Lecture 4: Dynamic Auction, Budgets, and Incentives

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Problem and Objective

Multiple heterogeneous items such as houses are to be sold to several bidders.

Bidders have valuations on their interested items and unit demands.

Bidders have quasi-linear preferences, hard budgets, and can be budget constrained.

Both valuations and budgets are private information. Bidders are strategic.

The question is how to design a dynamic auction that will allocate items as efficiently as possible, meet all budget constraints, and also induce bidders to bid sincerely.

Difficulty Issues

Budget constraints pose a serious obstacle to the efficient allocation of resources, because they can fail the competitive or Walrasian equilibrium (WE).

Two different types of private information makes the incentive issue more difficult to tackle. A similar issue is well-known in contract theory when agents have more than one-dimensional private information.

Motivation

The literature typically assumes that agents can pay up to their valuations on any good they want to buy. That is, if you participate in a market and valuate a house at $\$1,000,000$, you should have this amount of money.

Unfortunately, in almost every society, most people cannot afford to buy big houses or send their kids to private schools even if they very much like to do so.

Budget constraints become more severe when we have business downturns and financial crises.

Many organizations impose budget constraints to control their spending.

Salary caps are used by many professions to relax competition, etc.

Takeaway Points

This lecture discusses an efficient and strategy-proof dynamic auction for assignment markets under budget constraints with a WE or without a WE.

Sincere bidding is shown to be an ex post Nash equilibrium when both valuations and budgets are private information.

The auction finds an efficient outcome that is not only in the core but also strongly Pareto efficient. When no bidder is budget-constrained, the auction yields a WE with the minimal WE prices.

The set of WE price vectors is a lattice but can be open from below if the market has a WE.

This lecture is based on Yang and Yu (2025, accepted by **International Economic Review**).

Some Relevant References

One-Item Auctions include Laffont and Robert (1996), Che and Gale (1998, 2000), Zheng(2001), Maskin (2002), Pai and Vohra (2014),etc.

Assignment models with WE and budgets include Quinzii (1984), Demange and Gale (1985), Kaneko and Yamamoto (1986), Alkan and Gale (1990), Aggarwal et al. (2009), and Morimoto and Serizawa (2015), etc.

Assignment models with budgets and with or without WE include Talman and Yang (2015), Herings and Zhou (2022), etc.

Recent References on More General Models

Gul et al. (2024) consider a general model and prove the existence of a lottery WE over allocations, when agents have a budget of 0 and gross-substitutes preferences.

Nguyen and Vohra (2024) study a related model and prove the existence of an approximate WE with small excess demands when agents have some budgets, and of a lottery over approximate WE allocations with small excess demands when agents have no budgets, all under their Δ-substitutes.

Yang and Yu (2024) propose a dynamic auction for a general market where agents face hard budgets and general preferences. Their auction yields a strongly Pareto efficient core allocation. It finds a strong core allocation with an efficient assignment of items when bidders are not budget constrained.

The Model

- $N = \{1, 2, \ldots, n\}$–the set of indivisible items such as houses. Let $N_0 = N \cup \{0\}$, where 0 is the harmless dummy item.
- $M = \{1, 2, \ldots, m\}$–the set of bidders. Let $M_0 = M \cup \{0\}$, where 0 stands also for the seller.
- Every bidder $h \in M$ has a value function $v^h : N_0 \to \mathbb{Z}_+$ with $v^h(0) = 0$, where $\mathbb{Z}_+$ is the set of non-negative integers.
- Every bidder h has a hard budget $b^h \in \mathbb{Z}_+$ of money.
- Let $((v^h, b^h), h \in M, N)$ represent this model.
- For easy exposition, assume that the seller's reserve price of every item is 0.

Soft and Hard Budgets

- A hard budget b^h means that under no circumstances can bidder h pay more than his budget b^h. Otherwise, it is a soft budget.

