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Efficient iBF: Balanced Integration of Fragmented Matching Markets for Welfare Improvement

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Abstract. Matching markets often suffer from fragmentation, which leads to inefficiency. We model a fragmented market in a school-choice context and offer a practically relevant method for integration. Specifically, each student and school belong to a region, and we allow for inter-regional transfer of students with "balancedness" constraint: a matching is said to be balanced if, for each region, the outflow of students from that region to other regions is equal to the inflow of students from the latter to the former. Using a directed bipartite graph defined on students and schools, we characterize the set of Pareto efficient matchings among those that are individually rational, balanced and fair (efficient iBF). We also provide a class of polynomial-time algorithms to compute such matchings. When each region favors local students in their priority, the outcome of an algorithm from this class weakly improves student welfare upon the outcome where each region independently uses the deferred acceptance mechanism. Various real-life examples of fragmentation are discussed, and we illustrate how our method would address the issue.

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Keywords: fragmentation, balancedness, fairness, algorithm, efficiency, efficient iBF, school choice, matching with constraints

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1. Introduction

Matching theory has been applied in numerous real-life markets to centralize market transactions, but the centralized clearinghouses are still often organized at a small local level. School choice systems in many countries are run by individual cities rather than by larger entities. In kindergarten admission systems in major Chinese cities, the cities are divided into small districts, and a child in a given district can only be assigned to a kindergarten in the district. Under the Covid-19 pandemic, the Japanese government implemented a policy of initially distributing vaccines to municipalities, some of which are quite small, and those municipalities were then responsible for allocating the vaccines to their residents.

Fragmentation of matching markets is problematic because it limits the scope of choice for participants, causing inefficiency (see e.g., Robinson-Cortes (2019) for empirical work in the context of foster home assignment). To explore this problem more concretely, we describe a daycare allocation practice in some detail. In Japan, allocation of slots at accredited daycares is conducted by individual municipal governments. Each municipality assigns priority orders over the applicants, collects preferences from them, and uses those pieces of information to implement a centralized matching mechanism. A problem is that, with few exceptions, a child can only attend a daycare in the municipality of their residence. The City of Tokyo, for example, is divided into 23 small municipalities, each of which conducts a matching independently. Due to the small sizes of the municipalities, many families find that daycare centers outside the municipalities of their residence are among the closest. Moreover, it is often convenient to put their children to a daycare center close to their workplace while many people cross the boundary to commute.[1] For these reasons, inter-municipal admission is an appealing option for many families and it is indeed legal. However, inter-municipal admission is rare under the current practice. The rarity and difficulty of such practices is neither limited to daycare admission nor Japan. Quite the contrary, it is widespread. For instance, the high school choice program of Hebei province, China, abolished inter-municipal school choice in 2024. In other matching markets, as we will detail later, authorities try different policies but suffer from a variety of problems such as efficiency loss.

We study how to integrate fragmented matching markets. Specifically, we analyze mechanisms that improve upon mechanisms organized at the local level and achieve desirable fairness and efficiency properties. To do so, we depart from the standard model

[1]To get a sense of the magnitude, over 3.5 million people get on or get off a train in Tokyo's Shinjuku Station *each day*.

of matching between students and schools (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003) by assuming that each school belongs to exactly one region while each student is a resident of exactly one region. We consider a balancedness constraint that requires that, for each region, the number of residents of other regions who are matched to a school in it, called the inflow, must be equal to the number of its residents who are matched to a school in other regions, called the outflow.

Why is balancedness a reasonable requirement? In the context of Japanese daycare allocation, for instance, although inter-municipal admission is legal—in fact, the Japanese government amended its Children and Childcare Act to encourage this practice in 2018—, the law makes it clear that it is each municipality's responsibility to provide daycare services.[2] Therefore, it is impractical to organize a daycare allocation in a fully integrated manner across multiple municipalities. Moreover, each municipality heavily subsidizes daycares, so enrolling residents from other municipalities can be a severe financial burden. Our balancedness constraint is meant to alleviate municipalities' concerns by guaranteeing that no municipality carries an excessive burden. Similarly, full integration of the school choice programs of the cities in Hebei province of China has been problematic. Since it caused the loss of top students from certain cities, the province ended up abolishing the inter-municipal transfer policy.[3] Our balancedness constraint would limit the imbalance of top students across cities. Given those considerations, we design mechanisms that achieve desirable properties under the balancedness constraint. In other words, *we aim at partial integration of the regions instead of full integration.*

We find that there does not always exist a balanced matching that satisfies stability, a standard desideratum in school choice literature that is equivalent to individual rationality, fairness, and non-wastefulness. In fact, non-wastefulness alone is incompatible with balancedness. Given this observation, we weaken our desideratum to only require individual rationality, balancedness and fairness while not insisting on non-wastefulness.[4] Among the matchings satisfying all the three conditions (there always exists such a matching[5]), which we call the *iBF*s, we focus on the ones that cannot be further improved upon in terms of students' welfare, which we call *efficient iBFs*.[6]

[2]Revision to Article 14(4) of the Supplementary Provision to Children and Childcare Act of Japan, https://elaws.e-gov.go.jp/document?lawid=424AC0000000065.

[3]See a newspaper article, https://www.gov.cn/xinwen/2014-04/12/content_2657873.htm.

[4]We explain further justification behind this modeling choice in Remark 1 in Section 2.

[5]All the conditions are satisfied by the matching where no student is matched to any school.

[6]Since the schools' priority orders are not determined by preferences but rules in our intended applications, students' welfare (but not "schools' welfare") is the right measure of efficiency. See Remark 1 for more discussions.

We first characterize efficient iBFs. To this end, we define a novel bipartite graph called a fair improvement graph (FIG henceforth) on a matching, where the vertices on one side represent the students and those on the other side represent the schools, and existence of an arrow between two vertices depends on preferences, priorities, and the given matching. We show that an iBF is an efficient iBF if and only if there exists no "FIG cycle," a cycle on the FIG, for the matching.

Based on our characterization of efficient iBF, we then provide a polynomial-time algorithm that finds an efficient iBF. The algorithm is called the FIG cycles algorithm and is illustrated in some detail with an example in Section 1.1. Roughly, each step of the FIG cycles algorithm checks if the current iBF allows for a FIG cycle and, if so, finds a relocation of students that improves outcomes for students while retaining individual rationality, balancedness and fairness. This algorithm in particular finds a matching that weakly improves upon the matching where the deferred acceptance mechanism is used at each region, under the assumption that schools favor students in the same region (implying that the latter matching is fair). This result provides justification for using the FIG cycles algorithm in practical settings.

To better understand the markets with the balancedness constraint, we provide further discussions. First, we characterize when balancedness and stability are compatible with each other. Second, we examine strategic properties. Third, comparative statics are provided to evaluate the effect of merging and splitting regions. Fourth, we consider the case with weak priority, which often arises in school choice applications.

Although we used daycare allocation in Japan for illustrative purposes, we emphasize that our analysis is applicable in a wide variety of problems. Also in Japan, choice systems for public elementary and middle schools are organized at the small municipal level as well. Naturally, there exist much potential demand for enrolling in schools in other municipalities, but opportunities for inter-municipal school choice are severely limited.[7] The issue is not limited to Japan either. In the U.S., for example, school choice is basically organized at a highly local level, but some form of interdistrict school choice is practiced in 43 States.[8] The analysis of our paper, and particularly the FIG cycles algorithm, could be applied to improve efficiency of school choice mechanisms. Indeed, we study—as the fifth discussion topic—the present practices of kindergarten admission, high school admission, and college admission in China, as well as daycare admission in Japan in detail. We find

[7]For instance, Nerima Ward (one of the 23 municipalities of Tokyo City) declares that it admits a non-resident child "only in truly unavoidable circumstances." See Nerima's website, https://www.city.nerima.tokyo.jp/kosodatekyoiku/kyoiku/shochu/8jyo-9jyo-n1.html.

[8]See the website of the Education Commission of the States, https://ecs.my.salesforce-sites.com/mbdata/mbquest4e?rep=OE1705.

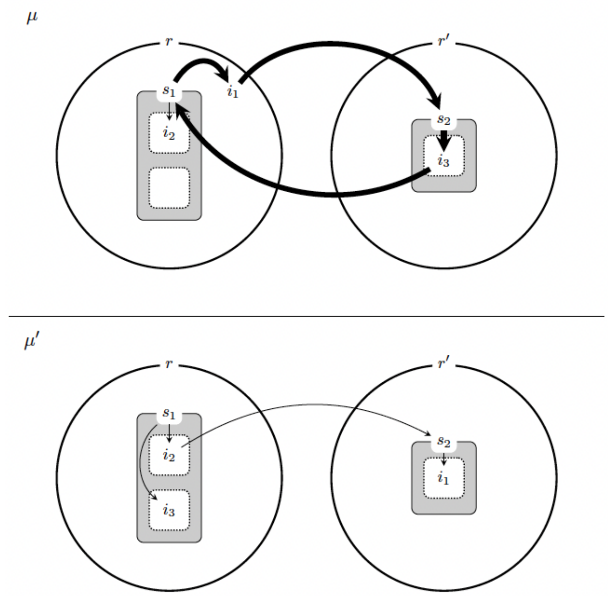


Figure 1. Illustrative Example. Each white dotted square in a school represents a seat in the school. The thick and thin arrows represent the FIG while the thick arrows represent a FIG cycle.

that each of those markets suffers from problems due to fragmentation, and we describe the routes through which our FIG cycles mechanisms can improve welfare.

At a high level, the fragmented nature of resource allocation is widespread beyond daycare allocation or school choice. In kidney exchange in the U.S., for instance, individual transplant centers often conduct exchanges on their own before sending remaining participants to national exchange, resulting in significant efficiency loss (Agarwal et al., 2019). In COVID-19 vaccine allocation in Japan in 2021, individual municipalities were charged with vaccinating their respective residents, which resulted in situations in which vaccine stocks run out quickly in one municipality while extra stocks remain unused in another.[9] We envision that research is called for to understand how to overcome inefficiency from the fragmented nature of the allocation problems in a practical manner when existing legal, institutional and other constraints prohibit full integration.

1.1. **Illustrative Example.** In this paper we introduce an algorithm of inter-regional transfer that improves students' welfare while respecting the balancedness condition and

[9]For instance, in an interview at the time with NHK, the national public broadcasting organization, an epidemiology expert pointed out that "adjustment of vaccine supply across municipalities is limited, and it is necessary to introduce a centralized system at a larger scale." See NHK's website, https://www3.nhk.or.jp/news/html/20210922/k10013272411000.html.

fairness. To gain intuition for why inter-regional transfer may improve welfare, consider the following simple environment (see Figure 1 for an illustration). There are two regions, r and r'. One school s_1 and two students i_1 and i_2 reside in r while one school s_2 and one student i_3 reside in r'. School s_1 has the capacity of two while school s_2 has the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_2, \qquad \succ_{s_1}: i_1, i_2, i_3,$$

$$\succ_{i_2}: s_2, s_1, \qquad \succ_{s_2}: i_3, i_1, i_2.$$

$$\succ_{i_3}: s_1, s_2,$$

If an assignment of students to schools is determined region by region and there is no inter-regional transfer, the efficient matching for students is given by

$$\mu = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2 & i_3 & i_1 \end{pmatrix},$$

which is realized by, for instance, running the student-proposing deferred acceptance algorithm in each region separately. However, with inter-regional transfer, a Pareto improvement for students is possible while respecting the balancedness condition. Specifically, if the two students i_1 and i_3 are sent to the schools in each other's regions, that is, if i_1 goes to s_2 and i_3 goes to s_1, then in the realized matching,

$$\mu' = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2, i_3 & i_1 & \emptyset \end{pmatrix},$$

these students are matched with their respective first-best school. In doing so, region r_1 takes in one student from outside (student i_3) while sending one student outside (student i_1), so the balancedness condition is satisfied.

Two things are noteworthy here. First, the number of students who are matched to some school is increased from μ to μ'. This is because, by swapping students between the two regions, the unmatched student i_1 was able to be matched with a school. We introduce an algorithm to make such an improvement possible.[10]

Second, there is another matching that respects the balancedness condition and Pareto dominates μ, which is

$$\mu'' = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_3 & i_2 & i_1 \end{pmatrix}.$$

[10] As we will explain in more detail later, this type of improvement is in a sharp contrast with most existing literature where the number of matched students is constant between before and after an improvement.

However, this matching is not fair according to our definition, as i_1 is ranked higher than i_2 at s_2. This suggests that care is needed about who can be moved to new schools across regions. The algorithm we introduce ensures that fairness is respected when an improvement is made.

