Lecture 3: A Universally Efficient Dynamic Auction for All Unimodular Demand Types

Time: 10:30-12:00, Wed, 30th July 2025

Venue: University of Tokyo

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Consider a general auction market where all kinds of indivisible goods, simply called objects or items, can be sold. They can be identical, heterogeneous, substitutes, complements, or any mixture of substitutes and complements.

Bidders may demand any number of items. Each bidder has his private valuation on every bundle of his interested items and can be strategic. The seller has her reserve price for every bundle of items.

How to design a dynamic auction mechanism with simple, practical and transparent rules that will allocate items efficiently, induce bidders to bid truthfully and require as little information from bidders as possible to protect their privacy?

A Particular Instance of Hayek's Problem

An auction design problem is a particular instance of Hayek's rational economic order problem with strategic agents. Hayek (1945) says: "The peculiar character of the problem of a rational economic order is determined precisely by the fact that the knowledge of the circumstances of which we must make use never exists in concentrated or integrated form but solely as the dispersed bits of incomplete and frequently contradictory knowledge which all the separate individuals possess..... To put it briefly, it is a problem of the utilization of knowledge which is not given to anyone in its totality...... Fundamentally, in a system in which the knowledge of the relevant facts is dispersed among many people, prices can act to coordinate the separate actions of different people...... This (price) mechanism would have been acclaimed as one of the greatest triumphs of the human mind."

A novel and universal dynamic auction design is introduced and applies to all unimodular demand types of Baldwin and Klemperer (2019, ECTA) which are a necessary and sufficient condition for the existence of competitive equilibrium and accommodate all kinds of indivisible goods.

The auction always induces bidders to bid truthfully and yields an efficient outcome. Sincere bidding is an ex post perfect Nash equilibrium.

The auction is privacy-preserving, robust against any regret, and independent of any probability distribution assumption.

The auction rules are simple, practical, transparent and detail-free.

This auction is developed by Fujishige and Yang (2025, Mathematics of Operations Research, appearing online).

The huge volume of the sale of spectrum licenses in the world since early 1990s; Klemperer (2004) and Milgrom (2004). Airport take-off and landing slots, cloud computing band and time allocation, networks, mining rights, treasury bills, key words, pollution permits, etc. The trading volume and value via auction is staggeringly high, involving billions and billions of dollars.

Substitutability and complementarity are fundamental properties of goods and services and are pervasive.

Baldwin and Klemperer (2019, ECTA) found surprisingly that contrary to popular belief, equilibrium is guaranteed for more classes of complements than of substitutes.

Milgrom (2017) says: "Markets for complements can be much harder than markets for substitutes and can require greater planning and coordination." See also Milgrom (2000, JPE), Jehiel and Moldovanu (2003), Klemperer (2004), Maskin (2005).

- $N = \{1, 2, \cdots, n\}$: the set of indivisible items. Each item $j \in N$ is also represented by the j-th unit vector $e(j) \in \mathbb{Z}^N$.
  Let $\{0,1\}^N$ denote the set of all bundles of items.

- B: a group of m bidders. Let $B_0 = B \cup \{0\}$ and 0 stand for the seller.

- Every bidder (he) $j \in B$ has a utility function $u^j : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$ specifying his valuation $u^j(x)$ (in units of currency, say, yen) on every bundle x.

- The seller (she) denoted by 0 has a reserve price function $u^0 : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$.

  Let $\mathrm{dom}(u^j) = \{x \in \{0,1\}^N \mid u^j(x) > -\infty\}$ denote the *effective domain* of u^j for all $j \in B_0$. A bundle x is *unacceptable* to an agent $j \in B_0$ if $u^j(x) = -\infty$, i.e., $x \notin \mathrm{dom}(u^j)$.

- Let $\mathcal{M} = (u^j, j \in B_0, N)$ or simply $\mathcal{M}$ represent the market.

- A price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^N$ specifies a price p_h for each item $h \in N$. This is a linear and anonymous pricing rule.

- At prices p, every bidder $j \in B$ tries to maximize his profit and his demand set $D^j(p)$ is given by

$$D^j(p) = \arg \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\}.$$

- At prices p, the seller chooses bundles to maximize her revenues and her demand set $D^0(p)$ is given by

$$\begin{aligned} D^0(p) &= \arg\max_{x \in \{0,1\}^N} \{u^0(x) + p \cdot (\textstyle\sum_{h \in N} e(h) - x)\} \\ &= \arg\max_{x \in \{0,1\}^N} \{u^0(x) - p \cdot x\} \end{aligned}$$

  The set $D^0(p)$ contains those bundles that the seller wishes to keep in hand and give her the highest revenues.

