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On Fundamental versus Strategic Uncertainty

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Abstract

In global games in which one player has better information than his rival, it may be that in the unique equilibrium, the better informed player has a lower payoff than the poorly informed player. The reason is that while the better informed player faces less (or even no) uncertainty about economic fundamentals, he may face greater strategic uncertainty.

1 Introduction

In strategic situations under incomplete information players face two kinds of uncertainty. The first concerns economic fundamentals that directly affect their payoffs and so is referred to as *fundamental* uncertainty. The second concerns the behavior of other players and is referred to as *strategic* uncertainty. The latter is particularly important if players face a coordination problem (as in Rubinstein, 1989).

It seems intuitive that a player who has better information about fundamentals—someone with inside information, say—is better suited to adapt his actions to circumstances and as a result, should have an advantage over other, poorly informed players. This intuition, however, does not take into account the strategic uncertainty faced by a well-informed player. Specifically, the actions of his opponent with noisy information about fundamentals will typically be noisy as well. This means that even a well-informed player will face substantial strategic uncertainty.

In this paper we study this trade-off between the two kinds of uncertainty—if you face large fundamental uncertainty then I face large strategic uncertainty—in the context of a canonical setting familiar from the theory of global games (Carlsson and van Damme, 1993). We have chosen to study this issue in the global-games setting

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because this framework has shed light on many interesting economic phenomena: currency attacks, bank runs, regime change, etc. (See Morris and Shin, 2003 and Angeletos and Lian, 2016 for surveys).

In our two-player setting, one of the players is perfectly informed about fundamentals while the other is very poorly informed. In other words, unlike most of the existing work, we study global games in which payoffs are symmetric but the quality of players' information is *asymmetric*.[1] We identify circumstances in which the effect mentioned above is so strong that it nullifies the advantage of the better informed player. In the unique equilibrium of the game, the poorly informed player has a *higher* expected payoff than the perfectly informed players.

The basic idea of the paper can be seen in the following two-player incomplete information game familiar from the theory of global games. Each of two players must choose whether to "invest" (I) or "not invest" (N). The payoffs depend on an underlying "fundamental state" and are as follows:

	I	N
I	θ, θ	$\theta - 1, 0$
N	$0, \theta - 1$	$0, 0$

Suppose that the payoff relevant state $\theta \in \{-1, \nu, 2\}$ where $\nu \in \left(\frac{1}{2}, 1\right)$ and that each of the three states is equally likely. Player 1 knows the realization of θ prior to choosing his action while player 2 receives only a binary signal $s \in \{b, g\}$ prior to choosing hers. The signals are distributed as follows:

$$\Pr[g \mid \theta] = \begin{cases} 0 & \text{if } \theta = -1 \\ \frac{1}{2} & \text{if } \theta = \nu \\ 1 & \text{if } \theta = 2 \end{cases}$$

The game can be "solved" by the iterated elimination of dominated strategies (as in the analysis of global games). First, note that it is dominant for player 1 to play N in state $\theta = -1$ and to play I in state $\theta = 2$. Now consider player 2. If her signal is b, then playing I is dominated because even if player 1 were to play I in state $\theta = \nu$, player 2's payoff from playing I is $\frac{2}{3}(-1) + \frac{1}{3}\nu < 0$. Her payoff from playing N is, of course, 0. If her signal is g, then playing N is dominated because even if player 1 were to play N in state $\theta = \nu$, player 2's payoff from playing I is $\frac{1}{3}(\nu - 1) + \frac{2}{3}(2) > 0$. Thus, player 2 should play N if her signal is b and I if it is g. Finally, consider player 1 in state $\theta = \nu$. He knows that player 2 will play N with probability $\frac{1}{2}$ and I with probability $\frac{1}{2}$. If he plays I in state $\theta = \nu$, his payoff is $\frac{1}{2}\nu + \frac{1}{2}(\nu - 1) > 0$ since $\nu > \frac{1}{2}$. Thus, I is optimal for player 1 when $\theta = \nu$.

To summarize, the *unique* equilibrium of the game is: player 1 chooses N if $\theta = -1$ and I if $\theta = \nu$ or 2. Player 2 chooses N if her signal $s = b$ and I if $s = g$. Note that, in

[1] Corsetti et al. (2004) also study asymmetric global games in the context of currency attacks but in their model there is payoff asymmetry as well as informational asymmetry.

equilibrium, the payoffs of the two players are the same in states $\theta = -1$ and $\theta = 2$. In state ν, however, player 1's expected payoff is $\frac{1}{2}\nu + \frac{1}{2}(\nu - 1) = \nu - \frac{1}{2}$ whereas player 2's payoff is $\frac{1}{2}\nu + \frac{1}{2}0 = \frac{1}{2}\nu$ and this is *higher* than player 1's payoff since $\nu < 1$. In an otherwise symmetric game, the player with better information—actually perfect information—about the payoff relevant parameter θ has a lower payoff than the player with very coarse information about θ.

