UTMD-107

Revisiting the Identification of the Conduct Parameter in Homogeneous Goods Markets

Yuri Matsumura
The Japan Fair Trade Commission

Suguru Otani
University of Tokyo

January 14, 2026

Revisiting the Identification of the Conduct Parameter in Homogeneous Goods Markets

Yuri Matsumura   Suguru Otani

January 14, 2026

Abstract

We revisit the identification of the conduct parameter in homogeneous goods markets. Lau (1982) shows that the conduct parameter is not identified if and only if the inverse demand function is separable, except for a specific separable function. This result has been regarded as an extension of the result in Bresnahan (1982) to more general settings. However, we show that Lau's claim is incorrect and provide a new characterization of the non-identification. Our characterization shows that the presence of demand rotation instruments in the demand function is the necessary and sufficient condition for identifying the conduct parameter. Therefore, our result properly generalizes the role of demand rotation instruments in identifying the conduct parameter, as highlighted by Bresnahan (1982), to more general settings.

Keywords: Identification, Conduct Parameter, Homogeneous Goods Market
JEL Codes: C5, C13, L1

1 Introduction

Empirical research in industrial organization examines a wide range of questions, including the measurement of market power and concentration, the evaluation of welfare, the assessment of cartels, and the prediction of merger impacts. In virtually all of these applications, empirical conclusions depend critically on assumptions about how firms compete, that is, on the specification of firm conduct. Researchers often impose a particular competitive model, such as price-taking, Cournot, or Bertrand, because conduct is rarely identified without additional information, such as detailed cost data or industry- or institution-specific information. Nevertheless, misspecifying conduct can distort the measurement or lead to misleading conclusions.

To address this problem, the *conduct parameter approach* has been used in the literature on industrial organization. This approach embeds a scalar measure of firm conduct into the marginal revenue function and allows researchers to estimate conduct directly from data, providing a flexible model to investigate several questions. The applications vary across industries and questions, including homogeneous-product markets (Porter 1983, Genesove and Mullin 1998, Okazaki, Onishi and Wakamori 2022), differentiated-product markets (Miller and Weinberg 2017, Ciliberto and Williams 2014, Sullivan 2020), pass-through analysis (Weyl and Fabinger 2013, Miller, Osborne and Sheu 2017), welfare evaluation under imperfect competition (Beringe and Whitmeyer 2025, Nocke and Schutz 2025), and merger evaluation (Aryal, Chattopadhyaya and Ciliberto 2025).

Despite its broad use, the basic identification result on the conduct parameter in a homogeneous goods market remains rooted in two classic contributions. First, Bresnahan (1982) shows that, with a linear demand and a linear marginal cost, a *demand rotation instrument* can identify the conduct parameter by moving the slope and intercept of inverse demand simultaneously because it can keep the equilibrium the same under the true conduct parameter while changing the equilibrium under the false conduct parameter. Second, Lau (1982) considers an environment with a more general inverse demand function and a marginal cost function and shows that conduct is not identified if and only if the inverse demand function is separable in demand shifters. Because demand rotation instruments break the separability of the inverse demand function, Lau's condition has been interpreted as a general characterization of identification and regarded as an extension of Bresnahan's idea in general settings.

The main contribution of this paper is to show that Lau's characterization is incomplete and to provide a new condition that fully characterizes if and only if conduct and marginal cost are not identified. Our main result shows that the non-identification of the conduct parameter and the marginal cost function arises if and only if demand shifters can change *only the slope* or *only the intercept* of the inverse demand function. In this case, the inverse demand function cannot have any demand rotation instrument, and hence conduct is identified precisely when the inverse demand function includes demand rotation instruments. This result is both simpler and more restrictive than separability in Lau's claim and clarifies the precise role of demand rotation instruments in identifying the conduct parameter in general environ-

ments.

The intuition is the following. Given a conduct parameter and a marginal cost function, the non-identification implies that we can transform the marginal cost function into another marginal cost function that, together with another conduct parameter, produces the same equilibrium quantity. However, this transformation is usually invalid because keeping the equilibrium points the same in both models requires the transformed marginal cost function to depend on demand shifters, which violates the exclusive demand shifters assumption. This dependence disappears only when demand shifters can change just the slope or just the intercept of the inverse demand function, in which case we can construct different pairs of conduct parameters and marginal cost functions that yield the same equilibrium for any value of the shifters without violating the exclusion restriction. Thus, when demand rotation instruments are present in the inverse demand function, the transformation remains invalid, preventing observational equivalence between alternative conduct and marginal-cost pairs and ensuring identification. The issue that makes Lau's proof incomplete is that while he also derives a transformation between marginal cost functions, he fails to eliminate the dependence of the transformed marginal cost function on demand shifters.

Taken together, our results refine the theoretical foundations of conduct parameter estimation and clarify the exact role that demand shifters must play to allow practitioners to implement empirical analysis with a flexible competition model. The rest of the paper is organized as follows. Section 2 describes the setting. Section 3 discusses Lau's result and our main result. Subsections 3.3 and 3.4 provide a new characterization of non-identification and show the necessary and sufficient condition for non-identification. Section 4 discusses some criticisms of the conduct parameter approach. Section 5 concludes. The Appendix includes the omitted proofs in the main text and a summary of Goldman and Uzawa (1964).

2 Setting

2.1 Conduct Parameter Model

Consider a homogeneous product market with an aggregate inverse demand and an aggregate marginal cost function denoted by $P(Q, \boldsymbol{X}^d)$ and $MC(Q, \boldsymbol{X}^s)$, respectively, where Q is the aggregate product quantity, $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$ are the vectors of demand shifters and cost shifters, respectively. Note that every vector is represented by a bold letter, and its element or scalar value is represented by a non-bold letter. Let K_d and K_s be the dimension of $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$, respectively.

Given the demand shifter $\boldsymbol{X}^d$ and the cost shifter $\boldsymbol{X}^s$, the equilibrium quantity Q^e solves the following equation:

$$P(Q, \boldsymbol{X}^d) + \theta Q \frac{\partial P}{\partial Q}(Q, \boldsymbol{X}^d) = MC(Q, \boldsymbol{X}^s), \tag{1}$$

where $\theta \in [0, 1]$ is called the conduct parameter. By rewriting the equilibrium condition (1), we have

$$\theta = \frac{P - MC}{P} \varepsilon^d,$$

where ε^d is the price elasticity of demand. Therefore, the conduct parameter is also regarded as the elasticity-adjusted Lerner index and represents the degree of market power of the firms. Depending on the value of θ, the condition can represent the first-order condition of several models between perfect competition ($\theta = 0$) and joint-profit maximization ($\theta = 1$).[1] Therefore, the left-hand side of (1) is regarded as a generalized marginal revenue with θ, and hence the condition is a generalized first-order condition with the conduct parameter.

For the identification analysis, we put some restrictions. First, we assume that the demand shifter and the cost shifter are mutually exclusive:

Assumption 1. *The set of all exogenous variables affecting the market equilibrium can be partitioned into (1) exclusive demand shifters, X^d, that affect the inverse demand function but not the marginal cost function, (2) exclusive cost shifters, X^s, that affect the marginal cost function but not the inverse demand function, and (3) common shifters, Z, that affect both the inverse demand function and the marginal cost function.*

The theoretical analysis of identification often simplifies the setting by assuming that the demand shifters and cost shifters are mutually exclusive, as per Assumption 1. We formalize our approach by conditioning the analysis on an arbitrary realization z of these common shifters. This means all subsequent functions are understood to be conditional functions, $P(Q, X^d) \equiv \tilde{P}(Q, X^d, Z = z)$ and $MC(Q, X^s) \equiv \tilde{MC}(Q, X^s, Z = z)$. By treating Z as fixed constants in this conditional analysis, we maintain the notational simplicity of focusing only on the exclusive shifters while ensuring that the derived non-identification characterization holds robustly across all values of the common variables in empirical settings.

The next assumption restricts the effectiveness of the shifters:

Assumption 2. *The given demand shifters and the cost shifters should affect the inverse demand function and the marginal cost function. Formally, we should have that for all $i = 1, \dots, K_d$ and $j = 1, \dots, K_s$,*

$$\frac{\partial P}{\partial X_i^d}(Q, X^d) \neq 0 \quad and \quad \frac{\partial MC}{\partial X_j^s}(Q, X^s) \neq 0$$

for any $Q > 0$, X^d, and X^s.

When this condition is not met, there could be areas where the inverse demand function and the marginal cost function cannot be identified. For example, when a demand shifter does not change the

---

[1]It can also nest Cournot competition when $\theta = 1/N$ under some marginal cost function such as constant marginal cost and linear marginal cost. In general, this holds when the aggregation of each firm's first-order condition results in the aggregate first-order condition.

inverse demand function for some interval of the demand shifter, the equilibrium quantity is not affected by the change in the demand shifter. Then, there is no variation to identify the inverse demand function on the interval. Because our identification result for the conduct parameter and the marginal cost function presumes the identification of the inverse demand function, to guarantee the identification of the inverse demand function, the assumption is necessary.

Next, we impose an assumption on the differentiability of the inverse demand function and the marginal cost function:

Assumption 3. *The inverse demand function is three times continuously differentiable, and the marginal cost function is twice continuously differentiable.*

As we will see, Lau (1982) only assumes twice-continuous differentiability of the inverse demand function and the marginal cost function. Hence, our result requires a stronger assumption on the inverse demand function. However, we will justify our assumption by showing that Lau's claim also needs three-times continuous differentiability later.

Finally, we impose the equilibrium existence condition:

Assumption 4. *Given an inverse demand function, a conduct parameter, and a marginal cost function, and given X^d and X^s, the derivative of the equilibrium condition with respect to Q at the equilibrium quantity Q^e is not zero, that is,*

$$(1+\theta)\frac{\partial P}{\partial Q}(Q^e, X^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, X^d) - \frac{\partial MC}{\partial Q}(Q^e, X^s) \neq 0.$$

Note that as the above equation consists of the derivative of the inverse demand and the marginal cost, by Assumption 3, the derivative of the equilibrium condition with respect to Q is finite, and hence the left-hand side is a finite value for any Q, X^d, and X^s.

2.2 The Data Generation Process

Suppose that the researcher observes the aggregate price P and the aggregate quantity Q, and the vector of exogenous variables X^d and X^s. Assume that the data is generated through the equilibrium condition (1). We assume that from the data, the reduced form of the equilibrium quantity and the equilibrium price are identified:

Assumption 5. *The reduced form of the equilibrium quantity and the equilibrium price, defined as*

$$Q^e = h_q(X^d, X^s), \quad P^e = h_p(X^d, X^s),$$

are identified.

The identification of the reduced forms follows directly from variation in the demand and cost shifters.

Remark 1. *While Bresnahan (1982) considers a model with error terms, by following Lau (1982), we do not consider any error term in the demand and the supply side, that is, our model does not have any unobserved characteristics to the econometrician. This is a restrictive assumption because there could be real data where X^d and X^s are the same in two time periods, but the equilibrium outcomes are different in the two periods. As the scope of this paper is to fully characterize the identification condition along with Lau's setting,. In other words, our identification analysis is independent of the presence of error terms and concerns whether the structural mapping from primitives (θ, MC) to the reduced-form equilibrium outcomes (h_q, h_p) is injective, that is, whether (θ, MC) can be uniquely recovered given (h_q, h_p). We do not include unobserved characteristics and leave it for future research.*

2.3 Definitions

Now, we introduce the definition of the identification problem in this setting. While our interest is the identification of the conduct parameter and the marginal cost function, by following Lau (1982), we take an indirect approach and specify the conditions under which the model is *not* identified. Our definition of non-identification is based on when two different models lead to observational equivalence:

Definition 1. *The conduct parameter and the marginal cost function are said to be non-identified if there are two distinct pairs of conduct parameters and marginal cost functions, denoted by (θ, MC) and (θ^*, MC^*), such that the corresponding reduced-form equilibrium quantity that satisfies the equilibrium condition (1) and the corresponding reduced-form equilibrium price are identical for every demand- and cost-shifter value:*

$$Q^e = h_q(X^d, X^s) = h_q^*(X^d, X^s) \text{ and } P^e = h_p(X^d, X^s) = h_p^*(X^d, X^s)$$

for any X^d and X^s.