- An *allocation* of items in N is a *redistribution* $X = (x^j, j \in B_0)$ of items among all agents in B_0 such that $\sum_{j \in B_0} x^j = \sum_{h \in N} e(h)$ and $x^j \in \{0,1\}^N$ for all $j \in B_0$.

- Allocation $X = (x^j, j \in B_0)$ is *feasible* if it satisfies $x^j \in \mathrm{dom}(u^j)$ for all $j \in B \cup \{0\}$.

- An allocation $X = (x^j, j \in B_0)$ is *efficient* if $\sum_{j \in B_0} u^j(x^j) \geq \sum_{j \in B_0} u^j(y^j)$ for every allocation $Y = (y^j, j \in B_0)$.

- Given an efficient allocation X, let $SV(B) = \sum_{j \in B_0} u^j(x^j)$. We call $SV(B)$ *the market value* of the items.

- A competitive or Walrasian equilibrium (p, X) consists of a price vector $p \in \mathbb{R}^N$ and an allocation X such that $x^j \in D^j(p)$ for every $j \in B_0$.

- Quasi-Linear Utilities: All agents $j \in B_0$ have the form of utilities $U^j(x, c) = u^j(x) + c$ for all bundles $x \in \{0, 1\}^N$ and money $c \in \mathbb{R}$.

- No Budget Constraints: Every agent has a limited but enough amount of budget so that she is not subject to any budget constraint.
  When a commodity is sold with a negative price, the seller will pay the price and the commodity can be bad.

- Feasibility: The market has at least one feasible allocation.

- The set $\mathrm{dom}(u^j)$ of every bidder $j \in B$ contains at least one nonzero vector and also the dummy bundle $\mathbf{0}$ with $u^j(\mathbf{0}) = 0$. So every bidder has the option of buying nothing and is interested in buying some items.

- The set $\mathrm{dom}(u^0)$ of the seller equals $\{0, 1\}^N$ with $u^0(\mathbf{0}) = 0$. The seller will not sell but retain a bundle if the price of the bundle is below her reserve price, and she will keep any bundle of her own items if the bundle is not sold.

- This market obviously has at least one feasible allocation.

The following two conditions will be imposed on the market $\mathcal{M}$:

(A1) *Integer Private Values*: Every bidder $j \in B$ knows his own utility function $u^j : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$ privately and is strategic. The seller knows her own utility function $u^0 : \{0,1\}^N \to \mathbb{Z} \cup \{-\infty\}$ privately and is honest.

(A2) *Common Unimodular Demand Type*: All agents $j \in B_0$ have the same unimodular demand type $\mathcal{D}$ for their utility functions u^j. This information is made known to the auctioneer.

- W.R.T. any utility function $u : S \to \mathbb{R}$, let the demand set at prices p be given by $D_u(p) = \arg\max_{x \in S}\{u(x) - p \cdot x\}$, where $S \subseteq \mathbb{Z}^N$.

- Following Baldwin and Klemperer (2019), we call the set $\mathcal{T}_u = \{p \in \mathbb{R}^N \mid \sharp D_u(p) > 1\}$ *the locus of indifference prices (LIP)* of the demand set D_u, where $\sharp D_u(p)$ denotes the number of elements in $D_u(p)$.

- This set $\mathcal{T}_u$ concerns those price vectors p at which there are at least two optimal bundles for any agent who has the utility function u.

- A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix A and an m-vector. A polyhedron is a *polytope* if it is bounded.

Demand Type

- A facet of a polyhedron of dimension n is a face that has dimension $n-1$.
- The LIP $\mathcal{T}_u$ is the union of $(n-1)$-dimensional linear pieces also called facets. These facets separate the unique demand regions, in each of which some bundle is the unique demand.
- The *normal vector* to a facet F is a vector which is perpendicular to F at a point in its relative interior.
- A non-zero integer vector is *primitive* if the greatest common divisor of its coordinates is one.

Demand Type (Baldwin and Klemperer 2019): A finite set $\mathcal{D}$ of nonzero primitive integer vectors in $\mathbb{Z}^N$ is a *demand type* $\mathcal{D}$ if $v \in \mathcal{D}$ implies $-v \in \mathcal{D}$ and every facet of the LIP $\mathcal{T}_u$ has a normal vector in the set $\mathcal{D}$.

