Note Title 7/8/2011

A crash course in Auction Theory 2

Risk Averse bidders

Bidders maximize expected utility $E[u(x)]$
where $u(0)=0$ $u'>0$ $u''<0$

Proposition: Suppose bidders are risk averse, then expected revenue in FPA is greater than in SPA

Proof: Note that bidding true value in a SPA is an equilibrium with risk aversion.

Suppose $\gamma(x)$ is the symmetric equilibrium in a FPA with risk aversion

$$\max_z \; G(z) \cdot u(x - \gamma(z))$$

FOC: $\left. g(z)\, u(x-\gamma(z)) - G(z)\, u'(x-\gamma(z))\, \gamma'(z) \right|_{z=x} = 0$

$$\gamma'(x) = \frac{u(x-\gamma(x))}{u'(x-\gamma(x))} \cdot \frac{g(x)}{G(x)}$$

Note that since $u(0)=0$ and $u''<0 \Rightarrow \frac{u(y)}{u'(y)} > y$ (why?)

And hence $\gamma'(x) > (x-\gamma(x)) \frac{g(x)}{G(x)}$

Denote eqm bid function with risk neutral bidders by β

If $\beta(x) \geq \gamma(x) \Rightarrow \gamma'(x) > \beta'(x)$

Since $\beta(0) = \gamma(0) = 0$, $\gamma(x) > \beta(x)$

Asymmetries Among Bidders

Bidding behavior in a SPA is unaffected.

Asymmetric FPAs with 2 bidders

For $i = 1, 2$, bidders i's valuation X_i is independently distributed over $[0, w_i]$ according to F_i.

Consider eqm bid functions β_1, β_2 with ϕ_1, ϕ_2

We can argue that $\beta_1(0) = \beta_2(0) = 0$ and $\beta_1(w_1) = \beta_2(w_2) = \bar{b}$

Given j follows β_j, i's expected payoff

$$u_i(x, b) = F_j(\phi_j(b))(x_i - b)$$

By differentiation

$$\phi_j'(b) = \frac{F_j(\phi_j(b))}{f_j(\phi_j(b))} \cdot \frac{1}{\phi_i(b) - b}$$

$\{i, j\} = \{1, 2\}$

ϕ_1, ϕ_2 have to satisfy above ~~set of~~ pair of linked differential equations.

Weakness lead to aggression

Suppose bidder 1 is "stronger" in the sense that F_1 dominates F_2 in terms of reverse hazard rate:

$$\frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)}$$

Proposition: If $f_1 / F_1 > f_2 / F_2$, then $\beta_2(x) > \beta_1(x)$ $\forall x \in (0, w_2)$

Proof: First, note that if for some $c \in (0, \bar{b})$,
$\phi_1(c) = \phi_2(c) = z$, then

$$\phi_2'(c) = \frac{F_2(z)}{f_2(z)} \cdot \frac{1}{z-c} > \frac{F_1(z)}{f_1(z)} \cdot \frac{1}{z-c} = \phi_1'(c)$$

$\Rightarrow$ if they intersect, β_1 is steeper than β_2.

Suppose $\beta_1(x) > \beta_2(x)$, $\exists x \in (0, w_2)$ then if $w_1 > w_2$ they have to intersect, and when they intersect β_2 has to be steeper than β_1.

Similar arguments for $w_1 = w_2$.

$\blacksquare$

On the other hand, denote $H_i(b) = F_i(\phi_i(b))$

$$\frac{H_2(b)}{h_2(b)} = \phi_1(b) - b > \phi_2(b) - b = \frac{H_1(b)}{h_1(b)}$$