We aim to achieve efficiency via inter-regional transfer like the one described in the above example. Specifically, we propose an algorithm that takes an arbitrary iBF as an input and achieves a Pareto improvement. The algorithm is based on a directed bipartite graph between students and schools that we call the *fair improvement graph* (or the *FIG*), and in each of its steps it "implements" a cycle in this graph—called a *FIG cycle*—, i.e., we move a student to a school that she points to. The outcome of repeatedly implementing FIG cycles turns out to be an *efficient iBF*. In fact, our main results characterize efficient iBF using cycles: we show that *an iBF is an efficient iBF if and only if there is no FIG cycle on it.*

Let us now discuss these results in the context of the aforementioned example. In our FIG, each school is pointed to by the top student (according to its priority) among those who strictly prefer the school to their current match and are acceptable to the school. This means that, under μ, s_1 is pointed to by i_3 while s_2 is pointed to by i_1. Note that i_2 cannot point to s_2 as he is not the "top student" for s_2 (i_1 is). Also, each school points to the students that are matched to the school. So, s_1 points to i_2 and s_2 points to i_3.

The above pointing rule is based on a standard idea in the previous literature that use cycles (e.g., Erdil and Ergin (2008) and Top Trading Cycles). However, in the present example, the graph constructed in this way does not have a cycle even though matching μ is not an efficient iBF. In order to achieve a Pareto improvement, we additionally require that each school with a vacancy points to each student matched to another school in the school's region as well as all unmatched students living in that region. In our example, this lets s_1 point to i_1.[11] With this, there is a cycle "$i_1 \rightarrow s_2 \rightarrow i_3 \rightarrow s_1 \rightarrow i_1$." Our characterization result shows that this implies μ is not an efficient iBF. Indeed, μ is not an efficient iBF because it is Pareto dominated by μ', which is an iBF. In fact, the Pareto-improvement μ' is obtained by "implementing" this cycle. On μ', the FIG now lets i_2 point to (only) s_2 and lets s_2 point to i_1, but does not let i_1 point to any school (because i_1 is matched to her first choice school). Since i_3 does not point to any school either (because i_3 is matched to her first choice school), there is no FIG cycle on μ'. Our characterization

[11] In this example, the school with a vacancy points to an unmatched student. The benefit from requiring a school to point to a student matched to another school in the same region does not appear in the current example. We will explain this point in "Example 2, Continued" in Section 3.2.

result shows that this implies μ' is an efficient iBF. Indeed, one can verify that there is no iBF that Pareto dominates μ'.

The remainder of this paper proceeds as follows. Section 2 provides a model where we define efficient iBF. Section 3 introduces the fair improvement graph (FIG) and FIG cycles. Section 4 provides our main theorem, which characterizes efficient iBF by non-existence of FIG cycles. Section 5 defines the FIG cycles algorithm which outputs an efficient iBF. Section 6 provides various discussions, and Section 7 concludes. Proofs of all results are provided in the Appendix unless stated otherwise.

2. Model

2.1. **Preliminary Definitions.** Let there be a finite set of students I and a finite set of schools S. Each student i has a strict preference relation $\succ_i$ over the set of schools and being unmatched (being unmatched is denoted by $\emptyset$). For any $s, s' \in S \cup \{\emptyset\}$, we write $s \succeq_i s'$ if and only if $s \succ_i s'$ or $s = s'$. Each school s has a strict priority order $\succ_s$ over the set of students and leaving a position vacant (which is denoted by $\emptyset$).[12] For any $i, i' \in I \cup \{\emptyset\}$, we write $i \succeq_s i'$ if and only if $i \succ_s i'$ or $i = i'$. Each school $s \in S$ is endowed with a (physical) **capacity** q_s, which is a nonnegative integer.

Student i is said to be **acceptable** to school s if $i \succ_s \emptyset$ (and unacceptable otherwise). Similarly, s is acceptable to i if $s \succ_i \emptyset$.[13] It will turn out that only rankings of acceptable partners matter for our analysis, so we often write only acceptable partners to denote preferences and priorities. For example,

$$\succ_i: s, s'$$

means that school s is the most preferred, s' is the second most preferred, and s and s' are the only acceptable schools under preferences $\succ_i$ of student i.

A **matching** μ is a mapping that satisfies (i) $\mu_i \in S \cup \{\emptyset\}$ for all $i \in I$, (ii) $\mu_s \subseteq I$ and $|\mu_s| \leq q_s$ for all $s \in S$, and (iii) for any $i \in I$ and $s \in S$, $\mu_i = s$ if and only if $i \in \mu_s$. That is, a matching simply specifies which student is assigned to which school (if any).

A matching is **individually rational** if no student or school is matched with an unacceptable partner.

[12]Strictness of priorities is assumed just for the sake of simplicity. In Section 6.4, we consider the case when indifferences are allowed and show that most results carry over to such a case.

[13]In some applications, all schools may regard all students as acceptable. None of our results will hinge on the assumption that some students can be unacceptable to some schools.

Given a matching μ, we say that a student i has **justified envy** to $j \in I$ if there is a school $s \in S$ such that (i) $\mu_j = s$, (ii) $s \succ_i \mu_i$, and (iii) $i \succ_s j$. We say that a matching μ is **fair** if there is no pair of students $(i, j) \in I^2$ such that i has justified envy to j.

A matching μ **weakly Pareto dominates** a matching μ' if $\mu_i \succeq_i \mu'_i$ for every $i \in I$. A matching μ **Pareto dominates** μ' if μ weakly Pareto dominates μ' and there exists $i \in I$ such that $\mu_i \succ_i \mu'_i$.

Finally, a **mechanism** is a function from the set of student preference profiles to the set of matchings.

2.2. **Regions, Balancedness, and Efficient iBF.** There is a set of regions, denoted R, which is a partition of $I \cup S$. Formally, R satisfies the following conditions:

(1) Each $r \in R$ is a nonempty subset of $I \cup S$.
(2) $r \cap r' = \emptyset$ for any $r, r' \in R$ such that $r \neq r'$.
(3) $\bigcup_{r \in R} r = I \cup S$.

The interpretation is that each s belongs to a single $r \in R$ and each i is a resident of a single $r \in R$. Let $r(s)$ be the region r such that $s \in r$, and similarly for $r(i)$.

We call $\mathcal{E} = (I, S, (\succ_a)_{a \in I \cup S}, (q_s)_{s \in S}, R)$ an **environment**.

We are now ready to introduce the key concept of this paper, "balancedness."

Definition 1. A matching μ is **balanced** if for each $r \in R$,

$$\underbrace{\sum_{s \in r} |\{i | i \in \mu_s, i \notin r\}|}_{\text{inflow to } r} = \underbrace{\sum_{s \notin r} |\{i | i \in \mu_s, i \in r\}|}_{\text{outflow from } r} \tag{2.1}$$

As (2.1) shows, balancedness means that for any given region r, the inflow of students to r is the same as the outflow of students. Note that balancedness is not a "pairwise" notion, that is, it does not necessarily require that for every pair of regions r and r', the number of students who live in r and are matched to a school in r' is the same as the number of students who live in r' and are matched to a school in r. The next example illustrates.

Example 1 (Three-way transfer)**.** Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $R = \{r_1, r_2, r_3\}$ and $r_k = \{i_k, s_k\}$ for each $k = 1, 2, 3$. Let

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_3 & i_1 & i_2 & \emptyset \end{pmatrix}.$$

See Figure 2 for a graphical representation. Note that under μ, the number of students who live in r_1 and are matched to a school in r_2 is 1 while the number of students who live

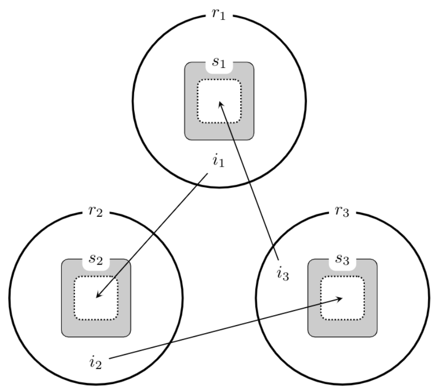


FIGURE 2. Three-way transfer (Example 1).

in r_2 and are matched to a school in r_1 is 0. Matching μ is, however, balanced because it satisfies (2.1): For each region, the inflow and outflow are both 1. □

We say that a matching is an **iBF** if it is individually rational, balanced, and fair. Below is the main solution concept of this paper.

Definition 2. A matching μ is said to be an **efficient iBF** if μ is an iBF and there is no iBF μ' that Pareto dominates μ.

We note that there always exists an iBF because the empty matching (i.e., the matching where no student is matched to any school) satisfies all the conditions for iBF. Moreover, the set of iBFs is finite because our problem is finite. Therefore, an efficient iBF is guaranteed to exist.

The standard environment without the balancedness constraint is subsumed by our model as a special case in which there is only one region. In that environment, there is a unique efficient iBF, which corresponds to a "student-optimal stable matching." In our setting, there may be multiple efficient iBFs. The following example illustrates.

Example 2 (Multiple efficient iBFs)**.** Let $I = \{i_1, i_2, i_3\}$, $S = \{s_1, s_2, s_3\}$, $R = \{r, r'\}$ where $r = \{i_1, s_1, s_2\}$ and $r' = \{i_2, i_3, s_3\}$. Each school has the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_3, s_1, \qquad \succ_{s_1}: i_1, i_2,$$

$$\succ_{i_2}: s_1, \qquad \succ_{s_2}: i_3,$$

$$\succ_{i_3}: s_2, \qquad \succ_{s_3}: i_1.$$

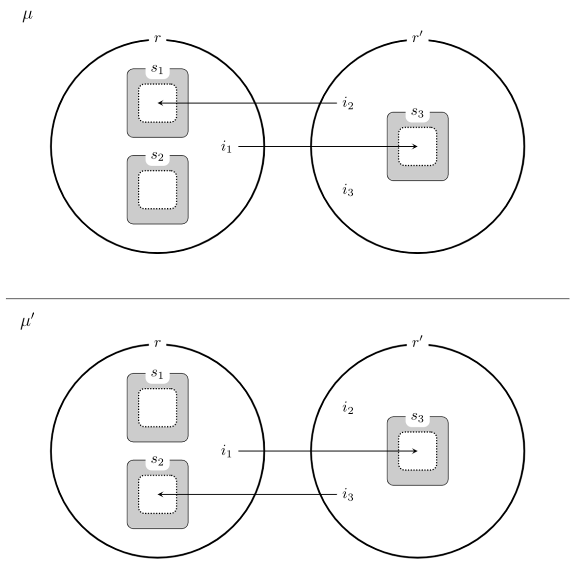


FIGURE 3. Multiple efficient iBFs (Example 2). Both μ and μ' are an efficient iBF.

Let

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_2 & \emptyset & i_1 & i_3 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ \emptyset & i_3 & i_1 & i_2 \end{pmatrix}.$$

See Figure 3 for a graphical representation. We show that μ and μ' are both efficient iBFs. To see this, first notice that there are only four individually rational and balanced matchings. They are μ, μ',

$$\mu'' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_1 & \emptyset & \emptyset & i_2, i_3 \end{pmatrix},$$

and the empty matching. The latter two matchings are Pareto dominated by μ and μ' while μ and μ' do not Pareto dominate each other, and one can verify by inspection that both μ and μ' are fair, and hence efficient iBFs. □

Let us compare our notion of iBF with the standard notion of stability. Specifically, we say that a matching μ is **non-wasteful** if there is no pair (i, s) of a student and a school such that $s \succ_i \mu_i$, $i \succ_s \emptyset$, and $|\mu_s| < q_s$, and that μ is **stable** if it is individually rational, fair, and non-wasteful (but not necessarily balanced). A matching is the **student-optimal stable matching** if it is stable and is weakly preferred to every stable matching by all students. In the environment of Example 2, the unique stable matching (and hence the student-optimal stable matching) would be:

$$\mu''' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_2 & i_3 & i_1 & \emptyset \end{pmatrix}.$$

But μ''' is not balanced because the inflow to region r is 2 while the outflow from r is 1.

This example demonstrates that the balancedness condition is not necessarily compatible with stability (recall that μ''' is a unique stable matching). In Section 6.1, we characterize when those properties are compatible with each other.

Remark 1. We discuss three issues regarding our modeling choice.

