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Abstract

We study a variation of the price competition model a la Bertrand, in which firms must offer menus of contracts that obey monotonicity constraints, e.g., wages that rise with worker productivity to comport with equal pay legislation. While such constraints limit firms' ability to undercut their competitors, we show that Bertrand's classic result still holds: competition drives firm profits to zero and leads to efficient allocations without rationing. Our findings suggest that Bertrand's logic extends to a broader variety of markets, including labor and product markets that are subject to real-world constraints on pricing across workers and products.

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1 Introduction

Bertrand's model of simultaneous price setting is a workhorse in pinning down equilibrium outcomes in markets with competing firms. When firms sell homogeneous goods and experience constant marginal cost of production, Bertrand competition between as few as two firms drives prices to marginal cost and profit to zero. Papers in the industrial organization literature (e.g. Einav et al., 2021; Thomadsen, 2005) nest Bertrand competition to determine prices (at marginal cost), papers in the labor literature (e.g. Cahuc et al., 2006; Postel-Vinay and Robin, 2002) nest Bertrand competition to determine wages (at marginal productivity), and papers in the education literature (e.g. Nei and Pakzad-Hurson, 2021) nest Bertrand competition to determine financial aid packages (at marginal university utility). The Bertrand model is so relied upon that its conclusions are often assumed as a primitive; Rothschild and Stiglitz (1976), Costrell and Loury (2004), and Azevedo and Gottlieb (2017) abstract away from the competitive process and begin with the premise that the presence of—or even the potential entry of—a competitor leads to zero profits.

In this paper, we study whether the conclusions of Bertrand's model hold in settings where firms compete over menus of products, wherein firms face a monotonicity constraint in pricing across products. Many competitive situations feature heterogeneous products–producers sell goods of varying quality, employers hire workers of different productivity, and universities provide aid to students of disparate ability. If firms were able to set prices independently across products on the menu, the standard Bertrand argument would imply that competition drives profit to zero for each product. However, under monotonic pricing, in which more productive workers, higher quality and cost goods, and higher ability students are associated with higher prices, a "product-by-product" Bertrand argument does not hold; a firm may be unable to change the price of one product in isolation depending on prices of others. This monotonicity constraint is realistic to capture important considerations across markets:

- If sellers cannot observe the types of different consumers, "satisficing" utility functions of the consumers require monotonicity in pricing to prevent adverse selection. Specifically, (the doctor of) a patient who seeks a remedy to a medical ailment will optimally select the cheapest intervention that corrects the ailment even if that intervention is costlier to provide for a hospital, and a firm seeking manufacturing equipment for a particular task will optimally select the cheapest machine capable of completing the task even if it is costlier to produce for a supplier.

- If agents can shirk, monotonicity in pricing is needed to prevent moral hazard. For example, although it may be difficult or impossible for a worker to convince an employer that she has a higher productivity than her "true" value by speaking a foreign language she does not know, she could hide her productivity by not speaking a foreign language she knows. Similarly, a student whose financial aid package is non-monotonic in her grades may intentionally reduce effort to secure more funding.[1] Our monotonicity assumption within a firm ensures robustness against an agent who considers reducing productivity.

- Legal restrictions may directly impose monotone pricing. Cowgill and Pakzad-Hurson (2025) argue that equal pay laws require wages within a firm to be monotone non-decreasing in productivity; otherwise, a firm would be in violation of the principle of "equal pay for equal (or better) work," a common legal standard.

[1] Shirking of this form is considered in the context of matching markets in Balinski and Sönmez (1999) and Sönmez (2013).

Formally, we study a model with two homogeneous firms that simultaneously announce menus of prices and quantities over a continuum of products, subject to a monotonicity constraint that each firm sets a weakly higher price for "more valuable" products. To fix ideas, we henceforth describe our model in the language of constant-returns-to-scale firms that seek to hire workers with different productivities. We allow for endogenous contracting (i.e. firms can elect not to hire workers of particular productivity levels) and rationing (i.e. firms can hire a strict subset of workers available at any productivity level).

Our main result finds that the set of equilibrium allocations corresponds exactly to the set of Bertrand allocations: the allocation is efficient (almost all workers are hired) and the wage of almost every worker equals her productivity. Efficiency implies both that firms collectively hire workers of all productivity levels and that all workers of each productivity level are hired, that is, firms do not collectively restrict the set of contracts offered, nor do they ration hiring. Our result suggests that a Bertrand allocation obtains in a broader variety of markets, and in particular, that multi-product firms price goods at marginal cost/price under competition, even with relevant constraints on pricing across products.

