Auction Theory

Lecture 2

UTMDC

Asymmetric Auctions

- Under symmetry and risk neutrality we have
  - revenue equivalence
  - efficiency
- How do asymmetries among bidders—different value distributions—affect
  - revenue?
  - efficiency?

Asymmetric Auctions

- Two risk neutral bidders
- Bidder 1 draws value X_1 from F_1 on $[0, \omega_1]$
- Bidder 2 draws value X_2 from F_2 on $[0, \omega_2]$
- Independence
- Bidder 1 is "strong"; bidder 2 is "weak"—$F_1 \leq F_2$

Asymmetric Auctions

- Again asymmetries have no effect on bidding in SPA—dominant strategy
- Suppose β_1, β_2 is an equilibrium of FPA.
- Inverses $\phi_1 \equiv \beta_1^{-1}$ and $\phi_2 \equiv \beta_2^{-1}$.
- Clearly, $\beta_1(0) = 0 = \beta_2(0)$.
- and let
$$\bar{b} \equiv \beta_1(\omega_1) = \beta_2(\omega_2)$$

Asymmetric FPA

- 1's expected payoff if he bids $b < \overline{b}$

$$\begin{aligned} \Pi_1(b, x) &= F_2(\phi_2(b))\,(x - b) \\ &= H_2(b)\,(x - b) \end{aligned}$$

- First-order condition

$$h_2(b)\,(x - b) = H_2(b)$$

- Or

$$\frac{d}{db} \ln F_2(\phi_2(b)) = \frac{1}{\phi_1(b) - b} \tag{1}$$

- Similarly,

$$\frac{d}{db} \ln F_1(\phi_1(b)) = \frac{1}{\phi_2(b) - b} \tag{2}$$

Weakness Leads to Aggression

- F_1 *dominates* F_2 *in terms of the reverse hazard rate*—that is, for all $x \in (0, \omega_2)$,

$$\frac{f_1(x)}{F_1(x)} > \frac{f_2(x)}{F_2(x)} \tag{3}$$

Proposition

Supppose (3) holds. Then in a FPA, (A) the "weak" bidder 2 *bids more aggressively than the "strong" bidder* 1*—that is,*

$$\beta_1(x) < \beta_2(x)$$

but (B) the distribution of bids for bidder 1 *stochastically dominates that of bidder* 2*, that is*

$$H_1(b) \leq H_2(b)$$

An Example

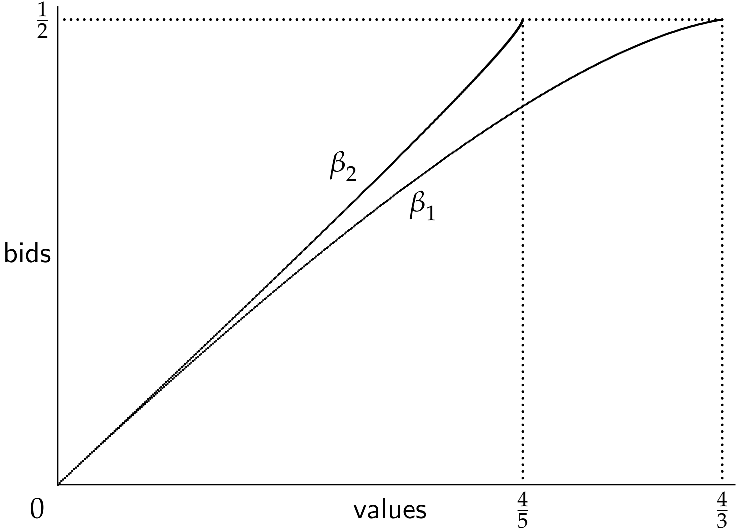

Figure: Equilibrium of an Asymmetric First-Price Auction

Asymmetric Auctions

Proposition

With asymmetries, FPA is inefficient. (SPA is efficient).

- With asymmetries revenue equivalence fails—allocation in FPA is different from allocation in SPA.
- In the example $E\left[R^{FPA}\right] > E\left[R^{SPA}\right]$ but in other examples the opposite ranking holds.
- Some partial results are available:
  - Suppose F_1 is log concave and that F_2 is a truncation of F_1, then $E\left[R^{FPA}\right] > E\left[R^{SPA}\right]$
- Asymmetric uniform distributions can be solved in closed-form.