(1) One might wonder why we require fairness (along with individual rationality and balancedness) for our main solution concept, iBF, while not insisting on non-wastefulness. We provide two reasons. First, fairness seems important in our intended applications discussed in Section 6.5. For example, Kamada and Kojima (2023) provide institutional details of daycare allocation in Japan, which suggests the importance of fairness. Second, and more fundamentally, non-wastefulness alone is incompatible with balancedness, even without requiring individual rationality or fairness. To see this, consider a market with two regions $r = \{i\}$ and $r' = \{s\}$, and i and s find each other acceptable and s has a capacity of one. The unique non-wasteful matching matches i with s but such a matching is not balanced. Thus, there is a sense in which non-wastefulness is too restrictive. By contrast, the remaining conditions that comprise stability, i.e., individual rationality and fairness, can be simultaneously satisfied together with balancedness.

(2) Our fairness notion does not allow any student to have justified envy to any other students. One could imagine weaker notions of fairness by allowing some justified envies to exist. For example, one might want to allow a student from region r to have justified envy to another student from a different region r' matched to a school in r'. A rationale for allowing such a situation would be that a student should be given precedence at schools in her own region compared to students from other regions. Our modeling choice is to reflect such a rationale to the priority of the schools instead of the fairness notion. That is, in the aforementioned situation, we imagine that the schools in r' rank students in r' higher than students in r while retaining our notion of fairness.

(3) Why do we focus on Pareto efficiency for students, excluding schools from welfare consideration? Our reply would be twofold. First, we are primarily interested in situations in which it is students that are real "individuals" whose preferences

are to be counted. This is not only because it is a standard assumption in the literature in school choice (Abdulkadiroğlu and Sönmez, 2003), but also because our applications such as public school choice and daycare allocation feature schools or daycare centers whose priority orders are decided by official rules, rather than being derived from preferences.[14] Second, Pareto efficiency for students implies Pareto efficiency when schools are considered to be agents as well, so our efficient iBF is also two-sided efficient among the iBFs.

3. FIG (Fair Improvement Graph) Cycles

3.1. **Definition of FIG Cycle.** The key steps of our analysis involve defining bipartite directed graphs over the sets of students and schools, and identifying cycles on them. A bipartite directed graph on I and S, or simply a **graph**, $\mathcal{G} \subseteq (I \times S) \cup (S \times I)$, is a set of ordered pairs of agents in $I \cup S$. An interpretation is that if $(i, s) \in \mathcal{G}$, then there is an arrow pointing from i to s. In this case, we say "i points to s." The case of $(s, i) \in \mathcal{G}$ is analogous. Given a graph $\mathcal{G}$, a **cycle in** $\mathcal{G}$ is any sequence of the form $(i_1, s_1, i_2, s_2, \ldots, i_m, s_m)$ such that for each $k \in \{1, \ldots, m\}$,

(1) i_k points to s_k, i.e., $(i_k, s_k) \in \mathcal{G}$,
(2) s_k points to i_{k+1}, i.e., $(s_k, i_{k+1}) \in \mathcal{G}$, where we set $i_{m+1} := i_1$,
(3) $i_k \neq i_{k'}$ for all $k' \neq k$, and
(4) $s_k \neq s_{k'}$ for all $k' \neq k$.

We will regard any two cycles as defined here, $(i_1, s_1, \ldots, i_m, s_m)$ and $(i_{k+1}, s_{k+1}, \ldots, i_m, s_m, i_1, s_1, \ldots, i_k, s_k)$, as identical to each other.

Let $D^{\mu}_s := \{i \in I | s \succ_i \mu_i\}$, and $Top_s(I')$ be the student $i \in I'$ who has the highest priority among those in I' at $\succ_s$.

Now we define a particular type of a graph and a cycle on it. This cycle will be used to characterize efficient iBF as well as to define our algorithm.

Definition 3. Given a matching μ, the **fair improvement graph (FIG)** for μ is a graph such that, for any $i \in I$ and $s \in S$,

(1) student $i \in I$ points to school $s \in S$ if $i = Top_s(D^{\mu}_s)$ and i is acceptable to s, and
(2) school $s \in S$ points to student $i \in I$ if either
  (a) $\mu_i = s$, or
  (b) $|\mu_s| < q_s$ and, $[i \in r(s)$ and $\mu_i = \emptyset]$ or $\mu_i \in r(s)$.

[14] In daycare allocation in Japan, for instance, priority orders are decided by each municipality. They are typically based on characteristics of the children or their parents, such as whether parents have full-time jobs and whether the parent is a single parent.

A **fair improvement graph cycle (FIG cycle)** on μ is a cycle in the FIG for μ.

A student i points to a school s when she finds s to be better than her current outcome μ_i, and if she is acceptable to s and the best student for s among those who find s to be an improvement (so, under our assumption of strict priorities, each school can be pointed to by at most one student). In this sense, the students point to the schools in the most fair manner, and this is why we call the graph the "fair improvement graph."

On the other hand, a school s can point to a student i in two different cases. The first case is as in other algorithms based on cycles in the literature such as the Top Trading Cycles (TTC) algorithm (Shapley and Scarf, 1974) and the stable improvement cycles algorithm (Erdil and Ergin, 2008). This is when i is currently matched to s, and it is described by part (2a) in Definition 3. The second case, described by part (2b) in Definition 3, depends on the notion of regions. School s can point to i if s has a vacancy and either i lives in the region of s and is unmatched, or i is matched to a school in the region. The need for this second case is illustrated in Section 1.1, and the logic behind the particular pointing rule is explained in Remark 3.

Given a matching μ and a cycle of the form $\mathcal{F} = (i_1, s_1, i_2, s_2, \ldots, i_m, s_m)$, call μ' the **matching generated by** $(\mu, \mathcal{F})$ if

$$\mu'_{i_k} = s_k \text{ for each } k \in \{1, \ldots, m\}, \text{ and } \mu'_j = \mu_j \text{ for all } j \in I \setminus \{i_1, \ldots, i_m\}.$$

Given a matching μ and a cycle $\mathcal{F}$, we say that we **implement** $\mathcal{F}$ on μ when we create the matching generated by $(\mu, \mathcal{F})$.

As we will show in Section 4, whether there exists a FIG cycle is crucial to the characterization of efficient iBF. Also, in the algorithm we define in Section 5, we repeatedly implement FIG cycles. But before stating the formal results that use the notion of FIG cycles, let us illustrate the concept of FIG cycle through a series of examples in the next subsection.

Remark 2. Erdil and Ergin (2017) consider a model with weak student preferences (and school priorities) and propose to improve students' welfare by using *chains* in addition to cycles. A chain can start from a matched or unmatched student and ends at a school that has a vacant seat. We could define a chain in our model too, while restricting the pointing from schools to students to the one described in part (2a) of Definition 3.

Let us be forthcoming about the similarity and difference between their chains and our FIG cycles. First, in our model, implementing all chains and cycles under such a pointing rule would violate balancedness (there is no such issue in Erdil and Ergin (2017) as they have no balancedness constraint). For example, in Example 2, there would be one chain

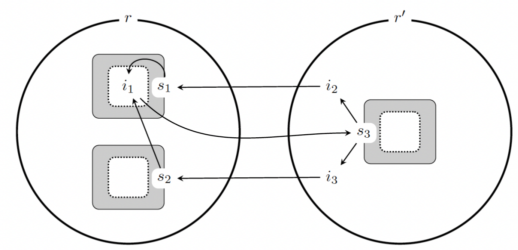


FIGURE 4. Example 2, Continued. The arrows represent the FIG. There are two FIG cycles.

going out of r ($i_1 \to s_3$) while there would be two chains going into r ($i_2 \to s_1$ and $i_3 \to s_2$). As we will see in Theorem 1, balancedness is respected if we implement any FIG cycle. Second, one might argue that "connecting" chains might work. That is, we would start from a chain, and at the end of the chain (which is a school), we would find an unmatched student in the same region and see if there was a chain originating from that student. If there was such an arrow, then we would follow the arrows. Continuing this way, if an arrow eventually pointed to a school that had already appeared, then we would call the closed set of arrows a cycle. Cycles constructed in this way turns out to be the same as our FIG cycles. One can view that our pointing rule from schools to students, especially the part described in part (2b) of Definition 3, correctly captures how this "connecting" should be done. Third, the reasons behind why there are chains are different. In Erdil and Ergin (2017), a chain is implemented on a stable matching. For the existence of chains it is necessary that the student preferences are weak: if instead the students' preferences are strict, then the last student on the chain would have strictly preferred to be matched to a vacant position in the last school in the chain under the original matching, so the original matching was not stable. In our model, there can exist a chain (defined in the absence of the pointing rule described in part (2b) of Definition 3) because iBF does not require non-wastefulness. □

3.2. **Examples of FIG Cycle.**

Example 2, Continued. Consider the same environment as in Example 2. The FIG for μ'' is drawn in Figure 4. Note that there are two FIG cycles:

$$\mathcal{F} := (i_1, s_3, i_2, s_1) \text{ and } \mathcal{F}' := (i_1, s_3, i_3, s_2).$$

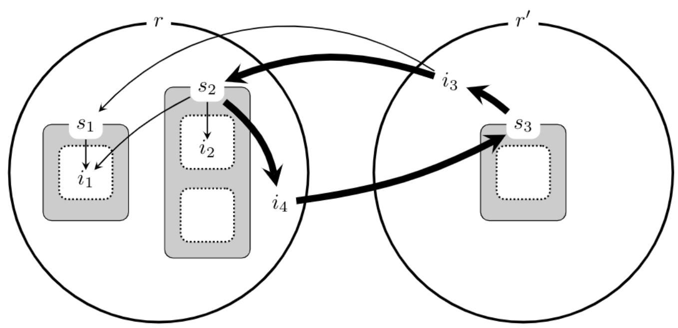


FIGURE 5. Example 3. The thick and thin arrows represent the FIG while the thick arrows represent a FIG cycle. School $s_2 \in r$ points to an unmatched student $i_4 \in r$ because s_2's capacity is currently not full.

Implementing the former cycle results in μ, and implementing the latter cycle results in μ'. Note that, in this example, it is important that the pointing rule for FIG lets a school point to a student matched to another school in the same region.[15] □

Example 3 (FIG cycle)**.** Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3\}$, $R = \{r, r'\}$ where $r = \{i_1, i_2, i_4, s_1, s_2\}$ and $r' = \{i_3, s_3\}$. Schools s_1 and s_3 have the capacity of one each while s_2 has the capacity of two. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_1, \qquad \succ_{s_1}: i_1, i_2, i_3, i_4,$$

$$\succ_{i_2}: s_2, s_1, \qquad \succ_{s_2}: i_2, i_3,$$

$$\succ_{i_3}: s_1, s_2, \qquad \succ_{s_3}: i_4.$$

$$\succ_{i_4}: s_1, s_3,$$

Consider the following matching:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_1 & i_2 & \emptyset & i_3, i_4 \end{pmatrix},$$

which, by inspection, one can show to be fair. This matching as well as the FIG for μ is drawn in Figure 5. We note two features of this FIG. First, s_2 points to an unmatched student i_4 even though it is already matched with student i_2. This is because s_2's capacity is not filled under μ: the capacity is 2 while it is matched to only one student (student i_2). Second, there are multiple arrows from student i_3. Although the rule for pointing in a FIG implies that each school is pointed to by at most one student, one student can

[15]This is the point alluded to in footnote 11.

point to multiple schools if she is the "number one" choice from multiple schools. In this example, i_3 ranks at the top among all students who prefer s_1 to their current match (i.e., among $\{i_3, i_4\}$), and she also ranks at the top among all students who prefer s_2 to their current match (i.e., among $\{i_3\}$). Note that there is a single FIG cycle:

$$\mathcal{F} := (i_3, s_2, i_4, s_3).$$

Implementing this cycle, we obtain the following matching:

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_1 & i_2, i_3 & i_4 & \emptyset \end{pmatrix},$$

which is an improvement over μ and is fair. One can show by inspection that μ' is also an efficient iBF. □

4. Characterization of Efficient iBF

This is the main section of this paper, and we are going to formalize the following claims:

(1) If we implement a FIG cycle on a given iBF μ, then it results in an iBF that Pareto dominates μ (Theorem 1).
(2) If there is no FIG cycle on a given iBF μ, then μ is an efficient iBF (Theorem 2).

These results in particular imply the following characterization of efficient iBF: *Given an iBF, it is an efficient iBF if and only if there is no FIG cycle on it* (Corollary 1). In what follows, we will examine each claim and explain their intuition in detail.

Theorem 1. *Let μ be an iBF. If there exists a FIG cycle $\mathcal{F}$ on μ, then a matching generated by $(\mu, \mathcal{F})$ is an iBF and Pareto dominates μ.*

An implication of this theorem is that, if we can find a FIG cycle on a given iBF, then that matching cannot be an efficient iBF. Thus, one can think of this theorem as providing a necessary condition for an iBF to be an efficient iBF.

