Lecture 1: **Dynamic Auctions for Substitutes and Complements**

Time: 10:30-12:00,Wed, 23rd July 2025
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Words of Caution

In the four lectures, I will focus on auction markets with multiple indivisible goods, dynamic auction mechanisms, basic concepts and essential results. (As you know, there have been and can be many important theories even on auctioning just a single item. But I shall assume you are familiar with most of them.) Due to time constraint, almost no proof will be given but I will make some hints or comments on some proofs. It seems that the proof can always be done as long as one can formulate the right model, the right assumptions, the right solutions, and the right mechanism.

I will leave about 10 minutes for discussion in the last part of every lecture.

Some Words on Dynamic Mechanism Design

It is well recognized that dynamic mechanisms have important advantages over direct/static mechanisms such as the Vickrey-Clarke-Groves (VCG) Mechanism in their capacity to alleviate agents' concern about privacy and to reduce computational complexity, payoff uncertainty, and information cost; see, e.g., Ausubel (2004, 2006), Ausubel and Milgrom (2007), Bergemann and Morris (2007), McMillan (1994), Milgrom (2007,2017), Perry and Reny (2005), Rothkopf (2007), Rothkopf et al. (1990), etc. I like to say:

Designing a direct mechanism is like to shot a single picture while designing a dynamic mechanism is to make a movie.

Some Principles for Designing Dynamic Mechanisms:

- Incentive-compatibility;
- Efficiency and privacy preservation: Allocative efficiency, information efficiency, and time efficiency, etc;
- Simplicity and transparency (procedural efficiency?);
- Detail free: reducing the assump. of common knowledge (Wilson doctrine);
- Robustness and error-tolerance .

Part 1: Assignment Markets

- Multiple houses for sale and many potential buyers.
- Every buyer has a private valuation on each house and can pay up to his valuation and demands at most one house, i.e., unit-demand, and has quasi-linear utilities in money.
- Every house has a reserve price below which it will not be sold.
- The basic question: Who should get which house at what price?
- This model is called the assignment or unit-demand market and was first studied by Koopmans and Beckmann (1957): Assignment problems and the location of economic activities, *Econometrica,* 25, 53-76; and by Shapley and Shubik (1972): The assignment game I: The core, *International Journal of Game Theory*, 1, 111-130.

An Example

Example 1: Four houses and five buyers. Valuations and reserve prices are given in the table:

Agents\Houses	House 0	House 1	House 2	House 3	House 4
Bidder 1	0	4	3	5	7
Bidder 2	0	7	6	8	3
Bidder 3	0	5	5	7	7
Bidder 4	0	9	4	3	2
Bidder 5	0	6	2	4	10
Seller	0	5	4	1	5

Private Information: Every buyer's valuations are private information which is only known to himself. The seller or anyone else does not have this information.

Assignment

- Let $N = \{1, 2, \dots, n\}$ be the set of items. Let 0 be the dummy good which has no value and does no harm and can be assigned to any number of buyers. Item $l \neq 0$ is a real one.

- $M = \{1,2, \dots, m\}$—the set of all buyers.

- $V^k(l)$ denotes the private valuation of buyer k on item l with $V^k(0) = 0$. $c(l)$ denotes the reserve price of item l with $c(0) = 0$. V^k and $c(.)$ are integer-valued.

- An assignment π allocates every buyer l an item $\pi(l)$ such that no real item is assigned to more than one buyer. An assignment may allocate the dummy good to several buyers and a real item may not be assigned to any buyer.

- An item $l \neq 0$ is unassigned at assignment π if the item is not assigned to any buyer. Let $U(\pi)$ denote the set of all unassigned items at π.

Efficiency

An assignment π^* is *efficient* if

$$\sum_{k \in M} V^k\big(\pi^*(k)\big) + \sum_{l \in U(\pi^*)} c(l) \geq \sum_{k \in M} V^k\big(\pi(k)\big) + \sum_{l \in U(\pi)} c(l)$$

for every assignment π. An efficient assignment allocates items in a way that generates the highest social value.

Let

$$SV(M) = \sum_{k \in M} V^k\big(\pi^*(k)\big) + \sum_{h \in U(\pi^*)} c(h)$$

be the value of an efficient assignment.

We call $SV(M)$ the *market value*.