Why is this? The reason is that even though player 1 faces no fundamental uncertainty—about θ—he faces substantial strategic uncertainty—about player 2's actions. The only state in which it is important for player 1 to know what player 2 is going to do is in state $\theta = \nu$ (in the other two states he has dominant actions). But in this state, player 1 only knows that player 2 will choose I or N with equal probability and so faces maximal strategic uncertainty.

Now consider player 2. When she gets the signal g, she faces *no* strategic uncertainty since she is sure that player 1 will choose I. She does face some fundamental uncertainty since she is unsure whether the state is $\theta = \nu$ or $\theta = 2$, but this is irrelevant for her decision. When player 2 gets the signal b, however, she faces both kinds of uncertainty but again, the payoff consequences of this are small.

Another way to dissect the forces at work is to formulate the information available to the two players in terms of partitions of the set of (Aumann) "ω-states" of the form $\omega = (\theta, s)$ which encode both the payoff relevant state θ and player 2's signal s. Define

$$\omega_1 = (-1, b), \ \omega_2 = (\nu, b), \ \omega_3 = (\nu, g) \text{ and } \omega_4 = (2, g)$$

and the prior probabilities of these are $\frac{1}{3}, \frac{1}{6}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Since player 1 knows θ, his information partition of the set of ω-states is

$$\mathcal{P}_1 = \{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$$

while that of player 2 is

$$\mathcal{P}_2 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$$

Note that the equilibrium strategy for player 1 is: $\{\omega_1\} \to N, \{\omega_2, \omega_3\} \to I, \{\omega_4\} \to I$, while for player 2 it is: $\{\omega_1, \omega_2\} \to N, \{\omega_3, \omega_4\} \to I$.

Consider the event $E_{II} = \{\omega_3, \omega_4\}$ in which both players choose I and the event $E_{NN} = \{\omega_1\}$ in which both players play N. In these events, the players' payoffs are the same.

In the event $E_{IN} = \{\omega_2\}$ in which player 1 plays I while player 2 plays N, player 1's payoff is $\nu - 1 < 0$ while player 2's payoff is 0. On the other hand, there is no ω-state in which player 1 plays N and 2 plays I and so $E_{NI} = \varnothing$. In other words, the event in which player 2 has a payoff advantage occurs while the event in which player 1 has an advantage never occurs.

Thus, even though player 2 is poorly informed relative to player 1, her payoff is higher than that of player 1.

In what follows we examine the robustness of the example above in a situation in which, as is typical in the theory of global games, there is a continuum of fundamental states θ. As in the example above, the value of the continuous parameter θ known to player 1 while player 2 still has very coarse (binary) information. In this sense, the discrepancy in the quality of information available to the players is even more extreme than in the example. Despite this extreme asymmetry of information, there are reasonable circumstances in which, in the unique equilibrium (actually the unique rationalizable outcome), player 2's payoff is higher than that of player 1.

2 Main result

We study the same game as in the introduction:

	I	N
I	θ, θ	$\theta - 1, 0$
N	$0, \theta - 1$	$0, 0$

but now assume that θ has a *normal* distribution with mean μ and variance σ^2. As usual, if a player knew that $\theta < 0$, then N would be a dominant action and if a player knew that $\theta > 1$, then I would be dominant. If $0 < \theta < 1$, and this fact were commonly known, both (I, I) and (N, N) constitute equilibria. In this case, (I, I) is the Pareto dominant equilibrium and is risk-dominant as long as $\theta > \frac{1}{2}$ (Harsanyi and Selten, 1988).

As above, the two players are *asymmetrically* informed. Player 1 knows the realization of θ and so faces no uncertainty about the fundamentals. Player 2, on the other hand, has very coarse information. Specifically, prior to choosing her actions, player 2 gets a binary signal $s \in \{b, g\}$ such that

$$\Pr [g \mid \theta] = \Phi (\theta, \nu, \tau)$$

where $\Phi (\cdot, \nu, \tau)$ is cumulative distribution function of a normal variable with mean ν and standard deviation τ.[2] We will assume throughout that

$$\tfrac{1}{2} < \nu < 1 \tag{1}$$

Note that the information about the fundamentals, that is θ, is very skewed. Player 1 knows the precise value of θ whereas player 2 knows only b or g. As in the example in the introduction, the only asymmetry is informational.

[2] The corresponding normal density will be denoted by $\phi (\cdot, \nu, \tau)$. The unit normal distribution function will simply be denoted by $\Phi (\cdot)$ so that $\Phi (x) \equiv \Phi (x, 0, 1)$ and the unit normal density by $\phi (\cdot)$.

