Resale in Asymmetric Auctions

UTMDC

Introduction

- Asymmetries lead to inefficiency in FPA
  - a motive for resale
- Two questions:
- Given resale, how do FPA and SPA compare?
- How does resale affect performance of FPA?
  - revenue
  - efficiency

Economic Environment

- Two bidders, independent private values (IPV)
- Values $X_1 \sim F_1$ and $X_2 \sim F_2$ (asymmetric)
- Bidder 1 is "strong" (and bidder 2 is "weak") in the sense that for all x, $F_1(x) \leq F_2(x)$.

First-Price Auctions without Resale (FPA)

- Equilibrium: exists in monotone strategies (γ_1, γ_2) (and is typically unique)
- Characterization:

$$\frac{d}{db} \ln F_1(\varphi_1(b)) = \frac{1}{\varphi_2(b) - b}$$
$$\frac{d}{db} \ln F_2(\varphi_2(b)) = \frac{1}{\varphi_1(b) - b}$$

- The pair of linked DE can be analytically solved only rarely
- Efficiency: since $\gamma_1(\cdot) \neq \gamma_2(\cdot)$, highest bid $\nRightarrow$ highest value (FPA is inefficient)

First-Price Auctions without Resale (FPA)

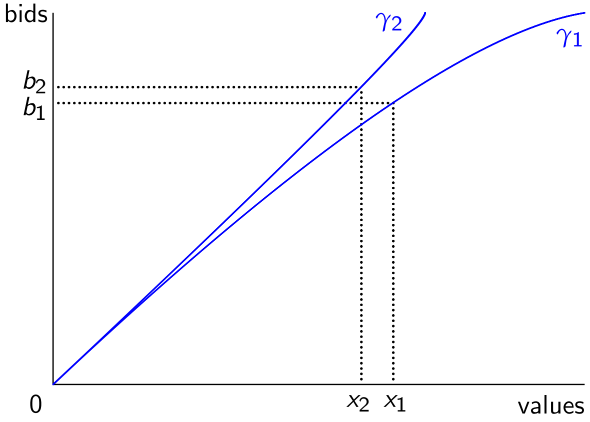
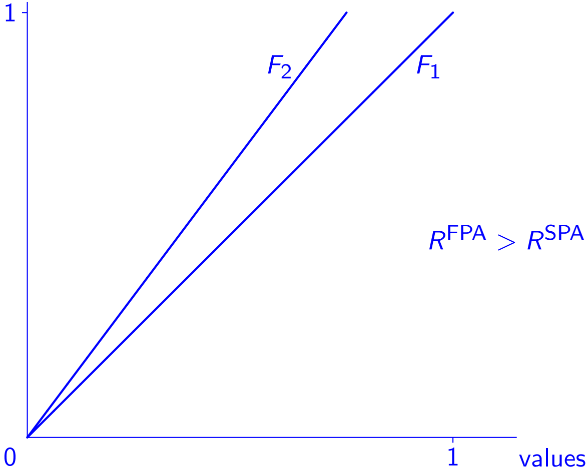


Figure: Asymmetric First-Price Auctions

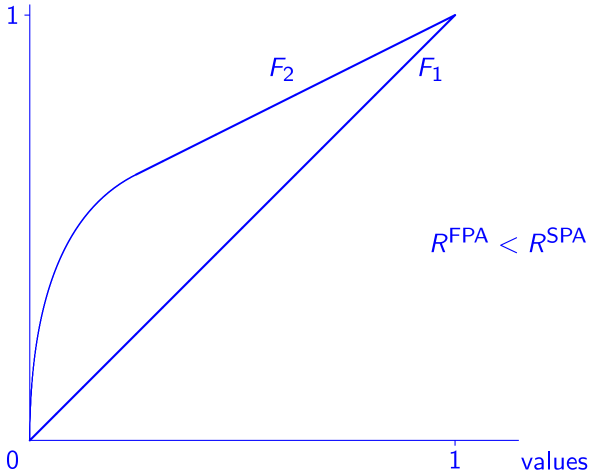
Comparing FPA and SPA

- Efficiency: SPA is, of course, efficient.
- Revenue: Ranking depends on distributions
  - $R^{FPA} \gtrless R^{SPA}$ even if F_1, F_2 are
    - stochastically ranked
    - regular
    - (truncated) Normals
  - Maskin and Riley (2000) classification

