Auction Theory

Lecture 1

UTMDC

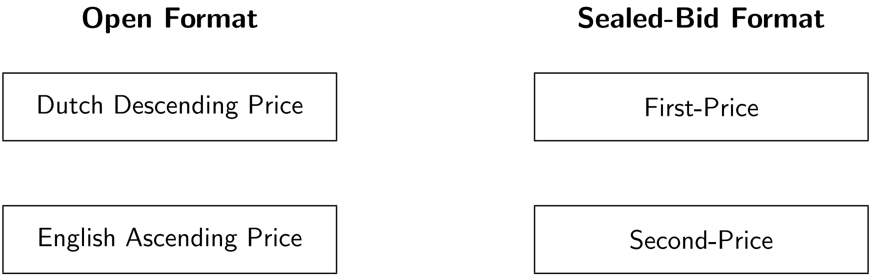


Figure: Open and Sealed-Bid Formats

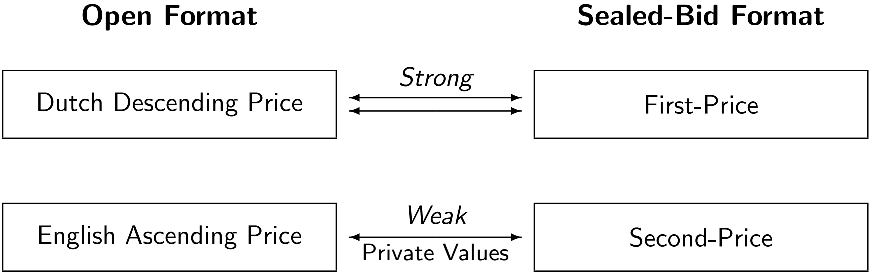


Figure: Equivalence of Open and Sealed-Bid Formats

Symmetric Independent Private Values

- Single indivisible object
- N risk-neutral bidders
- Each bidder i draws value X_i from F independently
- Realized value x_i is privately known
- N and F are commonly known

Basic questions

- Compare different auction formats in terms of
  - seller's revenue
  - efficiency

Bidding in SPA

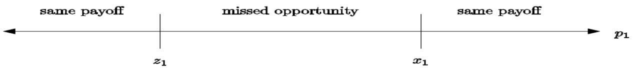
Proposition

It is a weakly dominant strategy to bid your value, i.e.
$\beta^{SPA}(x) = x$

Proof.

Other bids are never better, sometimes worse. □

Bid $z_1 < x_1$



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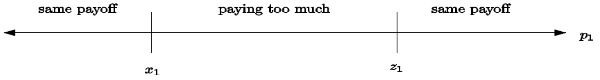

First-Price Auctions–An Example

- $N = 2$, F uniform on $[0, 1]$
- Guess that $\beta(x) = cx$ for some c

$$\max_{b \leq c} \Pr[b \text{ wins}](x_1 - b)$$

$$\max_{b \leq c} \Pr[\beta(X_2) \leq b](x_1 - b)$$

$$\max_{b \leq c} \frac{b}{c}(x_1 - b)$$

$$\beta(x_1) = \frac{x_1}{2}$$

FPA vs. SPA

- Which is better for the seller?
  - If $x_1 = 0.8$ and $x_2 = 0.7$, then $R^{SPA} = 0.7 > 0.4 = R^{FPA}$
  - If $x_1 = 0.8$ and $x_2 = 0.1$, then $R^{SPA} = 0.1 < 0.4 = R^{FPA}$
- *Expected* payments

$$m^{FPA}(x_1) = \Pr[X_2 \leq x_1] \times \beta(x_1) = x_1 \times \frac{1}{2} x_1$$

$$m^{SPA}(x_1) = \Pr[X_2 \leq x_1] \times E[X_2 \mid X_2 \leq x_1] = x_1 \times \frac{1}{2} x_1$$

- Expected revenues are the same!

$$E\left[R^A\right] = 2 \times E\left[m^A(X_1)\right] = 2 \int_0^1 \frac{1}{2} x^2 dx = \frac{1}{3}$$

Second-Price Auctions

- Define $Y_1 \equiv \max_{j \neq i} X_j$

$$\begin{aligned} G(y) &\equiv \Pr[Y_1 \leq y] \\ &= \prod_{j \neq i} \Pr[X_j \leq y] \\ &= F(y)^{N-1} \end{aligned}$$

- Expected payment of a bidder with value x

$$\begin{aligned} m^{SPA}(x) &= \Pr[Y_1 \leq x] \times E[Y_1 \mid Y_1 \leq x] \\ &= \int_0^x y g(y)\, dy \end{aligned}$$

FPA

Proposition

Symmetric equilibrium in the FPA is (also can be derived)

$$\beta(x) = E[Y_1 \mid Y_1 \leq x] = \frac{1}{G(x)} \int_0^x y g(y)\, dy$$

Proof.