For any $k \in M,\ let\ M_{-k} = M \setminus \{k\}$ and let $SV(M_{-k})$ be the value of an efficient assignment in the market without buyer k.

The Example

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Bidder 5	0	6	2	4	10
Seller	0	5	4	1	5

There are two efficient assignments:

$$\pi = \big(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)\big) = (0,2,3,1,4)$$
$$\rho = \big(\rho(1), \rho(2), \rho(3), \rho(4), \rho(5)\big) = (0,3,2,1,4)$$

Equilibrium

- Let $p = (p(1), \dots, p(n))$ denote *a feasible price vector* with always $p(0) = 0$ and $p(l) \geq c(l)$ for every $l \neq 0$. $p(l)$ is the price of item l.
- Given a price vector p, *the demand set* of buyer k is given by

  $D^k(p) = \{j \mid V^k(j) - p(j) \geq V^k(l) - p(l) \;\; for\ all\ l \in N \cup \{0\}\}.$

  Every item in the demand set gives the buyer the highest profit.
- *A Walrasian/competitive equilibrium (WE/CE)* consists of an assignment π and a feasible price vector p such that $\pi(k) \in D^k(p)$ for every buyer k and $p(l) = c(l)$ for every unsold item l at π.
- In equilibrium, every buyer receives an optimal item and the price of any unsold item equals its reserve price and all agents are in harmony.

The First Lattice Theorem

Proposition 1: If (p, π) is a Walrasian equilibrium, then π must be efficient. Moreover, if π is efficient, there must exist a feasible price vector p such that (p, π) is a Walrasian equilibrium.

Let $p, q \in R^n$. Define their meet $\wedge$ and join $\vee$ by

$$p \wedge q = (\min\{p(1), q(1)\}, \dots, \min\{p(n), q(n)\}$$

$$p \vee q = (\max\{p(1), q(1)\}, \dots, \max\{p(n), q(n)\}.$$

A set $S \subseteq R^n$ is a *lattice* if it holds $p \wedge q, p \vee q \in S \; for \; any \; p, q \in S$. A lattice is *complete* if it is compact.

Theorem 1 (Shapley and Shubik 1972): For the assignment market, the set of WE price vectors forms a nonempty complete lattice.

How to Find a Walrasian Equilibrium

Crawford and Knoer (1981) propose the first algorithm to find an approximate WE.

Demange, Gale, and Sotomayor (DGS) (1986) introduce the first dynamic auction which finds an exact WE. We discuss this in the lecture.

In the following, when we discuss any dynamic auction which finds a precise WE, we assume that all valuations are integral, and the size of price adjustment (increment) is equal to 1.

The DGS Auction

- A nonempty set of real items $S \subseteq N$ is *over-demanded* at a price vector p, if the number of bidders who demand only items in S is strictly greater than the number of items in S, i.e., $|S| < \left|\{k \in M \mid D^k(p) \subseteq S\}\right|$.
- An over-demanded set S is *minimal* if no strict subset of S is an over-demanded set.
- Step 1: The auctioneer announces the reserve prices $c(l)$ for all items. Set $t \coloneqq 0$ and p^t:=(c(1),…, c(n)). Then go to Step 2.
- Step 2: The auctioneer asks every bidder k to report his demand set $D^k(p^t)$ at the current prices p^t and checks if there is any over-demanded set at p^t. If there is no over-demanded set, the auction stops. Otherwise, there is an over-demanded set. Then choose a minimal over-demanded set S^t and increase the price of every item in S^t by one unit and keep the prices of all other items unchanged. Set $t \coloneqq t + 1$ and return to Step 2.

Illustration of the Auction

We use Example 1 to illustrate the auction.