We note that the proof shows a stronger result: Implementing a FIG cycle on an arbitrary matching μ (which is not necessarily an iBF) results in a matching that is (i) Pareto superior to μ, (ii) individually rational if μ is individually rational, (iii) fair if μ is fair, and (iv) balanced if μ is balanced.

Let us explain the intuition for the proof (of the stronger result above). For this, let μ be the original matching and μ' be the matching generated by $(\mu, \mathcal{F})$ where $\mathcal{F}$ is a FIG cycle on μ. The proof shows Pareto dominance, individual rationality, fairness and balancedness one by one. First, it is straightforward that μ' Pareto dominates μ by our

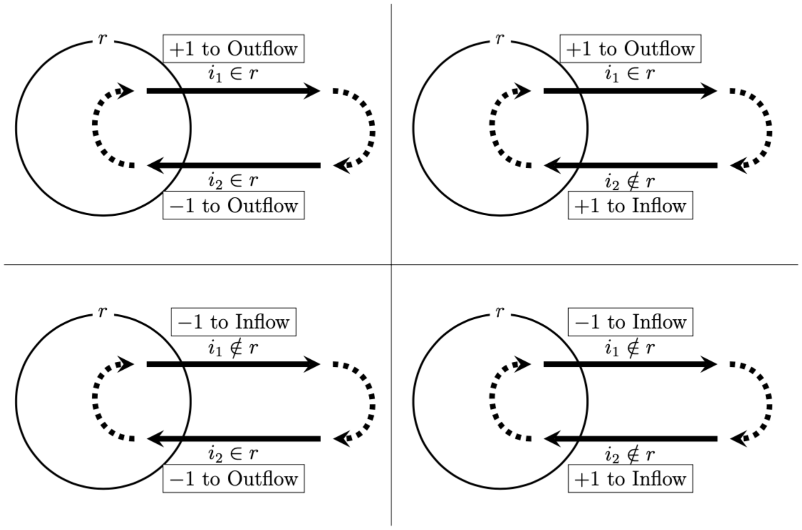


FIGURE 6. Why balancedness is maintained when a FIG cycle is implemented (proof intuition for Theorem 1).

pointing rule for students. Second, this observation also implies that μ' is individually rational if μ is individually rational.

Third, fairness of μ' is due to the pointing rule used in the FIG. A crucial step is to show that no student has justified envy to student i_k who is newly matched to s_k under μ', where i_k and s_k appear in the FIG cycle (i.e., i_k points to s_k under the FIG).[16] If some student i finds s_k to be better than her match under μ', then she should have also found s_k to be better than her match under μ (because μ' Pareto dominates μ). But the pointing rule for FIG implies that i_k is the best student (according to s_k's priority) among those who found s_k to be better than the match under μ. So, in particular, i is not higher than i_k under s_k's priority. This implies that i cannot have justified envy to i_k under μ'.

Finally, balancedness of μ' holds because the FIG cycle is "closed," that is, for any given region r, if an arrow from a student goes outside of r along the cycle, then another arrow from a student must come back to r and vice versa, which implies that the number of times the arrows go outside must be equal to the number of times the arrows come back to r. Whether balancedness is maintained by implementing a cycle may not be obvious due to the fact that an outgoing arrow may carry a student who lives in r or one who does not live in r, and similarly an incoming arrow may carry either type of a student. Figure 6 shows that in every possible case, balancedness is maintained when a

[16]Justified envy to other students matched to s_k or those that involve other schools can be shown not to exist by using fairness of μ and the fact that μ' Pareto dominates μ.

cycle is implemented.[17] We also note that feasibility would be violated under seemingly reasonable pointing rules. Part (2) of the next remark illustrates this point.

Remark 3.

(1) We note that Theorem 1 does not assert that implementing a FIG cycle on an iBF necessarily results in an efficient iBF. Indeed, Example 9 in the Online Appendix presents a case in which one needs to implement FIG cycles more than once to reach an efficient iBF.

(2) Theorem 1 asserts that, among other things, implementing a FIG cycle on an iBF results in a balanced matching. This property depends on a somewhat subtle manner in which we define the FIG. To illustrate, consider a region r and a school $s \in r$ that has a vacant seat. Under the pointing rule (2b) in the definition of FIG (Definition 3), s points to unmatched students in r and students matched to schools in r. If, for instance, s were to point to a student matched to a school outside r (which means that there is an arrow that originates from a school and goes across regions), then implementing a cycle containing such an arrow might violate balancedness. Example 10 in the Online Appendix provides a specific instance in which such violation occurs.

Theorem 2. *Let μ be an iBF. μ is an efficient iBF if there exists no FIG cycle on μ.*

Recall that Theorem 1 identifies a necessary condition for a given iBF to be an efficient iBF. Theorem 2 establishes that the same condition is in fact sufficient as well. Therefore, as we will formally state later in Corollary 1, combining those results provides a characterization of an efficient iBF.

The proof is by contraposition. That is, we take μ that is an iBF and assume that there is another iBF μ' that Pareto dominates μ. Then we show that there exists a FIG cycle on μ.

To find a FIG cycle, we construct a graph. In this graph, the students who are associated with arrows are those whose outcomes are different between μ and μ'. We denote by I' the set of those students. Meanwhile, the schools with arrows are the ones that are matched to students in I' under μ' (which means that these are the schools that are matched to

[17] Hafalir, Kojima and Yenmez (2022) study TTC under a variety of constraints, one of which is the balancedness constraint of the present paper. They verify that the balancedness constraint satisfies a condition called M-concavity, which Suzuki et al. (2023) showed is sufficient for the outcome of a certain version of TTC to satisfy balancedness. Although their TTC algorithm is substantially different from ours, one might also be able to use a similar indirect approach to establish balancedness of the outcome of our algorithm.

some new students under μ' relative to μ). We denote by $S(\mu')$ the set of those schools. Also, let $S(\mu)$ be the set of schools that students in I' are matched with under μ.

To understand the complication in the proof, consider first a simple case: there is a single region r (i.e., a standard environment without a balancedness constraint), and μ is a stable matching (in the standard sense). Then, since stability implies non-wastefulness and μ' Pareto dominates μ, it follows that $S(\mu') = S(\mu)$. We can have each school s point to students in $\mu_s \cap I'$, and allow a student in I' point to s if she is the top student among I' who regard s to be an improvement relative to μ. This way, each school is pointed to by one student, and each student is pointed to by one school. Thus, there is a cycle, and with some work, one can prove that such a cycle must be a FIG cycle. This is essentially the same method as what is used in Erdil and Ergin (2008).[18]

In our problem, complication arises for two reasons. First, fairness alone does not imply non-wastefulness, so μ may have some waste (for example, consider the empty matching). This means that $S(\mu')$ may not be equal to $S(\mu)$, so a graph constructed in the above manner might not have a cycle (schools in $S(\mu') \setminus S(\mu)$ would have no outgoing arrow, and those in $S(\mu) \setminus S(\mu')$ would have no incoming arrow). This suggests we need an alternative way of forming a graph. Second, arrows might go in and out of any region, hence in defining the alternative graph, we must make sure that the balancedness constraint would be respected when implementing a cycle in the graph.

We overcome these difficulties by constructing a graph, denoted $\mathcal{G}(\mu, \mu')$, in the following manner. First, each school in $S(\mu')$ is pointed to by the top student among those in I' who regard the school to be an improvement relative to μ, just as in the "simple case" above. Second, each student i in I' is pointed to by a school in the following three different ways depending on i's outcome under μ:

(1) If $\mu_i \in S(\mu')$, then we let μ_i point to i. This is the case that is analogous to the "simple case" explained above.

(2) If $\mu_i \in S \setminus S(\mu')$, then we can show, using individual rationality and balancedness, that we can find a school that resides in μ_i's region and belongs to $S(\mu')$ whose capacity is not filled under μ.[19] We let such a school point to i.

[18] We note that Erdil and Ergin (2008) allow school priorities to be weak, while we assume strict preferences here. However, as explained in Section 6.4, our analysis extends to the case with weak priorities without any significant change.

[19] Showing the existence of such a school is nontrivial. The proof constructs an additional graph for each region and uses that graph to establish the existence. See the Appendix for detail.

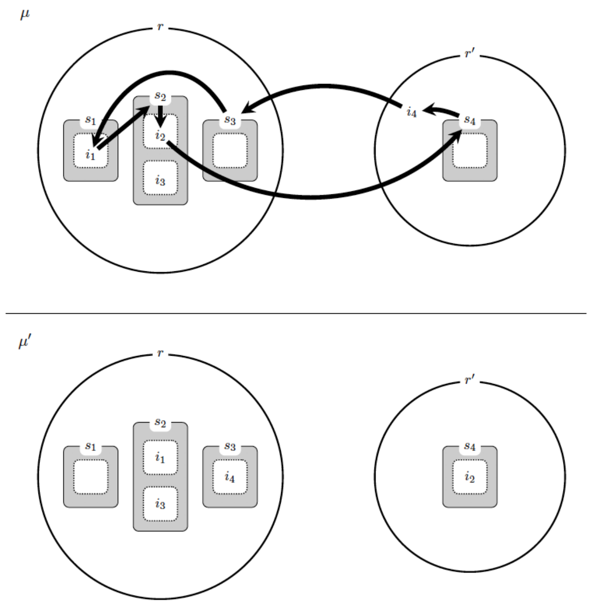


FIGURE 7. Example 4. Given an iBF μ and a Pareto dominating iBF μ', the proof of Theorem 2 constructs a cycle (represented by the thick arrows) and shows that the constructed cycle is in fact a FIG cycle.

(3) If $\mu_i = \emptyset$, then we can show, again using individual rationality and balancedness, that we can find a school that resides in i's region and belongs to $S(\mu')$ whose capacity is not filled under μ. We let such a school point to i.

Since each school is pointed to by one student, and each student is pointed to by one school, $\mathcal{G}(\mu, \mu')$ has a cycle. The proof shows that any cycle in this graph is a FIG cycle.

The next example illustrates how our construction works in a specific environment.

Example 4 (Construction of a FIG cycle)**.** Let $I = \{i_1, i_2, i_3, i_4\}$, $S = \{s_1, s_2, s_3, s_4\}$, $R = \{r, r'\}$ where $r = \{i_1, i_2, i_3, s_1, s_2, s_3\}$ and $r' = \{i_4, s_4\}$. School s_2 has the capacity of two while all other schools have the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_2, s_1, \qquad \succ_{s_1}: i_1,$$

$$\succ_{i_2}: s_4, s_2, \qquad \succ_{s_2}: i_2, i_3, i_1, i_4,$$

$$\succ_{i_3}: s_2, \qquad \succ_{s_3}: i_4,$$

$$\succ_{i_4}: s_2, s_3, \qquad \succ_{s_4}: i_2.$$

Consider the following matchings:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & \emptyset \\ i_1 & i_2, i_3 & \emptyset & \emptyset & i_4 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & \emptyset \\ \emptyset & i_1, i_3 & i_4 & i_2 & \emptyset \end{pmatrix}.$$

They are both iBFs, and μ' Pareto dominates μ.

Let us explain how to obtain the graph $\mathcal{G}(\mu, \mu')$ and a FIG cycle in this environment (see Figure 7 for a graphical illustration). First, note that $I' = \{i_1, i_2, i_4\}$, $S(\mu) = \{s_1, s_2\}$, and $S(\mu') = \{s_2, s_3, s_4\}$. Recall that a student in I' points to a school in $S(\mu')$ if and only if she is the top student in I' according to the school's priority among those who prefer the school to their current match. Hence, the edges originating at a student in $\mathcal{G}(\mu, \mu')$ are $(i_1, s_2), (i_2, s_4)$, and (i_4, s_3). Next, we illustrate how a student is pointed to by a school.

(1) $\mu_{i_1} = s_1 \in S \setminus S(\mu')$. This is case 2 of the aforementioned pointing rule from a school to a student. School s_3 is the only school that resides in μ_{i_1}'s region r and belongs to $S(\mu')$ whose capacity is not filled under μ. Thus, (only) s_3 points to i_1.

(2) $\mu_{i_2} = s_2 \in S(\mu')$. This is case 1 of the pointing rule, and thus s_2 points to i_2.

(3) $\mu_{i_4} = \emptyset$. This is case 3 of the pointing rule. School s_4 is the only school that resides in i_4's region r' and belongs to $S(\mu')$ whose capacity is not filled under μ. Thus, (only) s_4 points to i_4.