Bidding $\beta(z)$ when value is x results in payoff

$$\begin{aligned}\Pi(z,x) &= G(z)[x - \beta(z)] \\ &= G(z)x - \int_0^z y g(y)\, dy \\ &= G(z)(x - z) + \int_0^z G(y)\, dy\end{aligned}$$

$$\Pi(x,x) - \Pi(z,x) = G(z)(z - x) - \int_x^z G(y)\, dy \geq 0$$

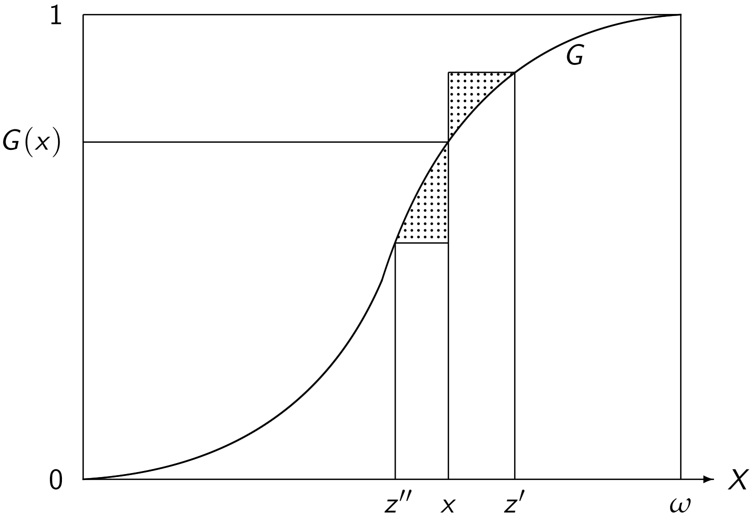

Figure: Losses from Over- and Under-Bidding in a First-Price Auction

First-Price Auctions

$$\beta(x) = \frac{1}{G(x)} \int_0^x yg(y)\ dy$$

- Shading decreases with competition N

$$\beta(x) = x - \int_0^x \frac{G(y)}{G(x)}\, dy = x - \int_0^x \left[\frac{F(y)}{F(x)}\right]^{N-1} dy$$

First-Price Auctions

- Equilibrium strategies

$$\beta(x) = \frac{1}{G(x)} \int_0^x yg(y)\ dy$$

- Expected payments in FPA

$$\begin{aligned} m^{FPA}(x) &= \Pr[Y_1 \leq x] \times \beta(x) \\ &= \int_0^x yg(y)\ dy \end{aligned}$$

First- and Second-Price Auctions

Proposition

With i.i.d. private values, the expected revenues in FPA and SPA are the same. (yet, expected selling price distribution are not the same)

Proposition

With i.i.d. private values, both FPA and SPA are efficient.

Reserve Prices

- Suppose seller sets a reserve price $r > 0$ in an SPA
- Expected payment of a bidder with value $x \geq r$

$$m^{SPA}(x, r) = rG(r) + \int_r^x yg(y)\, dy$$

Reserve Prices

- Ex ante expected payments

$$
\begin{aligned}
& E\left[m^{SPA}(x,r)\right] \\
&= \int_r^1 \left[rG(r) + \int_r^x yg(y)\ dy\right] f(x)\,dx \\
&= r(1-F(r))\,G(r) + \int_r^1 \int_r^x yg(y)\ dyf(x)\ dx \\
&= r(1-F(r))\,G(r) + \int_r^1 y(1-F(y))\,g(y)\ dy
\end{aligned}
$$

- Expected revenue

$$
N \times E\left[m^{SPA}(x,r)\right]
$$

Reserve Prices

- Seller's expected payoff Π from $r \geq 0$

$$\Pi = Nr\left(1 - F\left(r\right)\right)G\left(r\right) + N\int_r^{\omega} y\left(1 - F\left(y\right)\right)g\left(y\right)\,dy$$

-

$$\begin{aligned}\frac{1}{N}\frac{\partial \Pi}{\partial r} &= \left[1 - F\left(r\right) - rf\left(r\right)\right]G\left(r\right)\\ &= \left[1 - r\lambda\left(r\right)\right]\left(1 - F\left(r\right)\right)G\left(r\right)\end{aligned}$$

- When λ is increasing,

$$r^* - \frac{1}{\lambda\left(r^*\right)} = 0$$

$$r^* - \frac{1 - F\left(r^*\right)}{f\left(r^*\right)} = 0$$

Reserve Prices

Proposition

With i.i.d. private values, optimal $r^* > 0$*. The optimal reserve price* r^* *is independent of the number of bidders.*

- Exclusion principle
- Revenue Equivalance Principle

Risk Aversion

- Symmetric model as before
- But now bidders have a common vN-M utility function u
  - increasing, strictly concave with $u(0) = 0$
- How does risk aversion affect bidding?
  - No effect in SPA—dominant strategy to bid your value

Risk Aversion and FPA

- If bid $\gamma(z)$ with value x, expected utility is

$$\max_z G(z)u(x - \gamma(z))$$

First-order condition is

$$g(z) \times u(x - \gamma(z)) - G(z) \times \gamma'(z) \times u'(x - \gamma(z)) = 0.$$

In a symmetric equilibrium, $z = x$ and so

$$\gamma'(x) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \times \frac{g(x)}{G(x)}$$

- With risk neutrality, $u(x) = x$ and so

$$\beta'(x) = (x - \beta(x)) \times \frac{g(x)}{G(x)}$$

- If u is strictly concave and $u(0) = 0$, for all $y > 0$, $[u(y)/u'(y)] > y$. Thus,

$$\gamma'(x) = \frac{u(x - \gamma(x))}{u'(x - \gamma(x))} \times \frac{g(x)}{G(x)} > (x - \gamma(x)) \times \frac{g(x)}{G(x)}$$

- Recall

$$\beta'(x) = (x - \beta(x)) \times \frac{g(x)}{G(x)}$$

- Now $\gamma(0) = 0 = \beta(0)$ and if $\gamma(x) \leq \beta(x)$, then

$$\gamma'(x) > \beta'(x)$$

- So $\gamma(x) > \beta(x)$.

Risk Aversion

Proposition

In the symmetric IPV model, risk aversion leads to higher bids in FPA.

Corollary

In the symmetric IPV model, under risk aversion, exp. revenue in FPA exceeds that in SPA.