Asymmetric Uniform Distributions

$F_1 \sim U[0, w_1]$, $F_2 \sim U[0, w_2]$ $\quad w_1 > w_2 > 0$

$$\Rightarrow \phi_i'(b) = \phi_i(b) \cdot \frac{1}{\phi_j(b) - b} \qquad (1)$$

$$\Rightarrow (\phi_i' - 1)(\phi_j - b) = \phi_i - \phi_j + b$$

$$(\phi_j' - 1)(\phi_i - b) = \phi_j - \phi_i + b$$

By adding the two equations for $\{i, j\} = \{1, 2\}$

$$\frac{d}{db}\left((\phi_i - b)(\phi_j - b)\right) = 2b$$

$$(\phi_i - b)(\phi_j - b) = b^2 \quad \Big\} \quad (w_1 - \bar{b})(w_2 - \bar{b}) = \bar{b}^2$$

$$\Downarrow$$

Rewriting 1: $\quad \phi_i' = \phi_i \cdot \frac{\phi_i - b}{b^2} \qquad \bar{b} = \frac{w_1 w_2}{w_1 + w_2}$

Solution is = $\boxed{\phi_i(b) = \frac{2b}{1 + k_i b^2}}$

Since $\phi_1(\bar{b}) = w_1$ and $\phi_2(\bar{b}) = w_2$

$$\boxed{k_i = \frac{1}{w_i^2} - \frac{1}{w_j^2}}$$

Exercise: For $F_1 \sim U[0, \frac{1}{1-\alpha}]$, $F_2 \sim U[0, \frac{1}{1+\alpha}]$

show that FPA generates more revenue than SPA.

Answer: Distribution of eqm prices in a FPA is

$$L^I_\alpha(p) = \text{Prob}[\max\{\beta_1(x_1), \beta_2(x_2)\} \le p] \qquad p \in [0, 1/2]$$

$$= F_1(\phi_1(p)) \cdot F_2(\phi_2(p))$$

$$= (1-\alpha)\frac{2p}{1-4\alpha p^2} + (1+\alpha)\frac{2p}{1+4\alpha p^2}$$

$$= \frac{(1-\alpha^2)(2p)^2}{1-\alpha^2(2p)^2}$$ which is ~~α~~ decreasing in α.

$$L^{II}_p(\alpha) = F_1(p) + F_2(p) - F_1(p)\cdot F_2(p)$$
$$= (1-\alpha)p + (1+\alpha)p - (1-\alpha)(1+\alpha)p^2$$
$$= 2p - (1-\alpha^2)p^2$$

which is increasing in α.

$\Rightarrow$ FPA generates more revenue than SPA

Example 2: $F_1(x) = x-1$ over $[1,2]$
$F_2(x) = e^{\frac{1}{2}x-1}$ over $[0,2]$

(Verify!) $\beta_1(x) = x-1$ $\quad \beta_2(x) = \frac{1}{2}x$

Distribution of prices in FPA: $L^{I}(p) = p\,e^{p-1}$ $\Big\}$ ER: 0.632

in SPA: $\Big\}$ ER: 0.662

Mechanism Design

A selling mechanism (B, π, μ)

B: set of possible messages

π: Allocation rule

μ: payment rule

$\beta_i : [0, w_i] \to B_i$ is an equilibrium (B, π, μ) if given β_{-i},

β_i maximizes i's payoff

Revelation Principles

Defn: A direct mechanism (Q, M) which $B_i = X_i$

$Q : X \to \Delta$ $\qquad M : X \to \mathbb{R}^N$

A mechanism is a truthfull direct mechanism if it is an equilibrium to bid true values.

Proposition (Revelation principle) Given a selling mechanism and an equilibrium of that mechanism, there exist a truthful direct mechanism with outcomes equal the equilibrium of the original mechanism

Proof: $Q(x) = \pi(\beta(x))$ and $M(x) = \mu(\beta(x))$

$$\begin{tikzcd}
 & \text{Bids} \arrow[dr, "(\pi, \mu)"] & \\
\text{Value} \arrow[ur, "\beta"] \arrow[rr, "{(Q, M) = (\pi \circ \beta, \mu \circ \beta)}"'] & & \text{Outcomes}
\end{tikzcd}$$

