Lecture 2: Dynamic Auctions for Multiple Complements

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Venue: University of Tokyo

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Problem and Objective

In our previous lecture, we studied an auction market with *gross substitutes and complements.* In this lecture, we consider a related but quite different market where all items are complements or synergies. Synergies are the major sources of revenues for many firms. Examples include:
Different segments of transportation/telcommunication/pipeline networks;
Different volumes of a book;
Different parts of a machine or equipment; etc.

Our aim is to : clarify the scope of the problem concerned with complements; explore an appropriate solution; and develop an efficient and incentive compatible dynamic auction.

- Maskin (2005): *"Recent contributions to mechanism design: a highly selective review"* lists several major open questions
- The first open question is:
  How to design a dynamic auction for selling multiple complements?
- As a consequence, the proposed auction resolves the open problem.

Some Remarks

In theory and also in practice, anonymous pricing rules (i.e., the price of any bundle of goods is the same for all agents) are widely used and studied.

It should be noted that package auctions (see e.g., Ausubel and Milgrom 2002 and Porter, Rassenti, Roopnarine and Smith 2003) use the discriminatory pricing rule and can apply to more general environments. As argued by Milgrom (2004), discriminatory pricing fails to promote the law of one price and thus may be psychologically hard for some people to accept. Also in some countries, discriminatory prices are even illegal. In real-life auctions, people are more accustomed to anonymous prices.

Discriminatory and nonlinear pricing is common and natural for situations in which a seller is independently and separately negotiating with several buyers.

The Model

- An auctioneer (i.e. seller/she) wishes to sell a set $N = \{1, 2, \cdots, n\}$ of indivisible goods/items to a group $M = \{1, 2, \cdots, m\}$ of buyers/he.
- Let 0 represent the seller and $M(0) = M \cup \{0\}$ the set of all the agents.
- Each agent $h \in M(0)$ knows her own utility function of goods privately $u^h : 2^N \to R$ with $u^h(\emptyset) = 0$.

Assumptions

(A1) Integer private values: $u^h : 2^N \to Z_+$ for every agent $h \in M(0)$;

(A2) Quasilinear utility in money: net profit $= u^h(S) - p(S)$ when bidder h receives bundle S and pays $p(S)$ for every bidder;

(A3) Superadditivity for bidders: $\forall A, B \in 2^N$ with $A \cap B = \emptyset$;

$$u^h(A \cup B) \geq u^h(A) + u^h(B).$$

(the most general form of complementarity)

Special Cases

(1) A utility function $u^h : 2^N \to R_+$ is *supermodular* if $u^h(A \cup B) + u^h(A \cap B) \geq u^h(A) + u^h(B)$ for all $A, B \in 2^N$.

(2) Gross complements: Increasing the price of one good will decrease bidder h's demand for other goods.

A utility function u is *submodular* if $-u$ is supermodular.

Efficient Allocations

- An allocation of goods is a partition $\pi = (\pi(h) \mid h \in M(0))$ of goods such that $\pi(i) \cap \pi(j) = \emptyset$ for all $i \neq j$ and $\cup_{h \in M(0)} \pi(h) = N$.

- Allocation π assigns bundle $\pi(h)$ to agent h for $h \in M(0)$. Let $\mathcal{A}$ be the family of all allocations.

- An allocation π is efficient if for every allocation $\rho \in \mathcal{A}$,

$$\sum_{h \in M(0)} u^h(\pi(h)) \geq \sum_{h \in M(0)} u^h(\rho(h)).$$

- Does there exist a standard Walrasian equilibrium?

Non-existence of Standard WE

Example 1: Three bidders and three complementary items

Bidders' valuations

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	2	2	0	7	3	4	7
Bidder 2	0	2	0	2	3	6	3	6
Bidder 3	0	0	2	2	4	3	6	7
Seller	0	1	1	1	2	2	2	3

Two efficient allocations $(AB, C, \emptyset)$, $(AB, \emptyset, C)$. None of these allocations can be supported by a price vector $(p(A), p(B), p(C))$.

