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Endowment manipulations involving population variations in object exchange problems

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Abstract

This study examines the object exchange problem introduced by Shapley and Scarf (1974). We focus on two properties of allocation rules that require robustness to endowment manipulations involving population variations: *withdrawal-proofness* and *pre-delivery-proofness* (Thomson, 2014). We first show that no rule satisfies *individual rationality* and *withdrawal-proofness*. This impossibility result holds not only on the strict preference domain but also on well-studied restricted domains. However, this negative finding can be avoided by weakening *withdrawal-proofness*. We characterize the Top Trading Cycles rule (TTC) using *individual rationality*, *strategy-proofness*, and *weak withdrawal-proofness* under a richness condition on the domain. In contrast to *withdrawal-proofness*, several *individually rational* rules satisfy *pre-delivery-proofness*. Furthermore, a stronger version of *pre-delivery-proofness*, combined with *individual rationality*, uniquely characterizes TTC. Notably, this characterization holds on many natural restricted domains.

Keywords: top trading cycles rule; strategy-proofness; withdrawal-proofness; pre-delivery-proofness; housing market.

JEL codes: C78; D47.

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1 Introduction

We consider the object exchange problem introduced by Shapley and Scarf (1974), where each agent initially owns one indivisible object and has strict preferences over all objects. A "rule" maps each economy (which consists of a set of agents, a set of objects, their preferences, and individual endowments) to a feasible allocation that assigns exactly one object to each agent.

We study the immunity of a given rule to manipulation via endowments. Various forms of such manipulation have been explored in different models.[1] This paper focuses on two types of pairwise endowment manipulations involving changes in population. The first is "withdrawal": an agent withdraws with his endowment before the rule is applied; after the rule is applied without him, the withdrawing agent and one of the agents who participated exchange the first agent's endowment and the object allocated to the second agent in such a way that both agents are at least as well off as they would have been without the manipulation, and at least one is strictly better off. *Withdrawal-proofness*, introduced by Thomson (2014), requires that such manipulation be impossible. We derive an impossibility result: no rule satisfies both *individual rationality* (no agent is made strictly worse off after the reallocation) and *withdrawal-proofness* (Theorem 1). It is important to note that this impossibility result holds even when the preference domain is restricted to well-studied domains, such as the single-peaked (Bade, 2019), single-dipped (Tamura, 2023), and common-ranking preference (Nicolò and Rodríguez-Álvarez, 2017) domains.

To address this impossibility result, we weaken *withdrawal-proofness*. *Weak withdrawal-proofness* rules out any withdrawal manipulation that makes the withdrawing agent and one of the remaining participants strictly better off. Several rules, including the Top Trading Cycles rule (TTC), satisfy both *individual rationality* and *weak withdrawal-proofness* (Proposition 1 and Example 3). Moreover, these two axioms, together with *strategy-proofness* (no agent can benefit from misrepresenting his preferences), characterize TTC under a richness condition on the domain of preferences (Theorem 2).

The second manipulation is "pre-delivery": before the rule is applied, an agent pre-delivers to another agent the object that the second agent would receive

[1] Examples include destruction (Aumann and Peleg, 1974; Gale, 1974), withholding and hiding (Postlewaite, 1979), transfer (Gale, 1974), and augmentation of endowments (Thomson, 2024b). See Thomson (2023) for a comprehensive survey on endowment manipulations and the normative principles underlying these concepts.

if all agents participated, inducing that agent to withdraw; once the rule is applied without the agent who withdrew, the first agent ends up with a strictly preferred object compared to what he would have received otherwise. *Pre-delivery-proofness*, introduced by Thomson (2014), precludes this behavior. This property is relatively weak here, since it applies only when the pre-delivering agent initially owns the object that the withdrawing agent would receive under full participation. In fact, not only TTC but also many other rules satisfy this property (Proposition 2 and Example 4).

The incentive to withdraw may persist (or even intensify) when the withdrawing agent is pre-delivered an object that he finds at least as desirable as the one he would receive under full participation. We therefore introduce a stronger concept, *strict pre-delivery-proofness*, which eliminates pre-delivery manipulation where the withdrawing agent is weakly better off and the pre-delivering agent is strictly better off. TTC is characterized by *individual rationality* and *strict pre-delivery-proofness* (Theorem 3). Notably, this characterization holds on several natural restricted domains, including the single-peaked and single-dipped preference domains.

Our main axioms are relevant to kidney exchange, a real-world application of our model (Roth, Sönmez, and Ünver, 2004). Consider a hospital managing multiple patient–donor pairs. The hospital may strategically withdraw a pair from the centralized exchange pool and, after the centralized mechanism is executed, arrange an internal exchange between the withdrawn pair and another pair matched through the mechanism.[2] A similar concern arises with *pre-delivery-proofness*: two patients might exploit legal loopholes (e.g., fake marriage or adoption) to coordinate a pre-delivery scheme involving withdrawal, thereby obtaining higher-quality kidneys. Our results provide new insights into the design of mechanisms that are robust against such manipulations.

Related literature Both *withdrawal-proofness* and *pre-delivery-proofness* were first introduced by Thomson (2014) in the context of pure exchange economies. In that context, the Walrasian rule satisfies *pre-delivery-proofness* but violates *withdrawal-proofness*, whereas certain egalitarian and dictatorial rules violate both properties (Thomson, 2014, 2024a). These two axioms have also been applied to the

[2] Ashlagi and Roth (2012) observe a related form of strategic behavior in multi-hospital kidney exchange: hospitals withhold easily matchable donor–patient pairs for internal matching while enrolling only hard-to-match pairs (e.g., highly sensitized recipients) in the centralized pool.

problem of reallocating an infinitely divisible commodity among agents with single-peaked preferences (Bonifacio, 2024). For this problem, a certain class of reallocation rules, including the well-known uniform reallocation rule, satisfy *weak withdrawal-proofness* but not *pre-delivery-proofness*.[3] Our findings are consistent with these results: Theorem 1 and Proposition 2 parallel Thomson's conclusions for the Walrasian rule; and similar to Bonifacio (2024), Proposition 1 and Example 3 indicate the existence of *weak withdrawal-proof* rules in the object exchange problem. In addition, we provide characterizations of TTC based on *weak withdrawal-proofness* and *strict pre-delivery-proofness*. To the best of our knowledge, no characterization results based on *withdrawal-proofness* and *pre-delivery-proofness* (or their variants) have been established, even in other models.

Absence-proofness, first introduced by Doğan (2013) in the context of cooperative game theory, is considered a coalitional version of *withdrawal-proofness* because it concerns manipulation through withdrawal and reallocation by groups of arbitrary size. Doğan applies this axiom to the object exchange problem and shows that no rule satisfies it. Since both *individual rationality* and *withdrawal-proofness* are weaker than *absence-proofness*, Theorem 1 implies that Doğan's impossibility result continues to hold when *absence-proofness* is replaced with these two axioms.

Withdrawal-proofness and its weaker version may be related to *pair-efficiency* (Ekici, 2024) (or *reallocation-proofness*).[4] This is because, although the former is a variable-population property and the latter a fixed-population one, they share the underlying idea that no pair can benefit from reallocating their own objects after the rule is applied. Similarly, *pre-delivery-proofness* (and its variants) and *endowments-swapping-proofness* (Fujinaka and Wakayama, 2018) share the idea that no pair can benefit from swapping their own endowments before the rule is applied. Several studies have examined the implications of *pair-efficiency*, *reallocation-proofness*, or *endowments-swapping-proofness*.[5] TTC is characterized by *individual rationality*, *pair-efficiency*, and *strategy-proofness* (Ekici, 2024); it is also character-

[3]Bonifacio (2024) refers to *weak withdrawal-proofness* as "withdrawal-proofness."

[4]More precisely, the original definition of *reallocation-proofness* (e.g., Pápai (2000) and Fujinaka and Wakayama (2018, 2024)) incorporates preference manipulations into a form of strategic behavior.

[5]See, for example, Pápai (2000), Fujinaka and Wakayama (2018, 2024, 2025a), Chen and Zhao (2021), Tamura (2023), Ekici (2024), and Hu and Zhang (2024). Extensions to more general object reallocation problems include Atlamaz and Klaus (2007), Feng (2023), and Fujinaka and Wakayama (2025b). Additionally, Tamura (2022) introduces an invariance axiom on endowments-swapping to characterize the crawler (Bade, 2019) on the single-peaked preference domain.

ized by *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness* (Fujinaka and Wakayama, 2018). Theorem 2 and Theorem 3 serve as counterparts to the characterizations by Ekici (2024) and Fujinaka and Wakayama (2018), respectively. Notably, unlike the characterization involving *endowments-swapping-proofness*, Theorem 3 does not require *strategy-proofness*.

Since Ma (1994) characterizes TTC by means of *individual rationality*, *efficiency*, and *strategy-proofness*, several alternative characterizations, such as those by Ekici (2024) and Fujinaka and Wakayama (2018), have been proposed in the fixed-population setting.[6] Our theorems extend this literature by providing new characterizations of TTC in the variable-population setting.

Organization of the paper The rest of the paper is organized as follows. Section 2 introduces the preliminary notation and definitions. Section 3 examines the implications of *withdrawal-proofness*. Section 4 analyzes *pre-delivery-proofness* and its variants. Section 5 concludes by discussing remaining issues for future research. Omitted proofs of the main results are provided in the appendix.

2 Preliminaries

Let $\mathcal{I} = \{1, 2, \ldots, |\mathcal{I}|\}$ and $\mathcal{O} = \left\{o_1, o_2, \ldots, o_{|\mathcal{O}|}\right\}$ be a finite set of potential **agents** and a finite set of potential **objects**, respectively.[7]

An "economy" is formalized as follows. Let (I, O) be a pair of $I \subsetneq \mathcal{I}$ and $O \subsetneq \mathcal{O}$ with $1 \leq |I| = |O| < +\infty$. Each agent $i \in I$ has a strict preference relation $\succ_i$ over O. Given $\succ_i$, we denote the induced weak preference relation by $\succsim_i$; that is, for each $\{o, o'\} \subseteq O$, if $o \succsim_i o'$, then either $o \succ_i o'$ or $o = o'$. Let $\mathscr{P}_O$ be the set of strict preferences over O. For each $i \in I$, let $\omega_i \in O$ be agent i's **endowment**. An **economy** is a list $e = (I, O, \succ, \omega)$, where

- $I \subsetneq \mathcal{I}$ is a set of agents;
- $O \subsetneq \mathcal{O}$ is a set of objects such that $1 \leq |I| = |O| < +\infty$;
- $\succ = (\succ_i)_{i \in I} \in \mathscr{P}_O^I$ is a preference profile; and
- $\omega = (\omega_i)_{i \in I} \in O^I$ is an endowment profile such that for each $\{i, j\} \subseteq I$ with $i \neq j$, $\omega_i \neq \omega_j$.

[6] See also, for example, Svensson (1999), Takamiya (2001), and Miyagawa (2002).
[7] Given a set A, we denote the cardinality of A by $|A|$.

Let $\mathscr{E}$ be the set of all economies.

Given (I,O) with $I \subseteq \mathcal{I}$, $O \subseteq \mathcal{O}$, and $1 \leq |I| = |O| < +\infty$, an **assignment** for (I,O) is a function $x\colon I \to O$ such that for each $\{i,j\} \subseteq I$ with $i \neq j$, $x(i) \neq x(j)$. We write x_i for $x(i)$. Let $X(I,O)$ be the set of assignments for (I,O). Let

$$\mathcal{X} = \bigcup_{\substack{I \subseteq \mathcal{I}, O \subseteq \mathcal{O}:\\ 1 \leq |I| = |O| < +\infty}} X(I,O).$$

Let $\mathscr{D} \subseteq \mathscr{E}$ be the set of admissible economies, which we call a **domain**. A **rule** on $\mathscr{D}$ is a function $f\colon \mathscr{D} \to \mathcal{X}$ that maps $e = (I,O,\succ,\omega) \in \mathscr{D}$ to $f(e) \in X(I,O)$. We denote the object assigned to agent i at e by $f_i(e)$.

The Top Trading Cycles rule (TTC) on $\mathscr{D}$, denoted by $TTC\colon \mathscr{D} \to \mathcal{X}$, is central in the literature on object exchange problems. For each $e = (I,O,\succ,\omega) \in \mathscr{D}$, $TTC(e) \in X(I,O)$ is obtained by the following TTC algorithm:

- **Round 1.** Each agent points to the agent who owns his most preferred object. At least one "cycle" exists since the number of agents is finite. A **cycle** is a sequence of agents $(i_1(= i_{N+1}), i_2, \ldots, i_N)$ such that for each $n \in \{1,2,\ldots,N\}$, i_n points to i_{n+1}. Each agent in a cycle is assigned the object along the cycle and is removed from the economy with the assigned object. If any agent remains, the algorithm proceeds with the next round; otherwise, it terminates.

- **Round $t \geq 2$.** Each remaining agent points to the agent who owns his most preferred object among the remaining objects. At least one cycle exists. Each agent in a cycle is assigned the object along the cycle and is removed from the economy with the assigned object. If any agent remains, the algorithm proceeds to the next round; otherwise, it terminates.

The following two axioms are standard in the literature: no agent is strictly worse off after the reallocation; the chosen assignment cannot be changed to make some agent strictly better off without making another agent strictly worse off.

Individual rationality: For each $e = (I,O,\succ,\omega) \in \mathscr{D}$ and each $i \in I$, $f_i(e) \succsim_i \omega_i$.

Efficiency: For each $e = (I,O,\succ,\omega) \in \mathscr{D}$, there is no $x \in X(I,O)$ such that for each $i \in I$, $x_i \succsim_i f_i(e)$ and for some $j \in I$, $x_j \succ_j f_j(e)$.

3 Withdrawal-proofness

This section considers the following manipulation scenario: an agent withdraws from a given rule, taking his endowment with him; the rule is then applied to the subeconomy involving the remaining agents; afterward, the agent who withdrew and one of the agents who participated exchange the withdrawn endowment and the object assigned to that participant. *Withdrawal-proofness* rules out the possibility that this arrangement makes both agents at least as well off as they would have been without manipulation, and at least one of them strictly better off.

To formally define *withdrawal-proofness*, we introduce some notation. For each $e = (I, O, \succ, \omega) \in \mathscr{E}$ and each $i \in I$, let

$$e_{-i} = \left(I \setminus \{i\}, O \setminus \{\omega_i\}, \succ_{-i}^{O \setminus \{\omega_i\}}, \omega_{-i} \right),$$

where

- $I \setminus \{i\}$ is a set of agents;
- $O \setminus \{\omega_i\}$ is a set of objects such that $|I \setminus \{i\}| = |O \setminus \{\omega_i\}|$;
- $\succ_{-i}^{O \setminus \{\omega_i\}} = \left(\succ_k^{O \setminus \{\omega_i\}} \right)_{k \in I \setminus \{i\}} \in \mathscr{P}_{O \setminus \{\omega_i\}}^{I \setminus \{i\}}$ is a preference profile such that for each $k \in I \setminus \{i\}$ and each $\{o, o'\} \subseteq O \setminus \{\omega_i\}$ with $o \neq o'$,

  $$o \succ_k^{O \setminus \{\omega_i\}} o' \iff o \succ_k o';$$

  and
- $\omega_{-i} = (\omega_k)_{k \in I \setminus \{i\}} \in X(I \setminus \{i\}, O \setminus \{\omega_i\})$ is an endowment profile.