Mechanisms

- Setup:
  - N risk-neutral buyers
  - values X_i with support $[0, \omega_i]$
  - seller's value 0
- A selling *mechanism* is $(\mathcal{B}, \boldsymbol{\pi}, \boldsymbol{\mu})$
  - $\mathcal{B}_i$ — messages (bids)
  - $\pi_i(\mathbf{b})$ — probability of winning
  - $\mu_i(\mathbf{b})$ — expected payment
- Equilibrium strategy β_i

Direct Mechanisms

- In a *direct mechanism* $(\mathbf{Q}, \mathbf{M})$ each bidder reports a value (possibly false)
  - $Q_i(\mathbf{x})$ — i's probability of winning
  - $M_i(\mathbf{x})$ — i's expected payment
- A direct mechanism is *incentive compatible* (IC) if truthtelling is an eqm.
- Payoffs are

$$U_i(x_i) \equiv E_{\mathbf{X}_{-i}}\left[Q_i(x_i, \mathbf{X}_{-i})\, x_i - M_i(x_i, \mathbf{X}_{-i})\right]$$

The Revelation Principle

Theorem

Given any mechanism and any equilibrium of the mechanism, there exists an IC direct mechanism which is outcome equivalent.

Proof.

$$
\begin{tikzcd}
& \text{Bids} \arrow[dr, "(\boldsymbol{\pi}, \boldsymbol{\mu})"] & \\
\text{Values} \arrow[ur, "\boldsymbol{\beta}"] \arrow[rr, "(\boldsymbol{\pi}, \boldsymbol{\mu}) \circ \boldsymbol{\beta}"'] & & \text{Outcomes}
\end{tikzcd}
$$

Figure: The Revelation Principle

Incentive Compatibility

- Buyer i's payoff from reporting z_i is $q_i(z_i)\, x_i - m_i(z_i)$
- Equilibrium payoffs

$$U_i(x_i) \equiv q_i(x_i)\, x_i - m_i(x_i)$$

- Note that

$$U_i(x_i) = \max_z \{q_i(z)\, x_i - m_i(z)\}$$

  so U_i is convex
- Envelope Theorem implies

$$U_i'(x_i) = q_i(x_i)$$

  and so

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t)\, dt$$

- Convexity implies q_i is nondecreasing.

Incentive Compatibility

(Payoff Equivalence) Payoffs in an IC mechanism are determined by $\mathbf{Q}$ up to an additive constant

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t)\, dt$$

(Revenue Equivalence) Payments in an IC mechanism are determined by $\mathbf{Q}$ up to an additive constant

$$m_i(x_i) = m_i(0) + q_i(x_i)\, x_i - \int_0^{x_i} q_i(t)\, dt$$

Payoff Equivalence

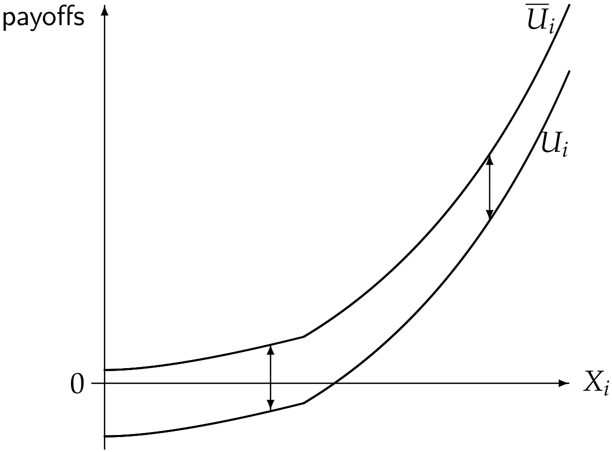

Figure: Payoff Equivalence

Incentive Compatibility

$(\mathbf{Q}, \mathbf{M})$ is *incentive compatible* (IC) if and only if (i) q_i is non-decreasing and (ii)