By taking the contraposition of Definition 1, we can characterize the identification of the conduct parameter and the marginal cost function:

Corollary 1. *The conduct parameter and the marginal cost are said to be identified if for any two distinct pairs of a conduct parameter and a marginal cost function, there exists a pair of demand shifters $\tilde{X}^d$ and $\tilde{X}^s$ where the equilibrium quantity Q^e is different, that is, $h_q(\tilde{X}^d, \tilde{X}^s) \neq h_q^*(\tilde{X}^d, \tilde{X}^s)$.*

Note that the non-identification and the identification assume that the inverse demand function is already identified. In other words, we study the identification of the conduct parameter and the marginal cost function given an identified inverse demand function.

3 Main Result: Characterization of the Non-identification

3.1 Lau's Claim

Lau (1982) investigates the identification of the conduct parameter and the marginal cost function and obtains the following claim, which is the quote of Theorem 1 in Lau (1982):

Claim 1. *Under the assumption that the inverse demand function and the marginal cost functions are twice continuously differentiable, the index of competitiveness θ cannot be identified from data on industry price and quantity and other exogenous variables alone if and only if the industry inverse demand function is separable in the demand shifter, that is,*

$$P(Q, \boldsymbol{X}^d) = P(Q, r(\boldsymbol{X}^d)), \tag{2}$$

but does not take the form

$$P(Q, r(\boldsymbol{X}^d)) = Q^{-\frac{1}{\theta}} r(\boldsymbol{X}^d) + s(Q). \tag{3}$$

The result states that the separability of the inverse demand function is crucial for the identification, but there is a type of separable inverse demand function that can lead to the identification. An inverse demand function satisfying (2) has weak separability defined in Goldman and Uzawa (1964).[2] The claim also emphasizes that the dimension of the demand shifter is important for the identification. When the demand shifter is a scalar, (2) nests any inverse demand function because we can set $\tilde{P}(Q, \boldsymbol{X}^d) = P(Q, r(\boldsymbol{X}^d))$. Thus, with a scalar demand shifter, the conduct parameter can be identified only when the inverse demand function takes the form (3).

To understand the intuition of Lau's claim, let us consider an example in Bresnahan (1982) without an error term. Bresnahan considers a market with a linear inverse demand and a linear marginal cost, where the linear inverse demand function is given as

$$P(Q, \boldsymbol{X}^d) = \alpha_0 - \alpha_1 Q + \alpha_2 X_1^d + \alpha_3 Q X_1^d + \alpha_4 X_2^d, \tag{4}$$

where X_1^d is a demand rotation instrument because it can change the slope and the intercept of the inverse demand function without changing the equilibrium quantity. It is easy to verify that the demand rotation instrument breaks the separability of the inverse demand function. Additionally, the dimension of the demand shifter is greater than one, and hence Lau's claim implies that even a nonlinear marginal cost function, (4), leads to the identification of the conduct parameter.

While Lau's claim has been regarded as an extension of Bresnahan (1982) to more general settings,

---

[2]See Definition B.1 in Appendix B. Note that the separability in Goldman and Uzawa (1964) is defined only when the dimension of the demand shifter is greater than two. When the demand shifter is a scalar, we cannot apply the definition of separability in Goldman and Uzawa (1964).

an important observation is that including demand-rotation instruments is inconsistent with imposing separability on the inverse demand function, but the separability assumption itself does not preclude the existence of demand-rotation shifters in an inverse demand function.

For example, consider an inverse demand function given as

$$P(Q, r(X^d)) = -Qr(X^d) + Q + r(X^d) = r(X^d)(1 - Q) - Q,$$

where the range of r is greater than one, that is, $r(X^d) \geq 1$ for any X^d. Assume that the dimension of X^d is greater than two. Note that in this inverse demand function, the demand shifter can change the slope and the intercept of the inverse demand function.

First, the function is verified to be separable because for any i and j, we have

$$\frac{\partial}{\partial Q} \left(\frac{\frac{\partial P}{\partial X_i^d}(Q, X^d)}{\frac{\partial P}{\partial X_j^d}(Q, X^d)} \right) = \frac{\partial}{\partial Q} \left(\frac{r_i(X^d)(1 - Q)}{r_j(X^d)(1 - Q)} \right) = \frac{\partial}{\partial Q} \left(\frac{r_i(X^d)}{r_j(X^d)} \right) = 0,$$

where $r_i(X^d) \equiv \frac{\partial r(X^d)}{\partial X_i^d}$.

Second, we can verify that the demand shifters can work as demand rotation instruments to break observational equivalence. Given the inverse demand function, the marginal revenue under θ is written as

$$MR(Q, X^d; \theta) = -r(X^d)(1 - (1 + \theta)Q) + Q(1 + \theta).$$

When the quantity is $Q' = \frac{1}{1+\theta}$, the marginal revenue is not affected by the change in the demand shifter X^d because it is equal to one. Suppose that there is a marginal cost function MC where Q' holds as an equilibrium for some $\tilde{X}^s$. Then, as the marginal revenue is a constant in the demand shifter, any change in the demand shifter does not affect the equilibrium quantity. In contrast, suppose that under another conduct parameter θ^* and another marginal cost function MC^*, Q' holds as an equilibrium for some $\tilde{X}^d$ and $\tilde{X}^s$, that is, Q' satisfies

$$r(\tilde{X}^d)(1 - (1 + \theta^*)Q') + Q'(1 + \theta^*) = MC^*(Q', \tilde{X}^s).$$

Here, the marginal revenue under θ^* is a function of the demand shifter at Q'. Thus, the change in X^d leads to a different equilibrium quantity from Q', which implies the violation of the observational equivalence. Therefore, the demand shifter X^d can work as the demand rotation instrument at Q' because it changes the slope and the intercept of the inverse demand function simultaneously without changing the equilibrium quantity under the true conduct, whereas the change leads to a different equilibrium quantity under any false conduct.

Furthermore, when we treat $r(X^d)$ as a scalar demand shifter, the second argument implies that

even a scalar demand shifter can break observational equivalence, although the inverse demand function does not satisfy (3). The example admits a separable inverse demand function with a demand rotation instrument, but Lau does not clearly explain how this situation fits his claim, and hence it is not clear why separability, rather than the availability of demand rotation instruments, is the key to identification.

3.2   Main Result

We now show the following theorem:

Theorem 1. *Given Assumption 1, 2, 3, and 4, and data on price, quantity, and other exogenous variables,*

(i) *if the conduct parameter θ and the marginal cost function MC are not identified, the industry inverse demand function must be given as*

$$P(Q, \boldsymbol{X}^d) = Q^\alpha r(\boldsymbol{X}^d) + s(Q) \tag{5}$$

where $\alpha \neq -\frac{1}{\theta}$.

(ii) *if the inverse demand function takes the form (5) with the properties described in (i), then the conduct parameter and the marginal cost are not identified.*

As in Lau's claim, when the demand shifter is a vector, the inverse demand function is separable because the demand shifters affect the inverse demand function only through r. However, our result tells more about under what type of separable function the non-identification holds and has a clear economic interpretation. As in Claim 1, our result has a special case where any change in demand shifters does not affect the marginal revenue under the true conduct but does under any false conduct, and hence the identification always holds ($\alpha = -\frac{1}{\theta}$). Except in this case, observe that the demand shifter changes only the slope of the inverse demand function when $\alpha \neq 0$ or changes only the intercept of the inverse demand function when $\alpha = 0$. Recall that demand rotation instruments can change the intercept and the slope of the inverse demand function simultaneously. Therefore, we cannot have any demand rotation instruments under (5). The result implies that the identification holds if and only if the inverse demand function has demand shifters that work as demand rotation instruments. Here, we are explicit that a demand rotation instrument is not necessarily a single variable, but could be a combination of changes in some demand shifters that alter the slope and the intercept of the inverse demand function simultaneously. Note also that Bresnahan graphically demonstrates that the conduct parameter is identified when a demand rotation instrument moves the demand curve in a way that leaves the equilibrium point unchanged at the true conduct but changes the equilibrium under any false conduct. However, this is an extreme case and it suffices that the instrument changes the slope and the intercept simultaneously. This clearly reinforces the idea of Bresnahan (1982) using demand rotation instruments to identify the conduct parameter in more general settings.

Hereafter, we provide the proof of Theorem 1. First, we derive the necessary condition for non-identification based on Definition 1. By rewriting Definition 1, we obtain a transformation that maps the derivatives of MC into the derivatives of another marginal cost function MC^* that yields observational equivalence given an inverse demand function (Lemma 2). However, the transformation is not valid because it consists of derivatives of the inverse demand function and hence depends on the demand shifter $\boldsymbol{X}^d$. Without additional restrictions, MC^* would therefore depend on $\boldsymbol{X}^d$, violating Assumption 1. Therefore, to make the transformation valid, we need to remove the effect of the demand shifter from the transformation, which puts restrictions on the inverse demand function. Then, the restriction leads to a differential equation that characterizes the inverse demand function that leads to non-identification (Lemma 4). Then, to show sufficiency, under the inverse demand function characterized in Lemma 4, we construct a transformation that leads to observational equivalence (Lemma 5).

3.3 Necessary Condition for the Non-identification

First, we characterize the non-identification condition based on Definition 1. The characterization is based on the fact that observational equivalence implies that the reduced forms of the equilibrium quantity h_q and h_q^* are identical for any $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$. Therefore, its derivative with respect to the demand and cost shifters, ∇h_q and ∇h_q^*, should also be identical. Then, by applying the implicit function theorem to the equilibrium condition, we can compute ∇h_q and ∇h_q^*. The following lemma characterizes the non-identification condition:

Lemma 1. *Non-identification implies that for any $\boldsymbol{X}^d$, $\boldsymbol{X}^s$, and Q^e under these exogenous variables, we have for $i = 1, \ldots, K_d$,*

$$
\frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d)
$$
$$
= \lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) \left[\frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d) \right], \tag{6}
$$

and for $j = 1, \ldots, K_s$,

$$
\frac{\partial MC^*}{\partial X_j^s}(Q^e, \boldsymbol{X}^s) = \lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) \frac{\partial MC}{\partial X_j^s}(Q^e, \boldsymbol{X}^s), \tag{7}
$$

where $\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s)$ is defined as

$$
\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) \equiv \frac{(1 + \theta^*) \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC^*}{\partial Q}(Q^e, \boldsymbol{X}^s)}{(1 + \theta) \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s)}.
$$

See Appendix A for the detailed proof. Equation (6) and (7) imply that when non-identification holds, the derivative of the marginal revenue with θ can be transformed into the derivative of the marginal revenue with θ^*, and the derivative of the marginal cost MC with respect to $\boldsymbol{X}^s$ can be transformed into

10

the derivative of the marginal cost MC^* with respect to $\boldsymbol{X}^s$ by $\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s)$.