That is, given an economy $e \in \mathscr{E}$, e_{-i} is the "reduced economy" obtained by having agent i withdraw with his endowment ω_i.

***Withdrawal-proofness*:** There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$, $\{i, j\} \subseteq I$ with $i \neq j$, and $\{y_i, y_j\} \subseteq \mathcal{O}$ such that $e_{-i} \in \mathscr{D}$, $\{y_i, y_j\} = \{\omega_i, f_j(e_{-i})\}$, for each $k \in \{i, j\}$, $y_k \succsim_k f_k(e)$, and for some $k \in \{i, j\}$, $y_k \succ_k f_k(e)$.

We present an impossibility theorem on domains that satisfy the following two conditions.

D1. For each $e = (I, O, \succ, \omega) \in \mathscr{D}$ with $|I| \geq 2$ and each $i \in I$, $e_{-i} \in \mathscr{D}$.

D2. There is an economy $e^* = (I, O, \succ^*, \omega^*) \in \mathscr{D}$ such that $I = \{i, j, k\}$ and

$$\begin{array}{ccc} \succ_i^* & \succ_j^* & \succ_k^* \\ \hline \omega_k^* & \omega_i^* & \omega_i^* \\ \omega_i^* & \omega_k^* & \omega_j^* \\ \omega_j^* & \omega_j^* & \omega_k^* \end{array}$$

D1 states that the domain includes all reduced economies obtained by the withdrawal of a single agent. D2 is a richness condition: the domain includes a three-agent economy in which agents j and k rank agent i's endowment as their most preferred and each prefers the other's endowment to his own, while agent i ranks one of j's and k's endowments as the most preferred and the other as the least preferred.

Theorem 1. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D1 *and* D2*. Then, no rule on $\mathscr{D}$ satisfies individual rationality and withdrawal-proofness.*

Before proving this theorem, we present a lemma.

Lemma 1. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D1*. If a rule f on $\mathscr{D}$ satisfies individual rationality and withdrawal-proofness, then for each $e = (I, O, \succ, \omega) \in \mathscr{D}$ with $|I| = |O| = 2$, $f(e) = TTC(e)$.*

Proof. Since $|I| = |O| = 2$, each $e \in \mathscr{D}$ falls into one of the following three categories:

$$\text{(i)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline \omega_j & \omega_i \\ \omega_i & \omega_j \end{array} \qquad \text{(ii)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline \omega_i & \omega_i \\ \omega_j & \omega_j \end{array} \qquad \text{(iii)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline \omega_i & \omega_j \\ \omega_j & \omega_i \end{array}$$

In cases (ii) and (iii), *individual rationality* implies $f(e) = \omega = TTC(e)$. We consider case (i) below. Suppose on the contrary that

$$\left(f_i(e), f_j(e)\right) = \left(\omega_i, \omega_j\right) \neq \left(\omega_j, \omega_i\right) = \left(TTC_i(e), TTC_j(e)\right).$$

Consider $e_{-i} \in \mathscr{E}$. Since $\mathscr{D}$ satisfies D1, $e_{-i} \in \mathscr{D}$. By $f_j(e_{-i}) = \omega_j$,

$$f_j(e_{-i}) = \omega_j \succ_i \omega_i = f_i(e) \quad \text{and} \quad \omega_i \succ_j \omega_j = f_j(e),$$

in violation of *withdrawal-proofness*. Hence, $f(e) = TTC(e)$. □

Proof of Theorem 1. Suppose on the contrary that there exists a rule $f\colon \mathscr{D} \to \mathcal{X}$ satisfying the two axioms. Since $\mathscr{D}$ satisfies D2, there is $e^* = (I, O, \succ^*, \omega^*) \in \mathscr{D}$ such that $I = \{i, j, k\}$ and

$$\begin{array}{ccc} \succ_i^* & \succ_j^* & \succ_k^* \\ \hline \omega_k^* & \omega_i^* & \omega_i^* \\ \omega_i^* & \omega_k^* & \omega_j^* \\ \omega_j^* & \omega_j^* & \omega_k^* \end{array}$$

There are two cases.

• **Case 1: $f_j(e^*) = \omega_i^*$.** Then, $f_k(e^*) \neq \omega_i^*$. Consider the pair $\{i, k\}$ and $e^*_{-k} \in \mathscr{E}$. Since $\mathscr{D}$ satisfies D1, $e^*_{-k} \in \mathscr{D}$. By Lemma 1,

$$\left(f_i(e^*_{-k}), f_j(e^*_{-k})\right) = \left(TTC_i(e^*_{-k}), TTC_j(e^*_{-k})\right) = \left(\omega_i^*, \omega_j^*\right).$$

Hence,

$$\omega_k^* \succsim_i^* f_i(e^*) \quad \text{and} \quad f_i(e^*_{-k}) = \omega_i^* \succ_k^* f_k(e^*),$$

in violation of *withdrawal-proofness*.

• **Case 2: $f_j(e^*) \neq \omega_i^*$.** Consider the pair $\{i, j\}$ and $e^*_{-i} \in \mathscr{E}$. Since $\mathscr{D}$ satisfies D1, $e^*_{-i} \in \mathscr{D}$. By Lemma 1,

$$\left(f_j(e^*_{-i}), f_k(e^*_{-i})\right) = \left(TTC_j(e^*_{-i}), TTC_k(e^*_{-i})\right) = \left(\omega_k^*, \omega_j^*\right).$$

Hence,

$$f_j(e^*_{-i}) = \omega_k^* \succsim_i^* f_i(e^*) \quad \text{and} \quad \omega_i^* \succ_j^* f_j(e^*),$$

in violation of *withdrawal-proofness*. □

Before proceeding, we introduce some notation. For each $i \in \mathcal{I}$, each $\succ_i \in \mathscr{P}_{\mathcal{O}}$, and each $O \subset \mathcal{O}$, let $\succ_i|_O \in \mathscr{P}_O$ be the restriction of $\succ_i$ over O; that is, the preference relation defined by setting for each $\{o, o'\} \subseteq O$ with $o \neq o'$,

$$o \succ_i|_O o' \iff o \succ_i o'.$$

In other words, $\succ_i|_O$ represents the induced preference relation over $O \subset \mathcal{O}$ from $\succ_i \in \mathscr{P}_{\mathcal{O}}$.

Remark 1. If $\mathscr{D}$ violates D2, Theorem 1 no longer holds. Precisely, TTC and at least one rule that differs from TTC satisfy the two axioms. Let $\succ \in \mathscr{P}_{\mathcal{O}}^{\mathcal{I}}$ be such that

$$
\begin{array}{cccc}
\succ_1 & \succ_2 & \succ_3 & \succ_{k\geq 4} \\
\hline
o_2 & o_3 & o_1 & o_k \\
o_1 & o_2 & o_3 & \vdots \\
o_3 & o_1 & o_2 & \\
\vdots & \vdots & \vdots &
\end{array}
$$

and $\omega = \left(o_1, o_2, \ldots, o_{|\mathcal{I}|}\right)$. In addition, let

$$\mathscr{D}(\succ, \omega) = \left\{e = (I, O, \succ', \omega') \in \mathscr{E} \colon \forall\, i \in I,\ \succ'_i = \succ_i|_O \text{ and } \omega'_i = \omega_i\right\}.$$

This domain $\mathscr{D}(\succ, \omega)$ satisfies D1 but not D2. We denote the **no-trade rule** on $\mathscr{D}(\succ, \omega)$ by $NT \colon \mathscr{D}(\succ, \omega) \to \mathcal{X}$; that is, for each $e = (I, O, \succ', \omega') \in \mathscr{D}(\succ, \omega)$, $NT(e) = \omega'$. Note that for each $e = (I, O, \succ', \omega') \in \mathscr{D}(\succ, \omega)$, if $\{1, 2, 3\} \not\subset I$, $TTC(e) = NT(e) = \omega'$; otherwise, for each $k \in I \setminus \{1, 2, 3\}$, $TTC_k(e) = NT_k(e) = o_k$ but

$$
\begin{aligned}
(TTC_1(e), TTC_2(e), TTC_3(e)) &= (o_2, o_3, o_1) \\
&\neq (o_1, o_2, o_3) \\
&= (NT_1(e), NT_2(e), NT_3(e)).
\end{aligned}
$$

Then, both TTC and NT satisfy *individual rationality* and *withdrawal-proofness*.[8] See Online Appendix B for the proof of this fact. $\lozenge$

Below, we provide examples of natural restricted domains that satisfy both D1 and D2.

Example 1. Let $\lhd$ be a linear order on $\mathcal{O}$ such that

$$o_1 \lhd o_2 \lhd \cdots \lhd o_{|\mathcal{O}|}.$$

Given an agent $i \in \mathcal{I}$ and a set of objects $O \subseteq \mathcal{O}$, agent i's preference relation $\succ_i \in \mathscr{P}_O$ is **single-peaked with respect to $\lhd$** if there exists an object $p(\succ_i) \in O$ such that

[8]Obviously, TTC on $\mathscr{D}(\succ, \omega)$ satisfies *efficiency*, while NT on $\mathscr{D}(\succ, \omega)$ violates *efficiency*.

- for each $o \in O \setminus \{p(\succ_i)\}$, $p(\succ_i) \succ_i o$; and
- for each $\{o, o'\} \subseteq O \setminus \{p(\succ_i)\}$, if either $o \lhd o' \lhd p(\succ_i)$ or $p(\succ_i) \lhd o' \lhd o$, then $o' \succ_i o$,

and $\succ_i \in \mathscr{P}_O$ is **single-dipped with respect to $\lhd$** if there is an object $d(\succ_i) \in O$ such that

- for each $o \in O \setminus \{d(\succ_i)\}$, $o \succ_i d(\succ_i)$; and
- for each $\{o, o'\} \subseteq O \setminus \{d(\succ_i)\}$, if either $o \lhd o' \lhd d(\succ_i)$ or $d(\succ_i) \lhd o' \lhd o$, then $o \succ_i o'$.

We denote the set of single-peaked preference relations over O with respect to $\lhd$ by $\mathscr{S}_{O,\lhd}$. Let

$$\mathscr{D}_\wedge = \{e = (I, O, \succ, \omega) \in \mathscr{E} \colon \forall\, i \in I,\ \succ_i \in \mathscr{S}_{O,\lhd}\}$$

be the **single-peaked domain**. The **single-dipped domain** $\mathscr{D}_\vee$ is defined analogously. Obviously, $\mathscr{D}_\wedge$ and $\mathscr{D}_\vee$ satisfy D1. To see that they satisfy D2, let $I = \{1, 2, 3\}$, $O = \{o_1, o_2, o_3\}$, $\omega^* = (o_1, o_2, o_3)$, and $\{\succ^*, \succ^{**}\} \subset \mathscr{P}_O$ be such that

$\succ_2^*$	$\succ_1^*$	$\succ_3^*$	$\succ_1^{**}$	$\succ_2^{**}$	$\succ_3^{**}$
ω_3^*	ω_2^*	ω_2^*	ω_3^*	ω_1^*	ω_1^*
ω_2^*	ω_3^*	ω_1^*	ω_1^*	ω_3^*	ω_2^*
ω_1^*	ω_1^*	ω_3^*	ω_2^*	ω_2^*	ω_3^*

Let $e^* = (I, O, \succ^*, \omega^*)$ and $e^{**} = (I, O, \succ^{**}, \omega^*)$. Then, $e^* \in \mathscr{D}_\wedge$ and $e^{**} \in \mathscr{D}_\vee$. ■

Remark 2. On the single-peaked domain, not only TTC but also many other rules satisfy desirable properties such as *individual rationality*, *efficiency*, and *strategy-proofness*. Examples include the crawler (Bade, 2019), the neighborhood TTC rules (Liu, 2025), and the r-neighborhood rules (Huang and Tian, 2023). Theorem 1 implies that none of these rules satisfy *withdrawal-proofness*. ♢

Example 2. Given an agent $i \in \mathcal{I}$, a set of objects $O \subseteq \mathcal{O}$, and his endowment $\omega_i \in O$, agent i's preference relation $\succ_i \in \mathscr{P}_O$ is a **common ranking preference relation** over O with respect to ω_i if for each $\{o_j, o_k\} \subseteq O$ such that $o_j \succ_i \omega_i$ and $o_k \succ_i \omega_i$,

$$j < k \iff o_j \succ_i o_k.$$

That is, a common ranking preference relation orders "acceptable" objects according to a ranking shared by all agents. Let $\mathscr{P}_O^{\omega_i} \subset \mathscr{P}_O$ be the set of common ranking preferences over O with respect to ω_i. Let

$$\mathscr{D}_{\text{cr}} = \{e = (I, O, \succ, \omega) \in \mathscr{E} \colon \forall\, i \in I,\ \succ_i \in \mathscr{P}_O^{\omega_i}\}$$

be the **common ranking domain**. Obviously, $\mathscr{D}_{\text{cr}}$ satisfies D1. To see that $\mathscr{D}_{\text{cr}}$ satisfies D2, let $I = \{1,2,3\}$, $O = \{o_1, o_2, o_3\}$, $\omega^* = (o_1, o_2, o_3)$, and $\succ^* \in \mathscr{P}_O$ be such that

$\succ_1^*$	$\succ_2^*$	$\succ_3^*$
ω_3^*	ω_1^*	ω_1^*
ω_1^*	ω_3^*	ω_2^*
ω_2^*	ω_2^*	ω_3^*

Let $e^* = (I, O, \succ^*, \omega^*)$. Then, $e^* \in \mathscr{D}_{\text{cr}}$. ■

In light of the above negative result, we consider a weaker version of *withdrawal-proofness* that pertains to manipulations in which both agents in the manipulating pair are strictly better off.

***Weak withdrawal-proofness*:** There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$, $\{i, j\} \subseteq I$ with $i \neq j$, and $\{y_i, y_j\} \subseteq \mathcal{O}$ such that $e_{-i} \in \mathscr{D}$, $\{y_i, y_j\} = \{\omega_i, f_j(e_{-i})\}$, $y_i \succ_i f_i(e)$ and $y_j \succ_j f_j(e)$.

Weak withdrawal-proofness allows us to escape the negative result: TTC is *weakly withdrawal-proof* regardless of whether the domain $\mathscr{D}$ satisfies D1 or D2.