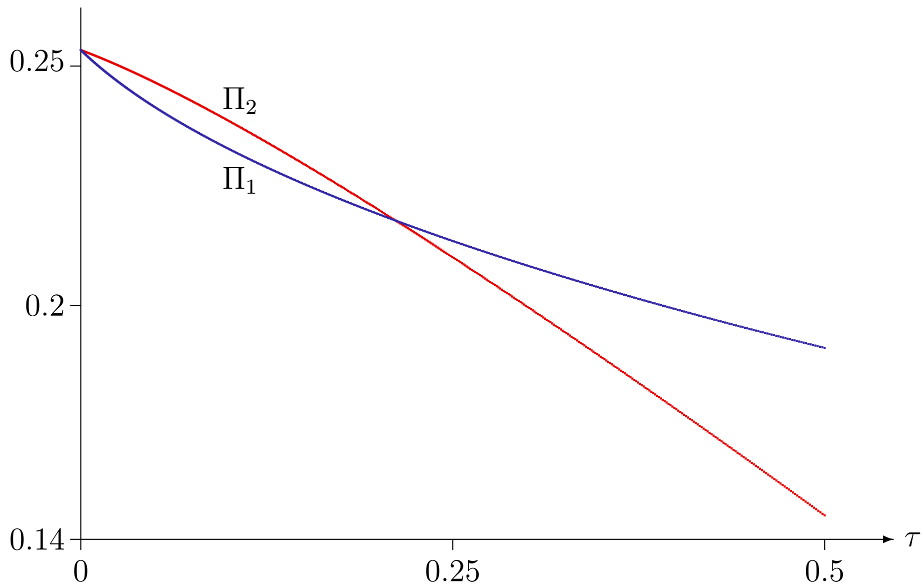

Figure 1: Equilibrium payoffs as a function of τ.

Here $\mu = 0.4, \sigma = 0.5$ and $\nu = 0.75$. For all $\tau < 0.5$, there is a unique equilibrium and the resulting payoffs are depicted. For all $\tau < 0.2$, player 2's payoff is greater than that of player 1.

Our main result is

Theorem 1 *For large enough σ and small enough τ, there is a unique equilibrium. The equilibrium payoff of the poorly informed player* 2 *is higher than the payoff of the perfectly informed player* 1*.*

See Figure 1 for an illustration of the theorem in an example. The proof of the theorem follows from Propositions 1, 2 and 3 below.

3 Equilibrium

In this section, we show that when the parameter τ is sufficiently small, there is an equilibrium in which player 2 follows her signal and plays I if her signal is g and N if it is b. Player 1 follows a threshold strategy and for some threshold k to be determined, plays I if $\theta \geq k$ and plays N if $\theta < k$.

Player 1 Suppose player 2 follows the strategy of playing I if and only if her signal is g. As usual, player 1's best response to this is to choose a threshold k such that he plays I if $\theta > k$ and N if $\theta < k$. At $\theta = k$, player 1 is indifferent between playing I and N. In state k, the probability that player 2 got the signal g (and so will play I) is $\Phi(k, \nu, \tau)$. Thus, player 1's payoff from playing I in state k is

$$\Phi(k, \nu, \tau) k + (1 - \Phi(k, \nu, \tau)) (k - 1)$$

while the payoff from playing N is 0. Thus, the threshold is the solution to

$$\Psi(k) \equiv k - (1 - \Phi(k, \nu, \tau)) = 0 \tag{2}$$

The assumption that $\nu > \frac{1}{2}$ implies that $\Psi\left(\frac{1}{2}\right) = \frac{1}{2} - \left(1 - \Phi\left(\frac{1}{2}, \nu, \tau\right)\right) < 0$ and also that $\Psi(\nu) = \nu - (1 - \Phi(\nu, \nu, \tau)) > 0$. Since for all k,

$$\Psi'(k) = 1 + \frac{1}{\tau}\phi(k, \nu, \tau) > 0$$

there is a unique solution lying between $\frac{1}{2}$ and ν. Since we will study how this threshold changes with τ, the solution to the equation above will be denoted by $k(\tau)$ and for future reference we record that

$$\tfrac{1}{2} < k(\tau) < \nu \tag{3}$$

Recall that at $\theta = \nu$, player 1 assigns probability $\frac{1}{2}$ each to player 2 playing I or N. His payoff from playing I in state $\theta = \nu$ is thus $\frac{1}{2}\nu + \frac{1}{2}(\nu - 1) = \nu - \frac{1}{2} > 0$ and this is, of course, strictly higher than the payoff from playing N. Since $k(\tau)$ is defined to be the state in which player 1 is indifferent between I and N, $k(\tau) < \nu$.