We prove our result via a comparison to a cooperative, matching-with-wages game in which firms can hire, fire, and poach workers, but cannot unilaterally lower wages of matched workers. We first show that the set of core allocations of the cooperative game corresponds to the set of Bertrand allocations, and second that the set of core allocations corresponds to the set of equilibrium allocations of our original, non-cooperative game. Therefore, our result suggests that a Bertrand allocation obtains "in the long run," even without the structure of and timing enforced by our non-cooperative game.

Our result contrasts with findings on Bertrand competition with endogenous contracting in the literature. Rothschild and Stiglitz (1976) show no contracting occurs for certain parameter values, while Azevedo and Gottlieb (2017) show, in general, not every contract is offered by firms in equilibrium. These distinctions are driven by adverse selection. Our monotonicity constraint guards against selection "downward" by workers, whereas the analogous conditions in Rothschild and Stiglitz (1976) and Azevedo and Gottlieb (2017) consider both "downward" and "upward" selection. With the restriction of monotonic wages, we show that it is always possible for firms to deviate from a non-Bertrand allocation such that wages remain monotone and profit increases; our proof considers exhaustive cases, and in each, a firm can hire or fire to increase profit, even when it must "sacrifice" by increasing the wages of productive, underpaid workers to abide by monotonicity.

Throughout, we describe our model in the context of two firms competing to hire a continuum of workers by simultaneously making weakly-positive wage offers. Instead, we could have described our model in the context of two firms competing to sell differentiated products to a continuum of consumers by simultaneously announcing weakly-positive prices. These two settings yield analogous results in equilibrium: in the former, almost every worker is hired at a wage equaling her marginal productivity. In the latter, almost every good is sold at a price equaling its marginal cost.

Section 2 presents our model and main result, Section 3 presents the proof, and Section 4 discusses robustness. Technical details are relegated to the Appendix.

2 Model and Result

There are two firms 1,2 and a continuum of workers. A worker type is identified by a pair $(v,\ell) \in [0,1]^2$, where v is the worker's productivity, and ℓ is an index. We assume there is a non-atomic measure μ that governs the distribution of worker types, where $f(v)$ denotes the density of workers with productivity v. Let $F(\cdot)$ denote the associated cumulative distribution function.[2] We additionally assume $0<\underline{f}\leq \bar{f}<+\infty$, where $\underline{f}:=\inf\{f(v)|v\in[0,1]\}$ and $\bar{f}:=\sup\{f(v)|v\in[0,1]\}$.

The game proceeds as follows. First, each firm i simultaneously announces a measurable set of workers $\mathcal{S}_i$ to which it makes job offers, as well as a measurable function $\mathscr{w}_i$ on $\mathcal{S}_i$ where $\mathscr{w}_i(v,\ell)$ is the wage offered to worker $(v,\ell)\in\mathcal{S}_i$. We require wage offers to be non-negative, that is, $\mathscr{w}_i(v,\ell)\geq 0$ for all i and (v,ℓ). Second, each worker observes the identity of the firm that made an offer to her (if any) and the associated wage offered to her and chooses to accept one of the offers or stay unassigned and receive the wage of zero. Each firm i is matched to the subset of workers $S_i\subset\mathcal{S}_i$ who accept its offer, and pays each such worker the offered wage.

Each worker's payoff is equal to her wage if she accepts an offer from a firm and zero otherwise. Firms have a constant-returns-to-scale production technology and seek to maximize profit. Formally, if S_i is measurable, then firm i obtains payoff

$$\int_{S_i}\left[v-\mathscr{w}_i(v,\ell)\right]d\mu.$$

If S_i is nonmeasurable, then i's payoff is -1. Our solution concept is pure-strategy subgame perfect Nash equilibrium ("equilibrium").

The main substantive restriction we make is that each firm's wage offers must be monotone non-decreasing in productivity. Formally, we assume that for any two workers (v,ℓ) and (v',ℓ') with $v\geq v'$ who receive wage offers from the same firm i, it is the case that $\mathscr{w}_i(v,\ell)\geq\mathscr{w}_i(v',\ell')$. Note that this implies all workers of the same productivity who receive offers from the same firm must receive the same wage. Therefore, we consolidate notation hereafter and describe wage offers as a function of productivity: $\mathscr{w}_i(v)$ represents the wage available to each worker (v,ℓ) who receives a job offer from i.