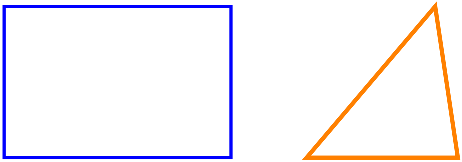

Figure 1: Blue and orange lines are facets of the rectangle and triangle, resp.

Demand types are derived from utility functions and given as sets of integer vectors. These demand types capture the quintessential and natural attributes of the commodities but do not reveal the values of the consumers. For instance, consumers view tables as something sharing the same physical property but they can each have different valuations on tables.

A square matrix M is *unimodular* if all its elements are integral and its determinant is $+1$ or -1. A matrix M is *totally unimodular* if every minor of M is 0 or ± 1. A set of n integer vectors in $\mathbb{R}^N$ is *a unimodular basis* for $\mathbb{R}^N$ if the $n \times n$ matrix which has the n integer vectors as its columns is unimodular.

Unimodular Demand Type (Baldwin and Klemperer 2019): A demand type $\mathcal{D}$ in $\mathbb{R}^N$ is *unimodular* if every linearly independent subset of $\mathcal{D}$ can be extended to a unimodular basis for $\mathbb{R}^N$.

Example 1: Two bidders and two substitutable items. Valuations:

$Bundle$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$Bidder1$	0	3	4	5
$Bidder2$	0	5	2	6

Both bidders have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$. See the blue dashed lines in the figure.

Example 2: Three bidders and two complementary items $A = (1,0)$ and $B = (0,1)$. Valuations:

$Bundle$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$Bidder1$	0	2	2	5
$Bidder2$	0	2	2	5
$Bidder3$	0	1	1	4
$Seller$	0	1	1	3

All agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$.

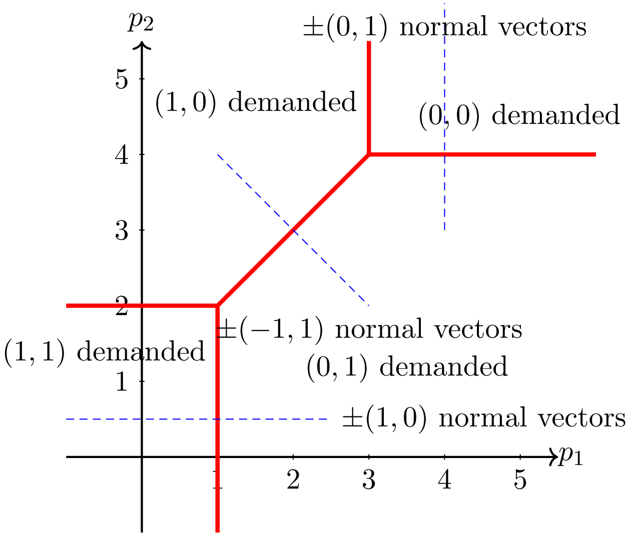

Figure 2: $u(0,0) = 0$, $u(1,0) = 3$, $u(0,1) = 4$, and $u(1,1) = 5$. The red lines denote LIP.

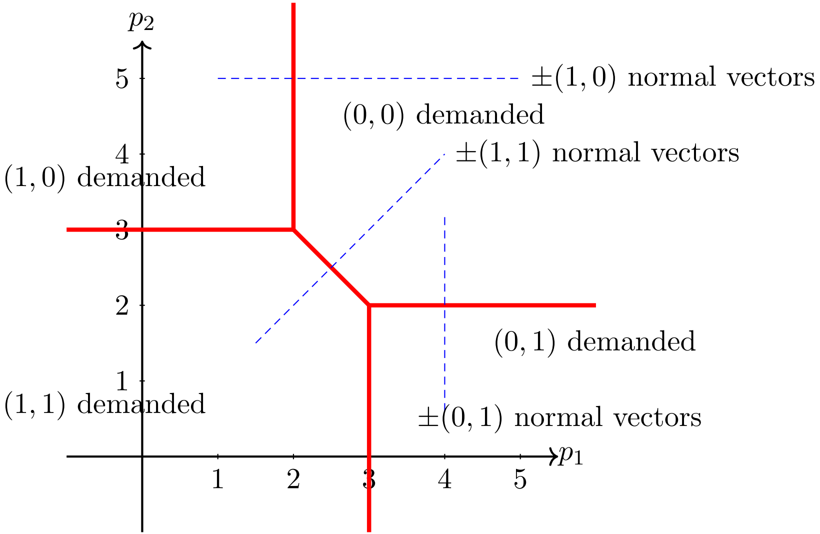

Figure 3: $u(0,0) = 0$, $u(1,0) = u(0,1) = 2$, and $u(1,1) = 5$. The red lines denote LIP.