FPA $>$ SPA



FPA $<$ SPA



First-Price Auctions with Resale (FPAR)

- Stage 1: First-price auction
  - losing bid is not announced
- Stage 2: Winner makes a take-it-or-leave-it offer to the loser (monopoly mechanism)

Equilibrium

- Unique monotonic equilibrium satisfies the system:

$$\frac{d}{db} \ln F_j\left(\phi_j\left(b\right)\right) = \frac{1}{p\left(b\right) - b}$$

where

$$p\left(b\right) = \arg\max_{p} \left[F_1\left(\phi_1\left(b\right)\right) - F_1\left(p\right)\right] p + F_1\left(p\right) \phi_2\left(b\right)$$

- Bid distributions are identical:

$$F_1\left(\phi_1\left(b\right)\right) = F_2\left(\phi_2\left(b\right)\right)$$

- If

$$F\left(p\right) = F_2\left(p - \frac{F\left(p\right) - F_1\left(p\right)}{f_1\left(p\right)}\right)$$

then

$$R^{FPAR} = \int_0^{\omega_1} \left(1 - F\left(p\right)\right)^2 dp$$

Resale Stage

- Suppose bidders follow ϕ_1 and ϕ_2.
- If j wins and $x_j < \phi_i\,(b)$ then he will set p to

$$\max[F_i(\phi_i(b)) - F_i(p)]p + F_i(p)x_j$$

- First-order condition:

$$p - \frac{F_i(\phi_i(b)) - F_i(p)}{f_i(p)} = x_j$$

- F_i regular $\Rightarrow$ first-order condition is sufficient

Equilibrium Characterization

- Fix b and suppose $\phi_j(b) < \phi_i(b)$ ("gains from trade").
- Payoff of j ("seller") from bidding b is

$$[F_i(\phi_i(b)) - F_i(p(b))]p(b) + F_i(p(b))x_j - F_i(\phi_i(b))\, b$$

- In equilibrium,

$$\frac{d}{db} \ln F_i(\phi_i(b)) = \frac{1}{p(b) - b}$$

  where $p(b)$ is monopoly price set by j.
- Payoff of i ("buyer") from bidding b is

$$(x_i - b)F_j(\phi_j(b)) + \int_{\phi_j(b)}^{\omega_j} [x_i - p(\beta_j(x_j))]_+ \, dF_j(x_j)$$

- In equilibrium,

$$\frac{d}{db} \ln F_j(\phi_j(b)) = \frac{1}{p(b) - b}$$

Equilibrium Characterization

Theorem

The differential equations

$$\frac{d}{db} \ln F_j \left(\phi_j \left(b\right)\right) = \frac{1}{p\left(b\right) - b}$$

are necessary and sufficient for equilibrium

Corollary

Equilibrium bid distributions in FPAR are identical

$$F_1 \left(\phi_1 \left(b\right)\right) = F_2 \left(\phi_2 \left(b\right)\right)$$

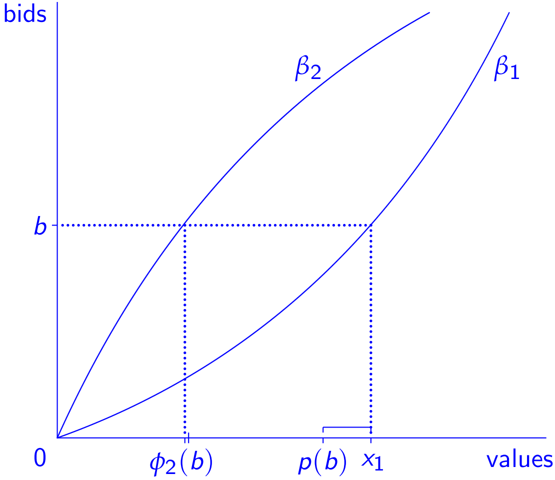
Resale "symmetrizes" first-price auction