Step t	Prices p^t	S^t	$D^1(p^t)$	$D^2(p^t)$	$D^3(p^t)$	$D^4(p^t)$	$D^5(p^t)$
0	(0,5,4,1,5)	{3}	{3}	{3}	{3}	{1}	{4}
1	(0,5,4,2,5)	{3}	{3}	{3}	{3}	{1}	{4}
2	(0,5,4,3,5)	{3}	{3,4}	{3}	{3}	{1}	{4}
3	(0,5,4,4,5)	{3}	{4}	{3}	{3}	{1}	{4}
4	(0,5,4,5,5)	{4}	{4}	{3}	{3,4}	{1}	{4}
5	(0,5,4,5,6)	{4}	{4}	{3}	{3}	{1}	{4}
6	(0,5,4,5,7)	{3}	{0,3,4}	{3}	{3}	{1}	{4}
7	(0,5,4,6,7)		{0,4}	{1,2,3}	{2,3}	{1}	{4}

The auction starts with $p^0 = (0,5,4,1,5)$ and ends at $p^7 = (0,5,4,6,7)$. At $t = 3$,{3} and {4} are minimal over-demanded sets but {3,4} is not. (p^7, π) is a Walrasian equilibrium with

$$\pi = \big(\pi(1), \pi(2), \pi(3), \pi(4), \pi(5)\big) = (0,2,3,1,4)\ or\ (0,3,2,1,4).$$

Convergence and Strategic Issue

Theorem 2 (DGS 1986): The DGS dynamic auction finds the minimum WE price vector in a finite of steps.

When facing an auction, every bidder has to think about how to report/bid his demand set at the current prices. Should he bid honestly according to his true valuations or manipulate his bid as if he had different valuations than his true ones? A bidder may manipulate only if doing so gives him extra profits.

Theorem 3 (Leonard 1983): For the assignment market, if trade takes place at the minimum WE equilibrium vector, then the payoff that every buyer k gets is equal to $SV(M) - SV(M_{-k})$, i.e., his marginal contribution to the grand coalitional value. The price he pays is equal to

$$p\big(\pi^*(k)\big) = V^k\big(\pi^*(k)\big) + SV(M_{-k}) - SV(M)$$

where $\pi^*(k)$ is the item assigned to the agent.

Sincere Bidding and Incentive Compatibility

Leonard (1983) considers a direct mechanism in which every buyer directly reports their valuations.

Theorem 4 (Leonard 1983): Truth revealing of their valuations is a dominant strategy for every buyer.

The DGS auction is a dynamic rule that implements the direct mechanism of Leonard (1983). DGS (1983) do not discuss the strategic issue regarding their dynamic mechanism. Nevertheless, Leonard's result for the associated direct mechanism implies that sincere bidding by every bidder constitutes an ex post Nash equilibrium of the DGS dynamic game, in the sense that the strategy for each player would remain optimal even if the private values of her opponents were revealed to her (so no regret). Sincere bidding by every bidder k means that at every step t of the dynamic auction, bidder k always reports his demand set $D^k(p(t))$ according to his true valuation V^k at prices $p(t)$.

Several Remarks

Theorems 3 and 4 are deep and elegant results. Their proof uses some basic results (primal-dual formulation) from linear programming and bi-partie graphs.

The DGS auction is an elegant design and a major improvement of Crawford and Knoer (1981)'s approximate algorithm. Minimal overdemanded sets play a key role in achieving the minimum prices in the DGS auction.

The key advantage of the DGS auction over the direct mechanism is that at least every winning bidder can avoid exposing some of their valuations. This is extremely important in practice, as no businessmen like to reveal their valuations or costs.

As the DGS auction must start from very low prices and is ascending, it cannot guarantee to find a WE after bidders have made mistakes.

Remarks on Assignment Markets

Historically, Koopman and Beckmann (1957) are the first to prove the existence of WE in the assignment market. It is interesting to note that Shapley and Shubik (1972) did not cite the previous paper.

There are numerous articles on the existence of WE in unit demand models with quasi-linear (QL) and non-QL utilities, including Quinzii (1984), Gale (1984), Kaneko and Yamamoto (1986) on NQL models, etc, and numerous articles on variations of the Crawford-Knoer adjustment process and the DGS auction.

The assignment market is a nice starting point but is restrictive in the sense that every bidder is allowed to buy only one item. What about more natural, more practical, and more general cases where every bidder may buy several items?

Auctions for Markets beyond Assignment Markets

The key driving force for designing dynamic auctions is the sale of radio spectrum licenses all over the world since 1990s, and the intellectual curiosity for understanding markets. The huge success of spectrum auctions was phenomenal, illuminating, and inspiring.

For instance, one of the 1st US spectrum auctions designed by Preston McAfee, Paul Milgrom, and Robert Wilson around 1995 was hailed by the New York Times as ``The greatest Auction Ever." The UK 3G auction in 2000 designed by Paul Klemperer with Ken Binmore generated 100 billion US dollars.