Example 1

Consider the efficient allocation $\pi = (AB, C, \emptyset)$.

Suppose that $(p(A), p(B), p(C))$ supports allocation π.

Then for bidder 1, it holds $7 \geq p(A) + p(B)$.

For bidder 2, we have $2 \geq p(C)$ and
$2 - p(C) \geq 6 - p(A) - p(C)$. So $p(A) \geq 4$.

For bidder 3, we have $0 \geq 2 - p(C)$ and
$0 \geq 6 - p(B) - p(C)$. Note that $p(C) = 2$. So $p(B) \geq 4$.

Combining $p(A) \geq 4$ and $p(B) \geq 4$ yields $p(A) + p(B) \geq 8$,
contradicting $7 \geq p(A) + p(B)$.

Anonymous and Nonlinear Pricing Rule

- An anonymous and nonlinear pricing system is a price system that assigns a price for each bundle of goods and asks every bidder to pay the same price for the same bundle of goods. Namely, it is a pricing function $p : 2^N \to R_+$ with $p(\emptyset) = 0$ and $p(S) = u^0(S)$ for the set S of all unsold items under prices p.

- A pricing function p is linear if $p(A) = \sum_{k \in A} p(\{k\})$ for all $A \subset N$.

- The auction discussed here uses an anonymous and nonlinear pricing system.

Anonymous and Nonlinear Pricing Walrasian Equilibria

- Define every bidder h's demand set:

$$D^h(p) = \arg\max_{S \subseteq N}\{u^h(S) - p(S)\}$$

- Let $K = \{1, \cdots, \max\{n, m\}\}$ and $L = \{0\} \cup K$, $\pi = (\pi(j) \mid j \in L)$: a partition of all goods among members in L. Let $\mathcal{B}$ be the collection of all such partitions.
- The seller tries to maximize her revenues and her supply family:

$$S(p) = \arg\max_{\pi \in \mathcal{B}}\{u^0(\pi(0)) + \sum_{j \in K} p(\pi(j))\}.$$

  A supply set $\pi \in S(p)$ is an allocation if $\cup_{j \in L}\pi(j) = \cup_{h \in M(0)}\pi(h)$.

Definition 1: *An anonymous and nonlinear pricing Walrasian equilibrium* (anpWE) consists of a price function $p^* : 2^N \to R_+$ and an allocation π^* such that $\pi^* \in S(p^*)$ and $\pi^*(h) \in D^h(p^*)$ for every bidder $h \in M$.

A Welfare Proposition

Observe that in any anpWE, for any sold bundle B, we always have $p(S \cup T) \geq p(S) + p(T)$ for any two disjoint sets S and T of the bundle B, because the seller chooses a supply set that maximizes her revenues.

Proposition 1 (Sun and Yang 2014): If (p^*, π^*) be an anonymous and nonlinear pricing Walrasian equilibrium, π^* is efficient. Furthermore, if p^* is an equilibrium pricing function and ρ is an efficient allocation, (p^*, ρ) is also an anonymous and nonlinear pricing Walrasian equilibrium.

The Scope of the anpWE

Can the concept of anpWE be too general to be interesting? The following example dispels such concern.

Example 1 (Bevia, Quinzii, and Silva 1999): The following market has no standard WE, although the function of every bidder is submodular and the seller valuates every bundle at zero. $(\pi(0), \pi(1), \pi(2), \pi(3)) = (\emptyset, B, A, C)$ is the unique efficient allocation. Assume that there are anpWe prices to support it, saying $p(AB) > p(A) + p(B)$. Then the seller will choose $\{AB, C\}$ instead of π, i.e., $\pi \notin S(p)$, thus no anpWE.

Bidders' valuations

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	10	8	2	13	11	9	14
Bidder 2	0	8	5	10	13	14	13	15
Bidder 3	0	1	1	8	2	9	9	10

An Equilibrium Existence Theorem

Theorem 1 (Sun and Yang 2014): The market under Assumptions A1, A2, and A3 has an anonymous and nonlinear pricing Walrasian equilibrium.