We have argued that $k(\tau)$ is player 1's unique best response to player 2's strategy. We begin by deriving some useful properties of player 1's threshold $k(\tau)$. Differentiating the equation $\Psi(k(\tau)) = 0$ with respect to τ we obtain

$$k'(\tau) + \phi(k(\tau), \nu, \tau)\, k'(\tau) - \frac{k(\tau) - \nu}{\tau}\phi(k(\tau), \nu, \tau) = 0$$

and so

$$k'(\tau) = \frac{k(\tau) - \nu}{\tau} \frac{\phi(k(\tau), \nu, \tau)}{1 + \phi(k(\tau), \nu, \tau)} < 0 \tag{4}$$

since by (3), $k(\tau) < \nu$. The next lemma shows that $k(\tau)$ converges to ν and that $k'(\tau)$ converges to a negative number. In other words, a small increase in τ away from 0 causes player 1 to become more aggressive in the sense that he plays I more often.

Lemma 3.1 *Player 1's threshold satisfies*

$$\lim_{\tau \to 0} k(\tau) = \nu$$

and

$$\lim_{\tau \to 0} k'(\tau) = \Phi^{-1}(1 - \nu) < 0$$

Proof. Since for all τ, $k(\tau) < \nu$ and $k(\tau)$ increases as $\tau \to 0$, it has a limit, say $k_0 \leq \nu$. Suppose that $k_0 < \nu$. Since for all τ,

$$k(\tau) = 1 - \Phi\left(\tfrac{k(\tau) - \nu}{\tau}\right) \tag{5}$$

and so in the limit,

$$k_0 = 1 - \Phi(-\infty) = 1$$

which is a contradiction since $\nu < 1$. Thus, $\lim_{\tau \to 0} k(\tau) = \nu$.

Using the fact that $\lim_{\tau \to 0} k(\tau) = \nu$ in (5) now immediately implies that

$$\nu = 1 - \Phi\left(\lim_{\tau \to 0} \tfrac{k(\tau) - \nu}{\tau}\right)$$

since Φ is continuous. Thus, we have

$$\lim_{\tau \to 0} \frac{k(\tau) - \nu}{\tau} = \Phi^{-1}(1 - \nu)$$

From (4) we also have

$$\begin{aligned} \lim_{\tau \to 0} k'(\tau) &= \lim_{\tau \to 0} \frac{k(\tau) - \nu}{\tau} \frac{\phi(k(\tau), \nu, \tau)}{1 + \phi(k(\tau), \nu, \tau)} \\ &= \Phi^{-1}(1 - \nu) \end{aligned}$$

which is negative since $\nu > \frac{1}{2}$. ∎

Player 2 Let $H_\nu(\theta)$ denote the step function at ν, that is,

$$H_\nu(\theta) = \begin{cases} 0 & \theta < \nu \\ 1 & \theta \geq \nu \end{cases} \tag{6}$$

which is just the cumulative distribution function of the Dirac measure at ν. Note that as $\tau \to 0$, $\Phi(\theta, \nu, \tau) \to H_\nu(\theta)$ for all $\theta \neq \nu$. In other words, $\Phi(\cdot, \nu, \tau)$ weakly converges to $H_\nu(\cdot)$.

Player 2's posterior beliefs on θ after she gets a signal g are

$$f(\theta \mid g) = \frac{\Phi(\theta, \nu, \tau)\,\phi(\theta, \mu, \sigma)}{\Pr[g]}$$

and so the expected payoff of player 2 with signal g when she plays I and player 1 uses the threshold $k(\tau)$ is

$$\frac{1}{\Pr[g]}\left[\int_{k(\tau)}^{\infty} \theta\Phi(\theta, \nu, \tau)\,\phi(\theta, \mu, \sigma)\,d\theta + \int_{-\infty}^{k(\tau)} (\theta - 1)\,\Phi(\theta, \nu, \tau)\,\phi(\theta, \mu, \sigma)\,d\theta\right] \tag{7}$$

and as $\tau \to 0$, the term in brackets converges to

$$\int_{\nu}^{\infty} \theta H_\nu(\theta)\,\phi(\theta, \mu, \sigma)\,d\theta + \int_{-\infty}^{\nu} (\theta - 1)\,H_\nu(\theta)\,\phi(\theta, \mu, \sigma)\,d\theta$$

which is positive since $H_\nu(\theta) = 0$ for $\theta < \nu$. This means that when τ is small enough, player 2 with signal g prefers to play I.

Similarly, the payoff of player 2 with signal b when she plays I is

$$\frac{1}{\Pr[b]}\left[\int_{k(\tau)}^{\infty}\theta\left(1-\Phi\left(\theta,\nu,\tau\right)\right)\phi\left(\theta,\mu,\sigma\right)d\theta+\int_{-\infty}^{k(\tau)}\left(\theta-1\right)\left(1-\Phi\left(\theta,\nu,\tau\right)\right)\phi\left(\theta,\mu,\sigma\right)d\theta\right]$$

and as $\tau\rightarrow 0$, the term in brackets converges to

$$\int_{\nu}^{\infty}\theta\left(1-H_{\nu}\left(\theta\right)\right)\phi\left(\theta,\mu,\sigma\right)d\theta-\int_{-\infty}^{v}\left(\theta-1\right)\left(1-H_{\nu}\left(\theta\right)\right)\phi\left(\theta,\mu,\sigma\right)d\theta$$

which is negative since $\nu<1$. This means that when τ is small enough, player 2 with signal b prefers to play N.