Part 2: The Double-Track Auction

This part focuses on auction markets where every bidder can demand several indivisible goods/items. There are two important cases.

- First, all items are substitutes, i.e. gross substitutes (GS).
- Second, all items can be split into two disjoint groups. All items in each group are substitutes but items across the two groups are complements, i.e., gross substitutes and complements (GSC).

We introduce the double-track auction for GSC, which automatically works for GS.

Kelso-Crawford Model of Gross Substitutes

The celebrated job-matching model of Kelso-Crawford (1982). Every firm can hire several workers, and every worker can work for at most one firm.

Kelso-Crawford prove that their market has a WE as long as every firm treats all workers as substitutes (Gross Substitutes (GS)), via a salary adjustment process. For each given increment of salary adjustment, their process finds a core allocation within a finite number of steps. Taking a sequence of increments converging to zero, the limit of the sequence of core allocations converges to a strict core allocation, i.e., a WE.

Gross Substitutes

Let $N = \{1, \ldots, n\}$ be a set of (indivisible) goods and 2^N a family of all subsets of N.

Let $u: 2^N \to R$ be a utility function and let $D(p)$ be the demand set given the prices $p \in R^N$ and the function u.

Def 1 (Kelso and Crawford 1982): The demand correspondence $D(\cdot)$ satisfies the gross substitutes condition (GS) if for any $p, q \in R^N$ with $q \geq p$ and any $S \in D(p)$, there exists $T \in D(q)$ such that

$$\{h \in S | p(h) = q(h)\} \subseteq T.$$

Single Improvement and No Complementarities

Def 2 (Gul and Stacchetti 1999): The demand correspondence $D(\cdot)$ has the single improvement property (SI) if for any $p \in R^N$ and any $S \notin D(p)$, there exists bundle T such that

$u(T) - \sum_{h \in T} p(h) > u(S) - \sum_{h \in S} p(h)$ and $|T \setminus S| \leq 1$ and $|S \setminus T| \leq 1$.

Def 3 (Gul and Stacchetti 1999): The demand correspondence $D(\cdot)$ has no complementarities (NC) if for any $p \in R^N$ and all $S, T \in D(p)$ and $B \subseteq S$, there is $C \subseteq T$ such that $(S \setminus B) \cup C \in \mathrm{D(p)}$.

Theorem 1 (Gul and Stacchetti 1999): Let function u be weakly increasing. Then GS, SI and NC are equivalent.

They prove $GS \Rightarrow SI \Longrightarrow NC \Longrightarrow GS$.

GS and $M^\natural - Concavity$

Def 4 (Murota and Shioura 1999): A function $u: 2^N \to R$ is $M^\natural$-concave if for any $S, T \subseteq N$ and $k \in S \setminus T$, the function satisfies

$$u(S) + u(T) \leq \max[u(S \setminus \{k\}) + u(T \cup \{k\}), \\ max_{l \in T \setminus S}\{u\big((S \setminus \{k\}) \cup \{l\}\big) + u\big((T \setminus \{l\}) \cup \{k\}\big)\}]$$

Theorem 2 (Fujishige and Yang 2003): Let function u be weakly increasing. Then GS and are $M^\natural - Concavity$ equivalent.

The line of proof is $GS \Leftrightarrow SI \Leftrightarrow M^\natural - Concavity$.

GS, SI, and NC are defined on demand sets, whereas Def 4 gives a precise and direct definition of utility functions.

GS and Submodularity

Def 5: Given a utility function $u: 2^N \to R$, define the indirect utility function $v: R^N \to R$ by

$$\mathrm{v}(p) = max_{S \subseteq N}\{u(S) - \sum_{h \in S} p(h)\}.$$

Def 6: A function $f: R^N \to R$ is submodular if for any $p, q \in R^N$, it holds $f(p) + f(q) \geq f(p \wedge q) + f(p \vee q)$.

The function is supermodular if $-f$ is submodular.

Theorem 3 (Ausubel and Milgrom 2002): Let function u be weakly increasing. Then the demand correspondence satisfies GS if and only if its indirect utility function is submodular.