These transformations cannot always be valid because the transformed marginal revenue is affected by the cost shifter $\boldsymbol{X}^s$ and the transformed marginal cost is affected by the demand shifter $\boldsymbol{X}^d$ through λ. These dependencies are not allowed by Assumption 1. Thus, to consider a valid transformation, we further rewrite (6) and (7). The next lemma provides a transformation of the derivative of MC and MC^* with respect to Q and $\boldsymbol{X}^s$ that leads to observational equivalence:

Lemma 2. *For any Q^e, $\boldsymbol{X}^d$, and $\boldsymbol{X}^s$, non-identification implies that the derivative of marginal cost MC can be transformed into the derivative of marginal cost MC^*: for $i = 1, \ldots, K_d$, and $j = 1, \ldots, K_s$,*

$$\frac{\partial MC^*}{\partial Q}(Q^e, \boldsymbol{X}^s) = D_i(Q^e, \boldsymbol{X}^d) + C_i(Q^e, \boldsymbol{X}^d)\frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s), \tag{8}$$

and

$$\frac{\partial MC^*}{\partial X_j^s}(Q^e, \boldsymbol{X}^s) = C_i(Q^e, \boldsymbol{X}^d)\frac{\partial MC}{\partial X_j^s}(Q^e, \boldsymbol{X}^s). $$

where $C_i(Q, \boldsymbol{X}^d)$ and $D_i(Q, \boldsymbol{X}^d)$ are defined as

$$C_i(Q, \boldsymbol{X}^d) \equiv \frac{\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) + \theta^* Q \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q, \boldsymbol{X}^d)}{\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) + \theta Q \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q, \boldsymbol{X}^d)}, \tag{9}$$

and

$$D_i(Q, \boldsymbol{X}^d) \equiv \frac{\theta^* - \theta}{\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) + \theta Q \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q, \boldsymbol{X}^d)} \left[\frac{\partial P}{\partial Q}(Q, \boldsymbol{X}^d)\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) \right.$$
$$\left. + Q\frac{\partial^2 P}{\partial Q^2}(Q, \boldsymbol{X}^d)\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) - Q\frac{\partial P}{\partial Q}(Q, \boldsymbol{X}^d)\frac{\partial^2 P}{\partial X_i^d \partial Q}(Q, \boldsymbol{X}^d) \right] \tag{10}$$

See Appendix A for the detailed proof. The subscript i in C_i and D_i indicates that C_i and D_i consist of the derivative of the inverse demand function with respect to X_i^d. Unlike Lemma 1, we have only the transformation relating to marginal cost, and hence we can see the potential obstacle in this transformation more clearly: for the transformations to hold universally for any Q^e and $\boldsymbol{X}^s$, they must be independent of $\boldsymbol{X}^d$. If the dependence holds, while a demand rotation instrument does not change the equilibrium quantity Q^e, it changes MC^* via C_i and D_i. However, this means that the marginal cost function is a function of the demand shifter $\boldsymbol{X}^d$, which is prohibited by Assumption 1. Therefore, in order for the transformations to be valid, both C_i and D_i must be independent of $\boldsymbol{X}^d$.

To see when the terms C_i and D_i are independent of $\boldsymbol{X}^d$, we take the derivative of C_i and D_i with respect to X_j^d and set them equal to zero for all $j = 1, \ldots, K_d$. A concern is that, as C_i and D_i already consist of the second-order derivative of the inverse demand function, taking the derivative of C_i and D_i leads to three-times derivative of the inverse demand function. This is why we need a stronger restriction

(Assumption 3) on the inverse demand function than Lau considers (the twice-continuous differentiability assumption in Lau's claim).

Case: The denominators of C_i and D_i are zero for some demand shifters Before we characterize the inverse demand function that makes C_i and D_i independent of $\boldsymbol{X}^d$, we first check when C_i and D_i are not well-defined. This implies that (9) and (10) are violated, and hence the identification holds by Corollary 1. All derivatives in C_i and D_i are finite by Assumption 3, and hence the numerator and the denominator of C_i and D_i cannot take an infinite value. In this case, regardless of the numerator of C_i or D_i, (9) and (10) are violated whenever at least one denominator for $i = 1, \ldots, K_d$ is zero. In this case, C_i and D_i will be infinite or indeterminate.

When the denominator of C_i or D_i is zero for some i, we have that for some $(\tilde{Q}, \tilde{\boldsymbol{X}}^d)$,

$$\frac{\partial P}{\partial X_i^d}(\tilde{Q}, \tilde{\boldsymbol{X}}^d) + \theta \tilde{Q} \frac{\partial^2 P}{\partial X_i^d \partial Q}(\tilde{Q}, \tilde{\boldsymbol{X}}^d) = 0. \tag{11}$$

Let $\mathcal{I}$ be the set of indices of the demand shifters where (11) holds. Note that (11) can be rewritten as

$$\frac{\partial}{\partial X_i^d}\left(P(\tilde{Q}, \tilde{\boldsymbol{X}}^d) + \theta \tilde{Q} \frac{\partial P}{\partial Q}(\tilde{Q}, \tilde{\boldsymbol{X}}^d)\right) = 0,$$

which is the derivative of the marginal revenue under θ with respect to X_i^d. Therefore, this implies that the marginal revenue under θ is not affected by the change in X_i^d at $(\tilde{Q}, \tilde{\boldsymbol{X}}^d)$. This also implies that the equilibrium quantity is not affected by the change in $\tilde{X}_i^d$ at $(\tilde{Q}, \tilde{\boldsymbol{X}}^d)$.

The non-identification implies that (11) does not hold for any Q^e and $\boldsymbol{X}^d$. Once we have $(\tilde{Q}, \tilde{\boldsymbol{X}}^d)$ for some i where (11) holds, it suffices for identification. While it is hard to characterize the inverse demand function that satisfies (11) for some $(\tilde{Q}, \tilde{\boldsymbol{X}}^d)$ and for some i, we can characterize the inverse demand function that satisfies (11) for all $(Q, \boldsymbol{X}^d)$. Note that (11) is a partial differential equation that can be solved analytically. Thus, we can characterize the inverse demand function that always leads to identification for any Q and $\boldsymbol{X}^d$:

Lemma 3. *Suppose that $\theta \neq 0$ and $Q > 0$. Then, the conduct parameter and the marginal cost function are identified when the inverse demand function is such that*

$$P(Q, \boldsymbol{X}^d) = Q^{-\frac{1}{\theta}} r(\boldsymbol{X}^d) + s(Q, \boldsymbol{X}_{-\mathcal{I}}^d), \tag{12}$$

where $\boldsymbol{X}_{-\mathcal{I}}^d$ is the vector of demand shifters whose index is not in $\mathcal{I}$, and $r : \mathbb{R}^{K_d} \to \mathbb{R}$ and $s : \mathbb{R}^{|\boldsymbol{X}_{-\mathcal{I}}^d|+1}$ are at least twice-continuously differentiable.

See Appendix A for the proof. Under (12), the equilibrium condition (14) is given as

$$s(Q, \boldsymbol{X}_{-\mathcal{I}}^d) + \theta Q s'(Q, \boldsymbol{X}_{-\mathcal{I}}^d) = MC(Q, \boldsymbol{X}^s),$$

Therefore, when (11) holds, the equilibrium quantity is not affected by the change in X_i^d. On the other hand, for any other $\theta^* \neq \theta$, the equilibrium condition becomes

$$Q^{-\frac{1}{\theta}} r(X^d) \left(1 - \frac{\theta}{\theta^*}\right) + s(Q, X_{-\mathcal{I}}^d) + Q s'(Q, X_{-\mathcal{I}}^d) = MC(Q, X^s).$$

As $r(X^d)$ is a function of X_i^d by Assumption 1, the equilibrium condition implies that the equilibrium quantity should depend on X_i^d under θ^*. Therefore, while the change in X_i^d does not change the equilibrium quantity under θ, it changes under θ^*, which violates observational equivalence. Therefore, the conduct parameter and the marginal cost function are always identified.

Recall that Lau's claim includes an edge case where a separable inverse demand function leads to the identification. The above lemma also includes his edge case when $\mathcal{I} = \emptyset$, that is, all demand shifters work as a demand rotation instrument. We will see that (12) is also the edge case in our result.

Case: The denominators of C_i and D_i are not zero Now, suppose that (11) does not hold for any Q and X^d and for all demand shifters. Then, we take the derivative of C_i and D_i with respect to X_j^d for all $j = 1, \ldots, K_d$ and put the derivative being equal to zero. This process leads to differential equations for the inverse demand function. Fortunately, these differential equations have analytical solutions, and we can characterize an inverse demand function that satisfies the independence of C_i and D_i of the demand shifter:

Lemma 4. *The non-identification of the conduct parameter and the marginal cost function implies* (11) *does not hold for all $i = 1, \ldots, K_d$, and $C_i(Q, X^d)$ and $D_i(Q, X^d)$ are independent of X^d for all i. The independence implies that the inverse demand function is given as*

$$P(Q, X^d) = Q^\alpha r(X^d) + s(Q)$$

where $\alpha \neq -\frac{1}{\theta}$, and $r : \mathbb{R}^{K_d} \to \mathbb{R}$ and $s : \mathbb{R}_+ \to \mathbb{R}$ are at least twice-continuously differentiable.

See Appendix A for the proof. Technically, the twice-continuously differentiability of r and s is required to meet Assumption 3. As we have discussed, this inverse demand function does not allow for any demand rotation instrument. Recall that Bresnahan (1982) provides the idea of using a demand rotation instrument to identify the conduct parameter. Then, Matsumura and Otani (2023) formalize his idea by deriving a sufficient condition that guarantees the inverse demand function includes demand rotation instruments for the identification in Bresnahan's setting. The contraposition of their sufficient condition implies that the non-identification holds only if the inverse demand function does not include any demand rotation instrument. Therefore, our characterization is the generalization of their result in general settings.

Another difference from Lau's claim is that the conduct parameter can be identified even when the demand shifter is a scalar as long as the inverse demand function includes a demand rotation instrument.

For example, consider an inverse demand function with a scalar demand rotation instrument:

$$P(Q, X_1^d) = \alpha_0 + \alpha_1 Q + \alpha_2 X_1^d + \alpha_3 Q X_1^d.$$

In this case, C_1 and D_1 are given as

$$C_1(Q, X_1^d) = \frac{\alpha_2 + \theta \alpha_3 Q}{\alpha_2 + \theta^* \alpha_3 Q} \quad \text{and} \quad D_1(Q, X_1^d) = (\theta^* - \theta)(-\alpha_1 + \alpha_3 X_1^d).$$

This implies that the transformation (8) is not valid for any marginal cost function because $D_1(Q, X_1^d)$ depends on X_1^d. Therefore, any MC^* that leads to observational equivalence with MC should depend on the demand shifter, under which the exclusion restriction is violated and hence identification is possible.

