# A crash course in Auction Theory

Note Title 7/7/2011

- ★ Auctions have been used in practice for a long time
- ★ We model it as a Bayesian games
- ★ Valuation structure: private values – common valuations
- ★ Primary types of auctions

| Open Formats | | Sealed bid |
|---|---|---|
| Dutch (Decending price) | $\Longleftrightarrow$ | First-Price |
| English (Ascending price) | $\longleftrightarrow$ | second-Price |

- ★ Compare auctions over Revenue and Efficiency
- ★ We will focus single-unit auctions

## Private Value Auctions

Symmetric model, $n$ bidders

Bidder $i$ assigns value $X_i$ to the object

$X_i$ is distributed according $F$ over $[0, \omega]$

## Second Price Auction (SPA)

$$U_i(b, x_i) = \begin{cases} x_i - \max_{j \neq i} b_j & \text{if } b > \max_{j \neq i} b_j \\ 0 & \quad\quad < \end{cases}$$

(ties are broken randomly)

**Proposition:** In a SPA, it is a weakly dominant strategy to bid according to $\beta(x) = x$.

**Proof:** Consider bidding $x' < x$ versus bidding $x$. For $p = \max_{j \neq i} b_j$, if $p < x'$ or $p > x$, then they bring the same utility. if $x' < p < x$, bidding $x$ is strictly better. Similar argment apply for $x < x''$.

**Corollary:** $\beta^{II}(x) = x$ is a BNE of SPA.

Note: independence, symmetry, risk-neutrality is not needed for this result, only private valuations is important

**Classroom Exercise:** Prove that there cannot be any other BNE of SPA, or give an exemple different from above.

**Answer:** There are other equilibria.

For $F = U[0,1]$ $\beta_1(x) = 0$ $\forall x$, $\beta_2(x) = 1$ $\forall x$ is a BNE.

Interim expected payment of a bidder with value $x$.

$$m^{II}(x) = \underbrace{F(x)^{n-1}}_{G(x)} \cdot E\left[ Y_1^{(n-1)} \mid Y_1^{(n-1)} < x \right]$$

## First-Price Auctions

$$U_i(x, b_i) = \begin{cases} x - b_i & \text{if} \quad b_i > \max_{j \neq i} b_j \\ 0 & \qquad\qquad < \end{cases}$$

A strategy for player $i$ $\beta_i(x)$

- First, suppose $n=2$ and $F = U[0,1]$ and conjecture that there is a symmetric equilibrium $\beta(x) = cx$

Given that $\beta_2(x) = cx$, 1 bids $b$ to maximize

$$(x-b)\, \underbrace{Pr[b > c X_2]}_{\frac{b}{c}} = \frac{1}{c} \max_b (x-b)\, b \Rightarrow b = \frac{1}{2} x$$

$$\Rightarrow c = 1/2$$

Second, $n=2$ and an arbitrary $F$ over $[0,1]$

Consider a symmetric equilibrium strategy $\beta(x)$

Given 2 follows $\beta(x)$, 1 bids $b$ to maximize

$$\max_b \quad (x-b) \cdot F(\beta^{-1}(b))$$

FOC: $$\left. -F(\beta^{-1}(b)) + (x-b)\, f(\beta^{-1}(b)) \frac{1}{\beta'(\beta^{-1}(b))} \right|_{b=\beta(x)} = 0$$

$$\Rightarrow \quad -F(x) + (x-\beta(x))\, f(x) \frac{1}{\beta'(x)} = 0$$

$$x f(x) = \underbrace{F(x)\beta'(x) + \beta(x) f(x)}_{\frac{d}{dx}(F(x)\beta(x))}$$

By integration $F(x)\beta(x) = \int_0^x y f(y)\,dy + \cancel{c}^{0}$

$$\beta(x) = \frac{1}{F(x)} \int_0^x y f(y)\,dy = E\left[X \mid X \leq x\right]$$

Lastly, lets consider $n$ bidders and $F$ over $[0,w]$

$$\max_{j \neq i} \beta(X_i) = \beta\left(\max_{j \neq i} X_i\right) = \beta(Y_1), \quad Y_1 \equiv Y_1^{(n-1)}$$

$Y_1$ is distributed according to $F(y)^{n-1} \equiv G(y)$

$$EU_1(x,b) = (x-b)\, G(\beta^{-1}(b))$$

FOC: $\left. -G(\beta^{-1}(b)) + (x-b)\, g(\beta^{-1}(b)) \cdot \frac{1}{\beta'(\beta^{-1}(b))} \right|_{b=\beta(x)} = 0$

$$-G(x) + (x-\beta(x)) \cdot g(x) \cdot \frac{1}{\beta'(x)} = 0$$

$$x\,g(x) = \underbrace{G(x)\,\beta'(x) + \beta(x)\,g(x)}_{\frac{d}{dx}\,G(x)\,\beta(x)}$$