A General Lattice Theorem for GS

Theorem 4 (Gul and Stacchetti 1999, Ausubel 2006): Let every agent's function be weakly increasing and the demand correspondence satisfy GS in an economy with m agents and n indivisible goods. Then the set of WE price vectors constitutes a nonempty complete lattice in R^N.

Dynamic Auctions for GS

Milgrom (2000) price adjustment process, which finds a core allocation wrt the given increment;

Gul and Stacchetti (2000) auction, which finds the minimum WE price vector, generalizing the DGS auction. They use matroid theory to prove the finite convergence to the WE price vector $p^* \in R^N$ with a WE assignment $\pi^*$and demonstrate that no dynamic auction can reveal sufficient information to implement the VCG mechanism for the GS market. Indeed, the minimum WE prices do not correspond to the VCG payments in the GS market. The VCG payment for bidder k equals

$$q\big(\pi^*(k)\big) = V^k\big(\pi^*(k)\big) + SV(M_{-k}) - SV(M)$$

where $\pi^*(k)$ is the bundle of items assigned to the agent. In the GS market, generally speaking, $q\big(\pi^*(k)\big) \neq \sum_{h\in\pi^*(k)} p^*(h)$ for at least one bidder k.

Ausubel (2006) proposes an ingenious dynamic auction design for the GS market which can converge globally to a WE and achieves incentive-compatibility. Based on the VCG framework, his idea is to run the original market plus all submarkets each dropping one bidder, i.e., $m+1$ markets in total when there are m bidders in the original market.

Problems with Complementarity

Milgrom (2000, 2017), Jehiel and Moldovanu (2003), Noussair (2003), and Klemperer (2004), and Maskin (2005) call for designing dynamic auctions which can handle complementarities. In fact, complementarities have long been a challenge for economic analyses; Scarf (1960) and Samuelson (1974) etc.

A casual observation is that complementarity/synergy is ubiquitous, pervasive, and overwhelming.

Substitutes are often observed. But substitutes and complements are more often jointly observed and more pervasive and more fundamental.

Interestingly, substitutes and complements often appear together in some regular and typical patterns. For instance, tables and chairs; computer hardware and software packages; workers and machines; left shoes and right shoes, students and teachers; buyers and sellers; suppliers and retailers, etc.

Gul and Stacchetti (1999) prove that GS is a ``necessary'' condition for existence of WE in some sense. The statement throws cold water on this very subject.

Gross Substitutes and Complements

- The above regular patterns have been called gross substitutes and complements (GSC) by Sun and Yang (2006,2009) in an equilibrium or auction model and same-side substitutes and cross-side complements by Ostrovsky (2008) in a vertical supply chain model (a matching model).
- Consider a market where indivisible goods can be split into two groups. Goods in the same group are substitutes and can be heterogeneous but goods across the two groups are complements.
- Every buyer has private valuation on every bundle of goods but may consume several goods.

An Example

Example 2: One table t and two chairs c_1 and c_2 are going to be sold to three buyers. Every agent views chairs as substitutes, but chairs and table as complements. The valuations of every agent are shown in the table and are private information.

	$\emptyset$	t	c_1	c_2	tc_1	tc_2	c_1c_2	tc_1c_2
Bidder 1	0	18	3	3	22	22	4	24
Bidder 2	0	1	11	11	13	13	20	23
Bidder 3	0	12	6	6	20	20	10	25

Who should get what at what prices?

The Formal Model

- $S_1 = \{1, 2, \ldots, s\}$: the set of items of type 1, e.g., tables.
  $S_2 = \{s + 1, s + 2, \ldots, n\}$: the set of items of type 2, e.g., chairs.

- Items of the same type can be also heterogeneous!

- $N = S_1 \cup S_2$: the union of S_1 and S_2--the set of all items

- $2^N = \{S \mid S \subseteq N\}$: the family of all bundles of items.

- $M = \{1,2, \ldots, m\}$—the set of all bidders.

- $V^k: 2^N \rightarrow Z_+$ is the utility function of bidder k, weakly increasing and integer-valued with $V^k(\emptyset) = 0$.

- V^k is private information.

Allocation and Efficiency

An *allocation* is a distribution of items among all bidders and can be denoted by $\pi = \big(\pi(1), \pi(2), \ldots, \pi(m)\big)$, where $\bigcup_{k \in M} \pi(k) = N$ and $\pi(k) \cap \pi(l) = \emptyset$ for $k \neq l$. π is a partition of goods among bidders. At π, bidder k gets the bundle $\pi(k)$ of items.