Remark 2. *The differentiability assumptions are imposed on the inverse demand function through Assumption 3. The functions r and s appear as components of the solution to a differential equation implied by the non-identification condition. Their differentiability is therefore inherited from that of P and is not imposed as an independent assumption. Cases in which r or s fail to be sufficiently differentiable are outside the scope of the theorem.*

3.4 Sufficient Condition for Non-identification

We turn to check the sufficiency of the inverse demand function characterized in Lemma 4. To see whether the inverse demand function (5) can lead to non-identification, we check whether there exists a transformation between MC and MC^* and the marginal revenue under θ and under θ^* such that any equilibrium in $\mathcal{E}$ can also be an equilibrium in $\mathcal{E}^*$ by using the mapping. Formally, suppose that there exists a transformation T such that

$$MC^*(Q, \boldsymbol{X}^s) = T\left(MC(Q, \boldsymbol{X}^s)\right).$$

and

$$P(Q, \boldsymbol{X}^d) + \theta^* Q \frac{\partial P}{\partial Q}(Q, \boldsymbol{X}^d) = T\left(P(Q, \boldsymbol{X}^d) + \theta Q \frac{\partial P}{\partial Q}(Q, \boldsymbol{X}^d)\right).$$

Then, suppose also that an equilibrium quantity Q^e satisfies the equilibrium condition under $\mathcal{E}$. When we have that for $\mathcal{E}^*$,

$$P(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) - MC^*(Q^e, \boldsymbol{X}^s)$$
$$= T\left(P(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d)\right) - T\left(MC(Q^e, \boldsymbol{X}^s)\right) = 0,$$

then Q^e also satisfies the equilibrium condition under $\mathcal{E}^*$, and hence the non-identification holds. In fact, under the inverse demand function (5), we can construct such transformation T by using (8). Therefore,

14

we can have the following sufficient condition for the non-identification:

Lemma 5. *Given the inverse demand function* (5) *where* $\alpha \neq -\frac{1}{\theta}$, *given a conduct parameter and a marginal cost function, we can construct another pair of a conduct parameter and a marginal cost function that leads to the same equilibrium point for any value of demand shifters and cost shifters. Therefore, non-identification of the conduct parameter and the marginal cost function holds.*

The proof is given in Appendix A. While Lau shows the separable inverse demand function is also a sufficient condition for non-identification, his sufficiency proof is incomplete because it can be verified that his proof implicitly assumes that there exists a transformation T under a separable inverse demand function (2). However, the separability itself does not guarantee the existence of the transformation. In contrast, Lemma 5 shows that the inverse demand function without any demand rotation instrument can lead to a transformation that leads to non-identification.

3.5 Discussion

Rethinking the role of demand rotation instruments: Lemma 4 emphasizes the role of demand rotation instruments. Intuitively, when a change in a demand shifter keeps the equilibrium quantity the same, we have two equilibrium conditions:

$$P(Q, X^d) + \theta Q \frac{\partial P}{\partial Q}(Q, X^d) = MC(Q, X^s)$$

$$P(Q, \tilde{X}^d) + \theta Q \frac{\partial P}{\partial Q}(Q, \tilde{X}^d) = MC(Q, X^s)$$

Here, we have two equations and two unknowns $(\theta, MC(Q, X^s))$, and hence we can solve the system of equation with respect to the conduct parameter and the marginal cost function. Therefore, the existence of a demand shifter that keep the equilibrium quantity the same is confirmed as the sufficient condition for the identification in more general setting.

To build intuition for how the instrument relates to the proof, consider an economy with a linear inverse demand with a demand rotation instrument and a linear marginal cost. Bresnahan (1982) considers this environment. Then, assume that the true conduct is the perfect competition, that is, $\theta = 0$, and the alternative conduct is the monopoly, that is, $\theta^* = 1$. Figure 1 illustrates this environment. Note that the equilibrium quantity under perfect competition is determined by the intersection of the inverse demand function and the marginal cost function MC. The equilibrium quantity under the monopoly is determined by the intersection of the marginal revenue MR^* and the marginal cost function MC^*.

In Figure 1a, the perfect competition and the monopoly lead to the same equilibrium E. Then, consider the change in a demand rotation instrument. In Figure 1b, the demand rotation instrument changes the intercept and slope of the demand function without changing the equilibrium point under the true model. Under the monopoly, the new equilibrium point is E^*, which is different from E, and hence the observational equivalence is violated. To obtain the same equilibrium after the change in the demand
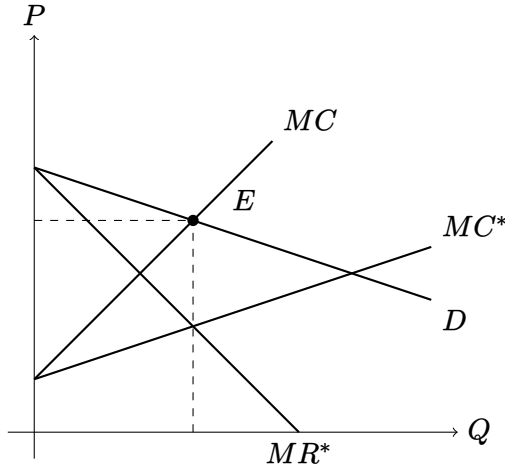
(a) Step 1: Observable equivalence holds.
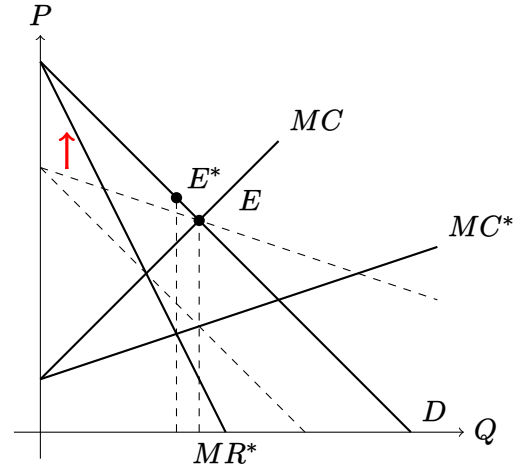
(b) Step 2: Demand rotation changes D and MR^*
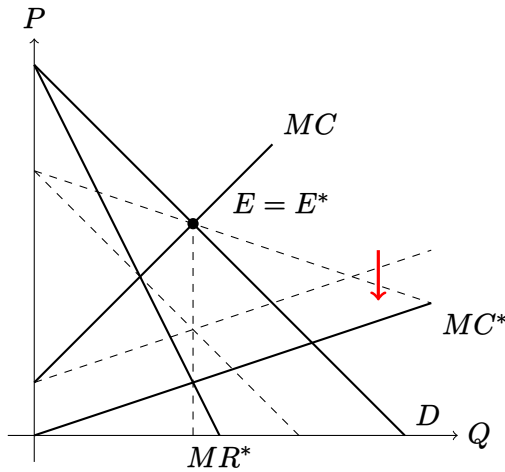
(c) Step 3.1: The intercept of MC^* changes along with the demand rotation to keep $E = E^*$.
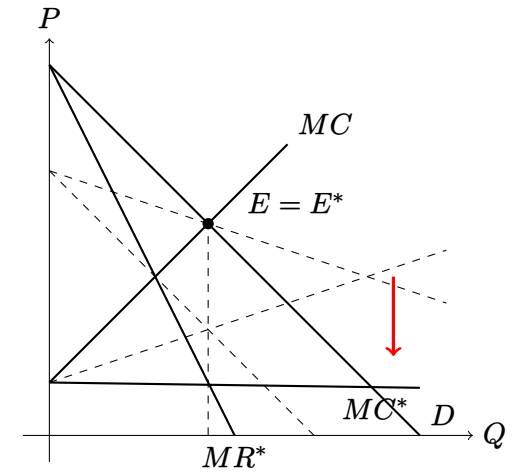
(d) Step 3.2: The slope of MC^* changes along with the demand rotation to keep $E = E^*$.

Figure 1: Intuition of the role of the demand rotation instrument and identification

Note: The figures illustrate the intuition of how the demand rotation instrument works to identify the conduct parameter. MC is the true marginal cost function, and MC^* is the marginal cost function that rationalizes the monopoly conduct. Step 1 illustrates that the monopoly and the perfect competition are observationally equivalent. In step 2, the demand rotation changes the intercept and slope of the demand function without changing the equilibrium point under the perfect competition, but changes the equilibrium point under the monopoly. Step 3 illustrates that to keep the equilibrium point under the monopoly, MC^* should change along with the demand rotation, which is impossible because the marginal cost function is independent of the demand shifter.

rotation instrument ($E = E^*$), MC^* should shift as in Figures 1c and 1d. Note that we changed only the demand shifter, and hence the cost shifter is unchanged. Thus, for the observational equivalence to hold, the shift in MC^* should be derived from the change in the demand rotation instrument, which implies that MC^* is a function of the demand shifter. However, this is impossible because it violates Assumption 1, that is, the demand shifter should not affect the marginal cost function. The transformation in Lemma 2 clearly shows the possibility of the dependence of the marginal cost function on demand shifters, and hence we need to shut down the effect of demand shifters, which leads to the inverse demand function (5).

The technical problem in Lau's proof: To show Claim 1, Lau follows the same logic as in Lemma 1, but he obtains a slightly different equation from (7):

$$\frac{\partial MC^*}{\partial X_j^s}(Q, X^s) = \mu(Q, X^s)\frac{\partial MC}{\partial X_j^s}(Q, X^s), \quad \forall Q, X^s, \tag{13}$$

where $\mu(Q, X^s)$ depends only on Q and X^s. This corresponds to Equation (15) in Lau (1982). Then he concludes that there is a transformation T such that

$$MC^*(Q^e, X^s) = T(MC(Q^e, X^s), Q^e).$$

The existence of the transformation T is shown by integrating (13) with respect to the cost shifter when it is a scalar, and by applying Lemma B.1 from Goldman and Uzawa (1964) when the cost shifter is a vector. Note also that in our approach, both (6) and (7) are necessary to derive the transformation between marginal cost functions. In contrast, while Lau derives both equations (6) and (7) in his proof, he relies solely on (7) to show the non-identification.

While Lau does not explicitly define μ in his formulation, it can be verified that his μ corresponds to our λ. However, he neglects a point that the function μ depends on the demand shifter X^d, and this dependence is not addressed in his analysis. As long as the effect of the demand shifter cannot be eliminated from μ, the transformation T between MC and MC^* could depend on X^d, which violates Assumption 1. Additionally, separability does not resolve the dependence of μ on the demand shifter. Unfortunately, Lau does not provide a justification for why he can assume that the effect of the demand shifter can be removed from μ, and hence it leaves a critical gap in the argument. Additionally, Lau only assumes that the inverse demand functions are twice continuously differentiable, which allows him to apply Lemma B.1 to justify the existence of the transformation T. However, to eliminate the effect of X^d from μ, it is necessary to differentiate μ with respect to X^d and require that the derivative is equal to zero as in our analysis. This step, in turn, requires that the inverse demand function be at least three-times continuously differentiable as in our proof. Therefore, twice-continuous differentiability is not enough to characterize the non-identification.

4 Other Discussions

Relationship with Firm Conduct Test   In recent years, several papers have developed tests for firm conduct (Backus, Conlon and Sinkinson 2021, Duarte, Magnolfi, Sølvsten and Sullivan 2024) that compare two different models and statistically determine which firm conduct is close to the true firm conduct. Dearing et al. (2024) investigate the mechanism how instrument variables can distinguish the true firm conduct by focusing on pass-through. Magnolfi and Sullivan (2022) clarify when firm conduct test outperforms firm conduct estimation. While these papers consider differentiated-product markets, the logic can be applied to homogeneous-product markets, and our result emphasizes that demand rotation instruments are essential in firm conduct test.

The results so far hold whenever we adopt the conduct parameter approach. However, the conduct parameter approach has been criticized on several fronts. In this section, we discuss its micro-foundations and its empirical accuracy.

Micro-foundations of Conduct Parameter Approach   The conduct parameter approach is based on the conjectural variation model. In the conjectural variation model, each firm has a conjecture about how the competitors will react to the firm's action. A major problem of the conjectural variation model is that the conjecture and the reaction function do not coincide at equilibrium. Bresnahan (1981, 1983) propose a Consistent Conjecture Equilibrium (CCE) that requires the conjecture and the reaction function to coincide at equilibrium. However, the existence and uniqueness of the CCE are not guaranteed in general (Klemperer and Meyer 1988, Robson 1983).

