \sigma_1(R_{-1}; x_1) = v_2(m) - v_2(m - x_1).$$

Note that the last equality follows from  $v_2(x_2) - v_2(x_2 - 1) > \delta$  for each  $x_2 \in M \setminus \{0\}$ . Furthermore,  $v_2(m) - v_2(m - x_1)$  corresponds to agent 1's payment under the Vickrey rule in the two-agent case.

Once we observe that, under the lower bound  $\delta$  on agent 2's incremental valuations, the object allocation and agent 1's payment under  $f$  remain the same as in the two-agent case, the remaining arguments presented in Parts 2 and 3—such as identifying the range of agent 2's incremental payments (see Lemma 10 in Section 6.2)—remain unchanged even with the presence of this lower bound  $\delta$ . Therefore, we can apply the same arguments from the two-agent case to the case with three or more agents, assuming that agent 2's quasi-linear preferences have incremental valuations greater than  $\delta$ .

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## Declarations

**Competing Interests** The authors declare that they have no competing interests relevant to the content of this article.

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