Proposition 1. *Let $\mathscr{D} \subseteq \mathscr{E}$. TTC on $\mathscr{D}$ satisfies weak withdrawal-proofness.*

Proof. See Appendix A. □

The following example shows that, in addition to TTC, several other rules satisfy *individual rationality* and *weak withdrawal-proofness*.

Example 3. Suppose that $|\mathcal{I}| = |\mathcal{O}|$. Let $\succ \in \mathscr{P}_{\mathcal{O}}^{\mathcal{I}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_{k\geq 4}$
o_3	o_1	o_1	o_k
o_1	o_3	o_2	$\vdots$
o_2	o_2	o_3	
$\vdots$	$\vdots$	$\vdots$	

and $\omega = \left(o_1, o_2, \ldots, o_{|\mathcal{I}|}\right)$. Let $e^\flat = (\mathcal{I}, \mathcal{O}, \succ, \omega)$. Let $f^\flat \colon \mathscr{E} \to \mathcal{X}$ be the rule such that for each $e \in \mathscr{E}$,

$$f^\flat(e) = \begin{cases} \left(o_3, o_1, o_2, o_4, \ldots, o_{|\mathcal{I}|}\right) & \text{if } e = e^\flat \\ TTC(e) & \text{otherwise.} \end{cases}$$

Note that

$$TTC(e^\flat) = \left(o_3, o_2, o_1, o_4, \ldots, o_{|\mathcal{I}|}\right) \neq \left(o_3, o_1, o_2, o_4, \ldots, o_{|\mathcal{I}|}\right) = f^\flat(e^\flat).$$

Obviously, $f^\flat$ is *individually rational*. It also satisfies *weak withdrawal-proofness*. To see why, let $e \in \mathscr{E}$. If $e = e^\flat$, since all agents except agent 3 receive their most preferred objects under $f^\flat$, no pair has an incentive to manipulate; otherwise, since $f^\flat(e) = TTC(e)$ and for each $i \in I$, $f^\flat(e_{-i}) = TTC(e_{-i})$, the claim immediately follows from the *weak withdrawal-proofness* of TTC. ■

Additionally, $f^\flat$ in Example 3 obviously satisfies *efficiency*. Hence, *individual rationality* and *weak withdrawal-proofness* combined with *efficiency* cannot characterize TTC. On the other hand, it is noteworthy that agent 3 benefits from misrepresenting his preferences under rule $f^\flat$ in Example 3. Let $\succ_3' \in \mathscr{P}_\mathcal{O}$ be such that

$\succ_3'$
o_1
o_3
o_2
$\vdots$

and $e' = (\mathcal{I}, \mathcal{O}, (\succ_3', \succ_{-3}), \omega)$. Then,

$$f_3^\flat(e') = TTC_3(e') = o_1 \succ_3 o_2 = f_3^\flat(e^\flat).$$

We can show that this is the case for any rule that differs from TTC satisfying *individual rationality* and *weak withdrawal-proofness*.

We consider the following incentive condition on preference revelation, which states that no agent can benefit from misrepresentation of his preferences.

Strategy-proofness: For each $e = (I, O, \succ, \omega) \in \mathscr{D}$, each $i \in I$, and each $e' = (I, O, (\succ_i', \succ_{-i}), \omega) \in \mathscr{D}$, $f_i(e) \succsim_i f_i(e')$.

We characterize TTC in terms of *individual rationality*, *strategy-proofness*, and *weak withdrawal-proofness* on any domain satisfying D1 and the following richness condition on its domain of definition:

D3. For each $e = (I, O, \succ, \omega) \in \mathscr{D}$, each $i \in I$, each $\{o, o'\} \subseteq O \setminus \{\omega_i\}$ with $o \neq o'$, there is a pair of preferences $\{\succ'_i, \succ''_i\} \subset \mathscr{P}_O$ such that

$$\begin{array}{cc} \succ'_i & \succ''_i \\ \hline o & o \\ o' & \omega_i \\ \omega_i & \vdots \\ \vdots & \end{array}$$

and $\{(I, O, (\succ'_i, \succ_{-i}), \omega), (I, O, (\succ''_i, \succ_{-i}), \omega)\} \subset \mathscr{D}$.

D3 states that for any pair of objects o and o', each agent can rank o as the most preferred, o' as the second most preferred, and his endowment as the third most preferred; and also rank o as the most preferred and his endowment as the second most preferred.

Theorem 2. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D1 *and* D3. *Then, a rule on $\mathscr{D}$ satisfies individual rationality, weak withdrawal-proofness, and strategy-proofness if and only if it is TTC.*

Proof. See Appendix A. $\square$

Remark 3. D3 requires the domain of preferences to be sufficiently rich. In fact, many natural restricted domains, such as the single-peaked domain, the single-dipped domain, and the common ranking domain, violate D3. $\Diamond$

Remark 4. Although Theorem 2 does not rely on any efficiency-related axioms, the proof of its "only if" part employs techniques developed by Ekici and Sethuraman (2024), who provide an alternative proof of Ekici's (2024) *pair-efficiency* characterization of TTC. See Appendix A for details. $\Diamond$

4 Pre-delivery-proofness

This section considers the following type of manipulation: agent j pre-delivers to agent i the object ω_j that i would receive under full participation; upon receiving this object, agent i withdraws; agent j, now holding i's endowment ω_i, participates; if the rule assigns j an object he strictly prefers to his original assignment, the manipulation is successful. We focus on rules that are immune to such manipulations, that is, *pre-delivery-proof* rules.

To formally define *pre-delivery-proofness*, we introduce some notation. For each $e = (I, O, \succ, \omega) \in \mathscr{E}$ and each $\{i, j\} \subseteq I$ with $i \neq j$, let

$$e^{i,j}_{-i} = \left(I \setminus \{i\}, O \setminus \{\omega_j\}, \succ^{O\setminus\{\omega_j\}}_{-i}, \omega^{i,j}_{-i} \right),$$

where

- $I \setminus \{i\}$ is a set of agents;
- $O \setminus \{\omega_j\}$ is a set of objects such that $|I \setminus \{i\}| = |O \setminus \{\omega_j\}|$;
- $\succ^{O\setminus\{\omega_j\}}_{-i} = (\succ^{O\setminus\{\omega_j\}}_{k})_{k \in I\setminus\{i\}} \in \mathscr{P}^{I\setminus\{i\}}_{O\setminus\{\omega_j\}}$ is a preference profile such that for each $k \in I \setminus \{i\}$ and each $\{o, o'\} \subseteq O \setminus \{\omega_j\}$ with $o \neq o'$,

  $$o \succ^{O\setminus\{\omega_j\}}_{k} o' \iff o \succ_k o';$$

  and
- $\omega^{i,j}_{-i} \in X(I \setminus \{i\}, O \setminus \{\omega_j\})$ is an endowment profile such that $\omega^{i,j}_{j} = \omega_i$ and for each $k \in I \setminus \{i, j\}$, $\omega^{i,j}_{k} = \omega_k$.

That is, given an economy $e \in \mathscr{E}$, $e^{i,j}_{-i}$ denotes the "reduced swapping economy" in which agents i and j first exchange their endowments, and then agent i withdraws.

***Pre-delivery-proofness*:** There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $\{i, j\} \subseteq I$ with $i \neq j$ such that $e^{i,j}_{-i} \in \mathscr{D}$, $\omega_j = f_i(e)$, and $f_j(e^{i,j}_{-i}) \succ_j f_j(e)$.

The next result shows that TTC satisfies *pre-delivery-proofness* on any domain.

Proposition 2. *Let $\mathscr{D} \subseteq \mathscr{E}$. TTC on $\mathscr{D}$ satisfies pre-delivery-proofness.*

Proof. See Appendix A. □

This result may seem appealing, but only because *pre-delivery-proofness* is quite weak in our setting. In fact, several rules that differ from TTC satisfy this property. For example, the no-trade rule satisfies *pre-delivery-proofness*. Moreover, the no-trade rule also satisfies *strategy-proofness*, so a characterization of TTC similar to Theorem 2 does not hold if *weak withdrawal-proofness* is replaced with *pre-delivery-proofness*. The following example shows that a rule that differs from TTC can satisfy *pre-delivery-proofness* even when both *individual rationality* and *efficiency* are imposed.

Example 4. Suppose that $|\mathcal{I}| = |\mathcal{O}|$. Let $\succ \in \mathscr{P}_{\mathcal{O}}^{\mathcal{I}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_{k\geq 4}$
o_2	o_1	o_1	o_k
o_3	o_2	o_2	$\vdots$
o_1	o_3	o_3	
$\vdots$	$\vdots$	$\vdots$	

and $\omega = \left(o_1, o_2, \ldots, o_{|\mathcal{I}|}\right)$. Let $e^\natural = (\mathcal{I}, \mathcal{O}, \succ, \omega)$. Let $f^\natural \colon \mathscr{E} \to \mathcal{X}$ be the rule such that for each $e \in \mathscr{E}$,

$$f^\natural(e) = \begin{cases} \left(o_3, o_2, o_1, o_4, \ldots, o_{|\mathcal{I}|}\right) & \text{if } e = e^\natural \\ TTC(e) & \text{otherwise.} \end{cases}$$

Note that

$$TTC(e^\natural) = \left(o_2, o_1, o_3, o_4, \ldots, o_{|\mathcal{I}|}\right) \neq \left(o_3, o_2, o_1, o_4, \ldots, o_{|\mathcal{I}|}\right) = f^\natural(e^\natural).$$

Obviously, $f^\natural$ satisfies *individual rationality* and *efficiency*. To see that $f^\natural$ satisfies *pre-delivery-proofness*, let $e \in \mathscr{E}$. There are two cases.

- **Case 1: $e \neq e^\natural$.** Then, $f^\natural(e) = TTC(e)$. Since for each $\{i, j\} \subseteq I$ with $i \neq j$, $e_{-i}^{i,j} \neq e^\natural$, we have $f^\natural(e_{-i}^{i,j}) = TTC(e_{-i}^{i,j})$. By Proposition 2, $f^\natural(e)$ satisfies *pre-delivery-proofness* in this case.

- **Case 2: $e = e^\natural$.** Since for each $k \notin \{1, 3\}$, $f_k^\natural(e) = \omega_k$ and agent 3 receives his most preferred object, it suffices to consider the case $j = 1$ and $i = 3$. Then, $\omega_1 = f_3^\natural(e)$ and $f_1^\natural(e_{-3}^{1,3}) = TTC_1(e_{-3}^{1,3}) = \omega_3 = f_1^\natural(e)$. Hence, $f^\natural(e)$ satisfies *pre-delivery-proofness* in this case. ∎

Pre-delivery-proofness addresses only manipulations in which the withdrawing agent is pre-delivered the object he would receive if all agents participated. Obviously, the agent may still choose to withdraw if he is pre-delivered an object he strictly prefers. To account for such manipulations, we propose a strengthening of *pre-delivery-proofness*.

Strict pre-delivery-proofness: There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $\{i, j\} \subseteq I$ with $i \neq j$ such that $e^{i,j}_{-i} \in \mathscr{D}$, $\omega_j \succsim_i f_i(e)$, and $f_j(e^{i,j}_{-i}) \succ_j f_j(e)$.

The conjunction of this axiom with *individual rationality* characterizes TTC on any domain that includes all reduced swapping economies.

D4. For each $e = (I, O, \succ, \omega) \in \mathscr{D}$ with $|I| \geq 2$ and each $\{i, j\} \subseteq I$ with $i \neq j$, $e^{i,j}_{-i} \in \mathscr{D}$.

Theorem 3. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D4. *Then, a rule f on $\mathscr{D}$ satisfies individual rationality and strict pre-delivery-proofness if and only if it is TTC on $\mathscr{D}$.*

Proof. See Appendix A. □

Remark 5. It is worth noting that Theorem 3 characterizes TTC without assuming either *efficiency* or *strategy-proofness*, which are often required in the fixed-population setting. Moreover, TTC satisfies *strict pre-delivery-proofness* regardless of whether the domain $\mathscr{D}$ satisfies D4. See the proof of the "if" part for details. ♢

Remark 6. *Strict pre-delivery-proofness* can be weakened to the following:

Endowments-swapping-and-withdrawal-proofness: There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $\{i, j\} \subseteq I$ with $i \neq j$ such that $e^{i,j}_{-i} \in \mathscr{D}$, $\omega_j \succ_i f_i(e)$, and $f_j(e^{i,j}_{-i}) \succ_j f_j(e)$.

Both *pre-delivery-proofness* and *endowments-swapping-and-withdrawal-proofness* are weaker than *strict pre-delivery-proofness*, but there is no logical relationship between the two axioms.[9] We note that Theorem 3 does not hold if *strict pre-delivery-proofness* is replaced with *endowments-swapping-and-withdrawal-proofness*.

[9] The no-trade rule NT satisfies *pre-delivery-proofness* but violates *endowments-swapping-and-withdrawal-proofness*. To see that NT violates *endowments-swapping-and-withdrawal-proofness*, let $e = (I, H, \succ, \omega) \in \mathscr{D}$ be such that $\{i, j\} \subseteq I$ with $i \neq j$, $\omega_j \succ_i \omega_i$, and $\omega_i \succ_j \omega_j$. Then, $\omega_j \succ_i \omega_i = NT_i(e)$ and $NT_j(e^{i,j}_{-i}) = \omega_i \succ_j \omega_j = NT_j(e)$. Rule $\widehat{f}$ defined in Example 5 below satisfies *endowments-swapping-and-withdrawal-proofness* but violates *pre-delivery-proofness*. To see that $\widehat{f}$ violates *pre-delivery-proofness*, consider economy $\widehat{e}$ and a pair of agents $\{1, 3\}$. Then, $\omega_3 = o_3 = \widehat{f}_1(\widehat{e})$ and $\widehat{f}_3(\widehat{e}^{1,3}_{-1}) = \omega^{1,3}_3 = o_1 \succ_3 o_2 = \widehat{f}_3(\widehat{e})$.