Several alternative micro-foundations have been proposed. For example, Escrihuela-Villar (2015) shows that the corporation coefficient approach is equivalent to the conjectural variation model in some specific settings. In this model, each firm maximizes its profit plus the weighted sum of its competitors' profits, where the weight is called the corporation coefficient. Menezes and Quiggin (2020) investigates the supply function approach in which firms choose not a quantity but a supply schedule based on price. Menezes and Quiggin (2023) show the relationship between the conduct parameter approach and the supply function approach in a model with linear demand and linear marginal cost, and Menezes and Quiggin (2020) briefly discuss how to estimate competitiveness given information on marginal cost.

Accuracy of the Conduct Parameter Approach   While this paper focuses only on the identification problem, Corts (1999) argues that the conduct parameter approach can be inaccurate when the data-generating process lies outside the conduct parameter model. For example, while a repeated game model can sustain the equilibrium quantity under joint-profit maximization, the conduct parameter estimate could be biased away from $\theta = 1$. As Corts (1999) and Magnolfi and Sullivan (2022) mention, this criticism can be mitigated by assuming that the data-generating process is a static competition derived from the equilibrium condition in (1).

5 Conclusion

This paper revisited the identification of the conduct parameter in homogeneous product markets. In the literature, Lau considers the identification problem in a fairly generalized setting. While Lau (1982) characterizes the non-identification condition, we point out some problems in Lau's paper and provide a novel characterization of the non-identification condition. Based on the new characterization, we find that the non-identification is equivalent to the absence of demand shifters that work as demand rotation instruments proposed in Bresnahan (1982). We then consider the identification of all primitives in the model and show that the inverse demand function that leads to the non-identification of the conduct parameter and the marginal cost function cannot be identified, which immediately implies that the conduct parameter and the marginal cost function are also not identified. Therefore, while a demand rotation instrument is key for identifying the conduct parameter, identifying the entire model requires an inverse demand function with a demand rotation instrument that is itself identifiable.

References

Aryal, Gaurab, Anirban Chattopadhyaya, and Federico Ciliberto, "Bridging Quasi-Experimental and Structural Approaches for Robust Evaluation of US Airline Mergers," 2025. 2

Backus, Matthew, Christopher Conlon, and Michael Sinkinson, "Common Ownership and Competition in the Ready-to-Eat Cereal Industry," *Working Paper*, 2021. 18

Beringe, Konstantin and Mark Whitmeyer, "Robust Welfare under Imperfect Competition," *Working Paper*, 2025. 2

Bresnahan, Timothy F., "Duopoly Models with Consistent Conjectures," *The American Economic Review*, 1981, *71* (5), 934–945. 18

_ , "The Oligopoly Solution Concept Is Identified," *Economics Letters*, January 1982, *10* (1), 87–92. 1, 2, 6, 7, 9, 13, 15, 19

_ , "Existence of Consistent Conjectures: Reply," *The American Economic Review*, 1983, *73* (3), 457–458. 18

Ciliberto, Federico and Jonathan W. Williams, "Does Multimarket Contact Facilitate Tacit Collusion? Inference on Conduct Parameters in the Airline Industry," *The RAND Journal of Economics*, 2014, *45* (4), 764–791. 2

Corts, Kenneth S., "Conduct Parameters and the Measurement of Market Power," *Journal of Econometrics*, February 1999, *88* (2), 227–250. 18

Dearing, Adam, Lorenzo Magnolfi, Daniel Quint, Christopher J. Sullivan, and Sarah B. Waldfogel, "Learning Firm Conduct: Pass-Through as a Foundation for Instrument Relevance," August 2024. 18

Duarte, Marco, Lorenzo Magnolfi, Mikkel Sølvsten, and Christopher Sullivan, "Testing Firm Conduct," *Quantitative Economics*, 2024, *15* (3), 571–606. 18

Escrihuela-Villar, Marc, "A Note on the Equivalence of the Conjectural Variations Solution and the Coefficient of Cooperation," *The B.E. Journal of Theoretical Economics*, July 2015, *15* (2), 473–480. 18

Genesove, David and Wallace P. Mullin, "Testing Static Oligopoly Models: Conduct and Cost in the Sugar Industry, 1890-1914," *The RAND Journal of Economics*, 1998, *29* (2), 355–377. 2

Goldman, S. M. and H. Uzawa, "A Note on Separability in Demand Analysis," *Econometrica*, 1964, *32* (3), 387–398. 3, 7, 17, 33

Klemperer, Paul D. and Margaret A. Meyer, "Consistent Conjectures Equilibria: A Reformulation Showing Non-Uniqueness," *Economics Letters*, January 1988, *27* (2), 111–115. 18

Lau, Lawrence J., "On Identifying the Degree of Competitiveness from Industry Price and Output Data," *Economics Letters*, January 1982, *10* (1), 93–99. 1, 2, 5, 6, 7, 17, 19

Magnolfi, Lorenzo and Christopher Sullivan, "A Comparison of Testing and Estimation of Firm Conduct," *Economics Letters*, March 2022, *212*, 110316. 18

Matsumura, Yuri and Suguru Otani, "Resolving the Conflict on Conduct Parameter Estimation in Homogeneous Goods Markets between Bresnahan (1982) and Perloff and Shen (2012)," *Economics Letters*, August 2023, *229*, 111193. 13

Menezes, Flavio M. and John Quiggin, "The Strategic Industry Supply Curve," *The Journal of Industrial Economics*, 2020, *68* (3), 523–555. 18

Menezes, Flavio M and John Quiggin, "Competition in Supply Functions and Conjectural Variations: A Unified Solution," *Working Paper*, 2023. 18

Miller, Nathan H. and Matthew C. Weinberg, "Understanding the Price Effects of the MillerCoors Joint Venture," *Econometrica*, 2017, *85* (6), 1763–1791. 2

__ , **Matthew Osborne, and Gloria Sheu**, "Pass-through in a Concentrated Industry: Empirical Evidence and Regulatory Implications," *The RAND Journal of Economics*, March 2017, *48* (1), 69–93. 2

Nocke, Volker and Nicolas Schutz, "Concentration Indices, Welfare Distortions, and Misallocation in Oligopoly," Working Paper March 2025. 2

Okazaki, Tetsuji, Ken Onishi, and Naoki Wakamori, "Excess Capacity and Effectiveness of Policy Interventions: Evidence from the Cement Industry," *International Economic Review*, 2022, *63* (2), 883–915. 2

Porter, Robert H., "A Study of Cartel Stability: The Joint Executive Committee, 1880-1886," *The Bell Journal of Economics*, 1983, *14* (2), 301–314. 2

Robson, Arthur J., "Existence of Consistent Conjectures: Comment," *The American Economic Review*, 1983, *73* (3), 454–456. 18

Sullivan, Christopher, "The Ice Cream Split: Empirically Distinguishing Price and Product Space Collusion," 2020. 2

Weyl, E Glen and Michal Fabinger, "Pass-Through as an Economic Tool: Principles of Incidence under Imperfect Competition," *Journal of Political Economy*, June 2013, *121* (3), 528–583. 2

Appendix

A Omitted Proofs

In this section, we provide the proofs of the lemmas and theorems in the main text. In what follows, we suppress the arguments of functions when no confusion arises.

A.1 Proof of Lemma 1

Assume that given $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$, we can solve the equilibrium condition for the equilibrium quantity Q^e. The non-identification implies that the reduced form quantity $Q^e = h_q(\boldsymbol{X}^d, \boldsymbol{X}^s) = h_q^*(\boldsymbol{X}^d, \boldsymbol{X}^s)$ satisfies

$$P(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) = MC(Q^e, \boldsymbol{X}^s). \tag{14}$$

and

$$P(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) = MC^*(Q^e, \boldsymbol{X}^s), \tag{15}$$

for any $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$. For notational simplicity, we define

$$F(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) \equiv P(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) - MC(Q^e, \boldsymbol{X}^s).$$

Then, by Assumption 4, we can apply the implicit function theorem, and for the reduced-form equation $Q^e = h_q(\boldsymbol{X}^d, \boldsymbol{X}^s)$, the gradient of the reduced-form function h_q with respect to $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$ is given by

$$\nabla h_q(\boldsymbol{X}^d, \boldsymbol{X}^s) = \left(-\frac{\frac{\partial F}{\partial X_i^m}(h_q(\boldsymbol{X}^d, \boldsymbol{X}^s), \boldsymbol{X}^d, \boldsymbol{X}^s)}{\frac{\partial F}{\partial Q}(h_q(\boldsymbol{X}^d, \boldsymbol{X}^s), \boldsymbol{X}^d, \boldsymbol{X}^s)} \right)_{\substack{m=d,s \\ i=1,\ldots,K_m}}. \tag{16}$$

The derivatives of F for each variable are for $i = 1, \ldots, K_d$ and $i = 1, \ldots, K_s$,

$$\frac{\partial F}{\partial X_i^d}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) = \frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d) Q^e,$$

$$\frac{\partial F}{\partial X_i^s}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) = -\frac{\partial MC}{\partial X_i^s}(Q^e, \boldsymbol{X}^s),$$

$$\frac{\partial F}{\partial Q}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) = (1 + \theta) \frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s).$$

Thus, the derivative of h with respect to X_i^d and X_i^s are given as for $i = 1, \ldots, K_d$ and $i = 1, \ldots, K_s$,

$$\frac{\partial h_q}{\partial X_i^d}(\boldsymbol{X}^d, \boldsymbol{X}^s) = -\frac{\frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d)}{(1+\theta)\frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s)},$$

and

$$\frac{\partial h_q}{\partial X_i^s}(\boldsymbol{X}^d, \boldsymbol{X}^s) = \frac{\frac{\partial MC}{\partial X_i^s}(Q^e, \boldsymbol{X}^s)}{(1+\theta)\frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s)}.$$

Note that the same argument can be applied to the equilibrium condition (15) of the alternative model. Recall that the non-identification implies that $Q^e = h_q(\boldsymbol{X}^d, \boldsymbol{X}^s) = h_q^*(\boldsymbol{X}^d, \boldsymbol{X}^s)$ for all $\boldsymbol{X}^d$ and $\boldsymbol{X}^s$, and hence we must have

$$\nabla h_q^*(\boldsymbol{X}^d, \boldsymbol{X}^s) = \nabla h_q(\boldsymbol{X}^d, \boldsymbol{X}^s) \quad \forall \boldsymbol{X}^d, \boldsymbol{X}^s.$$

From (16), this implies that for all Q^e, $\boldsymbol{X}^d$, and $\boldsymbol{X}^s$,

$$\begin{pmatrix} \frac{\partial F}{\partial X_1^d}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta^*, MC^*) \\ \vdots \\ \frac{\partial F}{\partial X_{K_d}^d}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta^*, MC^*) \\ \frac{\partial F}{\partial X_1^s}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta^*, MC^*) \\ \vdots \\ \frac{\partial F}{\partial X_{K_s}^s}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta^*, MC^*) \end{pmatrix} = \lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) \begin{pmatrix} \frac{\partial F}{\partial X_1^d}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) \\ \vdots \\ \frac{\partial F}{\partial X_{K_d}^d}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) \\ \frac{\partial F}{\partial X_1^s}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) \\ \vdots \\ \frac{\partial F}{\partial X_{K_s}^s}(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s; \theta, MC) \end{pmatrix}, \quad (17)$$

where $\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s)$ is defined as

$$\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) \equiv \frac{(1+\theta^*)\frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC^*}{\partial Q}(Q^e, \boldsymbol{X}^s)}{(1+\theta)\frac{\partial P}{\partial Q}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial Q^2}(Q^e, \boldsymbol{X}^d) - \frac{\partial MC}{\partial Q}(Q^e, \boldsymbol{X}^s)}.$$

Assumption 4 guarantees that $\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s)$ is nonzero and finite. From the first K_d rows in (17), (6) and for the later K_s rows, (7) hold.