Symmetrization

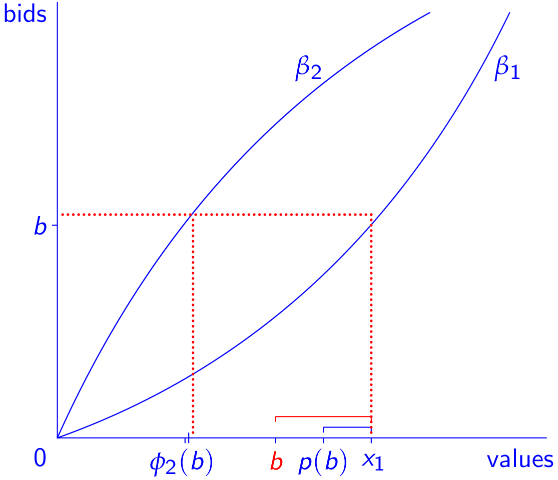
- In FPA, change from b to $b + \epsilon$ gains approximately $x_i - b$ from bidders $x_j \in \left[\varphi_j\left(b\right), \varphi_j\left(b + \epsilon\right)\right]$
- In FPAR, change from b to $b + \epsilon$
  - by bidder j (seller) gains approximately $p(b) - b$
  - by bidder i (buyer) gains approximately

$$(x_i - b) - (x_i - p(b)) = p(b) - b$$

Symmetrization



Symmetrization



Equilibrium in FPAR

Theorem

There exists a unique monotone equilibrium in the first-price auction with resale.

- Important that losing bid is not announced
  - otherwise, there is no (weakly) monotone equilibrium
- Unique in class of (weakly) monotone equilibria

Revenue in FPAR

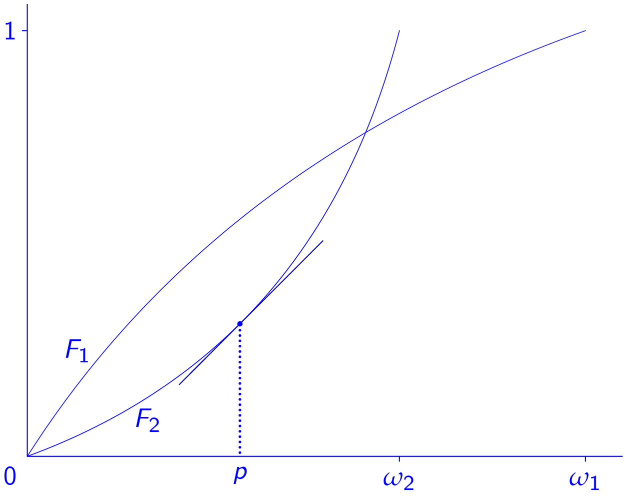
- Given F_1, F_2 construct F by: if $F_i(p) \leq F_j(p)$, then

$$F(p) = F_j\left(p - \frac{F(p) - F_i(p)}{f_i(p)}\right)$$

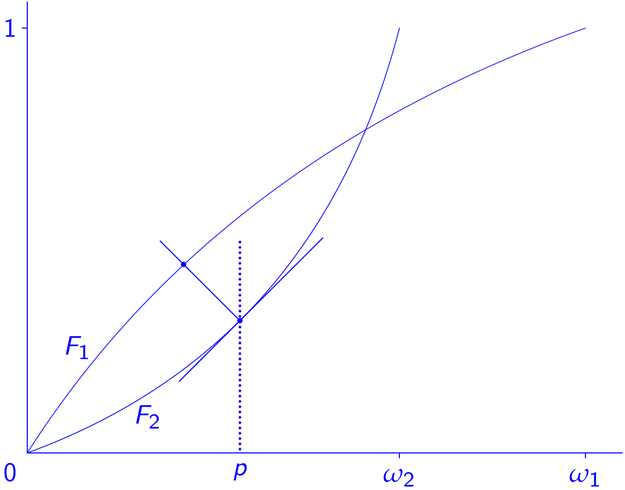
- Bid distributions in FPAR(F_1, F_2) are the same as in FPA(F, F)

$$\begin{aligned} R^{\text{FPAR}}(F_1, F_2) &= R^{\text{FPA}}(F, F) \\ &= R^{\text{SPA}}(F, F) \\ &= \int_0^{\bar{p}} (1 - F(p))^2 \, dp \end{aligned}$$