As shown in Example 5 below, a rule that differs from TTC satisfies *individual rationality* and *endowments-swapping-and-withdrawal-proofness*.[10,11] $\diamondsuit$

Example 5. Suppose that $|\mathcal{I}| = |\mathcal{O}|$. Let $\succ \in \mathscr{P}_{\mathcal{O}}^{\mathcal{I}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_{k\geq 4}$
o_3	o_1	o_1	o_k
o_2	o_2	o_2	$\vdots$
o_1	o_3	o_3	
$\vdots$	$\vdots$	$\vdots$	

and $\omega = \left(o_1, o_2, \ldots, o_{|\mathcal{I}|}\right)$. In addition, let

$$\mathscr{D}(\succ) = \left\{ e = (I, O, \succ', \omega') \in \mathscr{E} \colon \forall\, i \in I,\ \succ'_i = \succ_i|_O \right\}.$$

This domain $\mathscr{D}(\succ)$ satisfies D4. Let $\widehat{e} = (\mathcal{I}, \mathcal{O}, \succ, \omega)$. Let $\widehat{f} \colon \mathscr{D}(\succ) \to \mathcal{X}$ be the rule such that for each $e \in \mathscr{D}(\succ)$,

$$\widehat{f}(e) = \begin{cases} \left(o_3, o_1, o_2, o_4, \ldots, o_{|\mathcal{I}|}\right) & \text{if } e = \widehat{e} \\ TTC(e) & \text{otherwise.} \end{cases}$$

Note that

$$TTC(\widehat{e}) = \left(o_3, o_2, o_1, o_4, \ldots, o_{|\mathcal{I}|}\right) \neq \left(o_3, o_1, o_2, o_4, \ldots, o_{|\mathcal{I}|}\right) = \widehat{f}(\widehat{e}).$$

Obviously, $\widehat{f}$ is *individually rational*. For the proof of the *endowments-swapping-and-withdrawal-proofness* of this rule, see Online Appendix B. ■

The single-peaked and single-dipped domains satisfy D4. We provide another example of a domain that satisfies this condition. Thus, we characterize TTC as the unique rule satisfying the two axioms even on these restricted domains.[12]

[10]Rule $\widehat{f}$ in Example 5 satisfies *efficiency*. This implies that *individual rationality* and *endowments-swapping-and-withdrawal-proofness* cannot characterize TTC even under *efficiency*.

[11]*Endowments-swapping-and-withdrawal-proofness* does not cover situations in which the remaining participant is indifferent between the outcomes with and without manipulation. To address such cases, we define a stronger axiom, *strict endowments-swapping-and-withdrawal-proofness*, which requires immunity to manipulations where the withdrawing agent strictly benefits, while the participating agent may be indifferent. Unlike *strict pre-delivery-proofness*, this axiom leads to an impossibility result: no *individually rational* rule satisfies *strict endowments-swapping-and-withdrawal-proofness*. The proof is provided in Online Appendix C.

[12]The common ranking domain $\mathscr{D}_{\text{cr}}$ violates D4. To see this, suppose $\mathcal{I} = \{1, 2, 3, 4\}$ and $\mathcal{O} =$

Example 6. Suppose that $|\mathcal{I}| = |\mathcal{O}|$. Fix a preference profile $\succ^\star = (\succ_i^\star)_{i\in\mathcal{I}} \in \mathscr{P}_{\mathcal{O}}^{\mathcal{I}}$. Let

$$\mathscr{D}(\succ^\star) = \{e = (I, O, \succ, \omega) \in \mathscr{E} \colon \forall\, i \in I,\ \succ_i = \succ_i^\star|_O\}.$$

Roughly speaking, $\mathscr{D}(\succ^\star)$ is the set of economies obtained by reducing an economy where each agent i has the preference relation $\succ_i^\star$. Importantly, for each i, each $\succ_i^\star$ is arbitrary. Obviously, $\mathscr{D}(\succ^\star)$ satisfies D4. ■

5 Concluding remarks

We conclude by highlighting three directions for future research.

1. **Weak withdrawal-proofness.** We have characterized TTC by *individual rationality*, *weak withdrawal-proofness*, and *strategy-proofness*. Whether this characterization holds on natural restricted domains, such as the single-peaked preference domain, is an open question. Another direction is to characterize the class of rules that satisfy *individual rationality*, *weak withdrawal-proofness*, and additional punctual properties such as *efficiency*.

2. **Pre-delivery-proofness.** As stated above, *pre-delivery-proofness* is relatively weak in our model. Consequently, both TTC and the no-trade rule satisfy *individual rationality*, *pre-delivery-proofness*, and *strategy-proofness*. Characterizing the full set of rules that satisfy these three axioms is open. It would also be worthwhile to identify rules that satisfy *individual rationality*, *efficiency*, and *pre-delivery-proofness*.

3. **Multiple-object reallocation.** To the best of our knowledge, the implications of *withdrawal-proofness* and *pre-delivery-proofness* have not been examined in the context of multi-object reallocation problems. Since our model assumes each agent initially owns exactly one object, these axioms are not directly applicable. Extending them to multi-object settings is an open and important area for future work. Bu, Chen, and Thomson (2014) examine two related axioms—*endowments-splitting-proofness* and *endowments-merging-proofness*—introduced by Thomson (2014) in the context of pure exchange economies. They show that no rule satisfies *individual rationality*, *efficiency*,

$\{o_1, o_2, o_3, o_4\}$ and let $e = (\mathcal{I}, \mathcal{O}, \succ, \omega) \in \mathscr{D}_{\mathrm{cr}}$ be such that $\omega = (o_1, o_2, o_3, o_4)$ and $o_3 \succ_4 o_4 \succ_4 o_2 \succ_4 o_1$. Then, $e_{-1}^{1,4} \notin \mathscr{D}_{\mathrm{cr}}$.

and either of these. It would be worth exploring whether similar impossibility results hold for extended versions of *withdrawal-proofness* or *pre-delivery-proofness*.

A Appendix: Proofs of our main results

A.1 Additional notation and definitions

This appendix provides the proofs of our main results. Before proceeding, we introduce some additional notation and definitions. We begin with notation related to the TTC algorithm. Let $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $r \in \mathbb{N} = \{1, 2, \ldots\}$. For each $\{j, k\} \subseteq I$, we write

$$j \overset{(e,r)}{\rightarrow} k$$

to indicate that agent j points to agent k in Round r at e. Let $\mathbb{C}(e, r) \subset 2^I$ denote the set of groups of agents involved in cycles in Round r at e. We denote the set of agents involved in cycles in Round r at e by

$$I(e, r) = \bigcup_{C \in \mathbb{C}(e,r)} C,$$

and the set of objects assigned to agents in Round r at e by

$$\begin{aligned} O(e, r) &= \{o \in O \colon \exists\, i \in I(e, r),\ o = TTC_i(e)\} \\ &= \{o \in O \colon \exists\, i \in I(e, r),\ o = \omega_i\}. \end{aligned}$$

Additionally, define $I^r(e)$ and $O^r(e)$ as

$$I^r(e) = \bigcup_{t=1}^{r} I(e, t) \quad \text{and} \quad O^r(e) = \bigcup_{t=1}^{r} O(e, t).$$

For convenience, let $I^0(e) = O^0(e) = \emptyset$. With abuse of notation, $C = \{i_1 (= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, r)$ also represents the sequence $(i_1, i_2, \ldots, i_N)$ of agents in the cycle C. That is, for each $i_n \in C$,

- $i_n \in I \setminus I^{r-1}(e)$ and $\omega_{i_n} \in O \setminus O^{r-1}(e)$; and
- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_{n+1}}\}\right)$, $TTC_{i_n}(e) = \omega_{i_{n+1}} \succ_{i_n} o$.

We also use the following notion in our proofs. For each $i \in I$, each $\succ_i \in \mathscr{P}_O$, and each $o \in O$, let $U^+(\succ_i, o) = \{o' \in O\colon o' \succ_i o\}$ be the **strict upper contour set of o according to $\succ_i$**.

A.2 Two lemmas

In this subsection, we present two lemmas concerning TTC. These lemmas state that if an agent receives an object in a round of the TTC algorithm at e that occurs before both of the rounds in which the manipulating agents i and j receive their objects, then the agent receives the same object in the same round at all three economies, e, e_{-i}, and $e^{i,j}_{-i}$.[13]

Lemma 2. *Let $e = (I, O, \succ, \omega) \in \mathscr{D}$, $\{i, j\} \subseteq I$ with $i \neq j$, and $(r_i, r_j) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. Suppose that $e_{-i} \in \mathscr{D}$ and $\min\{r_i, r_j\} \geq 2$. Then, for each $t \in \{1, 2, \ldots, \min\{r_i, r_j\} - 1\}$,*

$$\mathbb{C}(e, t) \subseteq \mathbb{C}(e_{-i}, t);$$
$$\forall\, k \in I(e, t),\ TTC_k(e_{-i}) = TTC_k(e).$$

Lemma 3. *Let $e = (I, O, \succ, \omega) \in \mathscr{D}$, $\{i, j\} \subseteq I$ with $i \neq j$, and $(r_i, r_j) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. Suppose that $e^{i,j}_{-i} \in \mathscr{D}$ and $\min\{r_i, r_j\} \geq 2$. Then, for each $t \in \{1, 2, \ldots, \min\{r_i, r_j\} - 1\}$,*

$$\mathbb{C}(e, t) \subseteq \mathbb{C}(e^{i,j}_{-i}, t);$$
$$\forall\, k \in I(e, t),\ TTC_k(e^{i,j}_{-i}) = TTC_k(e).$$

Proofs of Lemma 2 and Lemma 3. The proofs of Lemma 2 and Lemma 3 are identical, differing only in whether the reduced economy is e_{-i} or $e^{i,j}_{-i}$ depending on the manipulation. Therefore, we present a single proof that applies to both cases.

Given $e = (I, O, \succ, \omega) \in \mathscr{D}$, let $\omega^{i,j} = (\omega^{i,j}_k)_{k \in I} \in X(I, O)$ be such that $\omega^{i,j}_i = \omega_j$, $\omega^{i,j}_j = \omega_i$, and for each $k \in I \setminus \{i, j\}$, $\omega^{i,j}_k = \omega_k$. Let

$$e' = (I, O, \succ, \omega') \in \left\{e = (I, O, \succ, \omega), e^{i,j} = \left(I, O, \succ, \omega^{i,j}\right)\right\}.$$

[13] This result also holds when replacing e_{-i} or $e^{i,j}_{-i}$ with $e^{i,j}$. Fujinaka and Wakayama (2025b) prove the corresponding lemma in the context of object exchange problems with private and social endowments.

Thus, when $e' = e$,

$$e'_{-i} = \left(I \setminus \{i\}, O \setminus \{\omega'_i\}, \succ_{-i}^{O\setminus\{\omega'_i\}}, \omega'_{-i}\right) = \left(I \setminus \{i\}, O \setminus \{\omega_i\}, \succ_{-i}^{O\setminus\{\omega_i\}}, \omega_{-i}\right) = e_{-i};$$

when $e' = e^{i,j}$,

$$e'_{-i} = \left(I \setminus \{i\}, O \setminus \{\omega'_i\}, \succ_{-i}^{O\setminus\{\omega'_i\}}, \omega'_{-i}\right) = \left(I \setminus \{i\}, O \setminus \{\omega_j\}, \succ_{-i}^{O\setminus\{\omega_j\}}, \omega^{i,j}_{-i}\right) = e^{i,j}_{-i}.$$

By the supposition of Lemma 2 (resp. Lemma 3), $e_{-i} \in \mathscr{D}$ (resp. $e^{i,j}_{-i} \in \mathscr{D}$).

Let $r = \min\{r_i, r_j\} \geq 2$. We show the claim by induction on $t \in \{1, 2, \ldots, r - 1\}$.

Base step. Let $t = 1$. Let $C = \{i_1 (= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, 1)$. Pick any $i_n \in C$. Then,

$$\forall\, o \in O \setminus \{\omega_{i_{n+1}}\},\ TTC_{i_n}(e) = \omega_{i_{n+1}} \succ_{i_n} o. \tag{1}$$

By $i_n \in I(e, 1)$ and $t = 1 \leq r - 1$, $i_n \notin \{i, j\}$, which implies that

$$i_n \in I \setminus \{i\} \quad \text{and} \quad \omega'_{i_n} = \omega_{i_n} \in O \setminus \{\omega'_i\}. \tag{2}$$

Thus,

$$C \subseteq I \setminus \{i\} \quad \text{and} \quad \bigcup_{k \in C} \{\omega'_k\} = \bigcup_{k \in C} \{\omega_k\} \subseteq O \setminus \{\omega'_i\}.$$

In addition, by (1) and (2),

$$\forall\, o \in (O \setminus \{\omega'_i\}) \setminus \{\omega_{i_{n+1}}\},\ \omega_{i_{n+1}} \succ_{i_n}^{O\setminus\{\omega'_i\}} o.$$

This implies that $C \in \mathbb{C}(e'_{-i}, 1)$ and

$$TTC_{i_n}(e'_{-i}) = TTC_{i_n}(e) = \omega_{i_{n+1}}.$$

Induction hypothesis. Let $t \in \{2, 3, \ldots, r - 1\}$. For each $s \in \{1, 2, \ldots, t - 1\}$, the following claim holds:

$$\begin{gathered} \mathbb{C}(e, s) \subseteq \mathbb{C}(e'_{-i}, s); \\ \forall\, k \in I(e, s),\ TTC_k(e'_{-i}) = TTC_k(e). \end{gathered} \tag{3}$$

Induction step. Let $t \in \{2, 3, \ldots, r-1\}$ and $C = \{i_1 (= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, t)$. Pick any $i_n \in C$. Then,

$$TTC_{i_n}(e) = \omega_{i_{n+1}}. \tag{4}$$

By $i_n \in I(e, t)$ and $t \le r - 1$, $i_n \notin \{i, j\}$, which implies that

$$i_n \in I \setminus \{i\} \quad \text{and} \quad \omega'_{i_n} = \omega_{i_n} \in O \setminus \{\omega'_i\}. \tag{5}$$

Let $r'_n \in \mathbb{N}$ be such that $i_n \in I(e'_{-i}, r'_n)$. We proceed in four steps.

Step 1: For each $i_n \in C$, $\omega_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e'_{-i})$. Let $i_n \in C$ and $o \in U^+(\succ_{i_n}, \omega_{i_{n+1}})$. By $i_n \in I(e, t)$ and (4), $o \in O^{t-1}(e)$. Then, there are $s' \in \{1, 2, \ldots, t-1\}$ and $\ell \in I(e, s')$ with $TTC_\ell(e) = o$. By the induction hypothesis, (3) holds for s' and thus,

$$\ell \in I(e'_{-i}, s') \quad \text{and} \quad TTC_\ell(e'_{-i}) = TTC_\ell(e) = o.$$

Hence, $TTC_{i_n}(e'_{-i}) \neq o$. That is, $\omega_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e'_{-i})$.

Step 2: There is $r^* \in \mathbb{N}$ such that for each $i_n \in C$, $r^* = r'_{i_n}$. By Step 1 and (5),

$$\forall\, i_n \in C,\ \omega'_{i_{n+1}} = \omega_{i_{n+1}} \succsim_{i_n}^{O\setminus\{\omega'_i\}} TTC_{i_n}(e'_{-i}). \tag{6}$$

This implies

$$r'_{i_1} \le r'_{i_N} \le \cdots \le r'_{i_2} \le r'_{i_1}.$$

Then, there is $r^* \in \mathbb{N}$ such that for each $i_n \in C$, $r^* = r'_{i_n}$.