A.2 Proof of Lemma 2

From (6), we have (we suppress the arguments of the functions for brevity)

$$
\left[\frac{\partial P}{\partial X_i^d} + \theta^* Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}\right]\left[(1+\theta)\frac{\partial P}{\partial Q} + \theta Q^e \frac{\partial^2 P}{\partial Q^2} - \frac{\partial MC}{\partial Q}\right]
$$

$$
= \left[\frac{\partial P}{\partial X_i^d} + \theta Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}\right]\left[(1+\theta^*)\frac{\partial P}{\partial Q} + \theta^* Q^e \frac{\partial^2 P}{\partial Q^2} - \frac{\partial MC^*}{\partial Q}\right]
$$

$$
\frac{\frac{\partial P}{\partial X_i^d} + \theta^* Q \frac{\partial^2 P}{\partial X_i^d \partial Q}}{\frac{\partial P}{\partial X_i^d} + \theta Q \frac{\partial^2 P}{\partial X_i^d \partial Q}}\left[(1+\theta)\frac{\partial P}{\partial Q} + \theta Q^e \frac{\partial^2 P}{\partial Q^2} - \frac{\partial MC}{\partial Q}\right]
$$

$$
= \left[(1+\theta^*)\frac{\partial P}{\partial Q} + \theta^* Q^e \frac{\partial^2 P}{\partial Q^2} - \frac{\partial MC^*}{\partial Q}\right]
$$

$$
\frac{\partial MC^*}{\partial Q} = \underbrace{(1+\theta^*)\frac{\partial P}{\partial Q} + \theta^* Q^e \frac{\partial^2 P}{\partial Q^2} - C_i(Q, \boldsymbol{X}^d)\left[(1+\theta)\frac{\partial P}{\partial Q} + \theta Q^e \frac{\partial^2 P}{\partial Q^2}\right]}_{(*)} + C_i(Q, \boldsymbol{X}^d)\frac{\partial MC}{\partial Q}.
$$

The terms in $(*)$ can be more simplified as

$$
(1+\theta^*)\frac{\partial P}{\partial Q} + \theta^* Q \frac{\partial^2 P}{\partial Q^2} - \frac{\frac{\partial P}{\partial X_i^d} + \theta^* Q \frac{\partial^2 P}{\partial X_i^d \partial Q}}{\frac{\partial P}{\partial X_i^d} + \theta Q \frac{\partial^2 P}{\partial X_i^d \partial Q}}\left[(1+\theta)\frac{\partial P}{\partial Q} + \theta Q \frac{\partial^2 P}{\partial Q^2}\right]
$$

$$
= \frac{1}{\frac{\partial P}{\partial X_i^d} + \theta \frac{\partial^2 P}{\partial X_i^d \partial Q} Q}\left[\left((1+\theta^*)\frac{\partial P}{\partial Q} + \theta^* Q \frac{\partial^2 P}{\partial Q^2}\right)\left(\frac{\partial P}{\partial X_i^d} + \theta Q \frac{\partial^2 P}{\partial X_i^d \partial Q}\right)\right.
$$

$$
\left. - \left((1+\theta)\frac{\partial P}{\partial Q} + \theta Q \frac{\partial^2 P}{\partial Q^2}\right)\left(\frac{\partial P}{\partial X_i^d} + \theta^* Q \frac{\partial^2 P}{\partial X_i^d \partial Q}\right)\right]
$$

$$
= \frac{\theta^* - \theta}{\frac{\partial P}{\partial X_i^d} + \theta \frac{\partial^2 P}{\partial X_i^d \partial Q} Q}\left[\frac{\partial P}{\partial Q}\frac{\partial P}{\partial X_i^d} + Q \frac{\partial P}{\partial X_i^d}\frac{\partial^2 P}{\partial^2 Q} - Q \frac{\partial P}{\partial Q}\frac{\partial^2 P}{\partial X_i^d \partial Q}\right],
$$

which corresponds to $D_i(Q, \boldsymbol{X}^d)$ in (10), and hence we have

$$
\frac{\partial MC^*}{\partial Q}(Q, \boldsymbol{X}^s) = D_i(Q, \boldsymbol{X}^d) + C_i(Q, \boldsymbol{X}^d)\frac{\partial MC}{\partial Q}(Q, \boldsymbol{X}^s).
$$

Next, (6) and (7) imply that for $i = 1, \ldots, K_d$ and $j = 1, \ldots, K_s$,

$$
\lambda(Q^e, \boldsymbol{X}^d, \boldsymbol{X}^s) = \frac{\frac{\partial MC^*}{\partial X_j^s}(Q^e, \boldsymbol{X}^s)}{\frac{\partial MC}{\partial X_j^s}(Q^e, \boldsymbol{X}^s)} = \frac{\frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta^* Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d)}{\frac{\partial P}{\partial X_i^d}(Q^e, \boldsymbol{X}^d) + \theta Q^e \frac{\partial^2 P}{\partial X_i^d \partial Q}(Q^e, \boldsymbol{X}^d)} \equiv C_i(Q, \boldsymbol{X}^d),
$$

Then, we have

$$
\frac{\partial MC^*}{\partial X_j^s}(Q, \boldsymbol{X}^s) = C_i(Q, \boldsymbol{X}^d)\frac{\partial MC}{\partial X_j^s}(Q, \boldsymbol{X}^s).
$$

A.3 Proof of Lemma 3

Let $u_i(Q, \boldsymbol{X}^d) \equiv \frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d)$. Due to Assumption 2, we have that $u_i(Q, \boldsymbol{X}^d) \neq 0$ for all Q and $\boldsymbol{X}^d$. Note that when $\theta = 0$, (11) implies that $u_i(Q, \boldsymbol{X}^d) = 0$ for all $i \in \mathcal{I}$, which cannot be held because of Assumption 2. Therefore, $\theta \neq 0$ to have (11). Additionally, (11) is reduced to $u_i(0, \boldsymbol{X}^d) = 0$ when $Q = 0$. Again, this cannot hold because of Assumption 2. Thus, we characterize the inverse demand function where (11) holds only on $Q > 0$ and $\theta \neq 0$.

Rewrite (11) for $i \in \mathcal{I}$ as

$$\frac{\frac{\partial u_i}{\partial Q}(Q, \boldsymbol{X}^d)}{u_i(Q, \boldsymbol{X}^d)} = -\frac{1}{\theta Q} \implies \frac{\partial}{\partial Q} \log |u_i(Q, \boldsymbol{X}^d)| = -\frac{1}{\theta Q}.$$

By integrating both sides with respect to Q, we have

$$\log |u_i(Q, \boldsymbol{X}^d)| = -\frac{1}{\theta} \log Q + R_i(\boldsymbol{X}^d),$$

where $R_i(\boldsymbol{X}^d)$ is a function of $\boldsymbol{X}^d$. By taking the exponential of both sides, we have

$$|u_i(Q, \boldsymbol{X}^d)| = Q^{-\frac{1}{\theta}} r_i(\boldsymbol{X}^d),$$

where $r_i(\boldsymbol{X}^d) = \exp(R_i(\boldsymbol{X}^d))$. Note that we can remove the absolute value for Q because it can be assumed that $Q \geq 0$. By removing the absolute value operator for u_i, we have two solutions

$$u_i(Q, \boldsymbol{X}^d) = \pm Q^{-\frac{1}{\theta}} r_i(\boldsymbol{X}^d).$$

Because r_i is an arbitrary function, we can unify these solutions and can simply put the solution as

$$u_i(Q, \boldsymbol{X}^d) = Q^{-\frac{1}{\theta}} r_i(\boldsymbol{X}^d).$$

This implies that the derivative of P with respect to X_i^d is a separable function of Q and $\boldsymbol{X}^d$. Hence, it is natural to think that $r_i(\boldsymbol{X}^d)$ is a derivative of a function of $\boldsymbol{X}^d$ with respect to X_i^d. In other words, there is a function $r(\boldsymbol{X}^d)$ of $\boldsymbol{X}^d$ such that $r_i(\boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}$ holds. Therefore, we have for $i \in \mathcal{I}$,

$$u_i(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d} Q^{-\frac{1}{\theta}}.$$

By integrating both sides with respect to X_i^d, we have

$$P(Q, \boldsymbol{X}^d) = Q^{-\frac{1}{\theta}} r(\boldsymbol{X}^d) + s_i(Q, \boldsymbol{X}_{-i}^d), \quad i \in \mathcal{I}, \tag{18}$$

where $s_i(Q, \boldsymbol{X}_{-i}^d)$ is a function of Q and $\boldsymbol{X}_{-i}^d$ where $\boldsymbol{X}_{-i}^d$ is the vector of $\boldsymbol{X}^d$ excluding X_i^d. To meet

Assumption 3, we must have that $r(\boldsymbol{X}^d)$ and $s_i(Q, \boldsymbol{X}^d_{-i})$ are at least twice-continuously differentiable.

When $\boldsymbol{X}^d$ is a scalar or when $\boldsymbol{X}^d$ is a vector but $|\mathcal{I}| \geq K_d - 1$, there is no argument in $\boldsymbol{X}^d_{-i}$, and hence we can remove the index of i from the function s_i. Therefore, $s(Q, \boldsymbol{X}^d_{-i}) = s(Q)$ holds, and hence we have (12). When $\boldsymbol{X}^d$ is a vector and $K_d - 2 \geq |\mathcal{I}| \geq 2$, we can remove the index i from s_i in the following way. Pick up any i and j in $\mathcal{I}$ such that $i \neq j$, and then the derivative of (18) with respect to X^d_j is given by

$$\frac{\partial P}{\partial X^d_j}(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X^d_j} Q^{-\frac{1}{\theta}} + \frac{\partial s_i(Q, \boldsymbol{X}^d_{-i})}{\partial X^d_j},$$

and

$$\frac{\partial P}{\partial X^d_j}(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X^d_j} Q^{-\frac{1}{\theta}}.$$

Since s_j is a function of the demand shifters without X_j, we do not have the term relating to s_j in the second equation. Therefore, by comparing the above two equations, we can conclude that the second term in the first equation vanishes:

$$\frac{\partial s_i(Q, \boldsymbol{X}^d_{-i})}{\partial X^d_j} = 0,$$

which implies that $s_i(Q, \boldsymbol{X}^d_{-i})$ is independent of X^d_j. By applying the same argument to all $i \in \mathcal{I}$, we can show that $s_i(Q, \boldsymbol{X}^d_{-i})$ is independent of the demand shifters whose indices are in $\mathcal{I}$. That is, we can reduce $s_i(Q, \boldsymbol{X}^d_{-i})$ to $s_i(Q, \boldsymbol{X}^d_{-\mathcal{I}})$ where $\boldsymbol{X}^d_{-\mathcal{I}}$ is the vector of $\boldsymbol{X}^d$ excluding the demand shifters whose indices are in $\mathcal{I}$. Then, by comparing (18) for all $i \in \mathcal{I}$, we have

$$s_i(Q, \boldsymbol{X}^d_{-\mathcal{I}}) = s_j(Q, \boldsymbol{X}^d_{-\mathcal{I}}),$$

for all $i, j \in \mathcal{I}$. This leads to the symmetry of s_i for all $i \in \mathcal{I}$, which implies that $s(Q, \boldsymbol{X}^d_{-\mathcal{I}}) = s_i(Q, \boldsymbol{X}^d_{-\mathcal{I}})$ for all $i \in \mathcal{I}$. Therefore, we can write the inverse demand function as

$$P(Q, \boldsymbol{X}^d) = Q^{-\frac{1}{\theta}} r(\boldsymbol{X}^d) + s(Q, \boldsymbol{X}^d_{-\mathcal{I}}).$$

When $|\mathcal{I}| = 1$, we have $\boldsymbol{X}^d_{-\mathcal{I}} = \boldsymbol{X}^d_{-i}$, and hence we can keep the index for s_i, but it is also fine to write it as $s(Q, \boldsymbol{X}^d_{-i}) = s_i(Q, \boldsymbol{X}^d_{-i})$.