We have thus established

Proposition 1 *There exists a $\overline{\tau}$ such that for all $\tau<\overline{\tau}$, there is an equilibrium in which player 1 chooses I if and only if $\theta\geq k\left(\tau\right)$, and player 2 chooses I if and only if his signal is g.*

The equilibrium of Proposition 1 is unique when σ is large enough and the uniqueness is "uniform" in τ. Precisely,

Proposition 2 *Fix any $\overline{\tau}>0$. Then there exists a $\underline{\sigma}$ such that for all $\sigma>\underline{\sigma}$ and for all $\tau<\overline{\tau}$, there is a unique equilibrium.*

Proof. The proposition is established using the iterated elimination of dominated strategies (similar to the arguments in standard global games). Details are in the Appendix. ■

4 Payoffs

We now show that in the equilibrium described above, player 2's payoff is higher than that of player 1. This finding is the same as in the discrete example of the introduction. There this somewhat counterintuitive payoff ranking emerged from the fact that in equilibrium, there were no circumstances in which (N,I) was played while there were circumstances in which (I,N) was played. In the equilibrium described in the previous section, both (N,I) and (I,N) are played with positive probability. The proposition below shows that the payoff ranking emerges even in these circumstances. Roughly, when τ is small, both (N,I) and (I,N) occur very rarely but the latter is much more likely.

Proposition 3 *There exists a $\overline{\tau}$ such that for all $\tau<\overline{\tau}$, there is an equilibrium in which the payoff of player 2 is higher than the equilibrium payoff of player 1.*

Proof. Consider the equilibrium of Proposition 1. The equilibrium payoff of player 1 is

$$\Pi_1 = \int_{k(\tau)}^{\infty} \theta\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta + \int_{k(\tau)}^{\infty} (\theta-1)(1-\Phi(\theta,\nu,\tau))\,\phi(\theta,\mu,\sigma)\,d\theta$$

The first term concerns the event in which both players choose I—this happens when $\theta \geq k(\tau)$ and player 2's signal is g. The second term concerns the event in which player 1 chooses I while player 2 chooses N—this happens when $\theta \geq k(\tau)$ and player 2's signal is b. Similarly, the payoff of player 2 is

$$\Pi_2 = \int_{k(\tau)}^{\infty} \theta\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta + \int_{-\infty}^{k(\tau)} (\theta-1)\,\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta$$

where, as above, the first term concerns the event in which both players choose I and the second term concerns the event in which player 1 chooses N while player 2 chooses I.

Define $\Delta(\tau) = \Pi_2 - \Pi_1$ to be the difference in payoffs (as a function of τ):

$$\begin{aligned}\Delta(\tau) &= \int_{-\infty}^{k(\tau)} (\theta-1)\,\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta - \int_{k(\tau)}^{\infty} (\theta-1)(1-\Phi(\theta,\nu,\tau))\,\phi(\theta,\mu,\sigma)\,d\theta \\ &= \int_{-\infty}^{\infty} (\theta-1)\,\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta - \int_{k(\tau)}^{\infty} (\theta-1)\,\phi(\theta,\mu,\sigma)\,d\theta\end{aligned}$$

First, note that since $\lim_{\tau\to 0} k(\tau) = \nu$,

$$\begin{aligned}\Delta(0) &= \int_{-\infty}^{\nu} (\theta-1)\,H_\nu(\theta)\,\phi(\theta,\mu,\sigma)\,d\theta - \int_{\nu}^{\infty} (\theta-1)(1-H_\nu(\theta))\,\phi(\theta,\mu,\sigma)\,d\theta \\ &= 0\end{aligned}$$

Differentiating Δ,

$$\begin{aligned}\Delta'(\tau) &= \underbrace{-\int_{-\infty}^{\infty} (\theta-1)\,\phi(\theta,\nu,\tau)\left(\frac{\theta-\nu}{\tau}\right)\phi(\theta,\mu,\sigma)\,d\theta}_{A(\tau)} \\ &\quad + \underbrace{(k(\tau)-1)\,\phi(k(\tau),\mu,\sigma)\,k'(\tau)}_{B(\tau)} \end{aligned} \tag{8}$$

and we will show that as $\tau \to 0$, the limit of the first term is zero while the limit of the second is positive. This will establish that $\lim_{\tau\to 0} \Delta'(\tau) > 0$ and so for small enough τ, $\Delta(\tau) > 0$ as well.