$$U_i\left(x_i\right) = U_i\left(0\right) + \int_0^{x_i} q_i\left(z\right) dz$$

Incentive Compatibility

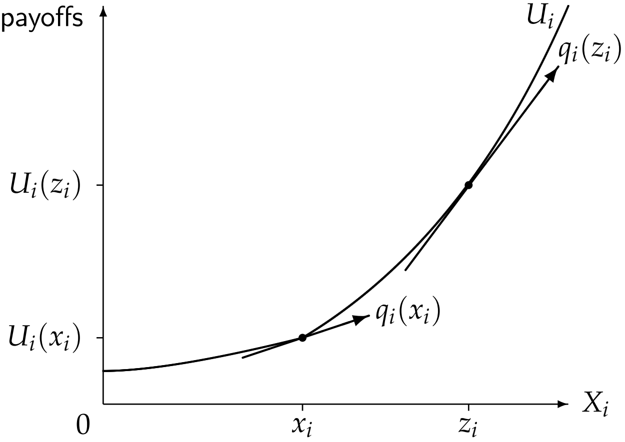

Figure: Implications of Incentive Compatibility

Individual Rationality

- $(\mathbf{Q}, \mathbf{M})$ is *individually rational* (IR) if

  $$U_i(x_i) \geq 0$$

  which is equivalent to $m_i(0) \leq 0$

Seller's Problem

- Choose $(\mathbf{Q}, \mathbf{M})$ to

$$\begin{aligned} & \max \sum_i E\left[m_i\left(X_i\right)\right] \\ & s.t.\ IC, IR \end{aligned}$$

- Revenue equivalence gives $E\left[m_i\left(X_i\right)\right] =$

$$\begin{aligned} & \int_0^{\omega_i} \left[m_i(0) + q_i\left(x_i\right) x_i - \int_0^{x_i} q_i(t)\ dt\right] f_i\left(x_i\right)\ dx_i \\ = \quad & m_i(0) + \int_0^{\omega_i} x_i q_i\left(x_i\right) f_i\left(x_i\right)\ dx_i + \int_0^{\omega_i} q_i(t)\left(1 - F_i(t)\right)\ dt \\ = \quad & m_i(0) + \int_0^{\omega_i} \left(x_i - \frac{1 - F_i\left(x_i\right)}{f_i\left(x_i\right)}\right) q_i\left(x_i\right) f_i\left(x_i\right)\ dx_i \\ = \quad & m_i(0) + \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i\left(x_i\right)}{f_i\left(x_i\right)}\right) Q_i(\mathbf{x}) f(\mathbf{x})\ \mathbf{dx} \end{aligned}$$

Seller's Problem

- Choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i \in \mathcal{N}} m_i(0) + \sum_{i \in \mathcal{N}} \int_{\mathcal{X}} \left(x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \right) Q_i(\mathbf{x}) f(\mathbf{x}) \, \mathbf{dx}$$

  subject to

  - IC: q_i nondecreasing
  - IR: $m_i(0) \leq 0$

Seller's Problem

- Choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left(\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \right) f(\mathbf{x})\, \mathbf{dx}$$

subject to IC and IR, where i's *virtual valuation* is

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$$

Seller's Problem

- Choose $(\mathbf{Q}, \mathbf{M})$ to maximize

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} \left(\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x}) \right) f(\mathbf{x}) \, \mathbf{dx}$$

  subject to IC and IR, where i's *virtual valuation* is

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}$$

- Ignoring IC for now
  - maximize $\sum_{i \in \mathcal{N}} \psi_i(x_i) Q_i(\mathbf{x})$ for every $\mathbf{x}$
  - set $m_i(0) = 0$
  - verify IC

Seller's Problem

- Choose $\mathbf{Q}$ to maximize

$$\sum_i \psi_i(x_i)\, Q_i(\mathbf{x})$$

- The *regular case*: ψ_i is increasing, so

$$Q_i(\mathbf{x}) > 0 \Leftrightarrow \psi_i(x_i) = \max_j \psi_j\left(x_j\right) \geq 0$$

is optimal and IC with a consistent payment rule

$$M_i(\mathbf{x}) = Q_i(\mathbf{x})\, x_i - \int_0^{x_i} Q_i(z, \mathbf{x}_{-i})\, dz$$

Optimal Mechanism

$$
\begin{aligned}
Q_i(\mathbf{x}) &= \begin{cases} 1 & x_i > y_i(\mathbf{x}_{-i}) \\ 0 & x_i < y_i(\mathbf{x}_{-i}) \end{cases} \\
M_i(\mathbf{x}) &= Q_i(\mathbf{x})\, y_i(\mathbf{x}_{-i})
\end{aligned}
$$

Winners pay their lowest winning value

$$
y_i(\mathbf{x}_{-i}) = \inf\left\{ z : \psi_i(z) \geq 0, \forall j \neq i, \psi_i(z) \geq \psi_j(x_j) \right\}
$$