Note that when $Q = 0$, we have that

$$u_i(0, \boldsymbol{X}^d) = 0^{-\frac{1}{\theta}} r_i(\boldsymbol{X}^d) = \infty,$$

because $-\frac{1}{\theta} < 0$. This is consistent with the argument that (11) does not hold at $Q = 0$. Thus, we have

characterized the inverse demand function where (11) holds.

A.4 Proof of Lemma 4

Because $C_i(Q, \boldsymbol{X}^d)$ and $D_i(Q, \boldsymbol{X}^d)$ should be independent of $\boldsymbol{X}^d$ simultaneously, we can first characterize the class of inverse demand functions that make $C_i(Q, \boldsymbol{X}^d)$ independent of $\boldsymbol{X}^d$. Then, we characterize the class of inverse demand functions that make $D_i(Q, \boldsymbol{X}^d)$ independent of $\boldsymbol{X}^d$ by substituting the derived inverse demand function into $D_i(Q, \boldsymbol{X}^d)$.

Step 1: Necessity for C_i is independent of $\boldsymbol{X}^d$

When $C_i(Q, \boldsymbol{X}^d)$ is independent of $\boldsymbol{X}^d$, the derivative of $C_i(Q, \boldsymbol{X}^d)$ with respect to $\boldsymbol{X}^d$ is zero. Let $u_i(Q, \boldsymbol{X}^d) \equiv \frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d)$. Due to Assumption 2, we have that $u_i(Q, \boldsymbol{X}^d) \neq 0$ for all Q and $\boldsymbol{X}^d$. Then, the derivative of C_i with respect to X_j^d for $j = 1, \ldots, K_d$ is given by

$$\frac{\partial C_i}{\partial X_j^d}(Q, \boldsymbol{X}^d) = \frac{(\theta^* - \theta)Q}{\left(u_i + \theta Q \frac{\partial u_i}{\partial Q}\right)^2} \left[\frac{\partial u_i}{\partial X_j^d \partial Q} u_i - \frac{\partial u_i}{\partial Q} \frac{\partial u_i}{\partial X_j^d}\right].$$

Note that because (11) does not hold, the denominator is not zero. As $\theta \neq \theta^*$, the derivative becomes zero when (1) $Q = 0$ or (2) the term in the bracket is zero. In the first case, for arbitrary inverse demand function, C_i is independent of $\boldsymbol{X}^d$. Therefore, we focus on the second case for $Q > 0$. Note that the bracket term is a partial differential equation:

$$\frac{\partial u_i}{\partial X_j^d \partial Q} u_i - \frac{\partial u_i}{\partial Q} \frac{\partial u_i}{\partial X_j^d} = 0. \tag{19}$$

Now, instead of using (19) directly, we use the following relationship:

$$\frac{\partial^2}{\partial X_j^d \partial Q} \log |u_i(Q, \boldsymbol{X}^d)| = \frac{\partial}{\partial Q}\left(\frac{1}{u_i}\frac{\partial u_i}{\partial X_j^d}\right) = \frac{1}{u_i^2}\left(u_i \frac{\partial^2 u_i}{\partial X_j^d \partial Q} - \frac{\partial u_i}{\partial X_j^d}\frac{\partial u_i}{\partial Q}\right).$$

Note that the assumption $u_i \neq 0$ implies that the inside of the log function is not zero and the denominator in the last equality is not zero. Therefore, to check the independence of C_i with respect to $\boldsymbol{X}^d$, it is sufficient to solve the following partial differential equation such that

$$\frac{\partial^2}{\partial X_j^d \partial Q} \log |u_i(Q, \boldsymbol{X}^d)| = 0, \quad i, j = 1, \ldots, K_d.$$

This implies that $\frac{\partial}{\partial Q} \log |u_i(Q, \boldsymbol{X}^d)|$ is independent of any element in $\boldsymbol{X}^d$, and hence we have a function $G(Q)$ such that

$$\frac{\partial}{\partial Q} \log |u_i(Q, \boldsymbol{X}^d)| = G(Q).$$

By integrating both sides with respect to Q, we have for $i = 1, \ldots, K_d$,

$$\log|u_i(Q, \boldsymbol{X}^d)| = \int G(Q)dQ + R_i(\boldsymbol{X}^d),$$

where the last term is a function that is an analogue of the constant of integration. By taking exponential on both sides, we have for $i = 1, \ldots, K_d$,

$$|u_i(Q, \boldsymbol{X}^d)| = g(Q)r_i(\boldsymbol{X}^d)$$

where $r_i(\boldsymbol{X}^d) \equiv \exp(R_i(\boldsymbol{X}^d))$ and $g(Q) = \exp\left(\int G(Q)dQ\right)$. This has two solutions, but again, because r_i is an arbitrary function, we can unify these solutions and put the solution as

$$u_i(Q, \boldsymbol{X}^d) = g(Q)r_i(\boldsymbol{X}^d).$$

Because u_i is a separable function of Q and $\boldsymbol{X}^d$, it is natural to think that $r_i(\boldsymbol{X}^d)$ is a derivative of a function of $\boldsymbol{X}^d$ with respect to X_i^d. In other words, there is a function $r(\boldsymbol{X}^d)$ of $\boldsymbol{X}^d$ such that $r_i(\boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}$ holds. Therefore, we have

$$u_i(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}g(Q).$$

Integrating both sides with respect to X_i^d, we have

$$P(Q, \boldsymbol{X}^d) = g(Q)r(\boldsymbol{X}^d) + s_i(Q, \boldsymbol{X}^d_{-i}), \quad i = 1, \ldots, K_d, \tag{20}$$

where $s_i(Q, \boldsymbol{X}^d_{-i})$ is an arbitrary function of Q and $\boldsymbol{X}^d_{-i}$ where $\boldsymbol{X}^d_{-i}$ is the vector of $\boldsymbol{X}^d$ excluding X_i^d. To meet Assumption 3, we must have that $R(\boldsymbol{X}^d)$ and $s_i(Q, \boldsymbol{X}^d_{-i})$ are at least twice-continuously differentiable and $g(Q)$ is at least continuously differentiable.

We can remove the index of s_i in the following way. Pick up any i, and then the derivative of (20) with respect to X_i^d is given by

$$\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}g(Q) + \frac{\partial s_i(Q, \boldsymbol{X}^d_{-i})}{\partial X_i^d} = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}g(Q),$$

and for any other $j \neq i$, the same derivative is given by

$$\frac{\partial P}{\partial X_i^d}(Q, \boldsymbol{X}^d) = \frac{\partial r(\boldsymbol{X}^d)}{\partial X_i^d}g(Q) + \frac{\partial s_j(Q, \boldsymbol{X}^d_{-j})}{\partial X_i^d}.$$

These imply that

$$\frac{\partial s_j(Q, \boldsymbol{X}_{-j}^d)}{\partial X_i^d} = 0 \quad \text{for all } j \neq i.$$

Therefore, $s_j(Q, \boldsymbol{X}_{-j}^d)$ is independent of X_i^d, and hence s_j is independent of X_j^d and X_i^d. By applying the same argument to all $i = 1, \ldots, K_d$, we can show that $s_i(Q, \boldsymbol{X}_{-i}^d)$ is independent of all elements in $\boldsymbol{X}_{-i}^d$, that is, $s_i(Q, \boldsymbol{X}_{-i}^d) = s_i(Q)$ for all $i = 1, \ldots, K_d$.

By comparing (20) for all $i = 1, \ldots, K_d$, it is easy to see that $s_i(Q) = s_j(Q)$ for all $i, j = 1, \ldots, K_d$. Therefore, we have a symmetry of s_i for all $i = 1, \ldots, K_d$, that is, $s_i(Q) = s(Q)$ for all $i = 1, \ldots, K_d$. Thus, $P(Q, \boldsymbol{X}^d)$ must be of the form

$$P(Q, \boldsymbol{X}^d) = g(Q)r(\boldsymbol{X}^d) + s(Q). \tag{21}$$

Note that when $g(Q) = 0$ for any $Q > 0$, the inverse demand function depends only on the aggregate quantity, which violate Assumption 2. Therefore, we assume that $g(Q) \neq 0$ for all $Q > 0$.

Step 2: Necessity for D_i is independent of $\boldsymbol{X}^d$

Next, given the inverse demand function (21), we further specify the form of the inverse demand function based on $D_i(Q, \boldsymbol{X}^d)$. By substituting (21) into (10), we have

$$
\begin{aligned}
D_i(Q, \boldsymbol{X}^d) = & \frac{\theta^* - \theta}{r_i(\boldsymbol{X}^d)(g(Q) + \theta Q g'(Q))} \Big[(g'(Q)r(\boldsymbol{X}^d) + s'(Q))g(Q)r_i(\boldsymbol{X}^d) \\
& + Q(g''(Q)r(\boldsymbol{X}^d) + s''(Q))g(Q)r_i(\boldsymbol{X}^d) \\
& - Q(g'(Q)r(\boldsymbol{X}^d) + s'(Q))g'(Q)r_i(\boldsymbol{X}^d) \Big] \\
= & \frac{\theta^* - \theta}{g(Q) + \theta Q g'(Q)} \Big[r(\boldsymbol{X}^d)[g'(Q)g(Q) + Q g''(Q)g(Q) - Q(g'(Q))^2] \\
& + s'(Q)g(Q) + Q s''(Q)g(Q) - Q s'(Q)g'(Q) \Big].
\end{aligned}
$$

The dependence of $D_i(Q, \boldsymbol{X}^d)$ on $\boldsymbol{X}^d$ comes only from the first term in the bracket. Therefore, $D_i(Q, \boldsymbol{X}^d)$ is independent of $\boldsymbol{X}^d$ if and only if

$$\frac{\partial D_i}{\partial X_j^d}(Q, \boldsymbol{X}^d) = \frac{(\theta^* - \theta)r_j(\boldsymbol{X}^d)}{g(Q) + \theta Q g'(Q)}[g'(Q)g(Q) + Q g''(Q)g(Q) - Q(g'(Q))^2] = 0.$$

As Assumption 2 implies that $r_j(\boldsymbol{X}^d)$ is nonzero, we must have the terms in the bracket to be zero:

$$g'(Q)g(Q) + Q g''(Q)g(Q) - Q(g'(Q))^2 = 0. \tag{22}$$

As $g(Q) \neq 0$ for $Q > 0$, let $v(Q) = \frac{g'(Q)}{g(Q)}$. Because $v'(Q) = \frac{g''(Q)g(Q)-(g'(Q))^2}{g(Q)^2}$, dividing (22) by $g(Q)^2$ gives

$$v(Q) + Qv'(Q) = 0. \tag{23}$$

This is a first-order linear differential equation. To solve this differential equation, we consider two cases.