First, consider the second term above. Lemma 3.1 implies that

$$\lim_{\tau\to 0} B(\tau) = (\nu-1)\,\Phi^{-1}(1-\nu)\,\phi(\nu,\mu,\sigma) > 0$$

Next, consider the first term in (8). By changing the variable to $x = \frac{\theta - \nu}{\tau}$,

$$\begin{aligned} A(\tau) &= -\int_{-\infty}^{\infty} (\tau x + \nu - 1)\, x\phi(x)\, \phi(\tau x + \nu, \mu, \sigma)\, dx \\ &= \int_{-\infty}^{\infty} \left[(\tau x + \nu - 1)\, \phi(\tau x + \nu, \mu, \sigma)\right] \phi'(x)\, dx \end{aligned}$$

using the fact that $x\phi(x) = -\phi'(x)$. Integrating by parts (treating term in brackets as one function)

$$A(\tau) = \left[(\tau x + \nu - 1)\, \phi(\tau x + \nu, \mu, \sigma)\right] \phi(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d}{dx} \left[\tau\phi(\tau x + \nu, \mu, \sigma)\right] \phi(x)\, dx$$

Since the first term is zero, we have that

$$A(\tau) = -\int_{-\infty}^{\infty} \left[\tau\phi(\tau x + \nu, \mu, \sigma) + \tau(\tau x + \nu - 1)\, \phi'(\tau x + \nu, \mu, \sigma)\right] \phi(x)\, dx$$

where $\phi'(z, \mu, \sigma)$ denotes the derivative of $\phi(z, \mu, \sigma)$ with respect to z. Now

$$\begin{aligned} |A(\tau)| &< \int_{-\infty}^{\infty} \tau\phi(\tau x + \nu, \mu, \sigma)\, \phi(x)\, dx + \int_{-\infty}^{\infty} \tau\, |\tau x + \nu - 1|\, |\phi'(\tau x + \nu, \mu, \sigma)|\, \phi(x)\, dx \\ &< \frac{\tau}{\sigma}\phi(0) + \frac{\tau}{\sigma^2}\phi(1) \int_{-\infty}^{\infty} |\tau x + \nu - 1|\, \phi(x)\, dx \\ &< \frac{\tau}{\sigma}\phi(0) + \frac{\tau}{\sigma^2}\phi(1) \int_{-\infty}^{\infty} |\tau x|\, \phi(x)\, dx + \frac{\tau}{\sigma^2}\phi(1)\, |\nu - 1| \\ &= \frac{\tau}{\sigma}\phi(0) + \frac{\tau^2}{\sigma^2}\phi(1) \sqrt{\frac{2}{\pi}} + \frac{\tau}{\sigma^2}\phi(1)\, (1 - \nu) \end{aligned}$$

where we have used the following facts: (1) $\frac{1}{\sigma}\phi(0)$ is the maximum value of $\phi(\tau x + \nu, \mu, \sigma)$; (2) $\frac{1}{\sigma^2}\phi(1)$ is the maximized value of $\phi'(\tau x + \nu, \mu, \sigma)$; and (3) $\int_{-\infty}^{\infty} |x|\, \phi(x)\, dx = \sqrt{\frac{2}{\pi}}$. Thus, $\lim_{\tau \to 0} |A(\tau)| = 0$.

To summarize, we have shown that derivative of the difference in payoffs $\Delta(\tau) = \Pi_2(\tau) - \Pi_1(\tau)$ is

$$\Delta'(\tau) = A(\tau) + B(\tau)$$

where $\lim_{\tau \to 0} A(\tau) = 0$ while $\lim_{\tau \to 0} B(\tau) > 0$. This implies that for τ small enough

$$\Delta(\tau) > 0$$

■

Some remarks on the main result are in order.

Remark 1 When τ is small, the choices of the two players are well-coordinated in states $\theta < \nu - \varepsilon$ since each player is very sure that the other will choose N. Their choices are also well-coordinated in states $\theta > \nu + \varepsilon$ since each player is very sure that the other will choose I. In states θ such that $\theta \in (\nu - \varepsilon, \nu + \varepsilon)$, player 1 faces a lot of strategic uncertainty since he believes that player 2 will play I and N with roughly equal probability. Player 2, on the other hand, is virtually certain that player 1 will play I if her own signal is g. Similarly, she is very sure that player 2 will play N if her signal is b. In this sense, player 2 faces less strategic uncertainty in the critical states than does player 1.

Remark 2 The assumption that $\nu > \frac{1}{2}$ plays a crucial role in our analysis. Note that if $\theta = \nu > \frac{1}{2}$, then player 1 strictly prefers to choose I over N. Because of this his threshold $k(\tau) < \nu$ and, in the limit, player 1 chooses I more often than player 2. Thus, player 1 plays too aggressively and as a result gets a lower payoff than player 2. If $\nu < \frac{1}{2}$, then by arguments analogous to those above, we would find that threshold $k(\tau) > \nu$ and now player 1 would play more conservatively. Moreover, if $\nu < \frac{1}{2}$, then player 1's payoff would be higher than player 2's payoff when τ is small enough. The significance of the assumption that $\nu > \frac{1}{2}$ can also be seen through the lens of "risk-dominance". Recall that when $\frac{1}{2} < \theta < 1$, and this fact is common knowledge, the (I, I) equilibrium risk-dominates the (N, N) equilibrium.