Step 3: $C \in \mathbb{C}(e'_{-i}, r^*)$ and $TTC_{i_n}(e'_{-i}) = TTC_{i_n}(e) = \omega_{i_{n+1}}$. By (5), Step 2 implies that

$$C \subseteq (I \setminus \{i\}) \setminus I^{r^*-1}(e'_{-i}) \quad \text{and} \quad \bigcup_{k\in C} \{\omega'_k\} = \bigcup_{k \in C} \{\omega_k\} \subseteq \left(O \setminus \{\omega'_i\}\right) \setminus O^{r^*-1}(e'_{-i}).$$

Let $i_n \in C$. Recall (6). If $\omega'_{i_{n+1}} = \omega_{i_{n+1}} \succ_{i_n}^{O\setminus\{\omega'_i\}} TTC_{i_n}(e'_{-i})$, then i_n does not receive the most preferred object among $(O \setminus \{\omega'_i\}) \setminus O^{r^*-1}(e'_{-i})$ according to $\succ_{i_n}^{O\setminus\{\omega'_i\}}$ in Round $r^* = r'_{i_n}$ of the TTC algorithm at e'_{-i}, which is a contradiction. Hence,

$$TTC_{i_n}(e'_{-i}) = \omega'_{i_{n+1}} = \omega_{i_{n+1}} = TTC_{i_n}(e).$$

Then, $C \in \mathbb{C}(e'_{-i}, r^*)$.

Step 4: $t = r^*$. Suppose on the contrary that $t \neq r^*$. There are two cases.

• **Case 1: $t < r^*$.** Then, for each $i_n \in C$, $i_n \in (I \setminus \{i\}) \setminus I^{t-1}(e'_{-i})$. This together with $C \notin \mathbb{C}(e'_{-i}, t)$ and Step 3 implies that there are $i_m \in C$, $k \in (I \setminus \{i\}) \setminus I^{t-1}(e'_{-i})$, and $o \in (O \setminus \{\omega'_i\}) \setminus O^{t-1}(e'_{-i})$ such that

$$i_m \overset{(e'_{-i},t)}{\rightarrow} k \quad \text{and} \quad \omega'_k = o \succ_{i_m}^{O\setminus\{\omega'_i\}} \omega_{i_{m+1}} = \omega'_{i_{m+1}} = TTC_{i_m}(e) = TTC_{i_m}(e'_{-i}).$$

By $\{o, \omega_{i_{m+1}}\} \subset O \setminus \{\omega'_i\} \subset O$,

$$o \succ_{i_m} \omega_{i_{m+1}} = TTC_{i_m}(e).$$

Further, by $i_m \in C \in \mathbb{C}(e, t)$, $o \in O^{t-1}(e)$. Then, there are $s' \in \{1, 2, \dots, t-1\}$ and $\ell \in I(e, s')$ with $TTC_\ell(e) = o$. By the induction hypothesis, (3) holds for s' and thus,

$$\ell \in I(e'_{-i}, s') \quad \text{and} \quad TTC_\ell(e'_{-i}) = TTC_\ell(e) = o \in O(e'_{-i}, s').$$

This contradicts $o \in (O \setminus \{\omega'_i\}) \setminus O^{t-1}(e'_{-i})$.

• **Case 2: $r^* < t$.** By $C \notin \mathbb{C}(e, t-1)$ and $C \in \mathbb{C}(e, t)$, Step 3 implies that there are $i_m \in C$, $k \in I(e, t-1)$, and $o \in O(e, t-1)$ such that

$$i_m \overset{(e,t-1)}{\rightarrow} k \quad \text{and} \quad \omega_k = o \succ_{i_m} \omega_{i_{m+1}} = \omega'_{i_{m+1}} = TTC_{i_m}(e) = TTC_{i_m}(e'_{-i}).$$

By $o \in O(e, t-1)$, there is $\ell \in I(e, t-1)$ with $TTC_\ell(e) = o$. By the induction hypothesis, (3) holds for $t-1$ and thus,

$$\ell \in I(e'_{-i}, t-1) \quad \text{and} \quad TTC_\ell(e'_{-i}) = TTC_\ell(e) = o \in O(e'_{-i}, t-1).$$

By $o \in O(e, t-1)$ and $t-1 < r$, $o \notin \{\omega_i, \omega_j\}$. Thus, $\{o, \omega_{i_{m+1}}\} \subset O \setminus \{\omega'_i\}$ and $o \succ_{i_m} \omega_{i_{m+1}}$ together imply $o \succ_{i_m}^{O\setminus\{\omega'_i\}} \omega_{i_{m+1}}$. Considering that

$$o \succ_{i_m}^{O\setminus\{\omega'_i\}} \omega_{i_{m+1}} = \omega'_{i_{m+1}} = TTC_{i_m}(e'_{-i}) \quad \text{and} \quad i_m \in C \in \mathbb{C}(e'_{-i}, r^*),$$

it holds that

$$o \in O^{r^*-1}(e'_{-i}),$$

which contradicts $o \in O(e'_{-i}, t-1)$ where $r^* \leq t-1$. $\square$

A.3 Proof of Proposition 1

Suppose on the contrary that there are $e = (I, O, \succ, \omega) \in \mathscr{D}$, $\{i, j\} \subseteq I$ with $i \neq j$, and $\{y_i, y_j\} \subseteq \mathcal{O}$ such that $e_{-i} \in \mathscr{D}$, $\{y_i, y_j\} = \{\omega_i, TTC_j(e_{-i})\}$, and

$$y_i \succ_i TTC_i(e) \quad \text{and} \quad y_j \succ_j TTC_j(e).$$

Since TTC is *individually rational*, $y_i \succ_i TTC_i(e) \succsim_i \omega_i$, which implies $y_i \neq \omega_i$. Thus,

$$TTC_j(e_{-i}) \succ_i TTC_i(e) \quad \text{and} \quad \omega_i \succ_j TTC_j(e).$$

Since $\omega_i \succ_j TTC_j(e)$,

$$r_i < r_j, \tag{7}$$

where $(r_i, r_j) \in \mathbb{N}^2$ is such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. Let $o = TTC_j(e_{-i})$. Since $o \succ_i TTC_i(e)$, $r_i \geq 2$ and $o \in O^{r_i - 1}(e)$. Then, there are $s' \in \{1, 2, \ldots, r_i - 1\}$ and $\ell \in I(e, s')$ such that $TTC_\ell(e) = o$. By Lemma 2 and (7),

$$\ell \in I(e_{-i}, s') \quad \text{and} \quad TTC_\ell(e_{-i}) = TTC_\ell(e) = o.$$

Since $s' < r_i < r_j$, $\ell \neq j$, which contradicts $TTC_j(e_{-i}) = o$. □

A.4 Proof of Theorem 2

We prove only the "only if" part. Let f be a rule on $\mathscr{D}$ satisfying the three axioms. Suppose on the contrary that $f \neq TTC$. For each $e = (I, O, \succ, \omega) \in \mathscr{D}$, let

$$\sigma(e) = \sum_{i \in I} \big| \{o \in O \colon o \succsim_i \omega_i\} \big|.$$

Fix an economy $\check{e} = (I, O, \check{\succ}, \omega) \in \mathscr{D}$ such that $f(\check{e}) \neq TTC(\check{e})$ and for each $e \in \mathscr{D}$,

$$\sigma(e) < \sigma(\check{e}) \implies f(e) = TTC(e). \tag{8}$$

Let

$$I_f = \{i \in I \colon f_i(\check{e}) \check{\succ}_i TTC_i(\check{e})\} \quad \text{and} \quad O_f = \left\{o \in O \colon \exists\, i \in I_f, o = \omega_i\right\};$$
$$I_t = \{i \in I \colon TTC_i(\check{e}) \check{\succ}_i f_i(\check{e})\} \quad \text{and} \quad O_t = \{o \in O \colon \exists\, i \in I_t, o = \omega_i\}.$$

By $f(\check{e}) \neq TTC(\check{e})$, $I_f \cup I_t \neq \emptyset$. Let

$$\overline{I} = I \setminus \left(I_f \cup I_t\right) \quad \text{and} \quad \overline{O} = O \setminus \left(O_f \cup O_t\right).$$

Note that for each $i \in \overline{I}$, $f_i(\check{e}) = TTC_i(\check{e})$ because $\check{\succsim}_i$ is strict. The next claim follows from the proof in Sethuraman (2016) or Ekici and Sethuraman (2024). For completeness, we provide the proof.

Claim 1 (Sethuraman (2016); Ekici and Sethuraman (2024)).

(i) *For each $i \in I_t$, $f_i(\check{e}) = \omega_i$ and for each $o \in O \setminus \{TTC_i(\check{e}), \omega_i\}$,*

$$TTC_i(\check{e}) \check{\succ}_i f_i(\check{e}) = \omega_i \check{\succ}_i o.$$

(ii) *$I_f = \emptyset$ and $I_t \neq \emptyset$.*

(iii) *For each $i \in I_t$, $TTC_i(\check{e}) \in O_t$.*

Proof of Claim 1. We first prove (i). Let $i \in I_t$. By the *individual rationality* of f,

$$TTC_i(\check{e}) \check{\succ}_i f_i(\check{e}) \check{\succsim}_i \omega_i.$$

Suppose on the contrary that there is $o \in O \setminus \{TTC_i(\check{e})\}$ such that $o \check{\succ}_i \omega_i$. Since $\mathscr{D}$ satisfies D3, there is $\succ_i^{\uparrow} \in \mathscr{P}_O$ such that

$$\begin{array}{c} \succ_i^{\uparrow} \\ \hline TTC_i(\check{e}) \\ \omega_i \\ \vdots \end{array}$$

and $\left(I, O, \left(\succ_i^{\uparrow}, \check{\succ}_{-i}\right), \omega\right) \in \mathscr{D}$. Let $e^{\uparrow} = \left(I, O, \left(\succ_i^{\uparrow}, \check{\succ}_{-i}\right), \omega\right)$. Note that $\sigma(e^{\uparrow}) < \sigma(\check{e})$. Then, by (8),

$$f(e^{\uparrow}) = TTC(e^{\uparrow}). \tag{9}$$

Also, we have $TTC_i(e^{\uparrow}) = TTC_i(\check{e})$; otherwise, $TTC_i(\check{e}) \succ_i^{\uparrow} TTC_i(e^{\uparrow})$, in violation of the *strategy-proofness* of TTC. Further, $f_i(e^{\uparrow}) \neq TTC_i(\check{e})$; otherwise, $f_i(e^{\uparrow}) = TTC_i(\check{e}) \check{\succ}_i f_i(\check{e})$, in violation of the *strategy-proofness* of f. That is, $f(e^{\uparrow}) \neq TTC(e^{\uparrow})$, which contradicts (9). Hence, $\check{\succ}_i$ is such that

$$\frac{\check{\succ}_i}{\begin{array}{c} TTC_i(\check{e}) \\ f_i(\check{e}) = \omega_i \\ \vdots \end{array}}$$

We next prove (ii) and (iii) in the following six steps.

Step 1: For each $i \in I_f$, $TTC_i(\check{e}) = \omega_i$ and for each $o \in O \setminus \{f_i(\check{e}), \omega_i\}$, $f_i(\check{e}) \check{\succ}_i TTC_i(\check{e}) = \omega_i \check{\succ}_i o$. It follows from the similar argument to (i).

Step 2: For each $i \in \overline{I}$, $f_i(\check{e}) = TTC_i(\check{e}) \in \overline{O}$. Let $i \in \overline{I}$. For each $j \in I_f$, by $TTC_j(\check{e}) = \omega_j$ (Step 1), $TTC_i(\check{e})(= f_i(\check{e})) \neq \omega_j$. For each $j \in I_t$, by (i), $f_j(\check{e}) = \omega_j$, which implies $f_i(\check{e})(= TTC_i(\check{e})) \neq \omega_j$. Hence, $f_i(\check{e}) = TTC_i(\check{e}) \in \overline{O}$.

Step 3: For each $i \in I_f$, $f_i(\check{e}) \in O_f$. Let $i \in I_f$. For each $j \in I_t$, by (i), $f_j(\check{e}) = \omega_j$, which implies $f_i(\check{e}) \neq \omega_j$. Since for each $j \in \overline{I}$, $f_j(\check{e}) \in \overline{O}$ (Step 2) and $|\overline{I}| = |\overline{O}|$, $f_i(\check{e}) \notin \overline{O}$. Hence, $f_i(\check{e}) \in O_f$.

Step 4: $I_f = \emptyset$. Suppose on the contrary that $I_f \neq \emptyset$. Let $i_1 \in I_f$. By $f_{i_1}(\check{e}) \neq \omega_{i_1}$ (Step 1) and $f_{i_1}(\check{e}) \in O_f$ (Step 3), there is $i_2 \in I_f \setminus \{i_1\}$ such that $f_{i_1}(\check{e}) = \omega_{i_2}$. Similarly, there is $i_3 \in I_f \setminus \{i_2\}$ such that $f_{i_2}(\check{e}) = \omega_{i_3}$, and so on. Since $|I_f|$ is finite, there is a set of agents $\{i_1(= i_{S+1}), i_2, \dots, i_S\} \subseteq I_f$ such that $S \geq 2$, for each $\{s, s'\} \subseteq \{1, 2, \dots, S\}$ with $s \neq s'$, $i_s \neq i_{s'}$, and $f_{i_s}(\check{e}) = \omega_{i_{s+1}}$. Since for each $s \in \{1, 2, \dots, S\}$, $f_{i_s}(\check{e})(= \omega_{i_{s+1}})$ is i_s's most preferred object according to $\check{\succ}_{i_s}$ (Step 1), $TTC_{i_s}(\check{e}) = \omega_{i_{s+1}} = f_{i_s}(\check{e})$, a contradiction.

Step 5: $I_t \neq \emptyset$. By $I_f \cup I_t \neq \emptyset$ and $I_f = \emptyset$ (Step 4), $I_t \neq \emptyset$.

Step 6: For each $i \in I_t$, $TTC_i(\check{e}) \in O_t$. Let $i \in I_t$. By $I_f = \emptyset$ (Step 4), $TTC_i(\check{e}) \notin O_f$. Since for each $j \in \overline{I}$, $TTC_j(\check{e}) \in \overline{O}$ (Step 2) and $|\overline{I}| = |\overline{O}|$, $TTC_i(\check{e}) \notin \overline{O}$. Hence, $TTC_i(\check{e}) \in O_t$. □

By $I_t \neq \emptyset$ (Claim 1(ii)), there is $i_1 \in I_t$. By $TTC_{i_1}(\check{e}) \neq \omega_{i_1}$ and $TTC_{i_1}(\check{e}) \in O_t$ (Claim 1(iii)), there is $i_2 \in I_t \setminus \{i_1\}$ such that $TTC_{i_1}(\check{e}) = \omega_{i_2}$. Similarly, there is $i_3 \in I_t \setminus \{i_2\}$ such that $TTC_{i_2}(\check{e}) = \omega_{i_3}$, and so on. Since $|I_t|$ is finite, there is a set of agents $\{i_1(= i_{S+1}), i_2, \dots, i_S\} \subseteq I_t$ such that $S \geq 2$, for each $\{s, s'\} \subseteq \{1, 2, \dots, S\}$ with $s \neq s'$, $i_s \neq i_{s'}$, and $TTC_{i_s}(\check{e}) = \omega_{i_{s+1}}$. There are two cases.