- It is essential to deal with hard budgets. In the soft budget b^h case, even if you can borrow, the possible amount l^h of loan is limited. In the end, you still face a hard budget of $b^h + l^h$.

- Bidder h is *budget constrained* if $b^h < \max_{a \in N_0} v^h(a)$. Otherwise, i.e., $b_h \geq \max_{a \in N_0} v^h(a)$, bidder h is *not budget constrained.*

- Caution! Many models contain budget constraints for agents. This does not mean that agents are budget constrained. See e.g. Quinzii (1984) and Alkan and Gale (1990), where agents have budget constraints but are not budget constrained.

Assumptions

The following two assumptions are imposed upon the model:

A1 *Private Information on Valuations and Budgets*: Every bidder $i \in M$ knows his valuation function v^i and budget b^i privately.

A2 *Quasilinear Utility*: Every agent $h \in M_0$ has quasi-linear utility in money. That is, bidder h will get the payoff of $v^h(a) - p(a)$ and the seller will get the payoff of $p(a)$ if item a is sold to the bidder at price $p(a) \in \mathbb{R}$.

The problem is how to assign items to bidders as efficiently as possible and to induce bidders to act honestly and also meet their hard budget constraints.

Assignment and Efficiency

- An *assignment* $\pi = (\pi(1), \ldots, \pi(m))$ assigns every bidder $i \in M$ exactly one item $\pi(i) \in N_0$ such that no real item $a \in N$ is assigned to more than one bidder and any item that is not assigned to a bidder is retained by the seller 0.
- At π, a real item $a \in N$ is *unassigned* if it is not assigned to any bidder, and we use $\pi(0) = N \setminus \left(\bigcup_{h \in M} \pi(h) \right)$ to denote the set of all unassigned items.
- Let $\mathcal{A}$ denote the family of all assignments.
- An assignment π is *efficient* if

$$\sum_{h \in M} v^h(\pi(h)) \geq \sum_{h \in M} v^h(\rho(h)) \quad \forall \rho \in \mathcal{A}.$$

Feasible Payments or Transfers

- A *transfer (payment) vector* $t = (t^1, \ldots, t^m)$ is feasible if $t^h \leq b^h$ for all $h \in M$. We use $t^0 = \sum_{h \in M} t^h$ to denote the seller's revenue.

- A pair (π, t) of an assignment π and a feasible payment t is called an allocation.

- At (π, t), agent $h \in M$ receives item $\pi(h)$ and pays t^h. The utility that the bidders and the seller obtain are given by

$$u^h(\pi, t) = v^h(\pi(h)) - t^h, \forall h \in M$$

$$u^0(\pi, t) = t^0 = \sum_{h \in M} t^h.$$

Walrasian Equilibrium

- A price vector $p = \big(p(a)\big)_{a \in N_0}$ specifies a price for each item with $p(0) = 0$.

- At prices p, the set of affordable items of bidder h is $B_p(b^h) = \{a \in N_0 \mid p(a) \leq b^h\}$ and
  the demand set of bidder h is defined by

  $$D^h(p) = \Big\{a \in B_p(b^h) \mid v^h(a) - p(a) \geq v^h(a') - p(a') \text{ for all } a' \in B_p(b^h)\Big\}.$$

- A *Walrasian equilibrium (WE)* is a pair (π, p) of assignment π and a price vector p such that $\pi(h) \in D^h(p)$ for every $h \in M$ and $p(a) = 0$ for every unassigned item $a \in \pi(0)$.

Example 1 with Equal Budgets

There are two bidders who compete for one item a. Their values are given by $v^1(a) = 20$ and $v^2(a) = 18$ and they have the same budgets $b^1 = b^2 = 6$. So both bidders are budget constrained. There is no Walrasian equilibrium, because the item is indivisible and bidders are budget constrained, some utilities cannot be transferred from one agent to another.

(1) If bidders are not budget constrained, they can compete the item by increasing the price up to their value.