By integration $$\beta^{I}(x) = \frac{1}{G(x)}\int_0^x y\,g(y)\,dy$$

$$= E[\,Y_1 \mid Y_1 \le x\,]$$

**Proposition:** In a FPA, $\beta^{I}(x) = E[\,Y_1 \mid Y_1 \le x\,]$ is a BNE.

**Proof:** Suppose all but player $i$ follows $\beta^{I}$. Consider bidder 1 with value $x$ and bid $b \equiv \beta^{I}(z)$

$$\Pi(x,z) = G(z)\left(x - \beta^I(z)\right)$$

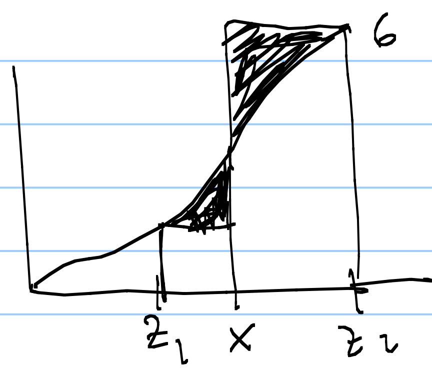
$$= G(z)\cdot x - \int_0^z y\, g(y)\,dy \quad \text{) integ by parts}$$

$$= G(z)\cdot x - G(z)\cdot z + \int_0^z G(y)\,dy$$

$$= G(z)(x-z) + \int_0^z G(y)\,dy$$

$$\Pi(x,x) - \Pi(x,z) = \int_0^x G(y)\,dy - G(z)(x-z) - \int_0^z G(y)\,dy$$

$$= (z-x)\,G(z) - \int_x^z G(y)\,dy \geq 0$$



$$\beta^{I}(x) = \frac{1}{G(x)}\int_0^x y\, g(y)\,dy = x - \frac{1}{G(x)}\int_0^x G(y)\,dy$$

$$= x - \int_0^x \left(\frac{F(y)}{F(x)}\right)^{n-1} dy$$

Example: $F(x) = x$

Confirm: $\beta(x) = \frac{n-1}{n}\, x$

Revenue Comparison

$$m^{I}(x) = G(x) \cdot \beta(x) = \int_0^x y\, g(y)\,dy = m^{II}(x)$$

$$\beta(x) = E[Y_1 \mid Y_1 \le x]$$

This implies FPA and SPA results in the same revenue

$$m^A(x) = \int_0^x y\, g(y)\, dy$$

$$m^A = \int_0^w m^A(x) \cdot f(x)\, dx = \int_0^w \left( \int_0^x y\, g(y)\, dy \right) f(x)\, dx$$

$$= \int_0^w \left( \int_y^w f(x)\, dx \right) y\, g(y)\, dy$$

interchanging order of integrals

$$= \int_0^w (1 - F(y))\, y\, g(y)\, dy$$

$$ER = n \times m^A = \boxed{n \int_0^w y\,(1 - F(y))\, g(y)\, dy}$$

Classroom Exercise: Show that $ER = E[Y_2^{(N)}]$

Answer: $\Pr[Y_2^{(N)} \leq y] = F(y)^n + n\,(1 - F(y))\, F(y)^{n-1}$

$$= n F(y)^{n-1} - (n-1) F(y)^n$$

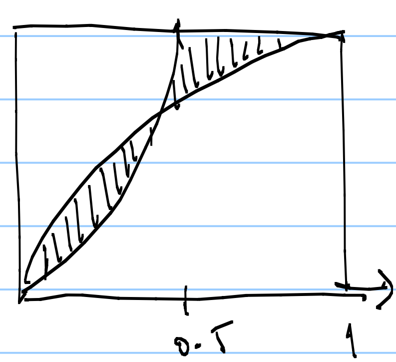
density: $n(n-1)\,F(y)^{n-2} f(y) - (n-1)\,n\,F(y)^{n-1} f(y)$

$$= n(n-1)\,F(y)^{n-2} f(y)\,\big(1 - F(y)\big)$$

$$E\left[Y_2^{(N)}\right] = \int_0^{\omega} y \cdot \big(n\,(n-1)\,F(y)^{n-2} f(y)\,(1-F(y))\big)\,dy$$

$$\stackrel{?}{=} \int_0^{\omega} n \cdot y \cdot (1-F(y))\; g(y)\,dy$$

This revenue equivalence does not follow expost arguments



**Proposition:** Distribution of equilibrium prices in SPA is a "mean preserving spread" of that in FPA. Therefore a risk averse sellers prefers FPA over SPA

**Proof:** $R^{II} = Y_2^{(N)}$ , $R^{I} = \beta\left(Y_1^{(N)}\right)$

$$E\left[R^{II} \mid R^{I} = p\right] = E\left[Y_2^{(N)} \mid Y_1^{(N)} = \beta^{-1}(p)\right]$$

$$= E\left[Y_1^{(N-1)} \mid Y_1^{(N-1)} < \beta^{-1}(p)\right]$$

$$= \beta(\beta^{-1}(p))$$

$$= p$$

$\Rightarrow R^{I}$ SOSD $R^{II}$ $\Rightarrow$ risk averse seller prefers FPA.