Remark 3 While ν, the state θ where the probability that player 2 receives the signal g switches from below $\frac{1}{2}$ to above $\frac{1}{2}$, is crucial, μ, the prior mean is not. Similarly, σ plays only a small role in ensuring the uniqueness of equilibrium. Overall, our result is rather independent of the prior distribution of the fundamentals. For instance, the result would hold if the prior distribution of θ were uniform on the interval $[-a, a]$ for $a > 1$.

5 Other information "paradoxes"

The nature of our finding is rather different from known examples showing that in multi-agent settings, information may have a negative value—that is, better information may make an agent worse off. Public information can have a negative value, as was pointed out by Hirshleifer (1971). Private information can have a negative value as well (see, for instance, Bassan et al., 2003). In a game, an improvement in the quality of information of a particular player may make that player worse off. Our main result is not about the value of information to a player; rather it compares the payoffs across players in an otherwise symmetric game.

A Appendix: Uniqueness

This appendix contains the proof of Proposition 2. As usual, uniqueness is established via the iterated elimination of dominated strategies. There are three steps.

1. It is dominant for player 1 to choose I if $\theta > 1$ and to choose N if $\theta < 0$.

2. Given Step 1, it is dominant for player 2 to choose I if her signal is g and to choose N if her signal is b. (This is the only step in which we will need σ^2 to be large.)

3. Given Step 2, the unique best response for player 1 is to choose a threshold $k(\tau)$ as defined in (2).

Step 1 is obvious and Step 3 has already been established in Section 3. So it remains to show the claim in Step 2.

Suppose player 2 with signal g plays I. Her payoff is no less than the payoff if player 1 plays N for all $\theta < 1$ and I for all $\theta \geq 1$. This is the same as player 1 choosing a threshold of $k = 1$ and so analogous to (7), this lower bound is

$$\frac{1}{\Pr[g]}\left[\int_1^{\infty}\theta\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta+\int_{-\infty}^{1}(\theta-1)\,\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta\right] \tag{9}$$

Similarly, suppose player 2 gets a b signal and plays I. Her payoff from playing I is at most the payoff if player 1 plays I for all $\theta \geq 0$ and N for all $\theta < 1$. This is the same as player 1 choosing a threshold of $k = 0$ and so this upper bound is

$$\frac{1}{\Pr[b]}\left[\int_0^{\infty}\theta\,(1-\Phi(\theta,\nu,\tau))\,\phi(\theta,\nu,\sigma)\,d\theta+\int_{-\infty}^{0}(\theta-1)\,(1-\Phi(\theta,\nu,\tau))\,\phi(\theta,\mu,\sigma)\,d\theta\right] \tag{10}$$

We now show that given any $\overline{\tau}$, there exists a $\underline{\sigma}$ such that for all $\sigma > \underline{\sigma}$ and $\tau < \overline{\tau}$, (9) is positive so that even if player 1 chooses a threshold $k = 1$, player 2 with signal g will choose I. We also show that (10) is negative so that even if player 1 chooses a threshold of $k = 0$, player 2 with signal b will choose N.

We will show that given any $\overline{\tau}$, there exists a $\underline{\sigma}$ such that for all $\sigma > \underline{\sigma}$ and $\tau < \overline{\tau}$, (9) is positive and (10) is negative.

The bracketed term in (9) is the same as

$$\int_{-\infty}^{\infty}\theta\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta-\int_{-\infty}^{1}\Phi(\theta,\nu,\tau)\,\phi(\theta,\mu,\sigma)\,d\theta \tag{11}$$

and player 2 would choose to play I whenever this is positive.

Since the second integral in (11) is at most 1, it is enough to show that the first term

$$\mathcal{I} = \int_{-\infty}^{\infty} \theta\Phi(\theta, \nu, \tau)\, \phi(\theta, \mu, \sigma)\, d\theta$$

is greater than 1. Using the fact that

$$\phi'(\theta, \mu, \sigma) = -\frac{1}{\sigma}\left(\frac{\theta - \mu}{\sigma}\right)\phi(\theta, \mu, \sigma)$$

we can write

$$\mathcal{I} = \underbrace{-\sigma^2 \int_{-\infty}^{\infty} \Phi(\theta, \nu, \tau)\, \phi'(\theta, \mu, \sigma)\, d\theta}_{\mathcal{J}} + \mu \int_{-\infty}^{\infty} \Phi(\theta, \nu, \tau)\, \phi(\theta, \mu, \sigma)\, d\theta$$

Integrating by parts

$$\mathcal{J} = \sigma^2 \int_{-\infty}^{\infty} \phi(\theta, \nu, \tau)\, \phi(\theta, \mu, \sigma)\, d\theta$$