(2) If the item is divisible, they can compete the item by reducing demand quantity to half instead of one and keeping payment unchanged (another way to transfer utility to the seller).

Unfortunately, neither is the case.

Example 2 with Unequal Budgets

A seller has two items $\{a_1, a_2\}$ for sale. There are three bidders 1, 2 and 3. Valuations and budgets are given in the Table. Budgets are different.

Bidder	$v^i(0)$	$v^i(a_1)$	$v^i(a_2)$	Budget b^i
1	0	8	6	9
2	0	7	0	5
3	0	0	6	3

To have a Walrasian equilibrium, we must have $p(a_1) \geq 5$ and $p(a_2) \geq 3$, but we have $D^2(p) = \{0\}$ for $p(a_1) > 5$ and $D^3(p) = \{0\}$ for $p(a_2) > 3$. There is no price vector to balance demand and supply for one item. So there is no WE.

Individual Rationality and Viable Coalitions

- An allocation (π, t) is *individually rational* if $u^i(\pi, t) \geq 0$ for every $i \in M_0$.

- A nonempty subset $S \subseteq M_0$ is called a viable coalition if S consists of either the seller with any number of bidders or a single bidder.

- Given a viable coalition S, an allocation (ρ^S, τ) is feasible for S, if $0 \in S$, $\tau^h \leq b^h$ for each bidder $h \in S$, $\rho^S(j) = 0$, and $\tau^j = 0$ for each bidder $j \in M \setminus S$, or if $S = \{h\}$ for some $h \in M$, $\tau^h = 0$ and $\rho^S(h) = 0$.

Blocking, Core, and Strong Core

- An allocation (π, t) is blocked by a viable coalition S if there exists a feasible allocation (ρ^S, τ) such that $u^h(\rho^S, \tau) > u^h(\pi, t)$ for all $h \in S$;

- An allocation (π, t) is weakly blocked by a viable coalition S if there exists a feasible allocation (ρ^S, τ) such that $u^h(\rho^S, \tau) \geq u^h(\pi, t)$ for all $h \in S$ and with at least one strict inequality.

- An allocation (π, t) is in the *core* and is called a core allocation if it is not blocked by any coalition. It is in the *strong core* and is called a strong core allocation if it cannot be weakly blocked by any coalition.

The Basic Idea of the New Dynamic Auction

The auction of Yang and Yu (2025) works roughly as follows.

- In the first round of the auction, on every item every bidder first makes a bid or no bid, the seller then chooses a set of bids yielding the highest revenue and asks every provisionally losing bidder to make new bids.
- In subsequent rounds, every provisionally losing bidder offers new bids and then the seller chooses an optimal set of bids. The auction process continues until no new bids are offered. When the auction ends, the chosen bids will finally be accepted.
- To avoid the extreme case that the auction may never terminate due to bidders' overbidding or manipulation, the auctioneer sets a price cap $\Omega^* \in \mathbb{Z}_+$ that significantly exceeds the number of $m \times n \times \max\{\max_{h\in M,\, a\in N} v^h(a), \max_{h\in M} b^h\}$.

The New Dynamic Auction: I

Initialization: Set $k = 1$ being the first round. On every item $a \in N_0$, every bidder $h \in M$ makes a bid $p_1^h(a) \in \mathbb{Z}_+$ or no bid. Go to the Provisional Assignment Stage.

Bidding Stage: After being offered to make new bids, every provisionally losing bidder i increases some of his previous bids by one, or makes bids upon items which have not been bid upon by him previously. In this round, bidder h must increase at least one of his previous bids or make a bid $p_k^h(a) \in \mathbb{Z}_+$ upon at least one item $a \in N_0$ which has not been bid upon by him before. Any other bidder j keeps his bids unchanged by setting $p_k^j = p_{k-1}^j$. Go to the Provisional Assignment Stage.