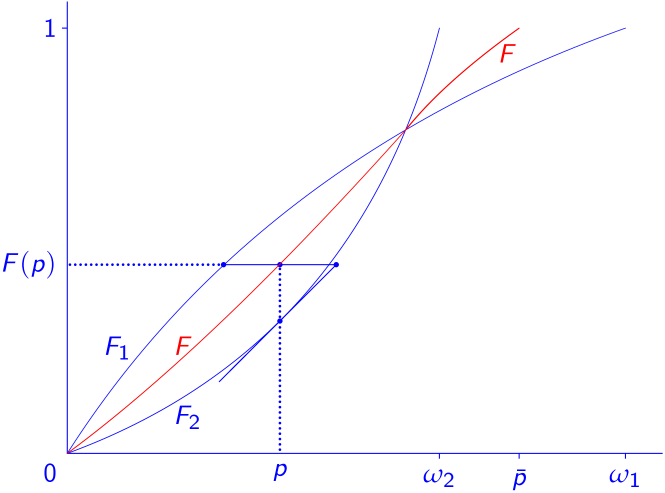
Construction of F



Construction of F



Construction of F



Second-price Auction with Resale (SPAR)

- In SPAR, losing bid (price) is known to winner.
- With resale, bidding value is not a dominant strategy.
- But bidding value is the unique (weak) ex post (or "robust") equilibrium
  - allocation is efficient and there is no resale
- There are other (inefficient) equilibria.
- Revenue in efficient equilibrium

$$\begin{aligned} R^{\text{SPAR}}(F_1, F_2) &= E\left[\min\{X_1, X_2\}\right] \\ &= \int_0^{\omega_2} (1 - F_1(p))(1 - F_2(p))\, dp \end{aligned}$$

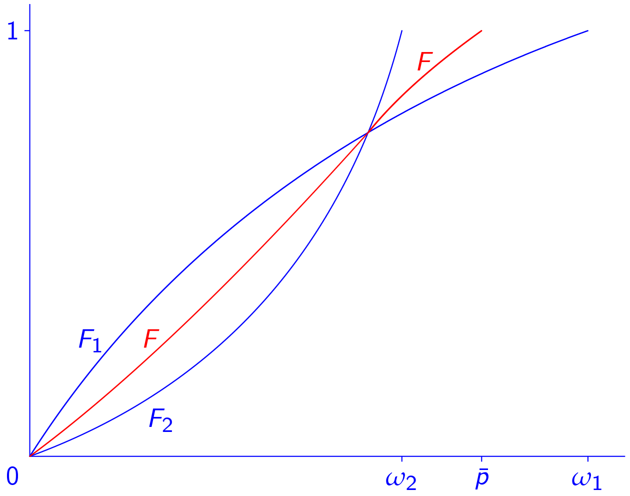
Comparing FPAR and SPAR

Theorem

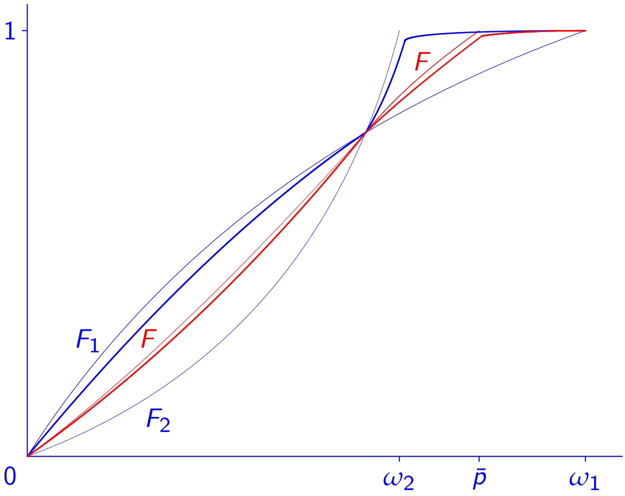
Suppose distributions are regular. Then the revenue from the first-price auction with resale exceeds that from the second-price auction (with or without resale).