Overall, the graph $\mathcal{G}(\mu, \mu')$ can be drawn as in Figure 7. There is a unique cycle on $\mathcal{G}(\mu, \mu')$, which is $(i_1, s_2, i_2, s_4, i_4, s_3)$. One can check that this is a FIG cycle as well.

We note a special feature of this example: Given an iBF μ and a Pareto dominating iBF μ', implementing a FIG cycle on μ constructed in the proof results in μ'. This feature turns out not to hold generally. This fact can be seen in Example 9 in the Online Appendix, where two steps of FIG cycle implementations may be necessary to move between two iBFs. □

Theorem 1 and Theorem 2 together imply the following characterization of efficient iBF.

Corollary 1. *Let μ be an iBF. μ is an efficient iBF if and only if there exists no FIG cycle on μ.*

5. FIG Cycles Algorithm

Let $\tilde{\mu}$ be an arbitrary iBF, e.g., the empty matching. Building on Corollary 1, we define a **FIG cycles algorithm** on $\tilde{\mu}$ as follows.

FIG Cycles Algorithm on $\tilde{\mu}$:

Step 0: Let $\mu^0 = \tilde{\mu}$ and go to Step 1.

Step l ($l \geq 1$): If there is no FIG cycle on μ^{l-1}, terminate the algorithm and output μ^{l-1}. Otherwise, choose a FIG cycle $\mathcal{F}$ on μ^{l-1} arbitrarily and let μ^l be the matching generated by $(\mu^{l-1}, \mathcal{F})$, and go to Step $l+1$.

We call a mechanism a **FIG cycles mechanism** if, for any given student preference profile, it outputs the outcome of a FIG cycles algorithm on some iBF.

Corollary 2. *Let $\tilde{\mu}$ be an arbitrary iBF. The FIG cycles algorithm on $\tilde{\mu}$ runs in polynomial time, and its output is an efficient iBF and weakly Pareto dominates $\tilde{\mu}$.*

Proof. Take an arbitrary iBF and denote it by $\tilde{\mu}$. Theorems 1 and 2 together with the definition of the Fig cycles algorithm show that the output of the FIG cycles algorithm on $\tilde{\mu}$ is an efficient iBF and weakly Pareto dominates $\tilde{\mu}$. To show that the algorithm runs in polynomial time, first note that each student can only become better off while running the algorithm, and at least one student must be made strictly better off at each step as long as the algorithm does not terminate in that step. Therefore, at most $|I| \times |S|$ steps are necessary for terminating the algorithm. Second, within each step, finding a FIG cycle can be done in polynomial time.[20] These two observations show that the algorithm runs in polynomial time, as desired. □

In some applications, a school may give a higher priority to residents of that school's region than non-residents. Let us now consider such a case. Formally, we say that **locals are favored** if for each $r \in R$, $s \in r$, $i \in r$, and $j \notin r$, we have $i \succ_s j$.

In this setting, there is a natural fair matching that seems to correspond well with the present practice in applications. It is the matching that is produced by the standard deferred acceptance mechanism of Gale and Shapley (1962), separately in each region. Formally, it is defined as follows.

Definition 4. A **region-wise student-optimal stable matching** μ^{RW} is a matching that satisfies the following:

(1) For each $r \in R$ and each $i \in r$, we have $\mu_i^{RW} \in r \cup \{\emptyset\}$.
(2) μ^{RW} is individually rational.
(3) For each $r \in R$, there is no pair of a student and a school $i, s \in r$ such that $s \succ_i \mu_i^{RW}$, $i \succ_s \emptyset$, and $|\mu_s^{RW}| < q_s$.

[20]There are well-known polynomial-time algorithms that identify a cycle if one exists and otherwise show that there is no cycle. The "depth-first search" algorithm, for example, has the running time of $O(|I| \times |S|)$ (Cormen et al., 2001).

(4) For each $r \in R$, there is no pair of students $i, i' \in r$ such that i has justified envy to i' under μ^{RW}.

(5) μ^{RW} weakly Pareto dominates all matchings that satisfy conditions (1), (2), (3) and (4).

Intuitively, a region-wise student-optimal stable matching requires that a matching restricted to each region (i.e., consider the students and schools that reside in that region) is a student-optimal stable matching in the standard sense (Conditions (2), (3), and (4) correspond to individual rationality, non-wastefulness, and fairness within each region). By Gale and Shapley (1962), a region-wise student-optimal stable matching always exists, and it can be obtained by their deferred acceptance algorithm in polynomial time.

We note that μ^{RW} is an iBF if locals are favored. To see this, first note that μ^{RW} is individually rational (Condition (2)). Second, it is balanced because no student is matched to a school outside of her region (Condition (1)), and thus both the inflow and outflow for any given region is equal to zero. Third, μ^{RW} is fair by Condition (4) and the assumption that locals are favored (so a student living in r cannot have justified envy to another student living in r' if the latter student is matched to a school in r').[21]

The next corollary considers the case when we run a FIG cycles algorithm starting from this matching.

Corollary 3. *Suppose that locals are favored. The output of a FIG cycles algorithm starting from the region-wise student-optimal stable matching* μ^{RW} *is an efficient iBF and weakly Pareto dominates* μ^{RW}*.*

Proof. Let μ^{RW} be the region-wise student-optimal stable matching. Note that it is an iBF as we have explained after providing the statement of Definition 4. Hence, Corollary 2 implies that the output of a FIG cycles algorithm starting from μ^{RW} is an efficient iBF. Finally, Theorem 1 and the definition of FIG cycles algorithm imply that the output Pareto dominates μ^{RW}. □

[21]We wrote Definition 4 following the most standard way, but some conditions and qualifiers are redundant if locals are favored. This is because condition (5) holds and, as we explained, μ^{RW} is fair. A simplified (equivalent) definition requires the following four conditions:

(1) For each $r \in R$ and each $i \in r$, we have $\mu^{RW}_i \in r \cup \{\emptyset\}$.

(2) μ^{RW} is individually rational.

(4') μ^{RW} is fair.

(5') μ^{RW} weakly Pareto dominates all matchings that satisfy conditions (1), (2), and (4').

6. Discussions

This section provides a number of discussions. In Section 6.1, we characterize when balancedness and stability are compatible with each other. Section 6.2 examines the strategic properties. In Section 6.3, we provide comparative statics to evaluate the effect of merging and splitting regions. Section 6.4 considers the case with weak priority, which often arises in school choice applications. Section 6.5 discusses fragmented matching markets in practice. In Section 6.6, we review the related literature.

6.1. **Balancedness and Stability.** As discussed in Section 2.2, Example 2 demonstrates that the balancedness condition is not necessarily compatible with stability. This observation leads to the question of when those properties are compatible with each other. The following result provides a characterization and establishes, in particular, that those cases exactly coincides with cases in which the unique efficient iBF is the student-optimal stable matching.

Theorem 3. *The following statements are equivalent.*

(1) *There exists a unique efficient iBF, and it coincides with the student-optimal stable matching.*
(2) *There exists a stable and balanced matching.*
(3) *Every stable matching is balanced.*

Proof. (1) $\rightarrow$ (2): Obvious.
(2) $\rightarrow$ (3): Let μ and μ' be two stable matchings. Define a graph as follows. The set of nodes are I and S. We let each $s \in S$ point to $i \in I$ if $\mu_i = s$ and let each $i \in I$ point to $s \in S$ if $\mu'_i = s$. By the rural hospital theorem (Roth, 1986), for each school s, the number of arrows that point to s is equal to the number of arrows that point from s and, for each student i, the number of arrows that point to i is equal to the number of arrows that point from i. Therefore, the graph is partitioned into sets of edges that correspond to a finite number of disjoint cycles such that μ' is generated by implementing all those cycles on μ.[22] Thus, by an argument analogous to the proof of Theorem 1, if μ is balanced, then μ' is balanced as well.
(3) $\rightarrow$ (1): Suppose that every stable matching is balanced. We first observe that the student-optimal stable matching μ^* is stable and hence balanced. By Wu and Roth (2018), for any matching μ that is individually rational and fair, $\mu^*_i \succeq_i \mu_i$ for all $i \in I$. Therefore, μ^* is a unique efficient iBF. □

[22]Formally, let $\mathcal{F}_1, \dots \mathcal{F}_k$ be the set of cycles. Let $\mu^0 := \mu$ and μ^ℓ be generated by $(\mu^{\ell-1}, \mathcal{F}_\ell)$ for each $\ell = 1, \dots, k$. Then it follows that $\mu' = \mu^k$.

6.2. **Strategic Property.** In this section, we investigate strategic properties. It turns out that our mechanism based on the FIG cycles algorithm is not strategy-proof. In fact, we show that there is no strategy-proof mechanism that always outputs an efficient iBF.

Formally, we say that mechanism φ is **strategy-proof** if

$$\varphi_i(\succ) \succeq_i \varphi_i(\succ'_i, \succ_{-i}),$$

for every student preference profile $\succ$, $i \in I$, and student preference $\succ'_i$.

Theorem 4. *There exists no strategy-proof mechanism that outputs an efficient iBF for all preference profiles.*

Proof. We prove the result by presenting an example. Let $I = \{i_1, i_2, i_3\}$ and $S = \{s_1, s_2, s_3\}$. Let there be two regions, $r = \{i_1, i_2, s_1\}$ and $r' = \{i_3, s_2, s_3\}$. Each school has the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_2, \qquad \succ_{s_1}: i_1, i_2, i_3,$$
$$\succ_{i_2}: s_3, \qquad \succ_{s_2}: i_3, i_2, i_1,$$
$$\succ_{i_3}: s_1, \qquad \succ_{s_3}: i_3, i_1, i_2.$$

In this environment, there are two efficient iBFs:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_3 & i_1 & \emptyset & i_2 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_3 & \emptyset & i_2 & i_1 \end{pmatrix}.$$

Fix a mechanism φ that outputs an efficient iBF for all preference profiles. It must be either $\varphi(\succ) = \mu$ or $\varphi(\succ) = \mu'$.

Suppose $\varphi(\succ) = \mu$. Then, consider $\succ'_{i_2}: s_3, s_2$. The unique efficient iBF at $\succ' := (\succ'_{i_2}, \succ_{-i_2})$ is μ', so $\varphi(\succ') = \mu'$. Noting that $\mu'_{i_2} = s_3 \succ_{i_2} \emptyset = \mu_{i_2}$, we have obtained that $\varphi_{i_2}(\succ) \not\succeq_{i_2} \varphi_{i_2}(\succ'_{i_2}, \succ_{-i_2})$, violating strategy-proofness.

Next, suppose $\varphi(\succ) = \mu'$. Then, by considering $\succ'_{i_1}: s_2, s_3$ and following a symmetric argument, we conclude that φ is not strategy-proof. This completes the proof. □

Theorem 4 offers a sense in which the lack of strategy-proofness is not a drawback unique to the FIG cycles mechanism, showing that any mechanism that outputs an efficient iBF necessarily fails to be strategy-proof.

6.3. **Comparative Statics.** Intuition suggests that merging regions will lead to a more efficient matching while splitting regions will have the opposite effect. To explore this intuition, first consider the case where all regions were merged, that is, the union of all regions in the original model is regarded as a single region after the mergers. In this case, there would be a unique efficient iBF, which coincides with the student-optimal stable matching. This matching weakly Pareto dominates any individually rational and fair matching, and hence any efficient iBF under any structure of regions. However, as we have argued in the Introduction, merging all regions may be impractical. A question of interest is whether mergers generally result in a more efficient matching even if there remain multiple regions after the mergers.

We say that an environment $\mathcal{E} = (I, S, (\succ_a)_{a \in I \cup S}, (q_s)_{s \in S}, R)$ is a **result of mergers from** another environment $\mathcal{E}' = (I', S', (\succ'_a)_{a \in I' \cup S'}, (q'_s)_{s \in S'}, R')$ if $I = I', S = S'$, $\succ_a = \succ'_a$ for every $a \in I \cup S$, $q_s = q'_s$ for every $s \in S$ and, for each $r \in R$, r is a union of (possibly one) regions of R'. That is, some regions in $\mathcal{E}'$ merge to form a region in $\mathcal{E}$, but otherwise all the primitives are unchanged between the two environments.

In the following simple example, merging regions Pareto-improves the outcomes for students.