The New Dynamic Auction: II

Provisional Assignment Stage: If bidder $h \in M$ has not made a bid on the dummy item, he is said to be *active* and his price of the dummy item is set as $p_k^h(0) = -2^{-h}$. Otherwise, bidder h is *inactive* and his price of the dummy item is set as $p_k^h(0) = 0$. If a bidder h has not yet made any bid upon an item $a \in N$ until round k, set an artificial price $p_k^h(a) = -\infty$. This gives the price system $P_k = (p_k^h)_{h \in M}$ in round k. If $p_k^h(a)$ equals the price cap Ω^* for $h \in M$ and some $a \in N$, go to the Final Assignment Extra. Otherwise, based on the current prices $P_k = (p_k^h)_{h \in M}$, the seller finds an optimal solution π_k to the problem:
$\max_{\rho \in \mathcal{A}} \sum_{h \in M} p_k^h(\rho(h))$.
At π_k, bidder h is said to be a *provisionally losing (PL) bidder*, if he is active and assigned the dummy item, i.e., $p_k^h(0) = -2^{-h}$ and $\pi_k(h) = 0$. If there is no PL bidder, go to the Final Assignment. Otherwise, the seller asks all PL bidders to submit new bids in the next round. Set $k = k + 1$ and go to the Bidding Stage.

The New Dynamic Auction: III

Final Assignment: The auction ends in round K. Every bidder $h \in M$ will get item $\pi_K(h)$ specified by the current provisional assignment π_K if he can pay his bid $t^h = p_K^h(\pi_K(h))$, otherwise, he will get nothing but have to pay a positive amount $\delta^* \in \mathbb{Z}_+$ of money as a penalty for failing to pay his bid t^h.

Final Assignment Extra: Bidder $h \in M$ will have to pay the price Ω^* but not get any item if his bid $p_k^h(a)$ equals Ω^* for some item $a \in N$. Otherwise he gets nothing and pays nothing.The auction stops.

Notes on the Auction: I

- At round k, if a bidder $h \in M$ has not bid on a real item $a \in N$, we set his price on the item as $p^h(a) = -\infty$ so that the auctioneer will never assign the item to the bidder, i.e., $\pi_k(h) \neq a$.

- When bidders are budget-constrained, discontinuity in demand can occur in those auctions such as Demange et al. (1986) in which the auctioneer first sets prices and subsequently bidders respond with their demands. In contrast, this new auction can preserve continuity in demand, because bidders can first offer their bids according to their valuations and budgets, and then the seller responds.

Notes on the Auction: II

- This auction has a new and special rule on the dummy item. When a bidder $h \in M$ does not make a bid on the item, its price is set to $p_k^h(0) = -2^{-h}$. Otherwise, its price is set to $p_k^h(0) = 0$. So every bidder just needs to indicate whether he wants to have a dummy item or not.

- Differentiating active bidders from inactive ones and setting $p_k^h(0) = -2^{-h}$ for every active bidder h gives a new and easily implementable tie-breaking rule.

- This rule also improves market efficiency in a way that if bidder h has bid on the dummy item (i.e., $p_k^h(0) = 0$), the bidder is indifferent between the dummy item and possibly some real items, so the auctioneer can give other bidders priorities to get real items without hurting bidder h.

A Lemma

Lemma 1: If bidder h is a provisionally losing bidder in an optimal assignment π of the problem:

$$\max_{\rho \in \mathcal{A}} \sum_{h \in M} p_k^h(\rho(h)), \tag{1}$$

i.e., $\pi(h) = 0$ and $p_h(0) = -2^{-h}$, the bidder must also be assigned a dummy item in any other optimal assignment ρ of the problem, i.e., $\rho(h) = 0$.

So the problem (1) has a unique set of provisionally losing bidders.