An allocation π^* is *efficient* if

$$\sum_{k \in M} V^k\big(\pi^*(k)\big) \geq \sum_{k \in M} V^k(\pi(k))$$

for every allocation π. An efficient allocation achieves the highest market value of the goods.

Equilibrium

- A price vector $p = (p_1, p_2, \dots, p_n)$ indicates a price p_j for every good/item $j \in N$.
- Given a price vector p, *the demand set* of bidder k is given by
$$D^k(p) = \{S \mid V^k(S) - \sum_{h \in S} p_h \geq V^k(T) - \sum_{h \in T} p_h \;\; for\ all\ T \subseteq N\}$$

Every bundle in the demand set gives the bidder the highest profit.

- *A Walrasian/competitive equilibrium* consists of a price vector p and an allocation π such that $\pi(k) \in D^k(p)$ for every bidder k.
- In equilibrium, every buyer receives an optimal bundle of items and all agents are in harmony.
- Proposition 1: If (π, p) is a WE, then π must be efficient.

Gross Substitutes and Complements

Definition 7: The demand set $D^k(p)$ satisfies *the Gross Substitutes and Complements (GSC) Condition* if, for any given prices p and increasing the price of one item of type $j = 1\ or\ 2$, the bidder who demands a bundle S at prices p will continue to demand items of type j in S whose prices do not change, and will not demand any item of the other type $l \neq j$ he does not demand at prices p.

In other words, bidder views items in the same set as substitutes but items across the two sets as complements.

GSC is introduced by Sun and Yang (2006). When $S_1 = \emptyset$ or $S_2 = \emptyset$, i.e. there is only one type of goods, GSC reduces to the GS Condition of Kelso and Crawford (1982).

Equilibrium Existence Theorem

Theorem 5 (Sun and Yang 2006): If GSC is satisfied by every agent in the market, there exists a competitive equilibrium.

Revisiting Example 2.

	$\emptyset$	t	c_1	c_2	tc_1	tc_2	c_1c_2	tc_1c_2
Bidder 1	0	18	3	3	22	22	4	24
Bidder 2	0	1	11	11	13	13	20	23
Bidder 3	0	12	6	6	20	20	10	25

A competitive equilibrium in this example is to set the price of each chair at 8 and the price of table at 16 and to assign the table to bidder 1 and two chairs to bidder 2.

The conventional strategy would sell the table and chairs as one package, resulting in the social value of only 25!

Auction Design

- Markets do not work automatically!
- They need Design and Maintenance—Lots of visible hands!
- Three major difficulties for the current auction design:
    1. Multiple Heterogeneous Items
    2. Complements & Substitutes Mixed Up
    3. Asymmetric and Private Information

Lyapunov Function

- The indirect utility function of bidder k:

$$v^k(p) = max_{S \subseteq N}\{V^k(S) - \sum_{h \in S} p(h)\}.$$

- The Lyapunov function:

$$L(p) = \sum_{k \in M} v^k(p) + \sum_{h \in N} p(h)\,.$$

The use of this function is well known in the literature (see, e.g., Arrow and Hahn 1971 and Varian 1981), but was first used by Ausubel (2006) in auction. His Proposition 1 shows that if a WE exists, then the set of equilibrium price vectors coincides with the set of minimizers of the Lyapunov function.

Integral Convexity, Generalized Lattice and Submodularity

Def 7: For any $p, q \in R^N$, we define their generalized meet and join by

$$p \wedge_g q = (\min\{p(1), q(1)\}, \dots, \min\{p(s), q(s)\}, \max\{p(s+1), q(s+1)\}, \dots, \max\{p(n), q(n)\})$$

$$p \vee_g q = (\max\{p(1), q(1)\}, \dots, \max\{p(s), q(s)\}, \min\{p(s+1), q(s+1)\}, \dots, \min\{p(n), q(n)\})$$

Def 8: A function $f: R^N \to R$ is g-submodular if for any $p, q \in R^N$, it holds

$$f(p) + f(q) \geq f\left(p \wedge_g q\right) + f\left(p \vee_g q\right).$$

The function is g-supermodular if $-f$ is g-submodular.