- Inefficient: sometimes not sold, sometimes misallocated

Optimal Mechanism

$$
\begin{aligned}
Q_i(\mathbf{x}) &= \begin{cases} 1 & x_i > y_i(\mathbf{x}_{-i}) \\ 0 & x_i < y_i(\mathbf{x}_{-i}) \end{cases} \\
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$$

- Inefficient: sometimes not sold, sometimes misallocated
- Not anonymous or distribution-free

Optimal Mechanism

$$y_i\left(\mathbf{x}_{-i}\right) = \inf\left\{z : \psi_i\left(z\right) \geq 0, \forall j \neq i, \psi_i\left(z\right) \geq \psi_j\left(x_j\right)\right\}$$

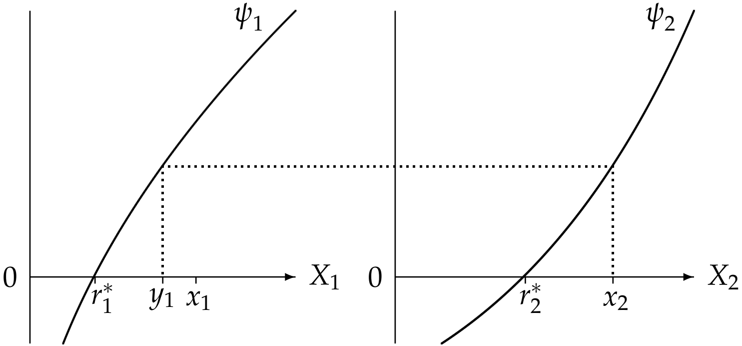

Figure: An Optimal Mechanism

Optimal Mechanism

A second-price auction with $r^* = \psi^{-1}(0)$ or equivalently

$$r^* - \frac{1}{\lambda(r^*)} = 0$$

$$\begin{aligned} y_i(\mathbf{x}_{-i}) &= \inf\left\{z : \psi_i(z) \geq 0, \forall j \neq i, \psi_i(z) \geq \psi_j(x_j)\right\} \\ &= \max\left\{\psi^{-1}(0), \max_{j \neq i} x_j\right\} \end{aligned}$$

Optimal Mechanism

- The bidder with the highest virtual valuation wins

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} = x_i - \frac{1}{\lambda_i(x_i)}$$

- If $\lambda_1 \leq \lambda_2$ and $\operatorname{supp} F_1 = \operatorname{supp} F_2$, then 2 is weaker but

$$\psi_1(x) = x - \frac{1}{\lambda_1(x)} \leq x - \frac{1}{\lambda_2(x)} = \psi_2(x)$$

Optimal Mechanism

- The share of i-bidders willing to buy at price p

$$q_i\left(p\right) = 1 - F_i\left(p\right)$$

  is their quantity demanded

- Revenue

$$p_i\left(q\right) \times q = qF_i^{-1}\left(1-q\right)$$

- Marginal revenue

$$\frac{d}{dq}\left[p_i\left(q\right) \times q\right] = F_i^{-1}\left(1-q\right) - \frac{q}{F_i'\left(F_i^{-1}\left(1-q\right)\right)}$$

Optimal Mechanism

- Marginal revenue from selling to i

$$MR_i(p) = p - \frac{1 - F_i(p)}{f_i(p)} = \psi_i(p)$$

- Marginal opportunity cost of selling to i

$$MC_i = \max\left\{0, \max_{j \neq i} MR_j\right\}$$

- A discriminating monopolist prices where $MR_i(p) = MC_i$

$$y_i(\mathbf{x}_{-i}) = \inf\left\{z : \psi_i(z) \geq 0, \forall j \neq i, \psi_i(z) \geq \psi_j\left(x_j\right)\right\}$$

- Buyer gets *informational rent*

$$E\left[X_i - y_i(\mathbf{X}_{-i})\right]$$

Optimal Mechanism

- Revenue from optimal negotiation

$$E\left[\max\left\{\psi\left(0\right),0\right\}\right]$$

- Revenue from
  - finding a second symmetric bidder
  - holding a second-price auction with no reserve

$$E\left[\max\left\{\psi\left(X_1\right),\psi\left(X_2\right)\right\}\right]$$

The auction gives higher expected revenue

- The auction is "detail-free"
  - universal — any object can be sold
  - anonymous — all bids are treated the same