Also, some algebra reveals that the integrand in $\mathcal{J}$,

$$\begin{aligned}
\phi(\theta, \nu, \tau)\, \phi(\theta, \mu, \sigma) &= \frac{1}{\sqrt{2\pi}\tau} \exp\left(-\frac{1}{2}\left(\frac{\theta - \nu}{\tau}\right)^2\right) \times \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\left(\frac{\theta - \mu}{\sigma}\right)^2\right) \\
&= \phi\left(\theta, \tfrac{\tau^2\mu + \sigma^2\nu}{\sigma^2 + \tau^2}, \sqrt{\tfrac{\sigma^2\tau^2}{\sigma^2 + \tau^2}}\right) \times \phi\left(0, \mu - \nu, \sqrt{\sigma^2 + \tau^2}\right)
\end{aligned}$$

and so

$$J = \sigma^2 \phi\left(0, \mu - \nu, \sqrt{\sigma^2 + \tau^2}\right)$$

Thus,

$$\mathcal{I} = \sigma^2 \phi\left(0, \mu - \nu, \sqrt{\sigma^2 + \tau^2}\right) + \mu \int_{-\infty}^{\infty} \Phi(\theta, \nu, \tau)\, \phi(\theta, \mu, \sigma)\, d\theta$$

and since the second term of $\mathcal{I}$ is positive,

$$\begin{aligned}
\mathcal{I} &> \frac{\sigma^2}{\sqrt{2\pi}\sqrt{\sigma^2 + \tau^2}} \exp\left(-\frac{(\mu - \nu)^2}{2(\sigma^2 + \tau^2)}\right) \\
&\equiv L(\sigma, \tau)
\end{aligned}$$

It is easily verified that $L(\sigma, \tau)$ is increasing in σ and $\lim_{\sigma \to \infty} L(\sigma, \tau) = \infty$. Moreover, it is also easily verified that for any $\sigma > |\mu - \nu|$, $L(\sigma, \tau)$ is decreasing in τ. Thus, we have shown that for any $\overline{\tau}$, $L(\sigma, \overline{\tau}) > 1$ once σ is large enough and moreover, $L(\sigma, \tau) > 1$ for all $\tau < \overline{\tau}$ as well.

Finally, consider the payoff of player 2 with signal b when she plays I and player 1 plays I as long as $\theta > 0$. This payoff can be written as

$$
\begin{aligned}
& \int_{-\infty}^{\infty} \theta \left(1 - \Phi\left(\theta, \nu, \tau\right)\right) \phi\left(\theta, \mu, \sigma\right) d\theta - \int_{-\infty}^{0} \left(1 - \Phi\left(\theta, \nu, \tau\right)\right) \phi\left(\theta, \mu, \sigma\right) d\theta \\
= \;& \mu - I - \int_{-\infty}^{0} \left(1 - \Phi\left(\theta, \nu, \tau\right)\right) \phi\left(\theta, \mu, \sigma\right) d\theta \\
< \;& \mu - \mathcal{I} \\
< \;& \mu - L\left(\sigma, \tau\right)
\end{aligned}
$$

since from above, $\mathcal{I} > L\left(\sigma, \tau\right)$. The properties of $L\left(\sigma, \tau\right)$ now guarantee that for any $\overline{\tau}$ this last expression is negative once σ is large enough and remains so for all $\tau < \overline{\tau}$ as well.

This completes the proof.

References

[1] Angeletos, G-M. and C. Lian (2016): "Incomplete Information in Macroeconomics: Accommodating Frictions in Coordination," in *Handbook of Macroeconomics*, Vol. 2, North-Holland.

[2] Bassan, B., O. Gossner, M. Scarsini, and S. Zamir (2003): "Positive Value of Information in Games," *International Journal of Game Theory*, 32 (1), 17–31.

[3] Corsetti, G., A. Dasgupta, S. Morris and H. Shin (2004): "Does One Soros Make a Difference? A Theory of Currency Crises with Large and Small Traders," *Review of Economic Studies*, 71, 81–113.

[4] Carlsson, H. and E. van Damme (1993): "Global Games and Equilibrium Selection," *Econometrica*, 61 (5), 989–1018.

[5] Harsanyi, J. and R. Selten (1988): *A General Theory of Equilibrium Selection in Games*, MIT Press, Cambridge, MA.

[6] Hirshleifer, J. (1971): "The Private and Social Value of Information and the Reward to Inventive Activity," *American Economic Review*, 61, 561–574.

[7] Morris, S, and H. Shin (2003): "Global Games: Theory and Application," Chapter 3 in *Advances in Economics and Econometrics*, Econometric Society Monographs, Cambridge University Press: Cambridge.

[8] Rubinstein, A. (1989): "The Electronic Mail Game: Strategic Behavior under “Almost Common Knowledge”," *American Economic Review*, 79 (3), 385–391.