Def 9: A set $S \subseteq R^N$ is a generalized lattice if $p \wedge_g q, p \vee_g q \in S$ for any $p, q \in S$.

Def 10 (Sun and Yang 2009): A set $S \subseteq R^N$ is *integrally convex* if it is convex and every point x in the set can be written as a convex combination of integer points contained by the set S and the set $N(x) = \{y \in Z^N | max_h |y(h) - x(h)| < 1\}$.

Favati and Tardella (1990) first introduce a similar concept for discrete sets.

Generalized Lattice Results

Lemma 1 (Sun and Yang 2009): For any model with quasi-linear utilities and indivisible goods, p^* is a WE price vector if and only if it is a minimizer of the Lyapunov function with its minimum value $L(p^*)$ equal to the market value $SV(M)$.

Theorem 6 (Sun and Yang 2009) : For the GSC model, the Lyapunov function is a continuous, convex and g-submodular function; and the set of equilibrium price vectors coincides with the set of minimizers of the Lyapunov function.

Theorem 7 (Sun and Yang 2009) : For the GSC model, the set of WE prices forms a nonempty, integrally convex, and complete generalized lattice.

Def 11: A demand correspondence $D(\cdot)$ has a generalized single improvement property (GSI) if any prices p and any $S \notin D(p)$, there is bundle T such that

$$V(S) - \sum_{h\in S} p(h) < V(T) - \sum_{h\in T} p(h)$$ and one of the following two holds: (a) $S \cap S_j = T \cap S_j$ and $|(S \setminus T) \cap S_j^c| \le 1$ and $|(T \setminus S) \cap S_j^c| \le 1$ for either $j = 1$ *or* 2. (b) either $T \subseteq S$ and $|(S \setminus T) \cap S_1| = |(S \setminus T) \cap S_2| = 1$ or S$\subseteq T$ and $|(T \setminus S) \cap S_1| = |(T \setminus S) \cap S_2| = 1$. Note that $S_1^c = S_2$ and $S_2^c = S_1$.

(b) says that we can strictly improve a suboptimal bundle by simultaneously adding one item to each set or simultaneously removing one item from each set.

The Law of Price Adjustment: Increase the price for any over-demanded item but decrease the price for any under-demanded item. (Tatonnement process)

The existing auctions are one-track auctions adjusting all prices only in one direction (either ascending or descending), of English or Dutch types. The exposure problem for the one-track auctions.

The basic idea is to try to find a minimizer of the Lyapunov function. Two hurdles: 1. every bidder's indirect utility function is private information, and the auctioneer cannot get the Lyapunov function. 2 Even if the function is known, choosing a proper search direction is also crucial. Otherwise, the auction may get stuck like what Scarf (1960) discovered 60 years ago. GSI plays an important role in establishing various properties of our auction.

The Double-Track Auction

- Sun and Yang (2009) propose the double-track auction, which uses only observable information. The idea of a simple version of the auction is given as follows:
- The auctioneer starts with very low prices for all items in one set but very high prices for items in the other set.
- Step 1: Every bidder is asked to report his demand set at the current prices.
- Step 2: The auction adjusts prices of items upwards in one set which are over-demanded but prices of items downwards in the other set which are over-supplied.
- Repeat Step 1 and Step 2 until the market clears.

The Precise Description of the Auction

- Let $p(t)$ be the price vector at time $t = 0,1,2,\ldots$. Let
  $\Delta = \{\delta \mid \delta = (\delta_1, \delta_2, \ldots, \delta_n), \delta_h \in \{0,1\}, h \in S_1;\ \delta_h \in \{0,-1\}, h \in S_2\}$.
- Step 0: Start with $p(0) = (L, \ldots, L, H, \ldots, H)$, low prices for all items in S_1 but high prices for items in S_2.
- Step 1: Every bidder reports his demand set at the current prices $p(t)$. The auctioneer adjusts $p(t)$ to $p(t+1) = p(t) + \delta(t)$ where $\delta(t)$ solves the following problem

$$\max_{\delta\in\Delta}\{\sum_{k\in M}(\min_{S\in D^k(p(t))} \sum_{h\in S}\delta_h) - \sum_{h\in N}\delta_h\}$$

As soon as $\delta(t) = 0$ is an optimal solution, stop. Otherwise, go back to Step 1.