Sincere Bidding Strategies: I

In Step $k = 1$, every bidder $h \in M$ sets an attainable utility $\hat{u}_1^h \in \mathbb{Z}_+$

$$\hat{u}_1^h = \max_{a \in N_0} v^h(a). \tag{2}$$

In subsequent step $k > 1$, if bidder h is a PL bidder, he will make new bids by reducing his attainable utility by a decrement

$$\Delta_k = \min \left\{ d \in \mathbb{Z}_{++} \mid v^h(a) - (\hat{u}_{k-1}^h - d) \in [0, b^h] \text{ for some } a \in N_0 \right\}.$$

That is, $\hat{u}_k^h = \hat{u}_{k-1}^h - \Delta_k$ is a possible attainable utility in step k and Δ_k is the minimal integer for this change. In most cases, Δ_k is 1. But if a bid reaches the budget, it can be larger than 1.

Sincere Bidding Strategies: II

In each step $k \geq 1$, for every item $a \in N_0$, every PL bidder $h \in M$ calculates a possible price $\hat{p}^h(a|\hat{u}_k^h) = v^h(a) - \hat{u}_k^h$ and makes a bid or not as

$$p_k^h(a) = \begin{cases} \hat{p}^h(a|\hat{u}_k^h) & \text{if } 0 \leq \hat{p}^h(a|\hat{u}_k^h) \leq b^h \\ b^h & \text{if } \hat{p}^h(a|\hat{u}_k^h) > b^h \\ \star & \text{if } \hat{p}^h(a|\hat{u}_k^h) < 0 \end{cases}$$

where '$\star$' stands for not bidding on item a. The bidder h will not bid more than his hard budget b^h when $\hat{p}^h(a|\hat{u}_k^h) > b^h$, because of a fixed penalty δ^*.

Sincere bidding will be shown to be an ex post Nash equilibrium of the dynamic auction game. When bidders bid sincerely, the auction finds a strongly Pareto-efficient core allocation when bidders are budget-constrained; otherwise, it finds a WE with the minimum WE prices, thus always yielding an efficient outcome.

Example 3 for Comparison

Example 3: A seller sells two items a_1 and a_2 to bidders 1, 2, 3, and 4. Valuations and budgets are given in Table 1. All bidders are budget constrained.

Table 1: Valuations and budgets

Bidder	$v^i(0)$	$v^i(a_1)$	$v^i(a_2)$	Budget b^i
1	0	10	2	5
2	0	10	4	5
3	0	2	7	4
4	0	7	7	3

Illustration of the DGS Auction

In step $k = 7$, bidder 3 gets item a_2 and pays 4 and the seller keeps item a_1. This yields a total increase of 7 in the utility of trade. Bidder 4 has a jump in demand from $t = 5$ to $t = 6$ and bidders 1 and 2 have a jump in demand from $t = 6$ to $t = 7$.

Table 2: Illustration of the DGS auction for Example 3.

k	p^k	S^k	$D^1(p^k)$	$D^2(p^k)$	$D^3(p^k)$	$D^4(p^k)$
1	$(0, 0)$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{a_1, a_2\}$
2	$(1, 0)$	$\{a_1, a_2\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{a_2\}$
3	$(2, 1)$	$\{a_1, a_2\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{a_2\}$
4	$(3, 2)$	$\{a_1, a_2\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{a_2\}$
5	$(4, 3)$	$\{a_1, a_2\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{a_2\}$
6	$(5, 4)$	$\{a_1\}$	$\{a_1\}$	$\{a_1\}$	$\{a_2\}$	$\{0\}$
7	$(6, 4)$	$\emptyset$	$\{0\}$	$\{0, a_2\}$	$\{a_2\}$	$\{0\}$

Illustration of the New Dynamic Auction: I

In $k = 1$, no bidder bids for the dummy item, so they are all active. Only bidders 2 and 4 are PL bidders. In $k = 2$, bidders 2 and 4 make new bids, and bidders 1 and 3 become PL. In $k = 7$, bidder 4 becomes a PL. In $k = 8$, bidder 4 makes a bid on the null item and keeps his bids on items a_1 and a_2 unchanged, and then becomes inactive. His attainable utility decreases from 4 to 0, as his previous prices reach his budget. Symbol '$-$' on a real item means $-\infty$ for its price.