An Efficient Dynamic Auction

We say that bidder h *bids sincerely* with respect to value function u^h if at every time $t \in Z_+$ and any price function $p(t)$ at time t, he reports a bid
$B_h(t) \in D^h(p(t)) = \arg\max_{S \subseteq N}\{u^h(S) - p(t,S)\}$ with
$B_h(t) = \emptyset$ when $\emptyset \in D^h(p(t))$. When $p(t)$ is a pricing function at time $t \in Z_+$, then $p(t,S)$ denotes the price of bundle $S \in 2^N$.

Given a pricing function $p : 2^N \to R_+$, the seller reports a supply set $\pi \in S(p)$ and each bidder $h \in M$ reports a demand bundle $B_h \in D^h(p)$.

W.r.t. the sets π and B_h, a bundle $B \in 2^N \setminus \{\emptyset\}$ is *over-demanded* if it is demanded by more than one bidder (i.e., $B_g = B_h = B$ for at least two bidders g, h) or demanded by some bidder h (i.e., $B_h = B$) but her bundle B_h is not in the seller's supply set π (i.e., $B_h \notin \pi$).

The Basic Dynamic Auction

- **Step 1:** The auctioneer announces an initial pricing function $p(0) : 2^N \to Z_+$ so that $p(0, S)$ equals the reserve price $u^0(S)$ for every bundle $S \subseteq N$. Set $t := 0$ and go to Step 2.

- **Step 2:** At each round t, every bidder h reports a demand bundle $B_h(t) \in D^h(p(t))$ at $p(t)$ (choose $B_h(t) = \emptyset$ when $\emptyset \in D^h(p(t))$). Then the auctioneer chooses a supply set $\pi(t) \in S(p(t))$ so that the market yields the least over-demanded bundles. If no bundle is over-demanded, go to Step 3. But if there is an over-demanded bundle, raises the price of each over-demanded bundle by one unit. Set $t := t + 1$ and return to Step 2.

The Basic Dynamic Auction

- **Step 3:** The auctioneer assigns the bundle $B_h(t)$ to bidder h who is asked to pay the price $p(t, B_h(t))$ in return, and in addition for any nonempty bundle $S \in \pi(t)$ which is not demanded by any bidder at $p(t)$, the auctioneer assigns the bundle to the seller if $p(t, S) = u^0(S)$, otherwise, the auctioneer assigns the bundle to some bidder who previously demanded the bundle but was the last to give up, and who is asked to pay $p(t, S)$. Then the auction stops.

Note: In Step 3, the auctioneer assigns $B_h(t) = B$ to bidder h and then may assign another bundle S to him, and asks him to pay the sum of current prices $p(t, B)$ and $p(t, S)$. This operation is called *the complementary activity rule*, which is a novel and important feature of this auction.

Revisiting Example 1

Example 1: Three bidders and three complementary items

Bidders' valuations

	$\emptyset$	A	B	C	AB	AC	BC	ABC
Bidder 1	0	2	2	0	7	3	4	7
Bidder 2	0	2	0	2	3	6	3	6
Bidder 3	0	0	2	2	4	3	6	7
Seller	0	1	1	1	2	2	2	3

Two efficient allocations $(AB, C, \emptyset)$, $(AB, \emptyset, C)$.

Table: Illustration of the basic dynamic auction for Example 1.