Proposition 1 (Fujishige and Yang 2025): Every unimodular demand type in $\mathbb{R}^N$ can be added with less than n new vectors so that the enlarged set is still a unimodular demand type and contains at least one basis.

This basic property of unimodular demand types is used in our auction design by naturally assuming that every given unimodular demand type spans the space $\mathbb{R}^N$. By using Proposition 1 and its proof, we have the following alternative characterization of unimodular demand types which is easy to verify.

Proposition 2: A demand type $\mathcal{D}$ in $\mathbb{R}^N$ is unimodular if and only if there exists a set $\mathcal{D}^* \supseteq \mathcal{D}$ of nonzero primitive integer vectors such that every $1 \leq k \leq n$ linearly independent vectors from the set $\mathcal{D}^*$ can find $n-k$ vectors from the set $\mathcal{D}^*$ to form a unimodular matrix.

Baldwin and Klemperer (2019) have identified a variety of demand types and shown the richness of complements.

A demand type $\mathcal{D}$ is *Gross Substitutes* if every vector $x \in \mathcal{D}$ has at most one 1 and at most one -1 and no other nonzero entries. Kelso and Crawford (1982).

Let (S_1, S_2) be a partition of the set N. A demand type $\mathcal{D}$ is *Gross Substitutes and Complements* (GSC) if every vector $x \in \mathcal{D}$ has at most two nonzero components of $+1$ or -1 and no other nonzero entries so that if two nonzero components of x have the same sign, then one nonzero component must be indexed by an element in S_1 and the other must be indexed by an element in S_2. Sun and Yang (2006, 2009).

A unimodular demand type $\mathcal{D}$ is *unimodular complements* if every vector $x \in \mathcal{D}$ implies either $x \in \{0, 1\}^N$ or $x \in \{0, -1\}^N$.

- A set $S \subseteq \mathbb{R}^N$ is a *polyhedron* if $S = \{x \in \mathbb{R}^N \mid Ax \leq b\}$ for some $m \times n$ matrix A and an m-vector. A bounded polyhedron is called *a polytope.*
- A polyhedron $S \subseteq \mathbb{R}^N$ is *integral* if all its vertices are integral.
- Given any $x, y \in \mathbb{R}^N$, define their meet $x \wedge y$ as the componentwise minimum of x and y and join $x \vee y$ as the componentwise maximum of x and y. A set $S \subset \mathbb{R}^N$ is *a lattice* if $x \wedge y \in S$ and $x \vee y \in S$ for any $x, y \in S$. A polyhedron is called *a polyhedron with a lattice structure* if it is also a lattice.
- A function f with a polyhedron domain in $\mathbb{R}^N$ is called *a polyhedral convex function* if it is given as

$$f(x) = \max\{B_j \cdot x + c_j \mid j = 1, \cdots, m\}$$

where B_j is an n-vector and c_j is a constant, $j = 1, \cdots, m$ for a given positive integer m.

- For every $j \in B_0$, define her indirect utility function by

$$V^j(p) = \max_{x \in \{0,1\}^N} \{u^j(x) - p \cdot x\}.$$

- Define the Lyapunov function $\mathcal{L} : \mathbb{R}^N \to \mathbb{R}$ by

$$\mathcal{L}(p) = \sum_{h \in N} p_h + \sum_{j \in B_0} V^j(p).$$

- A function $f : \mathbb{Z}^N \mapsto \mathbb{R}$ is *discrete concave* if for any $\lambda_j \geq 0$, $j = 1, \cdots, t$ and any $x^j \in \mathbb{Z}^N$ for $j = 1, \cdots, t$ with $\sum_{j=1}^t \lambda_j = 1$ and $\sum_{j=1}^t \lambda_j x^j \in \mathbb{Z}^N$ we have

$$f(\lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_t x^t) \geq \sum_{j=1} \lambda_j f(x^j).$$

Lemma 1: For the market model, the Lyapunov function $\mathcal{L}$ is a polyhedral convex function bounded from below.

Theorem 1 (Fujishige and Yang 2025): Assume that the market model satisfies Assumptions (A1) and (A2). Then the set of competitive equilibrium price vectors forms a nonempty integral polytope.