- No assumptions on distributions other than regularity.
- Proof uses a technique from calculus of variations
- This result holds for more general resale mechanisms: monopsony resale and random proposer mechanism

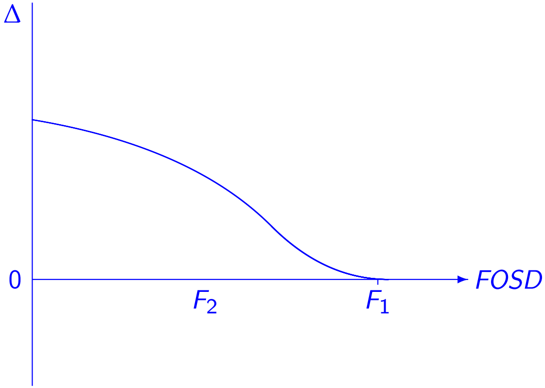
Variational Technique



Variational Technique



Variational Technique



Proof

(Assuming that F_1 is convex.) For any $t \in [0, \omega]$,

$$F^{-1}(t) > \frac{F_1^{-1}(t) + F_2^{-1}(t)}{2} \equiv G^{-1}(t)$$

Hence

$$R^{FPAR} = \int_0^{\omega} (1 - F(z))^2 \, dz > \int_0^{\omega} (1 - G(z))^2 \, dz$$

Let $G(z) = t$.

$$z = \frac{F_1^{-1}(t) + F_2^{-1}(t)}{2}$$

Differentiating

$$\frac{dz}{dt} = \frac{1}{2} \left(\frac{1}{f_1\left(F_1^{-1}(t)\right)} + \frac{1}{f_2\left(F_2^{-1}(t)\right)} \right)$$

and so

$$\int_0^1 (1 - G(z))^2 \, dz = \frac{1}{2} \int_0^1 (1-t)^2 \frac{1}{f_1\left(F_1^{-1}(t)\right)} dt + \frac{1}{2} \int_0^1 (1-t)^2 \frac{1}{f_2\left(F_2^{-1}(t)\right)} dt$$

Change variables using $t = F_1(z)$ and $t = F_2(z)$, resp.

$$\int_0^1 (1 - G(z))^2 \, dz = \frac{1}{2} \int_0^1 \left((1 - F_1(z))^2 + (1 - F_2(z))^2 \right) dz$$

Since arithmetic mean is greater than geometric mean

$$\frac{1}{2} \left((1 - F_1(z))^2 + (1 - F_2(z))^2 \right) > (1 - F_1(z))(1 - F_2(z))$$

$$R^{FPAR} > \int_0^1 (1 - G(z))^2 \, dz > R^{SPA}$$

Asymmetric Auctions

- Without resale, no revenue ranking possible

$$E\left[R^{FPA}\right] \lesseqgtr E\left[R^{SPA}\right]$$

- But once resale is considered,

$$E\left[R^{FPAR}\right] > E\left[R^{SPAR}\right]$$

- Sometimes adding a real-world feature simplifies theory!

FPAR vs FPA: Revenue

Conjecture: For all regular distributions, the revenue from the first-price auction with resale is higher than that from the first-price auction, that is,

$$R^{FPAR} > R^{FPA}$$

From previous result, if $R^{SPA} > R^{FPA}$, then conclusion holds.

Results

- For three families of distribution-pairs for which the equilibria in FPA is known, resale increases revenue

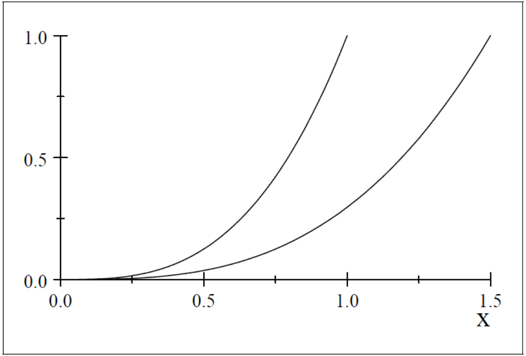
$$R^{FPAR} > R^{FPA}$$

- Presence of resale may actually decrease social surplus

$$S^{FPAR} \lesseqgtr S^{FPA}$$

Plum (1992)

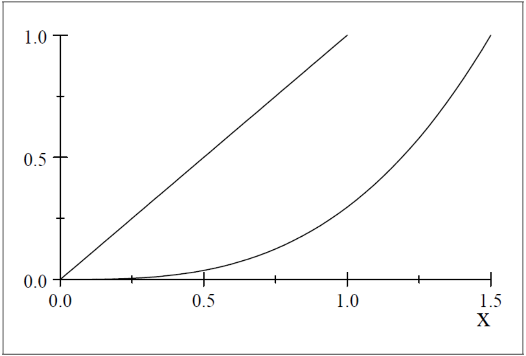
$$F_1(x) = (x/\omega_1)^a \text{ and } F_2(x) = (x/\omega_2)^a$$