Price Vector	Seller	Bidder 1	Bidder 2	Bidder 3
$p(0) = (1, 1, 1, 2, 2, 2, 3)$	$\{AB, C\}$	AB	AC	BC
$p(1) = (1, 1, 1, 2, 3, 3, 3)$	$\{AC, B\}$	AB	AC	ABC
$p(2) = (1, 1, 1, 3, 3, 3, 4)$	$\{AC, B\}$	AB	AC	ABC
$p(3) = (1, 1, 1, 4, 3, 3, 5)$	$\{AB, C\}$	AB	AC	BC
$p(4) = (1, 1, 1, 4, 4, 4, 5)$	$\{AB, C\}$	AB	AC	BC
$p(5) = (1, 1, 1, 4, 5, 5, 5)$	$\{AC, B\}$	AB	C	ABC
$p(6) = (1, 1, 2, 5, 5, 5, 6)$	$\{AB, C\}$	AB	A	ABC
$p(7) = (2, 1, 2, 5, 5, 5, 7)$	$\{AB, C\}$	AB	AC	BC
$p(8) = (2, 1, 2, 5, 6, 6, 7)$	$\{A, BC\}$	AB	$\emptyset$	$\emptyset$
$p(9) = (2, 2, 2, 6, 6, 6, 7)$	$\{AB, C\}$	AB	$\emptyset$	$\emptyset$

where the price vector is

$$p = (p(A), p(B), p(C), p(AB), p(AC), p(BC), p(ABC))$$

with $p(\emptyset) = 0$, seller's column indicates her supplies, whereas each bidder's column indicates its demands.

Bidder 1 gets AB by paying 6, bidder 2 gets C by paying 2, and bidder 3 gets nothing. Let $\pi^* = (AB, C, \emptyset)$ and $p^* = p^9$. Then (p^*, π^*) is a nonlinear pricing WE. Note that bidder 2 gave up C in Step 6.

Convergence Theorem

Theorem 2 (Sun and Yang 2014):
Suppose that Assumptions $(A1)$–$(A3)$ hold for the auction model. If all bidders bid sincerely, the basic dynamic auction yields an anonymous and nonlinear pricing Walrasian equilibrium, in a finite number of rounds.

Proof of Theorem 2: Part 1

The auction must stop at some step t^*, because the price of every bundle is weakly increasing, the value of every bundle is finite, and prices cannot increase forever.

Let $p^* = p(t^*)$ and let $B_h^* = B_h(t^*)$ that is demanded by bidder h, and let $\gamma^* = \gamma(t^*) \in S(p^*)$ that is the supply set of the seller. Recall that by definition γ^* is a partition of all the items N that maximizes the seller's revenues. We will construct an allocation π^* so that (p^*, π^*) constitutes an anpWE.

At p^*, no (nonempty) bundle is over-demanded. Thus, for any bidder $h \in M$, if his demand bundle B_h^* is not empty, it must be in the supply set γ^*. If $\cup_{h \in M} B_h^* = N$, let $\pi^*(h) = B_h^*$ for all $h \in M$ and $\pi^*(0) = \emptyset$, then clearly (p^*, π^*) is an anpWE and we are done.

Proof of Theorem 2: Part 2

Assume there is some bundle $B \in \gamma^*$ which is not demanded by any bidder at t^*. We call such a bundle a *squeezed out bundle*. First, consider $p^*(B) = u^0(B)$. Let $\gamma_0^* = \{B \in \gamma^* \mid p^*(B) = u^0(B) \text{ and } B \neq B_h^* \text{ for all } h \in M\}$ be the collection of all such bundles. Let $\pi^*(0) = \cup_{B \in \gamma_0^*} B$. We can assign $\pi^*(0)$ to the seller.

By superadditivity, we know that $p^*(\pi^*(0)) = u^0(\pi^*(0)) \geq \sum_{B \in \gamma_0^*} u^0(B) = \sum_{B \in \gamma_0^*} p^*(B)$. Note that $p^*(\pi^*(0)) \leq \sum_{B \in \gamma_0^*} p^*(B)$ because $\gamma^* \in S(p^*)$. Hence, we have

$$p^*(\pi^*(0)) = u^0(\pi^*(0)) = \sum_{B \in \gamma_0^*} p^*(B) = \sum_{B \in \gamma_0^*} u^0(B). \qquad (1)$$

Proof of Theorem 2: Part 3

Next, consider $p^*(B) > u^0(B)$. Then let l be the bidder who demanded B and was the last one to give up B. Let t be the step in which bidder l still demanded B but gave up B in the next step. Clearly, $t < t^*$. By the complementarity auction rule, we can assign B to bidder l and ask him to pay the current price $p^*(B)$.