Each strategy profile induces an *allocation*, which specifies the distribution of workers hired by each firm and the wage paid to each productivity level by each firm. Formally, an allocation for firm i is $A_i:=\{(f_i(v),w_i(v))\}_{v\in[0,1]}$, where $f_i(\cdot)$ and $w_i(\cdot)$ are measurable functions such that:

1. $f_i(v)\in[0,f(v)]$ is the density of workers of productivity v hired by i,
2. $w_i(v)\in[0,\infty)$ is the wage i pays to each worker of productivity v it hires, and
3. If $f_i(v)=0$, then we fix $w_i(v)=0$.

An allocation is a tuple $A:=(A_1,A_2)$ where A_i is an allocation for firm i such that $f_1(v)+f_2(v)\leq f(v)$ for each v, that is, total employment does not exceed the supply of workers (a feasibility requirement). Note that $w_i(\cdot)$ refers to the wages *paid* in an allocation, while our earlier notation $\mathscr{w}_i(\cdot)$ refers to the wages *offered*.

[2]More formally, we define a Borel measure $\tilde{\mu}^{\mathrm{p}}$ such that $\tilde{\mu}^{\mathrm{p}}([0,x])=F(x)$ for all $x\in[0,1]$, which exists and is unique (Royden and Fitzpatrick, 2010, Proposition 25, Section 20.3). Let μ^{p} be the unique measure defined on the Lebesgue measurable sets and coincides with $\tilde{\mu}^{\mathrm{p}}$ on Borel measurable sets: such μ^{p} exists and is unique because of the Caratheodory Extension Theorem and the Hahn Extension Theorem (see Stokey and Lucas, 1989, Theorems 7.3 and 7.2'). μ^{p} is the Lebesgue measure on Lebesgue sigma-algebra $\mathscr{B}^{\mathrm{p}}$ on $[0,1]$, representing the measure of productivity. Similarly, let μ^{w} be a measure on a sigma-algebra $\mathscr{B}^{\mathrm{w}}$ on $[0,1]$, representing the measure of indices. We assume that both μ^{p} and μ^{w} are non-atomic. We assume that the measure μ over worker types is given by the product measure of μ^{p} and μ^{w}, and the density function associated with μ is given by $f(v)\times g(\ell)$, where $f(v)$ is associated with measure μ^{p} and represents the density of workers with productivity v while $g(\ell)$ is associated with measure μ^{w} and represents the density of workers whose indices are ℓ.

An allocation A is a *Bertrand allocation* if for all $i \in \{1,2\}$ and almost all $v \in [0,1]$: $f_1(v)+f_2(v)=f(v)$, and $w_i(v)=v$ if $f_i(v)>0$.[3] Clearly, the set of Bertrand allocations is non-empty; consider the allocation in which firm 1 employs all workers and pays each worker a wage equal to her productivity (for all v, $f_1(v)=f(v)$, $w_1(v)=v$, $f_2(v)=0$, and $w_2(v)=0$).

The following is our main result:

THEOREM **1.** *An allocation can be induced by an equilibrium if and only if it is a Bertrand allocation.*

3 Proof

We prove Theorem 1 in two steps. We first demonstrate that the set of Bertrand allocations is equivalent to the set of core allocations of a cooperative version of our game (which will be formally defined later), and then we show that an allocation can be induced by an equilibrium of our non-cooperative game if and only if it is a core allocation of the cooperative game. This approach both aids in exposition (because the cooperative game considers only final allocations, and not strategies) and further generalizes our main finding on the focality of Bertrand allocations to a cooperative setting.

Step 1: Core of a Cooperative Game

Here, we describe the cooperative game, define the core, and characterize the set of core allocations.

Consider a cooperative game consisting of the same sets of players as the original non-cooperative game, where the distribution of worker types remains the same. The definition of an allocation is also the same as before. Corresponding to the monotonicity requirement of wages in the noncooperative games, we assume that, for any allocation, $w_i(v) \geq w_i(v')$ if $v \geq v'$ and $f_i(v)>0$.

Under an allocation for firm i, $A_i := \big\{(f_i(v),w_i(v))\big\}_{v\in[0,1]}$, i receives profit

$$\pi_i^{A_i} := \int_0^1 \big[v - w_i(v)\big] f_i(v)dv.$$

An allocation is *blocked* by a set of workers and a firm if there is an alternative wage schedule for a subset of workers such that both the firm and each worker in the subset obtain a higher payoff than in the present allocation. Formally, we say that an allocation $A := \big\{(f_i(v),w_i(v))\big\}_{v\in[0,1],i=1,2}$ is *blocked* by firm j via an alternative allocation (for j) $\tilde{A}_j := \big\{(\tilde{f}_j(v),\tilde{w}_j(v))\big\}_{v\in[0,1]}$ if $\pi_j^{\tilde{A}_j} > \pi_j^{A_j}$ and, for almost all $v\in[0,1]$, one of the following conditions hold (note that, because we define $\tilde{A}_j$ to be an allocation, it must satisfy all restrictions imposed on an allocation in addition to those listed below):