- **Case 1: $S = 2$.** Then, by Claim 1(i), $(\check{\succ}_{i_1}, \check{\succ}_{i_2})$ is such that

$$\begin{array}{cc} \check{\succ}_{i_1} & \check{\succ}_{i_2} \\ \hline TTC_{i_1}(\check{e}) = \omega_{i_2} & TTC_{i_2}(\check{e}) = \omega_{i_1} \\ f_{i_1}(\check{e}) = \omega_{i_1} & f_{i_2}(\check{e}) = \omega_{i_2} \\ \vdots & \vdots \end{array}$$

Consider $\check{e}_{-i_1} \in \mathscr{E}$. Since $\mathscr{D}$ satisfies D1, $\check{e}_{-i_1} \in \mathscr{D}$. By the *individual rationality* of f, $f_{i_2}(\check{e}_{-i_1}) = \omega_{i_2}$. Hence,

$$f_{i_2}(\check{e}_{-i_1}) = \omega_{i_2} \check{\succ}_{i_1} \omega_{i_1} = f_{i_1}(\check{e}) \quad \text{and} \quad \omega_{i_1} \check{\succ}_{i_2} \omega_{i_2} = f_{i_2}(\check{e}),$$

in violation of *weak withdrawal-proofness*.

• **Case 2: $S \geq 3$.** Then, by Claim 1(i), $(\check{\succ}_{i_s})_{s=1}^{S}$ is such that

$$\begin{array}{ccccc} \check{\succ}_{i_1} & \check{\succ}_{i_2} & \cdots & \check{\succ}_{i_{S-1}} & \check{\succ}_{i_S} \\ \hline TTC_{i_1}(\check{e}) = \omega_{i_2} & TTC_{i_2}(\check{e}) = \omega_{i_3} & \cdots & TTC_{i_{S-1}}(\check{e}) = \omega_{i_S} & TTC_{i_S}(\check{e}) = \omega_{i_1} \\ f_{i_1}(\check{e}) = \omega_{i_1} & f_{i_2}(\check{e}) = \omega_{i_2} & \cdots & f_{i_{S-1}}(\check{e}) = \omega_{i_{S-1}} & f_{i_S}(\check{e}) = \omega_{i_S} \\ \vdots & \vdots & & \vdots & \vdots \end{array}$$

Since $\mathscr{D}$ satisfies D3, there is $\hat{\succ}_{i_1} \in \mathscr{P}_O$ such that

$$\begin{array}{c} \hat{\succ}_{i_1} \\ \hline \omega_{i_2} \\ \omega_{i_3} \\ \omega_{i_1} \\ \vdots \end{array}$$

and $(I, O, (\hat{\succ}_{i_1}, \check{\succ}_{-i_1}), \omega) \in \mathscr{D}$. Let $\hat{e} = (I, O, (\hat{\succ}_{i_1}, \check{\succ}_{-i_1}), \omega)$. Then, $f_{i_1}(\hat{e}) \neq \omega_{i_2}$; otherwise, $f_{i_1}(\hat{e}) = \omega_{i_2} \check{\succ}_{i_1} \omega_{i_1} = f_{i_1}(\check{e})$, in violation of the *strategy-proofness* of f. By the *individual rationality* of f, $f_{i_1}(\hat{e}) \in \{\omega_{i_3}, \omega_{i_1}\}$. There are two subcases.

∘ *Subcase 2-1*: $f_{i_1}(\hat{e}) = \omega_{i_3}$. By $f_{i_2}(\hat{e}) \neq \omega_{i_3}$ and the *individual rationality* of f, $f_{i_2}(\hat{e}) = \omega_{i_2}$.

∘ *Subcase 2-2*: $f_{i_1}(\hat{e}) = \omega_{i_1}$. By $f_{i_S}(\hat{e}) \neq \omega_{i_1}$ and the *individual rationality* of f, $f_{i_S}(\hat{e}) = \omega_{i_S}$. Similarly, $f_{i_{S-1}}(\hat{e}) = \omega_{i_{S-1}}$, and so on. Hence, $f_{i_2}(\hat{e}) = \omega_{i_2}$.

In both subcases, we have $f_{i_2}(\hat{e}) = \omega_{i_2}$.

Now consider

$$\hat{e}_{-i_2} = \left(I \setminus \{i_2\}, O \setminus \{\omega_{i_2}\}, \left(\hat{\succ}_{i_1}^{O\setminus\{\omega_{i_2}\}}, \check{\succ}_{-\{i_1,i_2\}}^{O\setminus\{\omega_{i_2}\}} \right), \omega_{-i_2} \right).$$

Since $\mathscr{D}$ satisfies D1, $\hat{e}_{-i_2} \in \mathscr{D}$. Note that $\left(\hat{\succ}_{i_1}^{O\setminus\{\omega_{i_2}\}}, \check{\succ}_{-\{i_1,i_2\}}^{O\setminus\{\omega_{i_2}\}} \right)$ is such that

$\hat{\succ}_{i_1}^{O\setminus\{\omega_{i_2}\}}$	$\check{\succ}_{i_3}^{O\setminus\{\omega_{i_2}\}}$	$\cdots$	$\check{\succ}_{i_{S-1}}^{O\setminus\{\omega_{i_2}\}}$	$\check{\succ}_{i_S}^{O\setminus\{\omega_{i_2}\}}$
ω_{i_3}	ω_{i_4}	$\cdots$	ω_{i_S}	ω_{i_1}
ω_{i_1}	ω_{i_3}	$\cdots$	$\omega_{i_{S-1}}$	ω_{i_S}
$\vdots$	$\vdots$		$\vdots$	$\vdots$

Since $\sigma(\hat{e}_{-i_2}) = \sigma(\check{e}) - 2 < \sigma(\check{e})$, by (8), $f(\hat{e}_{-i_2}) = TTC(\hat{e}_{-i_2})$, which implies $f_{i_1}(\hat{e}_{-i_2}) = TTC_{i_1}(\hat{e}_{-i_2}) = \omega_{i_3}$. Hence, by $f_{i_1}(\hat{e}) \in \{\omega_{i_3}, \omega_{i_1}\}$ and $f_{i_2}(\hat{e}) = \omega_{i_2}$,

$$\omega_{i_2} \mathrel{\hat{\succ}_{i_1}} f_{i_1}(\hat{e}) \quad \text{and} \quad f_{i_1}(\hat{e}_{-i_2}) = \omega_{i_3} \mathrel{\check{\succ}_{i_2}} \omega_{i_2} = f_{i_2}(\hat{e}),$$

in violation of *weak withdrawal-proofness*. □

A.5 Proofs of Proposition 2 and Theorem 3

Since Proposition 2 follows from the "if" part of Theorem 3, we provide the proof of Theorem 3.

A.5.1 The "if" part

Since TTC clearly satisfies *individual rationality*, we only need to show the *strict pre-delivery-proofness* of TTC. Suppose on the contrary that there are $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $\{i, j\} \subseteq I$ with $i \neq j$ such that $e_{-i}^{i,j} \in \mathscr{D}$,

$$\omega_j \succsim_i TTC_i(e) \quad \text{and} \quad TTC_j(e_{-i}^{i,j}) \succ_j TTC_j(e).$$

Let $(r_i, r_j) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. By $\omega_j \succsim_i TTC_i(e)$,

$$r_j \leq r_i. \tag{10}$$

Let $TTC_j(e_{-i}^{i,j}) = o \in O \setminus \{\omega_j\}$. Since $o \succ_j TTC_j(e)$, $r_j \geq 2$ and $o \in O^{r_j - 1}(e)$. Then, there are $s' \in \{1, 2, \ldots, r_j - 1\}$ and $\ell \in I(e, s')$ with $TTC_\ell(e) = o$. By Lemma 3 and (10),

$$\ell \in I(e_{-i}^{i,j}, s') \quad \text{and} \quad TTC_\ell(e_{-i}^{i,j}) = TTC_\ell(e) = o.$$

Since $s' < r_j$, $\ell \neq j$, which contradicts $TTC_j(e_{-i}^{i,j}) = o$.

A.5.2 The "only if" part

We first prove an additional lemma, which states the following: if a set of agents $C = \{i_1, i_2, \ldots, i_N\}$ forms a cycle in Round r of the TTC algorithm at e, then the reduced set $C \setminus \{i_{m-1}\}$ forms a cycle in an earlier round than Round r at $e^{i_{m-1},i_m}_{-i_{m-1}}$, where i_m's endowment is $\omega_{i_{m-1}}$.

Lemma 4. *Let $e = (I, O, \succ, \omega) \in \mathscr{D}$, $r \in \mathbb{N}$, $C = \{i_1 (= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, r)$, and $m \in \{1, 2, \ldots, N\}$. Suppose that $N \geq 2$ and $e^{i_{m-1},i_m}_{-i_{m-1}} \in \mathscr{D}$. Then,*

- *there is $r'_{i_m} \leq r$ such that*

$$i_m \in I\left(e^{i_{m-1},i_m}_{-i_{m-1}}, r'_{i_m}\right) \quad \textit{and} \quad C_{-i_{m-1}} = C \setminus \{i_{m-1}\} \in \mathbb{C}\left(e^{i_{m-1},i_m}_{-i_{m-1}}, r'_{i_m}\right);$$

  and

- *for each $i_n \in C_{-i_{m-1}}$, $TTC_{i_n}(e^{i_{m-1},i_m}_{-i_{m-1}}) = TTC_{i_n}(e) = \omega_{i_{n+1}}$.*

Proof. For simplicity of notation, let

$$e' = e^{i_{m-1},i_m}_{-i_{m-1}} = \left(I \setminus \{i_{m-1}\}, O \setminus \{\omega_{i_m}\}, \succ^{O\setminus\{\omega_{i_m}\}}_{-i_{m-1}}, \omega^{i_{m-1},i_m}_{-i_{m-1}}\right) \in \mathscr{D}.$$

Note that

- for each $i_n \in C_{-i_{m-1}}$, $TTC_{i_n}(e) = \omega_{i_{n+1}}$;
- $C_{-i_{m-1}} \subseteq I \setminus \{i_{m-1}\}$;
- for each $i_n \in C_{-i_{m-1}} \setminus \{i_m\}$, $\omega^{i_{m-1},i_m}_{i_n} = \omega_{i_n} \in O \setminus \{\omega_{i_m}\}$; and
- $\omega^{i_{m-1},i_m}_{i_m} = \omega_{i_{m-1}} \in O \setminus \{\omega_{i_m}\}$.

There are two cases.

• **Case 1: $r = 1$.** By $C \in \mathbb{C}(e, 1)$,

$$\forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ \forall\, o \in O \setminus \{\omega_{i_{n+1}}\},\ \omega^{i_{m-1},i_m}_{i_{n+1}} = \omega_{i_{n+1}} = TTC_{i_n}(e) \succ_{i_n} o;$$
$$\forall\, o \in O \setminus \{\omega_{i_{m-1}}\},\ \omega^{i_{m-1},i_m}_{i_m} = \omega_{i_{m-1}} = TTC_{i_{m-2}}(e) \succ_{i_{m-2}} o.$$

Note that

$$\bigcup_{k \in C_{-i_{m-1}}} \left\{\omega^{i_{m-1},i_m}_k\right\} \subseteq O \setminus \{\omega_{i_m}\}.$$

Hence,

$$\forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ \forall\, o \in (O \setminus \{\omega_{i_m}\}) \setminus \{\omega_{i_{n+1}}\},\ \omega_{i_{n+1}}^{i_{m-1},i_m} = \omega_{i_{n+1}} \succ_{i_n}^{O\setminus\{\omega_{i_m}\}} o;$$
$$\forall\, o \in (O \setminus \{\omega_{i_m}\}) \setminus \{\omega_{i_{m-1}}\},\ \omega_{i_m}^{i_{m-1},i_m} = \omega_{i_{m-1}} \succ_{i_{m-2}}^{O\setminus\{\omega_{i_m}\}} o.$$

This implies that $C_{-i_{m-1}} \in \mathbb{C}(e',1)$ and

$$\forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ TTC_{i_n}(e') = \omega_{i_{n+1}}^{i_{m-1},i_m} = \omega_{i_{n+1}} = TTC_{i_n}(e);$$
$$TTC_{i_{m-2}}(e') = \omega_{i_m}^{i_{m-1},i_m} = \omega_{i_{m-1}} = TTC_{i_{m-2}}(e).$$

- **Case 2: $r \geq 2$.** By $C \in \mathbb{C}(e,r)$,

$$\{i_{m-1}, i_m\} \subseteq I(e,r). \tag{11}$$

Then, (11) and Lemma 3 together imply that for each $t \in \{1,2,\ldots,r-1\}$ and each $i \in I(e,t)$,

$$i \in I(e',t) \quad \text{and} \quad TTC_i(e') = TTC_i(e). \tag{12}$$

For each $i_n \in C_{-i_{m-1}}$, let $r'_{i_n} \in \mathbb{N}$ be such that $i_n \in I(e', r'_{i_n})$. We proceed in four steps.

Step 1: For each $i_n \in C_{-i_{m-1}}$, $\omega_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e')$. Let $i_n \in C_{-i_{m-1}}$ and $o \in U^+(\succ_{i_n}, \omega_{i_{n+1}})$. By $i_n \in C \in \mathbb{C}(e,r)$ and $TTC_{i_n}(e) = \omega_{i_{n+1}}$, $o \in O^{r-1}(e)$. Then, there are $s' \in \{1,2,\ldots,r-1\}$ and $\ell \in I(e,s')$ with $TTC_\ell(e) = o$. By (12), $\ell \in I(e',s')$ and $TTC_\ell(e') = TTC_\ell(e) = o$. Hence, $TTC_{i_n}(e') \neq o$. That is, $\omega_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e')$.