Table 3: Illustration of the new auction for Example 3.

k	$(\hat{u}_k^1, \hat{u}_k^2, \hat{u}_k^3, \hat{u}_k^4)$	$p_k^1(\cdot)$	$p_k^2(\cdot)$	$p_k^3(\cdot)$	$p_k^4(\cdot)$	$\pi_k(1,2,3,4)$
1	$(10, 10, 7, 7)$	$(-\frac{1}{2}, 0, -)$	$(-\frac{1}{4}, 0, -)$	$(-\frac{1}{8}, -, 0)$	$(-\frac{1}{16}, 0, 0)$	$(a_1, 0, a_2, 0)$
2	$(10, 9, 7, 6)$	$(-\frac{1}{2}, 0, -)$	$(-\frac{1}{4}, 1, -)$	$(-\frac{1}{8}, -, 0)$	$(-\frac{1}{16}, 1, 1)$	$(0, a_1, 0, a_2)$
3	$(9, 9, 6, 6)$	$(-\frac{1}{2}, 1, -)$	$(-\frac{1}{4}, 1, -)$	$(-\frac{1}{8}, -, 1)$	$(-\frac{1}{16}, 1, 1)$	$(a_1, 0, a_2, 0)$
4	$(9, 8, 6, 5)$	$(-\frac{1}{2}, 1, -)$	$(-\frac{1}{4}, 2, -)$	$(-\frac{1}{8}, -, 1)$	$(-\frac{1}{16}, 2, 2)$	$(0, a_1, 0, a_2)$
5	$(8, 8, 5, 5)$	$(-\frac{1}{2}, 2, -)$	$(-\frac{1}{4}, 2, -)$	$(-\frac{1}{8}, -, 2)$	$(-\frac{1}{16}, 2, 2)$	$(a_1, 0, a_2, 0)$
6	$(8, 7, 5, 4)$	$(-\frac{1}{2}, 2, -)$	$(-\frac{1}{4}, 3, -)$	$(-\frac{1}{8}, -, 2)$	$(-\frac{1}{16}, 3, 3)$	$(0, a_1, 0, a_2)$
7	$(7, 7, 4, 4)$	$(-\frac{1}{2}, 3, -)$	$(-\frac{1}{4}, 3, -)$	$(-\frac{1}{8}, -, 3)$	$(-\frac{1}{16}, 3, 3)$	$(a_1, 0, a_2, 0)$
8	$(7, 6, 4, 0)$	$(-\frac{1}{2}, 3, -)$	$(-\frac{1}{4}, 4, -)$	$(-\frac{1}{8}, -, 3)$	$(0, 3, 3)$	$(0, a_1, a_2, 0)$

Illustration of the New Dynamic Auction: II

The auction ends in $k = 17$ when there is no PL. Bidder 2 makes a bid for the null item and becomes inactive, while bidders 1 and 3 are still active. Although bidders 2 and 3 offer the same bid of 4 on item a_2, bidder 3 has higher priority than bidder 2, because bidder 2's bid on the null item is 0 but bidder 3's bid on it is $-\frac{1}{8}$. Clearly, $\pi^* = (a_1, 0, a_2, 0)$ is the unique optimal assignment. In the end, bidder 1 gets a_1 and pays 5, bidder 3 gets a_2 and pays 4. This is a core allocation. This outcome has improved the one found by the DGS auction by increasing the total utility from 7 to 17!