The following result sharpens the lattice results of Gul and Stacchetti (1999) and Ausubel (2006).

Proposition 3 (Fujishige and Yang 2025): Assume that the market model satisfies Assumption (A1) and all items are substitutes (GS). Then the set of competitive equilibrium price vectors forms a nonempty integral polytope with a lattice structure.

Dynamic Auction Design: Ideas and Challenges

The underlying principle of our auction design is to try to find a minimizer of the Lyapunov function. Two challenges.

How to Elicit Private Information? First, the auctioneer cannot get the Lyapunov function, which is based on every bidder's private information, i.e., his indirect utility function. How can we extract enough information from bidders?

How to Do Local Search? Second, typically, one uses the gradient of a function and does repeated local searches to find a minimiser. What can we do when the Lyapunov function is not available nor its gradient and we have to deal with all unimodular demand types?

The Answer is to use **prices and demand sets** (observable information) and to introduce the notion of **"search set"**.

Search Set (Fujishige and Yang 2025): A search set for a demand type $\mathcal{D}$ is the collection of the zero vector and all nonzero primitive integer vectors $\delta \in \mathbb{Z}^N$ such that we have $\delta \cdot d_i = 0$ for some $n-1$ linearly independent vectors $d_1, \cdots, d_{n-1} \in \mathcal{D}$. The search set is denoted by $\mathcal{SD}$.

Example 1: Two bidders and two substitutable items. Valuations:

Bundle	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
*Bidder*1	0	3	4	5
*Bidder*2	0	5	2	6

Both bidders have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,-1)\}$ whose search set equals $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,1)\}$.

Example 2: Three bidders and two complementary items $A = (1,0)$ and $B = (0,1)$. Valuations:

Bundle	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
*Bidder*1	0	2	2	5
*Bidder*2	0	2	2	5
*Bidder*3	0	1	1	4
Seller	0	1	1	3

All agents have the same unimodular demand type $\mathcal{D} = \{\pm(1,0), \pm(0,1), \pm(1,1)\}$ whose search set equals $\mathcal{SD} = \{(0,0), \pm(1,0), \pm(0,1), \pm(1,-1)\}$.

Every unimodular demand type $\mathcal{D}$ gives its own search set $\mathcal{SD}$.

Roughly speaking, if we adjust the prices of goods along the direction of an element from the search set, it will not cause dramatic change in demands on goods.

The auction is to search for a minimizer of the Lyapunov function $\mathcal{L}$. Given an integer price vector $p(t) \in \mathbb{Z}^n$ at time $t \in \mathbb{Z}_+$, the auctioneer asks every bidder i to report his demand $D^i(p(t))$. Then she uses every bidder's reported demand $D^i(p(t))$ to search for a price adjustment δ in the search set so as to reduce the value of the Lyapunov function $\mathcal{L}(p(t) + \delta)$ as much as possible, in hope of finding the minimum of the Lyapunov function.

The following proposition says when the auctioneer tries to adjust prices, she just needs to focus on the few choices in the search set $\mathcal{SD}$ rather than gropes around the entire convex hull of the set $\mathcal{SD}$.
Proposition 4 (Fujishige and Yang 2025): Under Assumptions (A1) and (A2) we have

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\}. \quad (1)$$

This result with its proof implies the next corollary, which says that if we can change prices slightly, the demand set of every bidder will not change.

Corollary 1 (Fujishige and Yang 2025): Under Assumptions (A1) and (A2), then for any $j \in B_0$, any $p \in \mathbb{Z}^N$, and any $\delta \in \mathcal{SD}$, we have $D^j(p+\varepsilon\delta) \subseteq D^j(p)$ for all $\varepsilon \in [0,1)$ and $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ lies in $D^j(p+\varepsilon\delta)$ for all $\varepsilon \in [0,1]$.