Then we must have $u^l(B) - p(t, B) \geq 1$, and $p^*(B) = p(t, B)$ or $p^*(B) = p(t, B) + 1$. For bidder l, it holds that

$$u^l(B) - p^*(B) \geq 0. \tag{2}$$

Proof of Theorem 2: Part 4

Case 1. When $B_l^* = \emptyset$, let $\pi^*(l) = B$. Because $B_l^* \in D^l(p^*)$ and $B_l^* = \emptyset$, we have $0 \geq u^l(B) - p^*(B)$. It follows from (2) that $u^l(B) - p^*(B) \geq 0$. These inequalities lead to $u^l(B) - p^*(B) = 0$, which implies $\pi^*(l) \in D^l(p^*)$.

Case 2. When $B_l^* \neq \emptyset$, let $\pi^*(l) = B_l^* \cup B$. For the seller, we know that

$$p^*(B_l^*) + p^*(B) \geq p^*(\pi^*(l)). \tag{3}$$

For bidder l, superadditivity implies that

$$u^l(\pi^*(l)) \geq u^l(B_l^*) + u^l(B). \tag{4}$$

Proof of Theorem 2: Part 5

(3) and (4) imply that

$$\begin{array}{rcl} u^I(\pi^*(I)) - p^*(\pi^*(I)) & \geq & u^I(\pi^*(I)) - (p^*(B_I^*) + p^*(B)) \\ & \geq & \Big(u^I(B_I^*) - p^*(B_I^*)\Big) + \Big(u^I(B) - p^*(B)\Big) \\ & \geq & u^I(B_I^*) - p^*(B_I^*) \end{array}$$

where the last inequality is derived from (2). Because $B_I^* \in D^I(p^*)$, we have $\pi^*(I) \in D^I(p^*)$.

Consequently, we have $u^I(\pi^*(I)) - p^*(\pi^*(I)) = u^I(\pi^*(I)) - (p^*(B_I^*) + p^*(B)) = u^I(B_I^*) - p^*(B_I^*)$, yielding

$$p^*(\pi^*(I)) = p^*(B_I^*) + p^*(B). \tag{5}$$

Proof of Theorem 2: Part 6

We can repeat this adjustment until every such squeezed out bundle B (i.e., $p^*(B) > u^0(B)$) in γ^* is assigned to some bidder. For any bidder h who is not assigned with any squeezed out bundle, let $\pi^*(h) = B^*_h$. So in the end each bidder h gets a bundle $\pi^*(h)$ in his demand set. Because γ^* is a seller's partition of N, $(\pi^*(0), \cdots, \pi^*(m))$ must be an allocation of N.

It follows from the formulas (1) and (5) that $\sum_{h \in M_0} p^*(\pi^*(h)) = \sum_{A \in \gamma^*} p^*(A)$. That is, the allocation $\pi^* \in S(p^*)$. Consequently, (p^*, π^*) is an anpWE. $\square$

The Dynamic Incentive Compatible (IC) Mechanism

This dynamic IC mechanism is based on the basic dynamic auction. Here we give a simplified version of the dynamic IC mechanism which has omitted an important component of bid withdrawals. See Sun and Yang (2014) for a complete version and detailed discussion.

- Basic notation:

  $\mathcal{M}$ denotes the original market;

  $\mathcal{M}_{-i}$ denotes the market $\mathcal{M}$ without bidder i, $(i \in M)$;

  $M_{-i} = M \setminus \{i\},\ M_{-0} = M,\ M(0) = M \cup \{0\},\ \mathcal{M}_{-0} = \mathcal{M}.$

- Every bidder i knows his own super-additive utility function u^i and can be strategic. The seller has her own super-additive utility function u^0 and acts honestly.