Example 5 (An instance in which merging regions makes every student better off)**.** Let $I = \{i\}$ and $S = \{s\}$. Let there be two regions, $r = \{i\}$, and $r' = \{s\}$.[23] School s has the capacity of one. Student i finds school s to be acceptable and school s finds student i to be acceptable. In this environment, there is a unique efficient iBF:

$$\mu = \begin{pmatrix} s & \emptyset \\ \emptyset & i \end{pmatrix}.$$

If regions r_1 and r_2 are merged, then there is a unique efficient iBF:

$$\mu' = \begin{pmatrix} s & \emptyset \\ i & \emptyset \end{pmatrix}.$$

Since $\mu'_i \succ_i \mu_i$ and i is the only student in the market, this means that merging the regions made every student strictly better off in this example. The intuition is simple: The merge reduced the constraint of balanced exchange between regions r and r', so after the merger, the student i can go to school s.

It turns out that the finding of this example is not a coincidence. Indeed, the next proposition establishes a comparative statics regarding any mergers.

Proposition 1. *Suppose that $\mathcal{E}$ is a result of mergers from $\mathcal{E}'$. Then, for any matching μ' that is an efficient iBF at $\mathcal{E}'$, there exists a matching μ that is an efficient iBF at $\mathcal{E}$ such that μ weakly Pareto dominates μ'.*

[23]In this example, region r does not have a school and region r' does not have a student. These features are not necessary to make our point.

Proof. We first show that μ' is an iBF at $\mathcal{E}$. Individual rationality and fairness are obvious. The following claim is proved in Appendix A.3:

Claim 1. *μ' is balanced at $\mathcal{E}$.*

The preceding arguments have established that μ' is an iBF at $\mathcal{E}$. Therefore, by the definition of an efficient iBF, there exists a matching μ that is an efficient iBF at $\mathcal{E}$ such that μ weakly Pareto dominates μ'. □

We note that a kind of converse of Proposition 1 does not hold. Specifically, Example 6 in Appendix A.4 presents an instance in which, starting at a certain efficient iBF, some student is made strictly better off in any efficient iBF after splitting a region.

To summarize, the analysis in this section confirmed the intuition that merging regions improves student welfare. We also found that, however, a precise sense in which this intuition goes through is somewhat subtle.

6.4. **Weak Priorities.** In applications such as daycare allocations and school choice, schools are sometimes endowed with weak priorities. Erdil and Ergin (2008) consider weak priorities and propose an algorithm based on cycles to improve upon the deferred acceptance algorithm with tie-breaking, albeit in a setting without our balancedness constraints. Accordingly, a natural question would be whether our analysis extends to cases where priorities are allowed to be weak in the presence of balancedness constraints. As it turns out, all of our results go through. In particular, the conclusions of Theorems 1 and 2 hold without any change.[24]

Here we illustrate how the proofs of Theorems 1 and 2 change with weak priorities. The only difference from the case of strict priorities is the following. In the FIG or $\mathcal{G}(\mu, \mu')$ (a graph which appears in the proof of Theorem 2), each school can be pointed to by at most one student. This was because a student can point to a school only if she is the top student among those who regard the school as an improvement. Under strict priorities, there is a unique "top" student, which was why each school can be pointed to by at most one student. Under weak priorities, however, there can be multiple "top" students, so a school can be pointed to by multiple students. But this change does not affect the remainder of the proof.

[24]Under weak priority, we say that a student is acceptable to a school if she is ranked *weakly* higher than the outside option, and modify the definitions of individual rationality, efficient iBF and FIG by adopting this definition of acceptability in the relevant parts of those concepts. The proofs change accordingly in a straightforward manner.

(a) Province-level regions in China's mainland (b) Cities in Hebei (and Beijing) (c) Districts in Beijing

FIGURE 8. Fragmentation in China at different levels

6.5. **Fragmented Matching Markets in Practice.** The issues of fragmentation and possible improvement through "partial integration" are prevalent. This section discusses several real-life examples, while relegating further details to the Online Appendix. Specifically, we describe three Chinese examples to illustrate that fragmentation can happen at a variety of scales: the nation fragmented into providences, a province fragmented into cities, and a city fragmented into districts. In each of our examples, our mechanism can address the problems such as inefficiencies caused by fragmentation. We also provide examples of Japanese daycare allocation, one of which was mentioned in the Introduction, to reinforce the point that our solution helps alleviate the problem with fragmentation.

Chinese college admission: In Chinese college admission, each college has a quota for students from each province-level region ("province" henceforth).[25] At each province, students are ranked in order of test scores, and serial dictatorship is run to determine which student takes a slot among those reserved for the province in the colleges across the country. By way of an example, the Online Appendix shows that this mechanism suffers from three problems due to its rigidity in determining the number of slots to reserve for each province: First, the number of seats reserved for students from other provinces may be too small relative to the number of students who want to enter that college from those provinces. Second, in contrast, the number of reserved seats may be too large, crowding out the students from the college's region. Third, balancedness or fairness may fail. Our FIG cycles mechanisms may address those shortcomings, as they endogenously determine the number of students who are allocated to schools outside the provinces

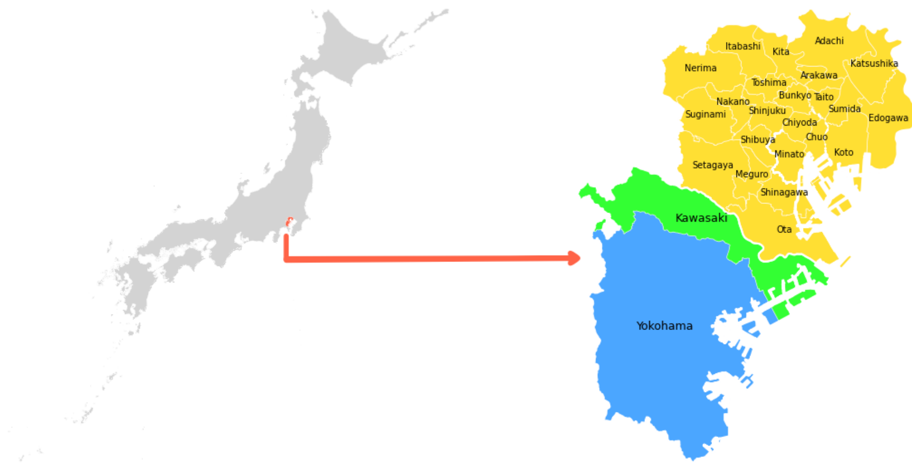
[25]Province-level regions of China consist of provinces, autonomous regions, municipalities, and special administrative regions.

of their residence based on the preferences of the market participants. In particular, balancedness of the output of the FIG cycles mechanisms would be desirable, as most colleges rely on funding from provincial governments. We note that, even though some provinces are large in area, it is common for college students to study across provincial borders (which motivates reserving seats for students from other provinces in the first place), and thus fragmentation at the national level is a problem that should be taken care of.

Chinese high school admission: In most provinces in China, high school admissions are conducted at the city level, and almost no inter-municipal transfer is allowed. In Hebei province, which has 11 cities, inter-municipal transfer was allowed before 2024, but the policy has been changed to disallow it because increased inter-municipal school choice allegedly led to the loss of top students from certain cities. This change has shut down the gain from inter-municipal school choice. By way of an example, the Online Appendix demonstrates that our FIG cycles mechanism achieves some gain from inter-municipal school choice while alleviating the problem of regional imbalance.

Chinese kindergarten admission: In major cities in China, kindergarten admissions are conducted at the district level. Some cities do not allow interdistrict transfers at all, while other cities do allow it, but only under limited conditions. Beijing, which has 16 districts, is one such example. In Beijing, the matching process is run by each district independently in an uncoordinated manner. This implies that a given child may receive offers from multiple districts and thus unnecessarily prevent others from receiving offers, possibly resulting in inefficiency. Balancedness would be desirable as kindergartens are locally funded, and our FIG cycles mechanism offers a way to achieve a welfare improvement while keeping the balance in a coordinated manner.

Japanese daycare admission: In Japan, admissions for accredited daycare centers are conducted by each individual municipality. For example, the City of Tokyo consists of 23 small municipalities, and many families live close to the border between those municipalities. Moreover, many parents commute across the border, making the daycare centers close to their workplace convenient. However, inter-municipal transfer has been difficult, with each municipality operating on their own and the daycare centers being heavily subsidized by the municipality (making balancedness desirable). Our FIG cycles mechanism has a potential to offer an improvement. Another example is the City of



(a) Japan (b) Municipalities in Tokyo, Yokohama, and Kawasaki

FIGURE 9. Japanese Daycare Admission: Fragmentation of Tokyo City and the long border between Yokohama City and Kawasaki City

Yokohama and the City of Kawasaki. They share a long border and adopted a policy such that each city opens daycare centers that have pre-set capacity for the other city, making some inter-municipal transfer possible. However, this policy has the same problem as the Chinese college admission discussed above. Therefore, our FIG cycles mechanisms may address shortcomings of the present policy, just as in the aforementioned case of the college admission. Specifically, the mechanism improves fairness and balance and achieve efficiency subject to those properties (along with individual rationality). Balancedness would be desirable since that the daycare centers in these cities are heavily subsidized by the respective cities.

6.6. **Related Literature.**

Welfare Improvement under Balancedness Constraint: To our knowledge, the balancedness condition was first introduced and studied in the matching literature by Dur and Ünver (2019) in the context of tuition exchange. The subsequent studies on matching markets with balancedness conditions include Dur, Kesten and Ünver (2015) on student exchange programs for European colleges, and Dur, Hammond and Ünver (2024) on student-athlete transferring college sports. The main difference between those contributions and ours is that they consider balancedness at the level of individual institutions, while the present paper considers balancedness at the regional level. In that particular respect, one could consider our model to allow for more generality because balancedness at each individual institution can be coded as a case in which each region happens to include only one institution. At the same time, we acknowledge that the desiderata studied by those papers besides balancedness are also different from ours, so their analysis and ours are logically unrelated with each other.

Our algorithm and the algorithms developed in the papers in the previous paragraph are based on a number of Pareto-improving cycles among students and schools. At a high level, this is a common idea and is shared by many other algorithms, including Gale's celebrated TTC algorithm in Shapley and Scarf (1974). The difference of our algorithm from TTC is that we construct cycles in a more subtle and nuanced manner, taking school priorities into account in particular, so that implementing the cycles will keep fairness of the original matching. Closer to our algorithm are those in groundbreaking works by Erdil and Ergin (2008, 2017) who, like us, provide iterative algorithms that improve efficiency while retaining fairness.[26] Similarities and differences between those studies and ours are illustrated in detail in Section 1.1, Remark 2 in Section 3.1, and Section 4.

Full Integration of Matching Markets: At a high level, the present paper is related to a burgeoning literature that considers "full integration" of multiple regions in matching problems. Ortega (2018) and Klein, Aue and Ortega (2024) study welfare effects of full integration under the deferred acceptance mechanism theoretically and empirically. Hafalir, Kojima and Yenmez (2022) study conditions under which the outcome of the standard (unconstrained) deferred acceptance mechanism satisfies the balancedness constraint, and Kamada and Kojima (2024) study conditions under which integration of matching markets benefits every student under a mechanism satisfying certain desirable properties.[27] By contrast, the present paper takes the balancedness constraint as given, thus precluding the full integration of multiple regions studied those papers, and considers how to improve participants' welfare under the constraint. In that sense, the present paper is complementary to the studies discussed here.

Matching with Constraints: This paper can be regarded as part of the literature in matching with constraints. Research in this literature include Abdulkadiroğlu (2005), Ergin and Sönmez (2006), and Hafalir, Yenmez and Yildirim (2013) for school choice,

[26] More detailed discussions of Erdil and Ergin (2008, 2017) are provided in Section 3.1. Algorithms based on analogous ideas have been adapted to other settings. Erdil and Kumano (2019) generalize the algorithm of Erdil and Ergin (2008) to the case in which the priority of each school does not necessarily satisfy the responsiveness condition. Combe, Tercieux and Terrier (2022) allow the initial matching to be unfair and offer a cycle-based algorithm that improves upon the initial matching in terms of both efficiency and fairness. Combe et al. (2022) offer a related algorithm and show its empirical performance on data of French teacher allocation.

[27] Hafalir, Kojima and Yenmez (2022) also consider conditions under which the output of the deferred acceptance mechanism satisfies other constraints.

Abraham, Irving and Manlove (2007) for project allocation, Westkamp (2013) and Aygün and Turhan (2020) for college admission, Pathak et al. (2021) on pandemic resource allocation, and Kamada and Kojima (2015, 2017, 2018, 2023), Goto et al. (2014), and Biró et al. (2010) on labor markets.[28] The main departure of the present paper is that we consider integration of multiple markets, while those earlier contributions treat the relevant market as given.