Case 1: $v(Q) = 0$ **for all** $Q > 0$**.** This happens when $g(Q)$ is a constant. In this case, (22) holds immediately, and hence (23) also holds. Let $g(Q) = C$ for some nonzero constant C. Then, we have

$$P(Q, \boldsymbol{X}^d) = Cr(\boldsymbol{X}^d) + s(Q).$$

For simplicity, we can absorb the constant C into $r(\boldsymbol{X}^d)$ and $s(Q)$, and hence we have

$$P(Q, \boldsymbol{X}^d) = r(\boldsymbol{X}^d) + s(Q). \tag{24}$$

Case 2: $v(Q) \neq 0$ **for all** $Q > 0$**.** In this case, (23) implies that

$$\frac{v'(Q)}{v(Q)} = -\frac{1}{Q}.$$

Then, it can be written as

$$\frac{d}{dQ} \log |v(Q)| = -\frac{d}{dQ} \log Q.$$

Integrating both sides with respect to Q, we have

$$\log |v(Q)| = -\log Q + a_1,$$

where $a_1 \in \mathbb{R}$ is a constant. By taking the exponential of both sides, we have

$$|v(Q)| = \frac{\alpha}{Q},$$

where $\alpha = \exp(a_1)$ is a positive constant. This has two solutions

$$v(Q) = \frac{g'(Q)}{g(Q)} = \pm\frac{\alpha}{Q}.$$

Since $\alpha > 0$, we redefine the constant as $\alpha \in \mathbb{R} \backslash \{0\}$ to absorb the sign into α. Then, we have

$$\frac{d}{dQ} \log |g(Q)| = \alpha \frac{d}{dQ} \log Q.$$

Again, by integrating both sides with respect to Q, we have

$$\log|g(Q)| = \alpha \log Q + a_2,$$

where $a_2 \in \mathbb{R}$ is a constant. By taking the exponential of both sides, we have

$$|g(Q)| = \alpha_2 Q^\alpha,$$

where $\alpha_2 = \exp(a_2)$ is a positive constant. Again, this has two solutions

$$g(Q) = \pm \alpha_2 Q^\alpha.$$

Since $\alpha_2 > 0$, we can define $\alpha_2 \in \mathbb{R}\backslash\{0\}$ to absorb the sign into α_2. Therefore, when C_i and D_i are independent of $\boldsymbol{X}^d$, the inverse demand function must be of the form

$$P(Q, \boldsymbol{X}^d) = Q^\alpha r(\boldsymbol{X}^d) + s(Q). \tag{25}$$

Here, α_2 is absorbed into r.

While α is nonzero in (25), by allowing that $\alpha = 0$, it can include (24) as a special case. At the same time, recall that we require that (11) does not hold for any i. However, when $\alpha = -\frac{1}{\theta}$, we have the inverse demand function is equal to (12) where $\mathcal{I} = \emptyset$, which implies that the conduct parameter and the marginal cost function are identified. Therefore, we should have $\alpha \neq -\frac{1}{\theta}$ to ensure the non-identification of the conduct parameter.

A.5 Proof of Lemma 5

From the proof of Lemma 4, we know that C_i and D_i are independent of $\boldsymbol{X}^d$ under (5). Under the inverse demand function (21), we have

$$\hat{C}_i(Q, \boldsymbol{X}^d) = \frac{Q^\alpha r_i(\boldsymbol{X}^d) + \theta^* \alpha Q^\alpha r_i(\boldsymbol{X}^d)}{Q^\alpha r_i(\boldsymbol{X}^d) + \theta \alpha Q^\alpha r_i(\boldsymbol{X}^d)} = \frac{1 + \theta^* \alpha}{1 + \theta \alpha}.$$

and

$$\begin{aligned}
\hat{D}_i(Q, \boldsymbol{X}^d) &= \frac{\theta^* - \theta}{Q^\alpha + \theta \alpha Q^\alpha} \left[s'(Q)Q^\alpha + Qs''(Q)Q^\alpha - s'(Q)\alpha Q^\alpha \right] \\
&= \frac{\theta^* - \theta}{1 + \theta \alpha} \left[\frac{d}{dQ} Qs'(Q) - \alpha s'(Q) \right].
\end{aligned}$$

Then, consider a transformation of the derivative of a function $f(Q, X)$ with respect to Q based on

(8) such that

$$
\begin{aligned}
T_Q\left(f(Q,X),Q\right) &\equiv \hat{D}_i(Q,\boldsymbol{X}^d) + \hat{C}_i(Q,\boldsymbol{X}^d)\frac{\partial f}{\partial Q}(Q,X) \\
&= \frac{\theta^* - \theta}{1 + \theta\alpha}\left[\frac{d}{dQ}Qs'(Q) - \alpha s'(Q)\right] + \frac{1 + \theta^*\alpha}{1 + \theta\alpha}\frac{\partial f}{\partial Q}(Q,X).
\end{aligned}
$$

This can be integrated with respect to Q, and hence we obtain a transformation of $f(Q,X)$ as

$$
T\left(f(Q,X),Q\right) \equiv \frac{\theta^* - \theta}{1 + \theta\alpha}\left[Qs'(Q) - \alpha s(Q)\right] + \frac{1 + \theta^*\alpha}{1 + \theta\alpha}f(Q,X).
$$

Note that we assume the integral constant is zero for simplicity.

Define a marginal cost function MC^* as

$$
MC^*(Q,\boldsymbol{X}^s) \equiv \frac{\theta^* - \theta}{1 + \theta\alpha}\left[Qs'(Q) - \alpha s(Q)\right] + \frac{1 + \theta^*\alpha}{1 + \theta\alpha}MC(Q,\boldsymbol{X}^s).
$$

Then, by substituting the marginal revenue under θ into T, we can obtain the marginal revenue under θ^*:

$$
\begin{aligned}
&\frac{\theta^* - \theta}{1 + \theta\alpha}\left[Qs'(Q) - \alpha s(Q)\right] + \frac{1 + \theta^*\alpha}{1 + \theta\alpha}\left[(1 + \theta\alpha)Q^\alpha r(\boldsymbol{X}^d) + s(Q) + \theta Qs'(Q)\right] \\
=&(1 + \theta^*\alpha)Q^\alpha r(\boldsymbol{X}^d) + \frac{(\theta^* - \theta + (1 + \theta^*\alpha)\theta)Qs'(Q) + (1 + \theta^*\alpha - (\theta^* - \theta)\alpha)s(Q)}{1 + \theta\alpha} \\
=&(1 + \theta^*\alpha)Q^\alpha r(\boldsymbol{X}^d) + \frac{(1 + \theta\alpha)s(Q) + \theta^*(1 + \theta\alpha)Qs'(Q)}{1 + \theta\alpha} \\
=&(1 + \theta^*\alpha)Q^\alpha r(\boldsymbol{X}^d) + s(Q) + \theta^* Qs'(Q).
\end{aligned}
$$

Then, suppose that Q^e satisfies the equilibrium condition under (θ, MC). Then, by using the definition of MC^* and the above observation on the marginal revenue, we can confirm that Q^e also satisfies the equilibrium condition under (θ^*, MC^*):

$$
\begin{aligned}
&P(Q^e,\boldsymbol{X}^d) + \theta^* Q^e\frac{\partial P}{\partial Q}(Q^e,\boldsymbol{X}^d) - MC^*(Q^e,\boldsymbol{X}^s) \\
=&\frac{\theta^* - \theta}{1 + \theta\alpha}\left[Qs'(Q) - \alpha s(Q)\right] + \frac{1 + \theta^*\alpha}{1 + \theta\alpha}\left[(1 + \theta\alpha)Q^\alpha r(\boldsymbol{X}^d) + s(Q) + \theta Qs'(Q)\right] \\
&\quad - \frac{\theta^* - \theta}{1 + \theta\alpha}\left[Qs'(Q) - \alpha s(Q)\right] - \frac{1 + \theta^*\alpha}{1 + \theta\alpha}MC(Q^e,\boldsymbol{X}^s) \\
=&\frac{1 + \theta^*\alpha}{1 + \theta\alpha}\left[(1 + \theta\alpha)Q^\alpha r(\boldsymbol{X}^d) + s(Q) + \theta^* Qs'(Q) - MC(Q^e,\boldsymbol{X}^s)\right] = 0.
\end{aligned}
$$

The last equality holds because Q^e satisfies the equilibrium condition under (θ, MC). Therefore, the two models are observationally equivalent, which implies that the non-identification holds.

B Summary of Goldman and Uzawa (1964)

Let n be the number of variables and $x = (x_1, \ldots, x_n)$ be a vector of n variables. Consider a partition of X into K parts, $\{x^1, \ldots, x^K\}$ such that $X = \bigcup_{k=1}^{K} x^k$ and $x^k \cap x^l = \emptyset$ for $k \neq l$.

Definition B.1. *A function is weakly separable with respect to the partition if*

$$\frac{\partial}{\partial x_l} \left(\frac{\frac{\partial f}{\partial x_i}(x^1, \ldots, x^K)}{\frac{\partial f}{\partial x_j}(x^1, \ldots, x^K)} \right) = 0, \quad i, j \in x^k, l \notin x^k.$$

This implies that the ratio of the derivative with respect to x_i and x_j, which are in the same category, is not affected by the change in the variables in other partitions. Intuitively, by taking the ratio, the component of f relating to x_l is canceled out, and hence the derivative of the ratio with respect to x_l becomes zero. When f is a utility function, this implies that the marginal rate of substitution between commodity i and j in the same partition is independent of the quantities of commodities outside x^k.

Then, Goldman and Uzawa (1964) specifies the functional form that a weak separable function should satisfy.

Theorem B.1. *A function $f(x)$ is weakly separable with respect to a partition $\{x^1, ..., x^K\}$ if and only if $f(x)$ is of the form:*

$$f(X) = \Phi(r^1(x^1), \ldots, r^K(x^K))$$

where $\Phi(r^1, \ldots, r^K)$ is a function of K variables and, for each k, $r^k(x^k)$ is a function of subvector x^k alone.

The next lemma is a key lemma in Lau's proof.

Lemma B.1. *Let $f(x)$ and $g(x)$ be two continuously twice-differentiable functions of n variables $x = (x_1, \ldots, x_n)$. If each indifference surface is connected, and if there exists a function $\lambda(x)$ such that*

$$\frac{\partial f}{\partial x_i}(x) = \lambda(x) \frac{\partial g}{\partial x_i}(x), \quad i = 1, \ldots, n, \quad \text{for all } x, \tag{26}$$

then $f(x)$ is a transformation of $g(x)$; namely, there exists a function T of one variable such that

$$f(x) = T(g(x)) \quad \text{for all } x.$$

Hence, in particular, the function $\lambda(x)$ satisfying (26) must be of the form:

$$\lambda(x) = \Lambda(g(x)) \quad \text{for all } x,$$

with some function Λ of one variable.

If there is a function T such that $f(x) = T(g(x))$, then by the chain rule, we have $\frac{\partial f}{\partial x_i}(x) = T'(g(x))\frac{\partial g}{\partial x_i}(x)$ where T' is the derivative of T. Thus, by defining $\lambda(x) = T'(g(x))$, we have $\frac{\partial f}{\partial x_i}(x) = \lambda(x)\frac{\partial g}{\partial x_i}(x)$. Intuitively, the lemma implies that the converse of the chain rule holds under additional conditions.