Price functions used in the auction mechanism

At each round t,

- $p^0(t)$: the open price function announced by the auctioneer;
- $p^i(t)$: the price function faced by bidder i;
- $p^{-l}(t)$: the price function faced by the seller in every market $\mathcal{M}_{-l}$,

Basic Idea: In each round t, the auctioneer announces $p^0(t)$ and informs every bidder i of $p^i(t)$. Every bidder i reports a bundle $A^i(t) \in 2^N$. Then the auctioneer adjusts price functions $p^k(t)$, $p^{-l}(t)$, $k, l \in M(0)$. When all markets are clear, every bidder i receives equili. bundle $\pi^{-0}(i)$ in $\mathcal{M}_{-0}$ and pays the difference q_i between the total equili. payments of his opponents in market $\mathcal{M}_{-i}$ and those in market $\mathcal{M}_{-0}$, i.e.,

$$\begin{array}{rcl} q_i & = & u^0(\pi^{-i}(0)) + \sum_{l \in M_{-i}} p^l(t^*, \pi^{-i}(l)) \\ & & -u^0(\pi^{-0}(0)) - \sum_{l \in M_{-i}} p^l(t^*, \pi^{-0}(l)) \end{array}$$

If $p^i(t+1, S) > U^*$ (a high price!) for some i and bundle S, the auctioneer assigns the whole bundle N to bidder i and asks him to pay U^*. Other bidders get nothing and pay nothing.

- A bundle S is *first-price-over-demanded by a bidder i* if bidder i is the unique bidder such that $A_i(t) = S$, $p^i(t, S) = p^{-0}(t, S)$, and $S \notin \pi^{-l}(t)$ for some $l \in M(0) \setminus \{i\}$.

- A bundle S is *second-price-over-demanded* if it is demanded by at least one bidder when $p^0(t, S) < p^{-0}(t, S)$, or by more than one bidder when $p^0(t, S) = p^{-0}(t, S)$.

- A bundle S is *over-demanded* whenever it is either first-price or second-price-over demanded.

The auctioneer adjusts the prices as follows:

For every bidder i, every bundle S, and every market $\mathcal{M}_{-j}$, let

$$q^0(t+1, S) = \begin{cases} p^0(t, S) + 1, & \text{if } S \text{ is 2nd-price-over-demanded,} \\ p^0(t, S), & \text{otherwise;} \end{cases}$$

$$q^i(t+1, S) = \begin{cases} p^i(t, S) + 1, \text{ if } S \text{ is 1st-price-over-demanded by bidder } i, \\ \max\{p^i(t, S), p^0(t+1, S)\}, & \text{otherwise;} \end{cases}$$

$$p^{-j}(t+1) = \bigvee_{l \in M_{-j}} p^l(t+1).$$

The IC dynamic auction mechanism

- **Step 1:** The auctioneer announces an initial pricing function $p^0(0) : 2^N \to Z_+$ so that $p^0(0, S)$ equals the reserve price $u^0(S)$ for every bundle $S \subseteq N$. Set $p^i(0) = p^{-i}(0) = p^0(0)$ for every $i \in M(0)$. Set $t := 0$ and go to Step 2.

- **Step 2:** At each round t, the auctioneer announces the price function $p^0(t)$ and informs every bidder i of his price function $p^i(t)$. Then, every bidder i, based on his own bidding history and the observed information, reports his bid $A_i(t) \in 2^N$. For every market $\mathcal{M}_{-j}$, the auctioneer chooses a supply set $\pi^{-j}(t) \in S(p^{-j}(t))$ so that the market $\mathcal{M}_{-j}$ has the least over-demanded bundles. If there is no over-demanded bundle in any $\mathcal{M}_{-j}$, go to Step 3. Otherwise, the auctioneer adjusts the price functions $p^0(t+1), p^i(t+1)$ and $p^{-j}(t+1)$ for all $i \in M$ and $j \in M(0)$. If $p^i(t+1, S) > U^*$ for some bidder i and some bundle S, then go to Step 4. Otherwise, set $t := t+1$ and return to Step 2.