Daycare Matching: One of the applications of the present paper is allocation of daycare seats. Research in daycare allocation in the matching literature include Kennes, Monte and Tumennasan (2014), Veski et al. (2017), Herzog and Klein (2018), Okumura (2019), Delacrétaz (2019), and Kamada and Kojima (2023). While those papers and ours share interest in daycare, the overlap is rather limited, as none of those papers analyzes models with multiple regions as we do.

7. Conclusion

This paper considered fragmented school-choice matching markets and proposed a way for integration. To accommodate practical concerns that full integration is not a viable option, we provided solutions for *partial* integration, namely a mechanism that satisfies the balancedness constraint. Given any matching, we defined a directed bipartite graph (the "FIG") in which the nodes represent students and schools while the edges are constructed using student preferences, school priorities, and the information about the current matching. Using this graph, we characterized the set of efficient iBFs (individually rational, balanced, and fair matchings) by non-existence of a cycle that would improve the welfare of the students involved in the cycle. This led us to define the FIG cycles algorithm that computes an efficient iBF in polynomial time. In terms of application, our analysis provided a way to improve upon mechanisms organized in a fragmented manner.

Market fragmentation is prevalent in real markets. We documented problems of fragmentation in kindergarten admission, high school admission, and college admission in China as well as daycare admission in Japan. To put our theory into practice, we started discussions with government officials about a possible implementation of our FIG cycles mechanism for their daycare allocation across the 23 municipalities in the City of Tokyo. We emphasize that our method does not require those municipalities to fully integrate with one another: It would be impractical to require that transfer of students be allowed between different municipalities without restrictions. We, by contrast, only require partial

[28]See also Kojima, Tamura and Yokoo (2018) who explore connection between matching with constraints and convexity and concavity notions in discrete mathematics.

integration where transfers can be made to the extent that the balancedness condition is respected. Moreover, it is desirable but not necessary for all the 23 municipalities to participate in the proposal: Even if only a subset of those municipalities participate, our FIG cycles mechanism achieves welfare gains for applicants in participating municipalities. Based on our experience with practitioners in daycare policies, we view such an approach as a practical solution.

We conclude by mentioning several possible directions for future research. The first is more study in incentive compatibility of our FIG cycles mechanisms. Although Theorem 4 shows that efficient iBFs cannot be implemented by a strategy-proof mechanism, manipulability of our FIG cycles mechanisms may be limited in contrast to, e.g., the Boston mechanism for which manipulations have been documented both in practice and in the lab (e.g., Abdulkadiroğlu and Sönmez (2003), Chen and Sönmez (2006)). It would not be straightforward to analyze this issue because there may be many possible formulations of degrees of manipulability (e.g., Erdil and Ergin (2008), Kojima and Pathak (2009)) and many different selection rules for FIG cycles in our algorithms to consider, and we are currently working on this issue.[29] Second, it may be of interest to consider other types of constraints than our balancedness constraint. For instance, although balancedness seems to be important in our intended applications, in some cases it may be politically acceptable to tolerate some imbalance as long as it is within some bound. A challenge is that we would need to consider different types of cycles than our current FIG cycles algorithm allows, a challenge we are investigating in our continuation work. Third, it would be important to quantify the magnitude of efficiency loss from fragmentation as well as how much of the loss can be eliminated by our proposal. In our ongoing project with Akira Matsushita (Kamada, Kojima and Matsushita, 2025), we use data from daycare allocation in the City of Tokyo to estimate the applicants' preferences and run counterfactual simulations of different mechanisms, including the present practice under fragmentation, our FIG cycles mechanisms, and the deferred acceptance mechanism without the balancedness constraint.

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[29]Another possibility may be to consider domain restriction under which strategy-proofness and efficient iBF are compatible.

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Appendix A. Proofs

A.1. **Proof of Theorem 1.**

Proof. Fix μ and a FIG cycle $\mathcal{F} = (i_1, s_1, i_2, s_2, \ldots, i_m, s_m)$ on μ. Let μ' be the matching that is generated by $(\mu, \mathcal{F})$. Clearly, μ' Pareto dominates μ.

To show that μ' is fair, notice that $\mu'_i \succeq_i \mu_i$ for all $i \in I$ by the definition of FIG cycle. This implies that, for every $i \notin \{i_1, \ldots, i_m\}$, no one has justified envy to i under μ' because μ is fair. Thus, it remains to show that for each $k \in \{1, \ldots, m\}$, no one has justified envy to i_k under μ'. To see this, note that for each $i \in I$ and $s \in S$, we have that $s \succ_i \mu_i$ holds if $s \succ_i \mu'_i$ because $\mu'_i \succeq_i \mu_i$. Hence, $D_s^{\mu'} \subseteq D_s^{\mu}$ for any $s \in S$ (and thus in particular for $s = s_k$).[30] Since $i_k = Top_{s_k}(D_{s_k}^{\mu})$ for each k by definition, this implies $i \not\succ_{s_k} i_k$ for every $i \in D_{s_k}^{\mu'}$. Hence, no one has justified envy to i_k under μ'.

To show that μ' is balanced, consider a sequence of regions, $(r_1, \ldots, r_m)$, such that $s_k \in r_k$ for each k. Fix $r \in R$ and let $K(r) = \{k \in \{1, \ldots, m\} | r_k = r\}$. If $K(r) = \emptyset$, then (2.1) in Definition 1 is satisfied for r under μ' because it is satisfied for r under μ. So suppose $K(r) \neq \emptyset$. Let

$$In_r = \{k \in K(r) | r_{k-1} \neq r\} \text{ and } Out_r = \{k \in K(r) | r_{k+1} \neq r\}.$$

Clearly we must have $|In_r| = |Out_r|$. Define $In'_r \subseteq In_r$ and $Out'_r \subseteq Out_r$ by

$$In'_r = \{k \in K(r) | r_{k-1} \neq r \text{ and } i_k \in r\} \text{ and } Out'_r = \{k \in K(r) | r_{k+1} \neq r \text{ and } i_{k+1} \in r\}.$$

On the one hand, the inflow to r has changed from μ to μ' by

$$\underbrace{(|In_r| - |In'_r|)}_{\text{the number of non-}r\text{ students coming to }r} \quad - \quad \underbrace{(|Out_r| - |Out'_r|)}_{\text{the number of non-}r\text{ students going out of }r}.$$

On the other hand, the outflow from r has changed from μ to μ' by

$$\underbrace{|Out'_r|}_{\text{the number of }r\text{ students going out of }r} \quad - \quad \underbrace{|In'_r|}_{\text{the number of }r\text{ students coming to }r}.$$

Note that these two values are equal because $|In_r| = |Out_r|$. Finally, since (2.1) holds for r under μ, this implies that (2.1) holds for r under μ'.

To show that individual rationality of μ implies individual rationality of μ', note first that $\mu'_i \succeq \emptyset$ for each $i \in I$ because μ is individually rational and μ' Pareto dominates μ. Moreover, $i \succ_s \emptyset$ for every $s \in S$ and $i \in \mu'_s$ because (i) $i \succ_s \emptyset$ for every $i \in \mu'_s \cap \mu_s$ by

[30] Recall that, for any matching $\tilde{\mu}$ and school s, $D_s^{\tilde{\mu}} = \{j \in I | s \succ_j \tilde{\mu}_j\}$.

the assumption that μ is individually rational, while (ii) $i \succ_s \emptyset$ for every $i \in \mu'_s \setminus \mu_s$ by the definition of FIG (more specifically, part (1) of Definition 3). $\square$

A.2. **Proof of Theorem 2.**

Proof. We prove the contraposition, so fix an iBF μ and assume that there exists an iBF μ' that Pareto dominates μ.

Lemma 1. *For any* $r \in R$, $|\{i \in I | \mu_i \in r, \mu'_i \notin r\} \cup \{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}|$ *is equal to* $|\{i \in I | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\} \cup \{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|$.

Proof. Since μ and μ' are both balanced, the change (from μ to μ') of the inflow of students to r and the change of the outflow are the same as each other. The change of the inflow is:

$$\underbrace{|\{i \in I \setminus r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{coming from an outside school to } r}} + \underbrace{|\{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{coming from being unmatched to } r}} - \underbrace{|\{i \in I \setminus r | \mu_i \in r, \mu'_i \notin r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{going out of } r}}.$$

The change of the outflow is:

$$\underbrace{|\{i \in r | \mu_i \in r, \mu'_i \in r' \text{ for some } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{going out of } r}} + \underbrace{|\{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{who was unmatched but is now matched to a non-}r\text{ school}}}$$
$$- \underbrace{|\{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \notin r' \text{ for all } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{who was matched to a non-}r\text{ school but now is not}}}.$$

Thus, we have:

(A.1)
$$\underbrace{|\{i \in r | \mu_i \in r, \mu'_i \in r' \text{ for some } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{going out of } r}} + \underbrace{|\{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{who was unmatched but is now matched to a non-}r\text{ school}}}$$
$$+ \underbrace{|\{i \in I \setminus r | \mu_i \in r, \mu'_i \notin r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{going out of } r}} =$$
$$\underbrace{|\{i \in I \setminus r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{coming from an outside school to } r}} + \underbrace{|\{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|}_{\substack{\text{the number of non-}r\text{ students} \\ \text{coming from being unmatched to } r}}$$
$$+ \underbrace{|\{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \notin r' \text{ for all } r' \neq r\}|}_{\substack{\text{the number of } r\text{ students} \\ \text{who was matched to a non-}r\text{ school but now is not}}}.$$

Now, recall that μ is individually rational. Hence, $\mu'_i \succeq_i \mu_i \succeq_i \emptyset$ for every i, and thus we have that $\mu_i \in r$ implies $\mu'_i \neq \emptyset$. Therefore,

$$\{i \in r | \mu_i \in r, \mu'_i \in r' \text{ for some } r' \neq r\} = \{i \in r | \mu_i \in r, \mu'_i \notin r\}$$

Also, for the same reason, we have

$$\{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \notin r' \text{ for all } r' \neq r\} = \{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}.$$

Hence, (A.1) is equivalent to

$$|\{i \in r | \mu_i \in r, \mu'_i \notin r\}| + |\{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}| + |\{i \in I \backslash r | \mu_i \in r, \mu'_i \notin r\}| =$$
$$|\{i \in I \setminus r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}| + |\{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|$$
$$+ |\{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}|.$$

Since $|\{i \in I \setminus r | \mu_i \in r, \mu'_i \notin r\}| = |\{i \in r | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}|$ holds due to the balancedness of μ and μ', we have

$$|\{i \in I | \mu_i \in r, \mu'_i \notin r\}| + |\{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}| =$$
$$|\{i \in I | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\}| + |\{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|.$$

Since the two terms in each side of the above equation are disjoint from each other, we have

$$|\{i \in I | \mu_i \in r, \mu'_i \notin r\} \cup \{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}| =$$
$$|\{i \in I | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\} \cup \{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}|.$$

This completes the proof. □

Consider the following graph, in which the only agents associated with arrows are the schools in r and students who are matched to a school in r under μ and students living in r who are unmatched under μ. First, each $s \in r$ points to each student $i \in \mu_s \setminus \mu'_s$. Then, for each student who was pointed to by some school in r and each student living in r who are unmatched under μ, let her point to the school μ'_i if $\mu'_i \in r$. Moreover, by Lemma 1, there is a one-to-one and onto mapping from $\{i \in I | \mu_i \in r, \mu'_i \notin r\} \cup \{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}$ to $\{i \in I | \mu_i \in r' \text{ for some } r' \neq r, \mu'_i \in r\} \cup \{i \in I \setminus r | \mu_i = \emptyset, \mu'_i \in r\}$. Take one such mapping ϕ. Then, for each $i \in \{i \in I | \mu_i \in r, \mu'_i \notin r\} \cup \{i \in r | \mu_i = \emptyset, \mu'_i \in r' \text{ for some } r' \neq r\}$, let i point to $\mu'_{\phi(i)}$. This defines a directed graph, denoted $\mathcal{G}(\mu, \mu', r)$. By construction, only schools in r, students matched to a school in r under μ, and the students of r who are unmatched under μ may be associated with arrows in this graph.

Let $I' := \{i \in I | \mu'_i \succ_i \mu_i\}$. By the assumption that μ' Pareto dominates μ, we have $I' \neq \emptyset$. For any matching $\tilde{\mu}$, let $S(\tilde{\mu}) := \{s \in S | s = \tilde{\mu}_i \text{ for some } i \in I'\}$.