# Reserve Prices

Reserve prices in SPA : It is still dominant strategy to bid true value

$$m^{II}(x,r) = \begin{cases} r\,G(r) + \int_r^x y\,g(y)\,dy & \text{if } x \geq r \\ 0 & \text{o.w} \end{cases}$$

Reserve prices in FPA: we can argue $\beta(x)=0$ if $x<r$

$\beta(r)=r$

Exercise: Confirm that for $x>r$

$$\beta^{I}(x,r) = E\left[\max\{Y_1, r\} \mid Y_1 < x\right]$$

$$= r\,\frac{G(r)}{G(x)} + \frac{1}{G(x)} \int_r^x y\,g(y)\,dy$$

$$m^{I}(x,r) = G(x)\,\beta(x,r) = r\,G(r) + \int_r^x y\,g(y)\,dy$$
$$= m^{II}(x)$$

## Revenue Effects of reserve prices

$$m^{A}(r) = \int_r^{w} m^{A}(x,r)\,f(x)\,dx$$
$$= r\,(1-F(r))\,G(r) + \int_r^{w} y\,(1-F(y))\,g(y)\,dy$$

$$R(r) = n \times m^{A}(r) + F(r)^{n} \cdot x_0$$

$$R'(r) = n \cdot \Big(\underbrace{(1-F(r))\,G(r) - f(r) \cdot r\,G(r)}_{G(r)\,(1-F(r)-r f(r))}\Big) + n\,G(r)\,f(r)\,x_0$$

Hazard rate $\lambda(x) = \dfrac{f(x)}{1-F(x)}$

$$R'(r) = n\left(1 - (r-x_0)\,\lambda(r)\right)(1-F(r))\,G(r)$$

For $x_0 > 0 \Rightarrow R'(r_0) > 0$, for $x_0 = 0$ (as long as $\lambda$ is bounded) $R(\varepsilon) > R(0)$

## Exclusion Principle

Optimal solution: $(r - x_0) = \dfrac{1}{\lambda(r)}$

for $x_0 = 0$ $\quad r^* - \dfrac{1-F(r^*)}{f(r^*)} = 0 \quad$ for $F \sim U[0,1]$

$$\boxed{r^* = \frac{1}{2}}$$

## Revenue Equivalence Principle (REP)

An auction is standard if the rules of the auction dictates that highest bidder is awarded the object.

**Proposition:** Suppose private values of $n$ bidders are i.i.d and bidders are risk neutral. Then any symmetric and increasing equilibrium of any standard auction, such that expected payment of bidder with value 0 is 0, yields the same revenue.

**Proof:** Consider any standard auction $A$, fix a symmetric equilibrium $\beta$. Let $m^A(x)$ be the interim expected payment. Note $m^A(0) = 0$

$$U(x, z) = G(z) \cdot x - m^A(z)$$

FOC: $g(z) \cdot x - m^{A\prime}(z)\Big|_{z=x} = 0 \Rightarrow g(x) \cdot x = m^{A\prime}(x)$

$$m^A(x) = \int_0^x y\, g(y)\, dy + \underbrace{m^A(0)}_{=0}$$

Some applications of REP

All-pay auctions: $u_i(x,b) = \begin{cases} x-b & \text{if } b > \max_{j\neq i} b_j \\ -b & \text{o.w} \end{cases}$

Using REP, we conclude $\beta(x) = m^A(x) = \int_0^x y\, g(y)\, dy$

<u>Third-Price Auctions</u>

$$m^{\text{III}}(x) = \int_0^x y\; g(y)\, dy$$

$$f_2^{(N-1)}(y \mid Y_1 < x) = \frac{1}{F(x)^{n-1}} (N-1)\big(F(x) - F(y)\big) f_1^{(n-2)}(y)$$

Also $m^{III}(x) = F(x)^{n-1} E[\beta^{III}(Y_2) \mid Y_1 < x]$

$$\int_0^x y\, g(y)\, dy = \int_0^x \beta(x)\,(n-1)\,(F(x)-F(y))\, f_1^{(n-2)}(y)\, dy$$

differentiate wrt $x$

$$x \underbrace{g(x)}_{\cancel{(n-1)} F(x)^{n-2} \cancel{f(x)}} = \int_0^x \beta(x) \cancel{(n-1)} \cancel{f(x)} f_1^{(n-2)}(y)\, dy$$

$\cancel{x}$ $\Rightarrow$ By differentiating once more

$$\beta^{III}(x) = x + \frac{F(x)}{(n-2) f(x)}$$

Exercise: Solve for eqm strategies of war of attrition loser's pay