Table 4: Illustration of the new auction for Example 3.

k	$(\hat{u}_k^1, \hat{u}_k^2, \hat{u}_k^3, \hat{u}_k^4)$	$p_k^1(\cdot)$	$p_k^2(\cdot)$	$p_k^3(\cdot)$	$p_k^4(\cdot)$	$\pi_k(1,2,3,4)$
9	$(6,6,4,0)$	$(-\frac{1}{2},4,-)$	$(-\frac{1}{4},4,-)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,0,a_2,0)$
10	$(6,5,4,0)$	$(-\frac{1}{2},4,-)$	$(-\frac{1}{4},5,-)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(0,a_1,a_2,0)$
11	$(5,5,4,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,-)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,0,a_2,0)$
12	$(5,4,4,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,0)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,0,a_2,0)$
13	$(5,3,4,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,1)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,0,a_2,0)$
14	$(5,2,4,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,2)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,0,a_2,0)$
15	$(5,1,4,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,3)$	$(-\frac{1}{8},-,3)$	$(0,3,3)$	$(a_1,a_2,0,0)$
16	$(5,1,3,0)$	$(-\frac{1}{2},5,-)$	$(-\frac{1}{4},5,3)$	$(-\frac{1}{8},-,4)$	$(0,3,3)$	$(a_1,0,a_2,0)$
17	$(5,0,3,0)$	$(-\frac{1}{2},5,-)$	$(0,5,4)$	$(-\frac{1}{8},-,4)$	$(0,3,3)$	$(a_1,0,a_2,0)$

Ex Post Nash Equilibrium

Sincere bidding in every round of an auction is an *ex post Nash equilibrium* if, for every bidder $i \in M$, using his sincere bidding strategy maximizes his utility as long as all other bidders in $M \setminus \{i\}$ follow their sincere bidding strategies.

This notion of equilibrium is robust against regret and independent of any probability distribution assumption. But it focuses only on the equilibrium path and cannot deal with the case of off-equilibrium path.

An Incentive Compatibility Theorem

Proposition 1 (Yang and Yu 2025): The proposed auction terminates within a finite number of steps.

Theorem 1 (Yang and Yu 2025): In the face of the proposed auction, sincere bidding is an ex post Nash equilibrium of the dynamic auction game.

The above result generalizes the incentive results of Leonard (1983) and Demange et al. (1986).

Three Convergence Theorems

Theorem 2 (Yang and Yu 2025): The proposed auction finds a strongly Pareto efficient core allocation.

The next result shows that in the presence of budget constraints, the price vector found by the proposed dynamic auction actually treats all the bidders better than the minimum integer Walrasian equilibrium price vector does provided a WE exists.

Theorem 3 (Yang and Yu 2025): If there exists a Walrasian equilibrium, the price vector generated by the proposed dynamic auction is not larger than the minimum integer Walrasian equilibrium price vector.

Theorem 4 (Yang and Yu 2025): When no bidder is budget constrained, the proposed auction finds a Walrasian equilibrium with the minimum equilibrium price vector, which is also in the strong core.

A Lattice Theorem

Theorem 5 (van der Laan, Talman, Yang, and Yu 2025): If a Walrasian equilibrium exists, the set of Walrasian equilibrium price vectors forms a lattice and contains a unique minimum integer Walrasian equilibrium price vector and a unique maximum integer Walrasian equilibrium price vector.

An intuitive display of this theorem is given by an simple example. The auctioneer sells one item a to bidders 1 and 2. Bidders have values $v^1(a) = 20$, $v^2(a) = 18$, and budgets $b^1 = 6$ and $b^2 = 10$, respectively. In this case, bidder 2 wins the item by paying a price $p(a) \in (6, 10]$. Both bidders are budget constrained. Clearly, Walrasian equilibrium prices form a nonempty lattice which is open from below but closed from above.

Two Questions

Question 1: Is it possible to design an efficient dynamic auction for the assignment market with possibly budget constrained bidders such that sincere bidding is an ex post perfect Nash equilibrium or even a dominant strategy for all bidders?

Question 2: Is it possible to design an efficient and strategy-proof dynamic auction when bidders are budget constrained and have gross substitutes preferences?

Thank You!