- **Step 3:** At the last round $t = t^*$, for every $j \in M(0)$ the auctioneer chooses an allocation π^{-j} for the market $\mathcal{M}_{-j}$ as the basic dynamic auction does. Finally, according to the allocation π^{-0} of the original market $\mathcal{M}$, the auctioneer assigns $\pi^{-0}(0)$ to the seller and $\pi^{-0}(i)$ to bidder i who is asked to pay the price

$$\begin{aligned} q_i &= u^0(\pi^{-i}(0)) + \textstyle\sum_{l \in M_{-i}} p^l(t^*, \pi^{-i}(l)) \\ &\quad - u^0(\pi^{-0}(0)) - \textstyle\sum_{l \in M_{-i}} p^l(t^*, \pi^{-0}(l)) \end{aligned}$$

  Then the auction stops.
- **Step 4:** The auctioneer assigns the whole bundle N to a bidder i with $p^i(t+1, S) > U^*$ for some bundle S and asks him to pay the price U^*. And all other bidders get nothing and pay nothing. The auction stops.

The Sealed-Bid VCG mechanism

- The (Sealed-Bid) VCG Mechanism

Every bidder $h \in M$ reports u^h. The seller computes an efficient allocation π and assigns $\pi(h)$ to bidder h who pays $u^h(\pi(h)) - SV(M) + SV(M_{-h})$, where $SV(M) = \sum_{h \in M(0)} u^h(\pi(h))$ and $SV(M_{-h}) = \sum_{h \in M(0)\setminus\{h\}} u^h(\rho(h))$ with ρ be an efficient allocation of items in the market without bidder h.

Bidder h's payoff: $SV(M) - SV(M_{-h})$ (marginal contribution)

Information and Strategies

Prior to the start of the auction, nature according to a joint probability distribution function $F(\cdot)$ draws a profile $\{u^i\}_{i\in M}$ with $u^i \in \mathcal{U}$ for all $i \in M$, and reveals to every player $i \in M$ only his own value function u^i of private information. Let $H_i(t)$ be the part of the information (or history) of play that player i has observed just before he takes action at time $t \in Z_+$. A natural and sensible specification can be that $H_i(t)$ comprises all observable price functions and his own past actions.

A (dynamic) *strategy* σ_i of player i is a function $\{(t, H_i(t), u^i) \mid t \in Z_+\} \to 2^N$, which tells him to submit a a bid $\sigma_i(t, H_i(t), u^i) = A_i(t) \in 2^N$ at each time $t \in Z_+$ when he observes $H_i(t)$.

Ex Post Perfect Nash Equilibrium

Ex Post Perfect Nash Equilibrium: An m-tuple $\{\sigma_i\}_{i \in M}$ of strategies is an *ex post perfect Nash equilibrium* if for any time $t \in Z_+$, following any history profile $\{H_i(t)\}_{i \in M}$, and for any realization $\{u^i\}_{i \in M}$ of profile of value functions of private information, the continuation strategy $\sigma_i(\cdot \mid t, H_i(t), u^i)$ of every player $i \in M$ constitutes his best response against the continuation strategies $\{\sigma_l(\cdot \mid t, H_l(t), u^l)\}_{l \in M_{-i}}$ of player i's opponents of the game even if the realization $\{u^i\}_{i \in M}$ becomes common knowledge.

In other words, this notion of equilibrium is not only robust against any regret but also independent of any probability distribution. It requires that the equilibrium strategy for every player should remain optimal at every node of the auction game even if the player were to learn his opponents' private values.

A Major Theorem

Theorem 3 (Sun and Yang 2014): Suppose that the market $\mathcal{M}$ satisfies Assumptions (A1)–(A3).

(1) When every bidder bids sincerely, the IC dynamic auction yields a VCG outcome for the market $\mathcal{M}$ in a finite number of rounds. Moreover, the VCG payment for every bidder is no less than the seller's reserve price for the bundle that the bidder receives.

(2) Sincere bidding is an ex post perfect Nash equilibrium in the auction game.