Distributions from Plum's family

Cheng (2006)

$$F_1(x) = (x/\omega_1)^{a_1} \text{ and } F_2(x) = (x/\omega_2)^{a_2} \text{ where } \omega_2 = \frac{a_2}{a_2+1}\frac{a_1+1}{a_1}\omega_1.$$

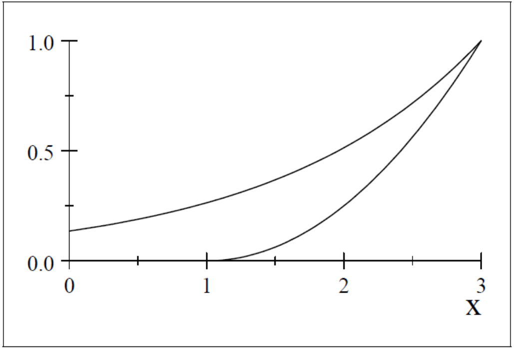


Distributions from Cheng's family 1

Cheng (2007)

$$F_1(x) = \left(\frac{x-1}{a}\right)^a \text{ over } [1, a+1]$$

$$F_2(x) = \exp\left(\frac{a}{a+1}x - a\right) \text{ over } [0, a+1]$$



Distributions from Cheng's class 2

FPAR > FPA for P

Proof.

The distribution of revenues—the highest bid—in the FPA is:

$$L^N(b) = F_1\left(\phi_1^N(b)\right) F_2\left(\phi_2^N(b)\right)$$
$$= \left(\frac{\phi_1^N(b)\,\phi_2^N(b)}{\omega_1\omega_2}\right)^a$$

distribution of revenues in the FPAR is

$$L^R(b) = \left(\frac{\phi_1^R(b)\,\phi_2^R(b)}{\omega_1\omega_2}\right)^a$$

We show that for all b, $L^N(b) > L^R(b)$ □

FPAR > FPA for C1 and C2

Proof.

For FPAR, consider F
For FPA, define G to be the distribution such that a symmetric FPA in which both bidders draw values from G is *revenue equivalent* to an asymmetric FPA in which the bidders draw values from F_1 and F_2.
We show that F *stochastically dominates* G; that is, $F(p) < G(p)$. □

Surplus under FPA

$$\int_0^{\omega_1} \left(\underbrace{\int_0^{\varphi_2\gamma_1(x_1)} x_1 dF_2(x_2)}_{\text{1 wins}} + \underbrace{\int_{\varphi_2\gamma_1(x_1)}^{\omega_2} x_2 dF_2(x_2)}_{\text{2 wins}} \right) dF_1(x_1)$$

Surplus under FPA

$$\int_0^{\omega_1}\left(\underbrace{\int_0^{\phi_2\beta_1(x_1)} x_1 dF_2(x_2)}_{\text{1 wins}} + \underbrace{\int_{\phi_2\beta_1(x_1)}^{P(x_1)} x_1 dF_2(x_2)}_{\text{2 wins but resells to 1}} + \underbrace{\int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2)}_{\text{2 wins and no resale}}\right) dF_1(x_1)$$

which can be written as

$$\int_0^{\omega_1}\left(\int_0^{P(x_1)} x_1 dF_2(x_2) + \int_{P(x_1)}^{\omega_2} x_2 dF_2(x_2)\right) dF_1(x_1)$$

An example

- Consider distributions identified by Plum (1992)

$$F_1(x) = \left(\frac{x}{\omega_1}\right)^a \text{ and } F_2(x) = x^a$$

- Can show that

$$\lim_{\omega_1 \to \infty} \frac{S^{FPAR}}{S^{FPA}} = 1 - \frac{(a+1)^{-\frac{a+1}{a}}}{a+2} < 1$$

- For the case when $a = 1$,

$$S^{FPA} > S^{FPAR}$$

  if and only if $\omega_1 > \omega_1^* \simeq 1.95$

An example

- Resale may decrease ex ante social surplus!
- Reason is speculation by weak bidder
- This occurs when the asymmetry is large because that is when the benefits to speculative bidding are also large.