From Proposition 4, we rewrite the maximand of (1) as

$$\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta) = \sum_{j \in B_0} (V^j(p(t)) - V^j(p(t) + \delta)) - \sum_{i \in N} \delta_i. \tag{2}$$

By Corollary 1, we can immediately infer the difference between $\mathcal{L}(p(t))$ and $\mathcal{L}(p(t) + \delta)$ just from the reported demands $D^j(p(t))$ and the price variation δ because $x^j \in \arg\min_{x \in D^j(p)} x \cdot \delta$ is in $D^j(p + \varepsilon\delta)$ for all $\varepsilon \in [0, 1]$. So, when prices move from $p(t)$ to $p(t) + \delta$, the reduction in indirect utility for every bidder j is uniquely given by

$$V^j(p(t)) - V^j(p(t) + \delta) = \min_{x^j \in D^j(p(t))} x^j \cdot \delta. \tag{3}$$

Dynamic Auction Design: Basic Relations III

From Proposition 4 and (3), we have:

Lemma 2 (Fujishige and Yang 2025): Under Assumptions (A1) and (A2) we further have:

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t)+\delta)\} = \max_{\delta \in \mathcal{SD}} \{\sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{h \in N} \delta_h\} \quad (4)$$

This relation (4) shows a dramatic change from the unobservable Lyapunov function $\mathcal{L}$ to the observable reported demands of bidders and integer price adjustment δ. A magic transformation!

The right-hand max-min formula says that when the auctioneer adjusts the prices from $p(t)$ to $p(t+1) = p(t) + \delta(t)$, she tries to balance two opposing forces by minimizing every bidder's loss for every possible price change δ in $\mathcal{D}$ and choosing one price change that maximizes the seller's gain from all possible price changes.

Theorem 2 (Fujishige and Yang 2025): Under Assumptions (A1) and (A2), $p^* \in \mathbb{Z}^N$ is a competitive equilibrium price vector if and only if $\mathcal{L}(p^*) \leq \mathcal{L}(p^* + \delta)$ for all $\delta \in \mathcal{SD}$.

In the auction process bidders do nothing but report their demand sets $D^j(p(t))$ and the auctioneer adjusts prices according to the right-hand formula of (4) in Lemma 2, i.e.,

$$\max_{\delta \in \mathrm{Conv}(\mathcal{SD})} \{\mathcal{L}(p(t)) - \mathcal{L}(p(t) + \delta)\} = \max_{\delta \in \mathcal{SD}} \{\sum_{j \in B_0} \min_{x^j \in D^j(p(t))} x^j \cdot \delta - \sum_{h \in N} \delta_h\}$$

Agent $j \in B_0$ **bids sincerely** w.r.t. her u^j if she submits her demand set $D^j(p(t)) = \arg\max_{x \in \{0,1\}^N} \{u^j(x) - p(t) \cdot x\}$ at $\forall t \in \mathbb{Z}_+$ and $\forall p(t) \in \mathbb{R}^N$.

The universally convergent dynamic (UCD) auction

Step 1: The auctioneer announces an (arbitrary) initial price vector $p(0) \in \mathbb{Z}^N$. Let $t := 0$ and go to Step 2.

Step 2: Every agent $j \in B_0$ reports her demand $D^j(p(t))$ at $p(t)$ to the auctioneer. Then based on reported demands $D^j(p(t))$, the auctioneer finds an integer solution $\delta(t)$ to the right side problem of (4). If the zero vector $\delta(t) = 0$ is an optimal solution, the auction stops. Otherwise, the auctioneer adjusts prices by setting $p(t+1) := p(t) + \delta(t)$ and $t := t + 1$. Return to Step 2.

Theorem 3 (Fujishige and Yang 2025): Under Assumptions (A1) and (A2), starting with any given initial price vector $p(0) \in \mathbb{Z}^N$, the UCD auction finds an integer competitive equilibrium vector in a finite steps when bidders bid sincerely.

Recall Example 2 with two items $A = (1, 0)$ and $B = (0, 1)$. Every agent knows her values privately. Agents' values are given in Table 1.

Table 1: *Valuations of Agents over Bundles.*

Agents\Bundles	$\emptyset$	$A = (1, 0)$	$B = (0, 1)$	$AB = (1, 1)$
Bidder 1	*0*	*2*	*2*	*5*
Bidder 2	*0*	*2*	*2*	*5*
Bidder 3	*0*	*1*	*1*	*4*
Seller	*0*	*1*	*1*	*3*

For this example, we have the demand type
$\mathcal{D} = \{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}$ and its search set
$\mathcal{SD} = \{(0, 0), \pm(1, 0), \pm(0, 1), \pm(1, -1)\}$.

Let us see how a simultaneously ascending auction would operate for this example. Start at $p(0) = (p_A(0), p_B(0)) = (0, 0)$. Every agent demands AB. As AB is overdemanded, the auction raises the two prices each by one unit. Prices are updated to $p(1) = (1, 1)$, to $p(2) = (2, 2)$, and to $p(3) = (3, 3)$. At $p(3)$ no bidder wants any item and the auction is stuck in a non-equilibrium state, causing an exposure problem. The new dynamic auction, however, finds a WE at $p(5) = (3, 2)$ in which AB is given to bidder 1 or 2 who pays 5.