Step 2: There is $r^* \in \mathbb{N}$ such that for each $i_n \in C_{-i_{m-1}}$, $r^* = r'_{i_n}$. By Step 1,

$$\begin{aligned} \forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ \omega_{i_{n+1}}^{i_{m-1},i_m} &= \omega_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e'); \\ \omega_{i_m}^{i_{m-1},i_m} &= \omega_{i_{m-1}} \succsim_{i_{m-2}} TTC_{i_{m-2}}(e'). \end{aligned} \tag{13}$$

Note that

$$\bigcup_{k \in C_{-i_{m-1}}} \left\{\omega_k^{i_{m-1},i_m}\right\} \subseteq O \setminus \{\omega_{i_m}\}.$$

It then follows from (13) that

$$\begin{aligned}\forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ \omega_{i_{n+1}}^{i_{m-1},i_m} &= \omega_{i_{n+1}} \succsim_{i_n}^{O\setminus\{\omega_{i_m}\}} TTC_{i_n}(e'); \\ \omega_{i_m}^{i_{m-1},i_m} &= \omega_{i_{m-1}} \succsim_{i_{m-2}}^{O\setminus\{\omega_{i_m}\}} TTC_{i_{m-2}}(e').\end{aligned} \tag{14}$$

Hence,

$$r'_{i_1} \le r'_{i_N} \le \cdots \le r'_{i_{m+1}} \le r'_{i_m} \le r'_{i_{m-2}} \le \cdots \le r'_{i_1}.$$

Then, there is $r^* \in \mathbb{N}$ such that for each $i_n \in C_{-i_{m-1}}$, $r^* = r'_{i_n}$.

Step 3: For each $i_n \in C_{-i_{m-1}}$, $TTC_{i_n}(e') = TTC_{i_n}(e) = \omega_{i_{n+1}}$ and $C_{-i_{m-1}} \in \mathbb{C}(e', r^*)$. Step 2 implies that

$$\begin{aligned} C_{-i_{m-1}} &\subseteq (I \setminus \{i_{m-1}\}) \setminus I^{r^*-1}(e'); \\ \bigcup_{k \in C_{-i_{m-1}}} \left\{\omega_k^{i_{m-1},i_m}\right\} &\subseteq (O \setminus \{\omega_{i_m}\}) \setminus O^{r^*-1}(e'). \end{aligned}$$

Recall (14). If either

- for some $i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\}$, $\omega_{i_{n+1}}^{i_{m-1},i_m} = \omega_{i_{n+1}} \succ_{i_n}^{O\setminus\{\omega_{i_m}\}} TTC_{i_n}(e')$, or
- $\omega_{i_m}^{i_{m-1},i_m} = \omega_{i_{m-1}} \succ_{i_{m-2}}^{O\setminus\{\omega_{i_m}\}} TTC_{i_{m-2}}(e')$,

then there is $j \in C_{-i_{m-1}}$ such that j does not receive the most preferred object among $(O \setminus \{\omega_{i_m}\}) \setminus O^{r^*-1}(e')$ according to $\succ_j^{O\setminus\{\omega_{i_m}\}}$ in Round $r^* = r'_j$ of the TTC algorithm at e', which is a contradiction. Hence,

$$\begin{aligned}\forall\, i_n \in C_{-i_{m-1}} \setminus \{i_{m-2}\},\ TTC_{i_n}(e') = \omega_{i_{n+1}}^{i_{m-1},i_m} &= \omega_{i_{n+1}} = TTC_{i_n}(e); \\ TTC_{i_{m-2}}(e') = \omega_{i_m}^{i_{m-1},i_m} &= \omega_{i_{m-1}} = TTC_{i_{m-2}}(e),\end{aligned}$$

and $C_{-i_{m-1}} \in \mathbb{C}(e', r^* = r'_{i_m})$.

Step 4: $r^* = r'_{i_m} \le r$. By Step 2, $r^* = r'_{i_m}$. We below show $r^* \le r$. Suppose on the contrary that $r < r^*$. Then, by $C_{-i_{m-1}} \in \mathbb{C}(e', r^*)$ (Step 3),

$$C_{-i_{m-1}} \subseteq (I \setminus \{i_{m-1}\}) \setminus I^{r-1}(e').$$

By $C_{-i_{m-1}} \notin \mathbb{C}(e', r)$, there are $i_{m'} \in C_{-i_{m-1}}$, $k \in (I \setminus \{i_{m-1}\}) \setminus I^{r-1}(e')$, and $o \in$

$(O \setminus \{\omega_{i_m}\}) \setminus O^{r-1}(e')$ such that

$$i_{m'} \overset{(e',r)}{\longrightarrow} k \quad \text{and} \quad \omega_k^{i_{m-1},i_m} = o \succ_{i_{m'}}^{O\setminus\{\omega_{i_m}\}} TTC_{i_{m'}}(e') = TTC_{i_{m'}}(e).$$

Note that by $\{o, TTC_{i_{m'}}(e') = TTC_{i_{m'}}(e)\} \subseteq O \setminus \{\omega_{i_m}\} \subset O$,

$$o \succ_{i_{m'}} TTC_{i_{m'}}(e). \tag{15}$$

By $i_{m'} \in C \in \mathbb{C}(e,r)$, $o \in O^{r-1}(e)$. Then, there are $s' \in \{1, 2, \ldots, r-1\}$ and $\ell \in I(e, s')$ with $TTC_\ell(e) = o$. This together with (12) implies that

$$\ell \in I(e', s') \quad \text{and} \quad TTC_\ell(e') = TTC_\ell(e) = o \in O(e', s').$$

This contradicts $o \in (O \setminus \{\omega_{i_m}\}) \setminus O^{r-1}(e')$. □

Proof of the "only if" part of Theorem 3. Suppose that $f : \mathscr{D} \to \mathcal{X}$ satisfies the two axioms. We show that for each $r \in \mathbb{N}$, each $e \in \mathscr{D}$, each $C \in \mathbb{C}(e,r)$, and each $i \in C$, $f_i(e) = TTC_i(e)$.[14]

Let $r \in \mathbb{N}$. Suppose that

$$\forall\, t \in \{1, 2, \ldots, r-1\},\ \forall\, e \in \mathscr{D},\ \forall\, C \in \mathbb{C}(e,t),\ \forall\, i \in C,\ f_i(e) = TTC_i(e). \tag{16}$$

Then,

$$\forall\, e = (I, O, \succ, \omega) \in \mathscr{D},\ O^{r-1}(e) = \{o \in O \colon \exists\, i \in I^{r-1}(e),\ o = f_i(e)\}. \tag{17}$$

We use induction on $|C|$.

Base step. Let $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $C = \{i\} \in \mathbb{C}(e,r)$ (that is, $|C| = 1$). Then,

$$\begin{aligned} \omega_i &\in O \setminus O^{r-1}(e); \\ \forall\, o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_i\}\right),\ &TTC_i(e) = \omega_i \succ_i o. \end{aligned} \tag{18}$$

[14] To prove this claim, it is necessary to employ induction on r. Since the cases $r = 1$ and $r \geq 2$ share similar arguments, we present a unified proof applicable to both cases. We note some distinctions in the case where $r = 1$: (16) is vacuously true when $r = 1$; (17) reduces to the statement that for each $e = (I, O, \succ, \omega) \in \mathscr{D}$, $O^{r-1}(e) = O^0(e) = \emptyset$; in the base step, (18) and *individual rationality* immediately imply $f_i(e) = \omega_i = TTC_i(e)$; and in the induction step, (20) directly follows from (19), and only Case 2 occurs.

By $i \in I(e,r)$ and (17), for each $o \in O^{r-1}(e)$, $f_i(e) \neq o$. Hence, by (18) and *individual rationality*, $f_i(e) = \omega_i = TTC_i(e)$.

Induction hypothesis. Let $N \in \{2,3,\ldots,|\mathcal{I}|\}$. For each $e \in \mathscr{D}$, each $C \in \mathbb{C}(e,r)$ with $|C| \leq N-1$, and each $i \in C$, $f_i(e) = TTC_i(e)$.

Induction step. Let $N \in \{2,3,\ldots,|\mathcal{I}|\}$, $e = (I,O,\succ,\omega) \in \mathscr{D}$, and $C = \{i_1(= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e,r)$. Then, for each $i_n \in C$,

$$\begin{aligned} \omega_{i_{n+1}} &\in O \setminus O^{r-1}(e); \\ \forall\, o \in O \setminus (O^{r-1}(e) \cup \{\omega_{i_{n+1}}\}),\ TTC_{i_n}(e) &= \omega_{i_{n+1}} \succ_{i_n} o. \end{aligned} \tag{19}$$

In addition, by $i_n \in I(e,r)$ and (17), for each $o \in O^{r-1}(e)$, $f_{i_n}(e) \neq o$. This, together with (19), implies

$$TTC_{i_n}(e) = \omega_{i_{n+1}} \succsim_{i_n} f_{i_n}(e). \tag{20}$$

Suppose on the contrary that

$$\exists\, i_m \in C,\ f_{i_m}(e) \neq \omega_{i_{m+1}} = TTC_{i_m}(e). \tag{21}$$

Let

$$e' = e^{i_{m-1},i_m}_{-i_{m-1}} = \left(I \setminus \{i_{m-1}\}, O \setminus \{\omega_{i_m}\}, \succ^{O\setminus\{\omega_{i_m}\}}_{-i_{m-1}}, \omega^{i_{m-1},i_m}_{-i_{m-1}}\right).$$

Since $\mathscr{D}$ satisfies D4, $e' \in \mathscr{D}$.[15] By Lemma 4, $TTC_{i_m}(e') = TTC_{i_m}(e) = \omega_{i_{m+1}}$, $C_{-i_{m-1}} \in \mathbb{C}(e', r'_{i_m})$, and $r'_{i_m} \leq r$. There are two cases.

- **Case 1: $r'_{i_m} < r$.** By (16) and $C_{-i_{m-1}} \in \mathbb{C}(e', r'_{i_m})$, $f_{i_m}(e') = TTC_{i_m}(e') = \omega_{i_{m+1}}$.
- **Case 2: $r'_{i_m} = r$.** By the induction hypothesis, $C_{-i_{m-1}} \in \mathbb{C}(e', r'_{i_m} = r)$ and $|C_{-i_{m-1}}| \leq N-1$, $f_{i_m}(e') = TTC_{i_m}(e') = \omega_{i_{m+1}}$.

That is, in both cases, $f_{i_m}(e') = TTC_{i_m}(e') = \omega_{i_{m+1}}$. Hence, by (20) and (21),

$$\omega_{i_m} \succsim_{i_{m-1}} f_{i_{m-1}}(e) \quad \text{and} \quad f_{i_m}(e') = \omega_{i_{m+1}} \succ_{i_m} f_{i_m}(e).$$

This contradicts *strict pre-delivery-proofness*. $\square$

[15]Considering that $N \geq 2$, $i_{m-1}(\neq i_m)$ surely exists.

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Online Appendix to "Endowment manipulations involving population variations in object exchange problems" by Fujinaka and Wakayama (October 26, 2025)

B Appendix: Omitted proofs in the main text

B.1 Remark 1

Since both TTC and NT clearly satisfy *individual rationality*, we now show that both rules satisfy *withdrawal-proofness*. Below, we show the *withdrawal-proofness* of TTC, as that of NT can be established by a parallel argument.

Let $e = (I, O, \succ', \omega') \in \mathscr{D}(\succ, \omega)$. There are two cases.

• **Case 1: $\{1,2,3\} \subseteq I$.** Then, no pair has an incentive to manipulate, as all agents receive their most preferred objects.

• **Case 2: $\{1,2,3\} \not\subseteq I$.** Consider a pair $\{i,j\} \subseteq I$ with $i \neq j$. Since $\{1,2,3\} \not\subseteq I$ and $\{1,2,3\} \not\subseteq I \setminus \{i\}$, $TTC(e) = \omega'$ and $TTC(e_{-i}) = \omega'_{-i}$. Then,

$$(TTC_i(e), TTC_j(e)) = (o_i, o_j) \quad \text{and} \quad \{\omega'_i, TTC_j(e_{-i})\} = \{o_i, o_j\}.$$

Suppose that there are $k \in \{i,j\}$ and $y_k \in \{\omega'_i, TTC_j(e_{-i})\}$ such that $y_k \succ'_k o_k = TTC_k(e)$. Then, $k \in \{1,2,3\}$; if $k \geq 4$, then $TTC_k(e) = o_k \succsim'_k y_k$, a contradiction. We only consider the case where $k = 1$ since we can consider the other cases similarly. By $y_1 \succ'_1 o_1 = TTC_1(e)$, $y_1 = o_2$ and $\{i,j\} = \{1,2\}$. Then, $TTC_2(e) = o_2 \succ'_2 o_1$, which implies that this pair has no incentive to manipulate.

B.2 Example 5

We now show that $\widehat{f}$ satisfies *endowments-swapping-and-withdrawal-proofness*. Let $e = (I, O, \succ', \omega') \in \mathscr{D}(\succ)$. There are two cases.

• **Case 1: $e \neq \widehat{e}$.** Note that $\widehat{f}(e) = TTC(e)$ and for each $\{i,j\} \subseteq I$ with $i \neq j$, by $e^{i,j}_{-i} \neq \widehat{e}$, $\widehat{f}_j(e^{i,j}_{-i}) = TTC_j(e^{i,j}_{-i})$. Since TTC satisfies *strict pre-delivery-proofness*, there is no pair $\{i,j\} \subseteq I$ with $i \neq j$ such that $\omega_j \succ'_i \widehat{f}_i(e) (= TTC_i(e))$ and $\widehat{f}_j(e^{i,j}_{-i}) (= TTC_j(e^{i,j}_{-i})) \succ'_j \widehat{f}_j(e) (= TTC_j(e))$.

• **Case 2:** $\boldsymbol{e = \widehat{e}}$**.** Since all agents except agent 3 receive their most preferred objects under $\widehat{f}(\widehat{e})$, there is no pair $\{i,j\} \subseteq I$ with $i \neq j$ such that $\omega_j \succ_i \widehat{f}_i(\widehat{e})$ and $\widehat{f}_j(\widehat{e}^{i,j}_{-i}) \succ_j \widehat{f}_j(\widehat{e})$.

C Appendix: Impossibility result for strict endowments-swapping-and-withdrawal-proofness

We begin by defining *strict endowments-swapping-and-withdrawal-proofness*.

***Strict endowments-swapping-and-withdrawal-proofness*:** There are no $e = (I, O, \succ, \omega) \in \mathscr{D}$ and $\{i,j\} \subseteq I$ with $i \neq j$ such that $e^{i,j}_{-i} \in \mathscr{D}$, $\omega_j \succ_i f_i(e)$, and $f_j(e^{i,j}_{-i}) \succsim_j f_j(e)$.

Strict endowments-swapping-and-withdrawal-proofness rules out the possibility that swapping endowments makes the agent who withdrew strictly better off, while the agent who participated may be indifferent between the outcomes with and without manipulation.

In contrast to *strict pre-delivery-proofness*, no rule satisfies both *individual rationality* and *strict endowments-swapping-and-withdrawal-proofness*. As with *strict pre-delivery-proofness*, if a rule satisfies both properties, it must be TTC, and the proof parallels the "only if" part of Theorem 3. However, if domain $\mathscr{D}$ satisfies the following condition in addition to D4, TTC violates *strict endowments-swapping-and-withdrawal-proofness*.