Efficiency

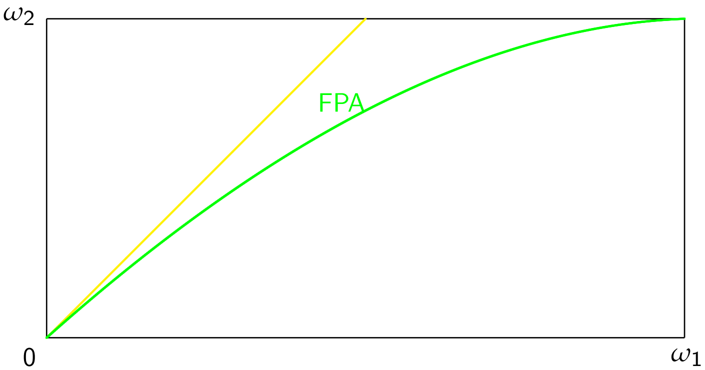


Figure: Resale may decrease Efficiency

Efficiency

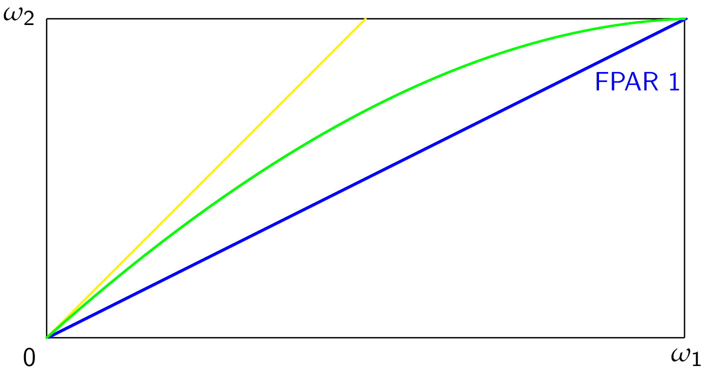


Figure: Resale may decrease Efficiency

Efficiency

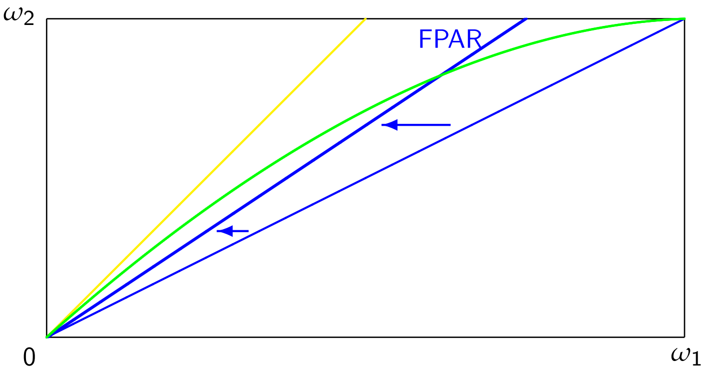


Figure: Resale may decrease Efficiency

Conclusion

- Including resale leads to unambiguous revenue ranking of FPA vs. SPA
- For three main families $\mathcal{P}$, $\mathcal{C}_1$ and $\mathcal{C}_2$, FPAR is revenue superior to FPA
- Conjecture: this holds in general (under regularity)
- Resale does not restore efficiency; in fact, it may decrease surplus!

Extensions

- More than 2 bidders
  - many ways of modelling resale
  - symmetry does not hold
- Interdependent values
  - symmetry still holds
  - revenue ranking between FPAR and SPAR does not
- Affiliated private values
  - conditional bid distributions of the two bidders are identical
- Multi-unit auctions

Multi-Unit Auctions with Resale

- Multi-unit auctions, single unit demand, discriminatory price auction
  - has a symmetric (efficient) equilibria

$$\beta(x) = E\left[Y_k^{(n-1)} \mid Y_k^{(n-1)} < x\right]$$

- When bidders can resell, they can bid for more than one item
  - if resale is a (centralized) uniform price auction

$$\beta^R(x) = E\left[Y_k^{(n-1)} \mid Y_1^{(n-1)} < x\right]$$

    for all items is another NE
  - revenue equivalence holds
- If resale is a uniform price auction with reserve price, original equilibrium is no longer an equilibrium!