Table 2: *Illustration of the New Dynamic Auction.*

time t	*prices* $p(t)$	$\delta(t)$	$D^0(p(t))$	$D^1(p(t)) = D^2(p(t))$	$D^3(p(t))$
0	$(0,0)$	$(1,0)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
1	$(1,0)$	$(0,1)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
2	$(1,1)$	$(1,0)$	$\{AB\}$	$\{AB\}$	$\{AB\}$
3	$(2,1)$	$(0,1)$	$\{AB, B, \emptyset\}$	$\{AB\}$	$\{AB\}$
4	$(2,2)$	$(1,0)$	$\{\emptyset\}$	$\{AB\}$	$\{AB, \emptyset\}$
5	$(3,2)$	$(0,0)$	$\{\emptyset\}$	$\{AB, B, \emptyset\}$	$\{\emptyset\}$

Recall $\mathcal{M}$ stands for the (original) market with m bidders and the seller with the set N of n items. For every bidder $j \in B$, let $\mathcal{M}_{-j}$ denote the market $\mathcal{M}$ *without bidder* j and $B_{-j} = B_0 \setminus \{j\}$. Let $\mathcal{M}_{-0} = \mathcal{M}$ and $B_{-0} = B_0$. So, for every $k \in B_0$, the sub-market $\mathcal{M}_{-k}$ comprises the set B_{-k} of agents and the set N of n items.

VCG Direct Mechanism: Every agent $j \in B_0$ reports her value function u^j. The auctioneer computes an efficient allocation X with respect to all reported u^j and assigns bundle x^j to bidder $j \in B$ and charges him a payment of $\beta^*_j = u^j(x^j) - SV(B) + SV(B_{-j})$, where $SV(B)$ and $SV(B_{-j})$ are the market values of the items in N in the markets $\mathcal{M}$ and $\mathcal{M}_{-j}$ for all $j \in B$, respectively. Bidder j's VCG payoff equals $SV(B) - SV(B_{-j})$, $j \in B$.

It is known from Green and Laffont (1977) and Holmström (1979) that in the setting of transferable utility any strategy-proof mechanism must generate the VCG outcome.

The basic idea of the IC dynamic auction to implement the universally convergent dynamic (UCD) auction for every sub-market $\mathcal{M}_{-k}$ ($k \in B_0$) simultaneously from the same starting price vector. This will create $m+1$ paths of price vectors. By using the bids of every bidder and the generated price vectors the auction will converge to a WE, generating a VCG outcome. As mentioned previously, Ausubel (2006) has explored this idea for the market with gross substitutes and Sun and Yang (2014) for the market with superadditive utilities.

It will be shown that sincere bidding is an ex post perfect Nash equilibrium.

Let $p^k(t)$ denote the prices of each market $\mathcal{M}_{-k}$ ($k \in B_0$) at time $t \in \mathbb{Z}_+$. Then at $t \in \mathbb{Z}_+$ and with respect to $p^k(t) \in \mathbb{Z}^N$, every bidder $j \in B_{-k}$ submits his bid $B_k^j(t) \subseteq \{0,1\}^N$ which may differ from his true demand set $D^j(p^k(t))$, but the seller's bid $B_k^0(t)$ always equals her true demand set $D^0(p^k(t))$. The auctioneer solves the problem (4), i.e.,

$$\max_{\delta \in \mathcal{SD}} \{ \sum_{j \in B_{-k}} \min_{x^j \in B_k^j(t)} x^j \cdot \delta - \sum_{h \in N} \delta_h \} \tag{5}$$

When $\delta^k(t) = \mathbf{0}$ is a solution to (5), the auction finds an "equili. allocation" $X^k = (x^{k,j}, j \in B_{-k})$ in the market $\mathcal{M}_{-k}$ in the sense that $x^{k,j} \in B_k^j(t)$ for every $j \in B_{-k}$ and $\sum_{j \in B_{-k}} x^{k,j} = \sum_{h \in N} e(h)$. As long as $\delta^k(t) \neq \mathbf{0}$, the auctioneer updates prices $p^k(t+1) = p^k(t) + \delta^k(t)$.