The Exposure Problem

- The existing auctions cannot handle the situation.
- *Example:* Bidders know their valuations privately.

	Ø	A	B	AB
Bidder 1	0	1	1	5
Bidder 2	0	2	2	5
Bidder 3	0	0	0	4

Price vectors	Price variations	Bidder 1	Bidder 2	Bidder 3
p(0)=(0,0)	δ(0)=(1,1)	{AB}	{AB}	{AB}
P(1)=(1,1)	δ(1)=(1,1)	{AB}	{AB}	{AB}
P(2)=(2,2)	δ(2)=(1,1)	{AB}	{AB}	{AB,Ø}
P(3)=(3,3)	δ(3)=(0,0)	{Ø}	{Ø}	{Ø}

Overcoming the Exposure Problem

- The double-track auction overcomes the exposure problem and finds a WE.

Price vectors	Price variation	Bidder 1	Bidder 2	Bidder 3
p(0)=(0,6)	δ(0)=(1,-1)	{A}	{A}	{A,Ø}
p(1)=(1,5)	δ(1)=(0,-1)	{A, Ø}	{A}	{Ø}
p(2)=(1,4)	δ(2)=(1,-1)	{A,AB,Ø}	{A}	{Ø}
p(3)=(2,3)	δ(3)=(0,0)	{AB,Ø}	{Ø,A,AB}	{Ø}

The Illustrative Example

- Every bidder knows his valuations privately. The auctioneer knows only that all values are no more than 26.

	Ø	t	c1	c2	tc1	tc2	c1c2	tc1c2
Bidder 1	0	18	3	3	22	22	4	24
Bidder 2	0	1	11	11	13	13	20	23
Bidder 3	0	12	6	6	20	20	10	25

- The auction starts with prices p(0)=(0,0,26), i.e., the price of each chair is 0 and the price of the table is 26.

The Illustration of the Auction

Prices $p(t)$	Variation $\delta(t)$	$D^1(p(t))$	$D^2(p(t))$	$D^3(p(t))$
$p(0) = (0,0,26)$	(1,1,-1)	$\{c_1c_2\}$	$\{c_1c_2\}$	$\{c_1c_2\}$
$p(1) = (1,1,25)$	(1,1,-1)	$\{c_1, c_2, c_1c_2\}$	$\{c_1c_2\}$	$\{c_1c_2\}$
$p(2) = (2,2,24)$	(1,1,-1)	$\{c_1, c_2\}$	$\{c_1c_2\}$	$\{c_1c_2\}$
$p(3) = (3,3,23)$	(1,1,-1)	$\{c_1, c_2, \emptyset\}$	$\{c_1c_2\}$	$\{c_1c_2\}$
$p(4) = (4,4,22)$	(1,1,-1)	$\{\emptyset\}$	$\{c_1c_2\}$	$\{c_1, c_2, c_1c_2\}$
$p(5) = (5,5,21)$	(1,1,-1)	$\{\emptyset\}$	$\{c_1c_2\}$	$\{c_1, c_2\}$
$p(6) = (6,6,20)$	(0,0,-1)	$\{\emptyset\}$	$\{c_1c_2\}$	$\{\emptyset, c_1, c_2\}$
$p(7) = (6,6,19)$	(0,0,-1)	$\{\emptyset\}$	$\{c_1c_2\}$	$\{\emptyset, c_1, c_2\}$
$p(8) = (6,6,18)$	(0,0,0)	$\{t, \emptyset\}$	$\{c_1c_2\}$	$\{\emptyset, c_1, c_2\}$

Convergence and Strategic Issue

- In this example, bidder 1 gets the table and pays 18, bidder 2 gets the two chairs and pays 12, and bidder 3 gets nothing and pays nothing.
- Here we only present the simple version of the double-track auction. In Sun and Yang (2009), the auction can start from anywhere and will converge globally to a WE.

- In our third lecture, we will study the dynamic auction of Fujishige and Yang (2025), which works for all unimodular demand types (i.e. the necessary and sufficient condition of Baldwin and Klemperer 2019). In contrast to their model with complete information and price-taking agents, our auction model is a setting with incomplete information and strategic bidders. It will be shown that sincere bidding is an ex post perfect Nash equilibrium of the auction game.