Lemma 2. *Suppose $s \in S(\mu) \setminus S(\mu')$. Then there exists a school $s' \in r(s) \cap S(\mu')$ such that $|\mu_{s'}| < q_{s'}$.*

Proof. Take an arbitrary s such that $s \in S(\mu) \setminus S(\mu')$ (If there is no such school, we are done). Starting from this school, follow the arrows in $\mathcal{G}(\mu, \mu', r(s))$ in an arbitrary manner without passing the same student twice (note that there is an outgoing arrow from s). Since there are a finite number of students, there is s' such that there is no more outgoing arrow from s' to a student who has not appeared in the path (note that, by definition, this path cannot end at any student). Because $s \notin S(\mu')$ and hence $s' \neq s$, the number of incoming arrows to s' is greater than that of outgoing arrows from s' along this path. This implies that the number of students who are in $\mu'_{s'} \setminus \mu_{s'}$ is larger than the number of students who are in $\mu_{s'} \setminus \mu'_{s'}$ by at least one. Hence, we have $q_{s'} \geq |\mu'_{s'}| > |\mu_{s'}|$. Since $s' \in r(s)$ and $s' \in S(\mu')$ by the definition of the graph, this completes the proof. □

Lemma 3. *Suppose $\mu_i = \emptyset$ and $i \in I'$. Then there exists a school $s' \in r(i) \cap S(\mu')$ such that $|\mu_{s'}| < q_{s'}$.*

Proof. Suppose there is i such that $\mu_i = \emptyset$ and $i \in I'$. Starting from this student i, follow the arrows in $\mathcal{G}(\mu, \mu', r(i))$ in an arbitrary manner without passing the same student twice (note that there is an outgoing arrow from i). Since there are a finite number of students, there is s' such that there is no more outgoing arrow from s' to a student who has not appeared in the path (note that, by definition, this path cannot end at any student). This implies that the number of students who are in $\mu'_{s'} \setminus \mu_{s'}$ is larger than the number of students who are in $\mu_{s'} \setminus \mu'_{s'}$ by at least one. Hence, we have $q_{s'} \geq |\mu'_{s'}| > |\mu_{s'}|$. Since $s' \in r(i)$ and $s' \in S(\mu')$ by the definition of the graph $\mathcal{G}(\mu, \mu', r(i))$, this completes the proof. □

Next, we define a graph, denoted $\mathcal{G}(\mu, \mu')$, as follows. In this graph, only students in I' and schools in $S(\mu')$ may be associated with arrows. Formally, for any $s \in S(\mu')$, consider the set of students in I' who strictly prefer s to their match at μ, i.e., $D_s^{\mu}(I') := D_s^{\mu} \cap I'$ (or, equivalently, $D_s^{\mu}(I') := \{i \in I' | s \succ_i \mu_i\}$), and let $Top_s(D_s^{\mu}(I'))$ point to s. Note that $D_s^{\mu}(I')$ is nonempty by the definitions of I', μ and μ', and thus for any $s \in S(\mu')$, there exists some $i \in I'$ who points to s. Next, consider any $i \in I'$.

(1) If $\mu_i \in S(\mu')$, then let μ_i point to i.
(2) If $\mu_i \in S \setminus S(\mu')$, then $\mu_i \in S(\mu) \setminus S(\mu')$ by the definition of $S(\mu)$ and hence by Lemma 2, there exists a school $s' \in r(\mu_i) \cap S(\mu')$ such that $|\mu_{s'}| < q_{s'}$. Let any such school s' point to i.
(3) If $\mu_i = \emptyset$, then by Lemma 3, there exists $s \in r(i) \cap S(\mu')$ such that $|\mu_s| < q_s$. Let any such school s point to i.

The graph must have a cycle because each school is pointed to by a single student, and each student is pointed to by at least one school.[31] Pick an arbitrary cycle and call it $\mathcal{F}^*$.

Lemma 4. *$\mathcal{F}^*$ is a FIG cycle.*

Proof. Let $\mathcal{F}^* = (i_1, s_1, \ldots, i_m, s_m)$.

It is straightforward to check that the last two conditions of a cycle are satisfied: By construction, each student appears only once in $\mathcal{F}^*$. Given this, since the pointing rule for $\mathcal{G}(\mu, \mu')$ implies that each school is pointed to only by a single student, each school appears at most once in $\mathcal{F}^*$.

To complete the proof, it suffices to show that $\mathcal{G}(\mu, \mu')$ is a subset of the FIG on μ. To show this, first we establish that, for any $k \in \{1, \ldots, m\}$, i_k points to s_k according to the definition of pointing used for FIG. To do so, it suffices to show that $i_k = Top_{s_k}(D^{\mu}_{s_k})$ and $i_k \succ_{s_k} \emptyset$ (part (1) of Definition 3). The latter holds for the following reason: we have $i \in D^{\mu}_{s_k}(I')$ for some $i \in \mu'_{s_k}$ because of the definitions of $S(\mu')$ and I' and the fact that $s_k \in S(\mu')$. Hence, by individual rationality of μ', it follows that $i_k = Top_{s_k}(D^{\mu}_{s_k}(I')) \succeq_{s_k} i \succ_{s_k} \emptyset$. To show the former, note first that, by construction, $i_k = Top_{s_k}(D^{\mu}_{s_k}(I'))$ and hence $i_k \succ_{s_k} i$ for any $i \in (D^{\mu}_{s_k} \cap I') \setminus \{i_k\}$. Next, consider any $i \in D^{\mu}_{s_k} \setminus I'$. Because $\mu'_{i'} = \mu_{i'}$ for any $i' \in I \setminus I'$ by the definition of I', it follows that $i \in D^{\mu'}_{s_k}$. This and the assumption that μ' is fair imply $j \succ_{s_k} i$ for every $j \in \mu'_{s_k}$. By the construction of the cycle, $i_k \succeq_{s_k} j$ for every $j \in \mu'_{s_k} \setminus \mu_{s_k} \neq \emptyset$ (the nonemptiness holds because $s_k \in S(\mu')$). Thus, we have $i_k \succ_{s_k} i$. Therefore, we have $i_k \succ_{s_k} i$ for any $i \in D^{\mu}_{s_k} \setminus \{i_k\}$, which implies $i_k = Top_{s_k}(D^{\mu}_{s_k})$.

Second, we consider three cases of the definition of $\mathcal{G}(\mu, \mu')$ to show that, for any $k \in \{1, \ldots, m\}$, s_k points to i_{k+1} according to the definition of pointing used for FIG, where $i_{m+1} = i_1$. Suppose first that i_{k+1} and s_k satisfy the condition described in Case 1 of the definition of $\mathcal{G}(\mu, \mu')$. This implies that i_{k+1} and s_k satisfy the assumption in part (2a) of Definition 3. Next, consider Case 2 of the definition of $\mathcal{G}(\mu, \mu')$. In this case, $|\mu_{s_k}| < q_{s_k}$ and $\mu_{i_{k+1}} \in r(s_k)$ hold, which satisfies the condition in part (2b) of Definition 3. Finally, consider Case 3 of the definition of $\mathcal{G}(\mu, \mu')$. In this case, $|\mu_{s_k}| < q_{s_k}$, $i_{k+1} \in r(s_k)$, and $\mu_{i_{k+1}} = \emptyset$ hold, which again satisfies the condition in part (2b) of Definition 3. This completes the proof. □

Lemma 4 completes the proof. □

[31] To find a cycle, take an arbitrary school and find the student pointing to that school. Then find the school pointing to that student. Then find the student pointing to that school, etc., until we find a school or a student that has already been visited. Since there are only finitely many students, this procedure ends in finite steps.

A.3. **Proof of Claim 1.**

Proof. We first observe that the balancedness of μ' at $\mathcal{E}'$ is equivalent to the property that, for each $r \in R'$,

$$\underbrace{\sum_{s\in r} |\{i|i \in \mu'_s, i \notin r\}|}_{\text{inflow to } r} - \underbrace{\sum_{s\notin r} |\{i|i \in \mu'_s, i \in r\}|}_{\text{outflow from } r} = 0.$$

Consider any $r^* \in R$. Then,

$$\sum_{r\in R', r\subseteq r^*} \left[\underbrace{\sum_{s\in r} |\{i|i \in \mu'_s, i \notin r\}|}_{\text{inflow to } r} - \underbrace{\sum_{s\notin r} |\{i|i \in \mu'_s, i \in r\}|}_{\text{outflow from } r} \right] = 0. \tag{A.2}$$

The left-hand side of (A.2) is equal to

$$\begin{aligned}
&\sum_{r\in R', r\subseteq r^*} \left[\underbrace{\sum_{s\in r} \left(|\{i|i \in \mu'_s, i \notin r^*\}| + |\{i|i \in \mu'_s, i \in r^* \setminus r\}|\right)}_{\text{inflow to } r} - \underbrace{\left(\sum_{s\notin r^*} |\{i|i \in \mu'_s, i \in r\}| + \sum_{s\in r^*\setminus r} |\{i|i \in \mu'_s, i \in r\}| \right)}_{\text{outflow from } r} \right] \\
&= \sum_{r\in R', r\subseteq r^*} \left[\sum_{s\in r} |\{i|i \in \mu'_s, i \notin r^*\}| - \sum_{s\notin r^*} |\{i|i \in \mu'_s, i \in r\}| \right] + \sum_{r\in R', r\subseteq r^*} \left[\sum_{s\in r} |\{i|i \in \mu'_s, i \in r^* \setminus r\}| - \sum_{s\in r^*\setminus r} |\{i|i \in \mu'_s, i \in r\}| \right] \\
&= \left[\sum_{s\in r^*} |\{i|i \in \mu'_s, i \notin r^*\}| - \sum_{s\notin r^*} |\{i|i \in \mu'_s, i \in r^*\}| \right] + \sum_{r,r'\in R', r,r'\subseteq r^*, r\neq r'} \left[\sum_{s\in r} |\{i|i \in \mu'_s, i \in r'\}| - \sum_{s\in r'} |\{i|i \in \mu'_s, i \in r\}| \right] \\
&= \underbrace{\sum_{s\in r^*} |\{i|i \in \mu'_s, i \notin r^*\}|}_{\text{inflow to } r^*} - \underbrace{\sum_{s\notin r^*} |\{i|i \in \mu'_s, i \in r^*\}|}_{\text{outflow from } r^*},
\end{aligned}$$

where the third equality comes from the symmetry or r and r' in the second of the two terms in the third line of the above equation.

By (A.2), this implies that

$$\underbrace{\sum_{s\in r^*} |\{i|i \in \mu'_s, i \notin r^*\}|}_{\text{inflow to } r^*} - \underbrace{\sum_{s\notin r^*} |\{i|i \in \mu'_s, i \in r^*\}|}_{\text{outflow from } r^*} = 0.$$

Hence the inflow and outflow for region r^* are equal to each other. Since this relation holds for every $r^* \in R$, μ' is balanced at $\mathcal{E}$. □

A.4. **Comparative Statics: An Example.** The following example shows that the converse of Proposition 1 does not hold. That is, starting at a certain efficient iBF, some student can be made strictly better off in any efficient iBF after splitting a region.

Example 6 (An instance in which splitting a region inevitably makes some student better off)**.** Let $I = \{i_1, i_2\}$ and $S = \{s_1, s_2, s_3\}$. Let there be two regions, $r = \{i_1, s_1, s_2\}$ and $r' = \{i_2, s_3\}$. Each school has the capacity of one. Student preferences and school priorities are given as follows:

$$\succ_{i_1}: s_2, s_3, \qquad \succ_{s_1}: i_2,$$

$$\succ_{i_2}: s_1, \qquad \succ_{s_2}: i_1,$$

$$\succ_{s_3}: i_1.$$

In this environment, there is an efficient iBF:

$$\mu = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ \emptyset & i_1 & \emptyset & i_2 \end{pmatrix}.$$

If region r is split into two regions, $r_1 = \{i_1, s_1\}$ and $r_2 = \{s_2\}$, then there is a unique efficient iBF:

$$\mu' = \begin{pmatrix} s_1 & s_2 & s_3 & \emptyset \\ i_2 & \emptyset & i_1 & \emptyset \end{pmatrix}.$$

Note that i_2 is better off as a result of the split. The intuition is the following: Before the split, i_2 in r' was unable to match with s_1 in r as there was no student who wanted to come from r to his region r'. However, the split made it impossible for i_1 in r_1 (which was part of r) to go to s_2, and she is now interested in coming to r'. This made it possible to implement a swap between r_1 and r'.

Notice that, in the above example, both μ and μ' are efficient iBFs before region r was split. In fact, multiplicity of efficient iBFs is necessary for the failure of the converse of Proposition 1: Indeed, Proposition 1 implies its converse if there exists a unique efficient iBF before the split of a region.