The Incentive Compatible Universal Dynamic (ICUD) Auction

Step 1: The auctioneer initially announces a common price vector $p^k(0) = p(0) \in \mathbb{Z}^N$ for all markets $\mathcal{M}_{-k}$, $k \in B_0$. Let $t := 0$ and go to Step 2.

Step 2: At each time $t \in \mathbb{Z}_+$ and prices $p^k(t) \in \mathbb{Z}^N$, every agent $j \in B_{-k}$ submits her bid $B_k^j(t) \subseteq \{0,1\}^N$. Based on reported bids, if the auctioneer finds an equilibrium allocation X^k in any market $\mathcal{M}_{-k}$ at the current step, she records the current prices as $p^k(T^k) \in \mathbb{Z}^n$ and the step as $T^k \in \mathbb{Z}_+$. For any other market $\mathcal{M}_{-k}$ which is not in equilibrium, the auctioneer calculates a price change $\delta^k(t)$ according to (5) and updates prices $p^k(t+1) := p^k(t) + \delta^k(t)$ for the market $\mathcal{M}_{-k}$. The UCD auction goes back to Step 2 with $t := t + 1$. When the auction has found an equilibrium allocation X^k in every market $\mathcal{M}_{-k}$, $k \in B_0$, go to Step 3. Otherwise, go to Step 4.

Step 3: All markets now clear. For every market $k \in B_0$ and every agent $j \in B_{-k}$ at every time $t = 0, 1, \cdots, T^k - 1$, based on her reported bids $B_k^j(t)$ and the price change $\delta^k(t)$ the auctioneer calculates agent $j's$ 'indirect utility reduction' $\Delta_j^k(t)$ when prices are changed from $p^k(t)$ to $p^k(t+1)$ in the market $\mathcal{M}_{-k}$, where

$$\Delta_j^k(t) = \min_{x^j(t) \in B_k^j(t)} x^j(t) \cdot \delta^k(t) \tag{6}$$

Every bidder $j \in B$ will be assigned the bundle $x^{0,j}$ of the allocation $X^0 = (x^{0,j}, j \in B_0)$ found in the original market $\mathcal{M}_{-0} = \mathcal{M}$ and asked to pay β_j, with the option to decline and walk away, when his payoff becomes negative, where

where

$$\beta_j = \sum_{h \in B_{-j}} \left[\left(\sum_{t=0}^{T^0-1} \Delta_h^0(t) - \sum_{t=0}^{T^j-1} \Delta_h^j(t) \right) + x^{j,h} \cdot p^j(T^j) - x^{0,h} \cdot p^0(T^0) \right] \quad (7)$$

The seller keeps the bundle $x^{0,0}$ of the allocation X^0 and receives the total payment $\sum_{j \in B} \beta_j$. The auction stops.

Step 4: In this case the auction does not find an allocation in every market $\mathcal{M}_{-k}$, $k \in B_0$. In the end, every bidder $j \in B$ gets nothing and pays nothing.

The first term of β_j is the accumulation of 'indirect utility reduction' of bidder $j's$ all opponents in B_{-j} along the path from $p(0)$ to $p^0(T^0)$ in the market $\mathcal{M}$ and along the path from $p^j(T^j)$ to $p(0)$ in the market $\mathcal{M}_{-j}$; The second term is the total equilibrium payment by all bidders in the market $\mathcal{M}_{-j}$; The third term is the total equilibrium payment by all opponents of bidder j in the market $\mathcal{M}$.

Ausubel (2004, 2006) and Sun and Yang (2014) have used *ex post perfect Nash equilibrium* to dynamic auction games of incomplete information which requires that the strategy for every player should remain optimal if the player were to get to know private information of his opponents at every node of the dynamic auction games (on and off equilibrium paths). This equilibrium is robust against any regret and also independent of any probability distribution.

An auction mechanism is said to be *ex post individually rational*, if, for every bidder, no matter how his opposing bidders act in the auction, as long as he is sufficiently able to judge whether his payoff is negative or nonnegative, he will never end up with a negative payoff.

An Incentive Compatibility Theorem

Theorem 4 (Fujishige and Yang 2025): Suppose that the market $\mathcal{M}$ satisfies Assumptions (A1) and (A2).

(1) If bidders bid sincerely, the ICUD auction converges to a competitive equilibrium, yielding a VCG outcome and a revenue for the seller not less than her reserve price.

(2) Sincere bidding by every bidder is an ex post perfect Nash equilibrium in the ICUD auction.

(3) The ICUD auction is ex post individually rational.

Thank You!