Incentive Compatibility

$$q_i(z_i) = \int_{X_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}$$

$$m_i(z_i) = \int_{X_{-i}} M_i(z_i, x_{-i}) f_{-i}(x_{-i}) \, dx_{-i}$$

$q_i(z_i)\,x_i - m_i(z_i)$

(IC) $\quad U_i(x_i) \equiv q_i(x_i)\, x_i - m_i(x_i) \geq q_i(z_i)\, x_i - m_i(z_i) \quad \forall x_i, z_i$

Alternatively $\quad U_i(x_i) = \max_{z_i} \{ q_i(z_i)\, x_i - m_i(z_i) \}$

1) Since U_i is maximum of affine functions, it is convex

2) $\quad U_i(z_i) \geq q_i(x_i)\, z_i - m_i(x_i)$

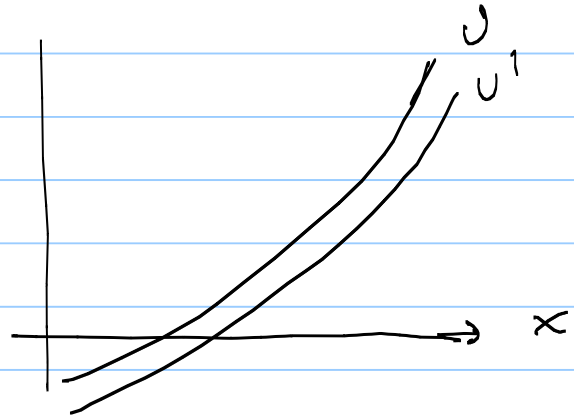
$\quad U_i(z_i) - U_i(x_i) \geq q_i(x_i)(z_i - x_i)$

$$\Rightarrow U_i'(x_i) = q_i(x_i)$$

3) Since U is absolutely continous

$$\boxed{U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)\, dt_i}$$

Payoff equivalence



4) IC = PE + q_i is increasing

Proposition (REP) Payments in an IC mechanism are determined by Q up to an additive constant

$$\boxed{m_i(x_i) = m_i(0) + q_i(x_i)\,x_i - \int_0^{x_i} q_i(t)\,dt}$$

Pf: Obvious

Individual Rationality: $U_i(x_i) \geq 0 \quad \forall x_i.$

Expected Revenue $E[R] = \sum_{i \in N} E[m_i(X_i)]$

$$E[m_i(X_i)] = \int_0^{w_i} m_i(x_i)\, f_i(x_i)\, dx_i$$

$$= m_i(0) + \int_0^{w_i} q_i(x_i)\, x_i\, f_i(x_i)\, dx_i - \underbrace{\int_0^{w_i}\int_0^{x_i} q_i(x_i)\, f_i(x_i)\, dt_i\, dx_i}_{\int_0^{w_i} (1 - F_i(x_i))\, q_i(x_i)\, dx_i}$$

$$= m_i(0) + \int_0^{w_i} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) q_i(x_i)\, f_i(x_i)\, dx_i$$

$$= m_i(0) + \int_X \underbrace{\left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right)}_{\psi_i(x_i)} Q_i(x)\, f(x)\, dx$$

$$ER = \sum_{i \in N} m_i(0) + \sum_{i \in N} \int_X \psi_i(x_i)\, Q_i(x)\, f(x)\, dx$$

subject to $m_i(0) \leq 0$, and q_i is increasing

we assume $\psi_i(x)$ is increasing.

Solution $\quad Q_i(x) > 0$ iff $\psi_i(x_i) \geq \max_{j \neq i} \psi_j(x_j)$
≥ 0

$$m_i(x_i) = Q_i(x) \cdot x_i - \int_0^{x_i} Q_i(z_i, x_{-i})\, dz_i$$

if $y_i(x_{-i}) = \inf \{ z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \max_{j \neq i} \psi_j(x_j) \}$

$$m_i(x_i) = \begin{cases} y_i(x_{-i}) & \text{if } Q_i(x) = 1 \\ 0 & \text{o.w.} \end{cases}$$

Symmetric Case: A second price auction with a reserve price $r = \psi^{-1}(0)$ is to optimal selling mechanism.