D5. There are $e = (I, O, \succ, \omega) \in \mathscr{D}$, $r \in \mathbb{N}$, $C = \{i_1 (= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, r)$ with $N \geq 2$, and $j \in I \setminus I^r(e)$ such that

- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_1}\}\right)$, $\omega_{i_1} \succ_j o$; and
- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_1}, \omega_j\}\right)$, $\omega_{i_1} \succ_{i_N} \omega_j \succ_{i_N} o$.

If such a cycle $C = \{i_1, i_2, \ldots, i_N\}$ and agent j exist, the pair $\{i_1, j\}$ can manipulate TTC: swapping their endowments followed by withdrawal makes agent j strictly better off since $\omega_{i_1} \succ_j TTC_j(e)$; C is also formed in the TTC algorithm at $e^{i_1,j}_{-j}$, and agent i_1 receives the same object ω_{i_2} under $TTC(e)$ and $TTC(e^{i_1,j}_{-j})$. Thus, TTC violates *strict endowments-swapping-and-withdrawal-proofness*.

We establish the following incompatibility between *individual rationality* and *strict endowments-swapping-and-withdrawal-proofness*.

Theorem 4. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D4 *and* D5*. Then, no rule on $\mathscr{D}$ satisfies individual rationality and strict endowments-swapping-and-withdrawal-proofness.*

We prove the following two lemmas. The first lemma states that if a rule satisfies *individual rationality* and *strict endowments-swapping-and-withdrawal-proofness*, it must be TTC. The second lemma shows that TTC violates *strict endowments-swapping-and-withdrawal-proofness*. Theorem 4 follows from these two lemmas.

Lemma 5. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D4*. If a rule f on $\mathscr{D}$ satisfies individual rationality and strict endowments-swapping-and-withdrawal-proofness, it is TTC on $\mathscr{D}$.*

Proof. Suppose that $f\colon \mathscr{D} \to \mathcal{X}$ satisfies the two axioms. We show that for each $r \in \mathbb{N}$, each $e \in \mathscr{D}$, each $C \in \mathbb{C}(e,r)$, and each $i \in C$, $f_i(e) = TTC_i(e)$.[16]

Let $r \in \mathbb{N}$. Suppose that

$$\forall\, t \in \{1,2,\ldots,r-1\},\ \forall\, e \in \mathscr{D},\ \forall\, C \in \mathbb{C}(e,t),\ \forall\, i \in C,\ f_i(e) = TTC_i(e). \tag{22}$$

Then,

$$\forall\, e = (I,O,\succ,\omega) \in \mathscr{D},\ O^{r-1}(e) = \{o \in O\colon \exists\, i \in I^{r-1}(e),\ o = f_i(e)\}. \tag{23}$$

We use induction on $|C|$.

Base step. Let $e = (I,O,\succ,\omega) \in \mathscr{D}$ and $C = \{i\} \in \mathbb{C}(e,r)$ (that is, $|C| = 1$). Then,

$$\begin{gathered} \omega_i \in O \setminus O^{r-1}(e); \\ \forall\, o \in O \setminus (O^{r-1}(e) \cup \{\omega_i\}), TTC_i(e) = \omega_i \succ_i o. \end{gathered} \tag{24}$$

By $i \in I(e,r)$ and (23), for each $o \in O^{r-1}(e)$, $f_i(e) \neq o$. Hence, by (24) and *individual rationality*, $f_i(e) = \omega_i = TTC_i(e)$.

Induction hypothesis. Let $N \in \{2,3,\ldots,|\mathcal{I}|\}$. For each $e \in \mathscr{D}$, each $C \in \mathbb{C}(e,r)$ with $|C| \leq N-1$, and each $i \in C$, $f_i(e) = TTC_i(e)$.

Induction step. Let $N \in \{2,3,\ldots,|\mathcal{I}|\}$, $e = (I,O,\succ,\omega) \in \mathscr{D}$, and $C = \{i_1(=$

[16] The proof of this claim follows the same lines as the proof of the corresponding claim in the "only if" part of Theorem 3. As in the "only if" part of Theorem 3, we present a unified proof applicable to the cases $r = 1$ and $r \geq 2$.

$i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, r)$. Then, for each $i_n \in C$,

$$\begin{aligned} \omega_{i_{n+1}} &\in O \setminus O^{r-1}(e); \\ \forall\, o \in O \setminus (O^{r-1}(e) \cup \{\omega_{i_{n+1}}\}),\, TTC_{i_n}(e) &= \omega_{i_{n+1}} \succ_{i_n} o. \end{aligned} \tag{25}$$

In addition, by $i_n \in I(e, r)$ and (23), for each $o \in O^{r-1}(e)$, $f_{i_n}(e) \neq o$. This, together with (25), implies

$$TTC_{i_n}(e) = \omega_{i_{n+1}} \succsim_{i_n} f_{i_n}(e). \tag{26}$$

Suppose on the contrary that

$$\exists\, i_m \in C,\ f_{i_m}(e) \neq \omega_{i_{m+1}} = TTC_{i_m}(e). \tag{27}$$

Let

$$e' = e^{i_m, i_{m+1}}_{-i_m} = \left(I \setminus \{i_m\}, O \setminus \{\omega_{i_{m+1}}\}, \succ^{O \setminus \{\omega_{i_{m+1}}\}}_{-i_m}, \omega^{i_m, i_{m+1}}_{-i_m} \right).$$

Since $\mathscr{D}$ satisfies D4, $e' \in \mathscr{D}$.[17] By Lemma 4, $TTC_{i_{m+1}}(e') = TTC_{i_{m+1}}(e) = \omega_{i_{m+2}}$, $C_{-i_m} = \{i_1, \ldots, i_{m-1}, i_{m+1}, \ldots, i_N\} \in \mathbb{C}(e', r'_{i_{m+1}})$, and $r'_{i_{m+1}} \leq r$. There are two cases.

- **Case 1: $r'_{i_{m+1}} < r$.** By (22) and $C_{-i_m} \in \mathbb{C}(e', r'_{i_{m+1}})$, $f_{i_{m+1}}(e') = TTC_{i_{m+1}}(e') = \omega_{i_{m+2}}$.
- **Case 2: $r'_{i_{m+1}} = r$.** By the induction hypothesis, $C_{-i_m} \in \mathbb{C}(e', r'_{i_{m+1}} = r)$ and $|C_{-i_m}| \leq N - 1$, $f_{i_{m+1}}(e') = TTC_{i_{m+1}}(e') = \omega_{i_{m+2}}$.

That is, in both cases, $f_{i_{m+1}}(e') = TTC_{i_{m+1}}(e') = \omega_{i_{m+2}}$. Hence, by (26) and (27),

$$\omega_{i_{m+1}} \succ_{i_m} f_{i_m}(e) \quad \text{and} \quad f_{i_{m+1}}(e') = \omega_{i_{m+2}} \succsim_{i_{m+1}} f_{i_{m+1}}(e).$$

This contradicts *strict endowments-swapping-and-withdrawal-proofness*. □

Lemma 6. *Let $\mathscr{D} \subseteq \mathscr{E}$ be a domain satisfying* D4 *and* D5. *Then, TTC on $\mathscr{D}$ violates strict endowments-swapping-and-withdrawal-proofness.*

Proof. Since $\mathscr{D}$ satisfies D5, there are $e = (I, O, \succ, \omega) \in \mathscr{D}$, $r \in \mathbb{N}$, $C = \{i_1(= i_{N+1}), i_2, \ldots, i_N\} \in \mathbb{C}(e, r)$, and $j \in I \setminus I^r(e)$ such that

- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_1}\}\right)$, $\omega_{i_1} \succ_j o$; and
- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_1}, \omega_j\}\right)$, $\omega_{i_1} \succ_{i_N} \omega_j \succ_{i_N} o$.

[17] Considering that $N \geq 2$, $i_{m+1} (\neq i_m)$ surely exists.

By $C \in \mathbb{C}(e,r)$, for each $i_n \in C$,

- $i_n \in I \setminus I^{r-1}(e)$ and $\omega_{i_n} \in O \setminus O^{r-1}(e)$; and
- for each $o \in O \setminus \left(O^{r-1}(e) \cup \{\omega_{i_{n+1}}\}\right)$, $TTC_{i_n}(e) = \omega_{i_{n+1}} \succ_{i_n} o$.

By $j \in I \setminus I^r(e)$ and $\omega_{i_1} \in O(e,r)$,

- $j \in I \setminus I^{r-1}(e)$ and $\omega_j \in O \setminus O^{r-1}(e)$; and
- $\omega_{i_1} \succ_j TTC_j(e)$.

Let $r_j \in \mathbb{N}$ be such that $j \in I(e, r_j)$. By $j \notin I^r(e)$, $r < r_j$.

Let

$$e' = e^{j,i_1}_{-j} = \left(I \setminus \{j\}, O \setminus \{\omega_{i_1}\}, \succ^{O\setminus\{\omega_{i_1}\}}_{-j}, \omega^{j,i_1}_{-j}\right).$$

Since $\mathscr{D}$ satisfies D4, $e' \in \mathscr{D}$. There are two cases.

- **Case 1: $r = 1$.** By $C \in \mathbb{C}(e,1)$ and $O^{r-1}(e) = O^0(e) = \emptyset$,

$$\begin{aligned} \forall\, i_n \in C \setminus \{i_N\},\ \forall\, o \in O \setminus \{\omega_{i_{n+1}}\},\ \omega^{j,i_1}_{i_{n+1}} = \omega_{i_{n+1}} = TTC_{i_n}(e) \succ_{i_n} o; \\ \forall\, o \in O \setminus \{\omega_{i_1}, \omega_j\},\ \omega^{j,i_1}_{j} = \omega_{i_1} = TTC_{i_N}(e) \succ_{i_N} \omega^{j,i_1}_{i_1} = \omega_j \succ_{i_N} o. \end{aligned} \tag{28}$$

Note that

$$C \subseteq I \setminus \{j\} \quad \text{and} \quad \bigcup_{k \in C} \left\{\omega^{j,i_1}_k\right\} \subseteq O \setminus \{\omega_{i_1}\}.$$

By (28),

$$\begin{aligned} \forall\, i_n \in C \setminus \{i_N\},\ \forall\, o \in (O \setminus \{\omega_{i_1}\}) \setminus \{\omega_{i_{n+1}}\},\ \omega^{j,i_1}_{i_{n+1}} = \omega_{i_{n+1}} \succ^{O\setminus\{\omega_{i_1}\}}_{i_n} o; \\ \forall\, o \in (O \setminus \{\omega_{i_1}\}) \setminus \{\omega_j\},\ \omega^{j,i_1}_{i_1} = \omega_j \succ^{O\setminus\{\omega_{i_1}\}}_{i_N} o. \end{aligned}$$

These imply that $C \in \mathbb{C}(e',1)$ and for each $i_n \in C \setminus \{i_N\}$, $TTC_{i_n}(e') = \omega^{j,i_1}_{i_{n+1}} = \omega_{i_{n+1}}$ and $TTC_{i_N}(e') = \omega^{j,i_1}_{i_1} = \omega_j$.

- **Case 2: $r \geq 2$.** For each $o \in O^{r-1}(e)$, there are $s' \in \{1, 2, \ldots, r-1\}$ and $\ell \in I(e, s')$ with $TTC_\ell(e) = o$. By Lemma 3 and $r < r_j$, $\ell \in I(e', s')$ and $TTC_\ell(e') = TTC_\ell(e) = o$. Hence, for each $i_n \in C$, $TTC_{i_n}(e') \neq o$. Additionally, by $TTC_{i_N}(e') \in O \setminus \{\omega_{i_1}\}$, $TTC_{i_N}(e') \neq \omega_{i_1}$. These imply that

$$\begin{aligned} \forall\, i_n \in C \setminus \{i_N\},\ TTC_{i_n}(e) = \omega_{i_{n+1}} = \omega^{j,i_1}_{i_{n+1}} \succsim_{i_n} TTC_{i_n}(e'); \\ TTC_{i_N}(e) = \omega_{i_1} = \omega^{j,i_1}_{j} \succ_{i_N} \omega_j = \omega^{j,i_1}_{i_1} \succsim_{i_N} TTC_{i_N}(e'). \end{aligned} \tag{29}$$

Note that
$$\bigcup_{k \in C} \left\{ \omega_k^{j,i_1} \right\} \subseteq O \setminus \{\omega_{i_1}\}.$$
It then follows from (29) that

$$\begin{aligned} \forall\, i_n \in C \setminus \{i_N\},\ \omega_{i_{n+1}} = \omega_{i_{n+1}}^{j,i_1} &\succsim_{i_n}^{O \setminus \{\omega_{i_1}\}} TTC_{i_n}(e'); \\ \omega_j = \omega_{i_1}^{j,i_1} &\succsim_{i_N}^{O \setminus \{\omega_{i_1}\}} TTC_{i_N}(e'). \end{aligned} \tag{30}$$

For each $i_n \in C$, let $r'_{i_n} \in \mathbb{N}$ be such that $i_n \in I(e', r'_{i_n})$. By (30),

$$r'_{i_1} \leq r'_{i_N} \leq r'_{i_{N-1}} \leq \cdots \leq r'_{i_2} \leq r'_{i_1},$$

that is, there is $r' \in \mathbb{N}$ such that for each $i_n \in C$, $r' = r'_{i_n}$. This also implies that

$$C \subseteq (I \setminus \{j\}) \setminus I^{r'-1}(e') \quad \text{and} \quad \bigcup_{k \in C} \left\{ \omega_k^{j,i_1} \right\} \subseteq (O \setminus \{\omega_{i_1}\}) \setminus O^{r'-1}(e').$$

Recall (30). If there is $i_n \in C$ such that $\omega_{i_{n+1}}^{j,i_1} \succ_{i_n}^{O \setminus \{\omega_{i_1}\}} TTC_{i_n}(e')$, then i_n does not receive the most preferred object among $(O \setminus \{\omega_{i_1}\}) \setminus O^{r'-1}(e')$ according to $\succ_{i_n}^{O \setminus \{\omega_{i_1}\}}$ in Round $r' = r'_{i_n}$ of the TTC algorithm at e', which is a contradiction. Hence, $C \in \mathbb{C}(e', r')$ and for each $i_n \in C \setminus \{i_N\}$, $TTC_{i_n}(e') = \omega_{i_{n+1}}^{j,i_1} = \omega_{i_{n+1}}$ and $TTC_{i_N}(e') = \omega_{i_1}^{j,i_1} = \omega_j$.

In both cases, we obtain $TTC_{i_1}(e') = \omega_{i_2}$. Hence,

$$\omega_{i_1} \succ_j TTC_j(e) \quad \text{and} \quad TTC_{i_1}(e') = \omega_{i_2} = TTC_{i_1}(e).$$

This implies that TTC violates *strict endowments-swapping-and-withdrawal-proofness*.

□