1. $\tilde{w}_j(v) \geq w_j(v)$ and $\tilde{w}_j(v) > w_{-j}(v)$,
2. $\tilde{w}_j(v) \geq w_j(v)$ and $\tilde{f}_j(v)+f_{-j}(v) \leq f(v)$,
3. $\tilde{w}_j(v) > w_{-j}(v)$ and $\tilde{f}_j(v)+f_j(v) \leq f(v)$, or

[3] Throughout, we write "almost all" to mean "except for a measure-zero set," as is commonly used in measure theory.

4. $\tilde{f}_j(v)+f_j(v)+f_{-j}(v)\leq f(v)$.

Intuitively, Condition 1 states a "no wage cuts" requirement; if a firm j weakly raises the wages of all workers involved, and strictly raises wages for workers employed by the other firm, then these workers are all willing to work for j. Condition 2 considers the case in which firm j does not need to poach workers from firm $-j$ to construct the blocking allocation, so the only constraint on wages is that existing workers' wages are not reduced. Condition 3 considers the case in which firm j does not need to keep any existing workers to construct the blocking allocation, so the only restriction on wages is that the wage paid to poached workers is higher than those paid by $-j$ to the same workers. Condition 4 considers the case in which firm j can hire from unemployed workers to construct the blocking allocation, so there is no restriction on the wage for these workers.

An allocation A is said to be a *core allocation* if there exists no firm j and alternative allocation $\tilde{A}_j$ for j that blocks A.

PROPOSITION **1.** *In the cooperative game, the set of core allocations coincides with the set of Bertrand allocations.*

It is straightforward to show that any Bertrand allocation is a core allocation: no firm j can hire unemployed workers (since $f_1(v)+f_2(v)=f(v)$ for almost all v) nor can it poach workers from the competing firm without incurring a loss (since $w_{-j}(v)=v$ for almost all v such that $f_{-j}(v)>0$, and Conditions 1 and 3 of the definition of block require poached workers to earn strictly more than they were at firm $-j$). It is more complicated to show that any core allocation is a Bertrand allocation. To do so, the appendix considers six exhaustive cases to show that any non-Bertrand allocation admits a firm with a blocking allocation. Here, we present the argument for one of the cases (this case is relatively simple but showcases some of the main proof ideas).

REMARK **1.** *Consider any allocation A in which $w_i(v)\leq v$ for all v and all i, and in which there exists a firm j and a subset of productivities V with positive (Lebesgue) measure such that $f_1(v)+f_2(v)<f(v)$ and $w_j(v)=v$ for all $v\in V$. Then A is not a core allocation.*

The red curve in Figure 1 depicts a wage function satisfying the conditions in Remark 1, where $V\subset[\underline{V},\bar{V}]$. Naively, firm j may consider firing all workers in set V who earn their full productivity in wage, and replacing them with workers of the same productivity who were previously unemployed at the minimum wage of 0 (i.e. a blocking outcome $\tilde{A}_j$ such that $\tilde{w}_j(v)=0$ and $\tilde{f}_j(v)=f(v)-f_1(v)-f_2(v)$ for all $v\in V$). But this is not possible, because of the monotonicity constraint, without also firing all workers with productivity strictly less than $\underline{V}$, which could possibly result in lower overall profits. Instead, could the firm fire all existing employees with productivity $v\in[\underline{V},\bar{V}]$ and replace them with workers of the same productivity by paying a wage equal to $\underline{V}$ (i.e. a blocking outcome $\tilde{A}_j$ such that $\tilde{w}_j(v)=\underline{V}$ and $\tilde{f}_j(v)=f(v)-f_1(v)-f_2(v)$ for all $v\in[\underline{V},\bar{V}]$)? Doing so would satisfy monotonicity, but would also result in the loss of profit from workers of productivity $v\in[\underline{V},\bar{V}]$ who earn strictly less than their productivity. Indeed, such losses in profit may be unavoidable, as there may not exist an interval of productivities such that all workers within the interval earn wages equal to their productivity—recall that V is only required to be measurable, and the measurability of V does not imply the existence of an interval subset of V.

Our proof below constructs a blocking outcome where workers within a certain interval $I:=[\underline{v},\bar{v}]$ of productivities are fired, such that any losses from the fired workers previously earning less than their productivity within this interval are small relative to the gain in profit

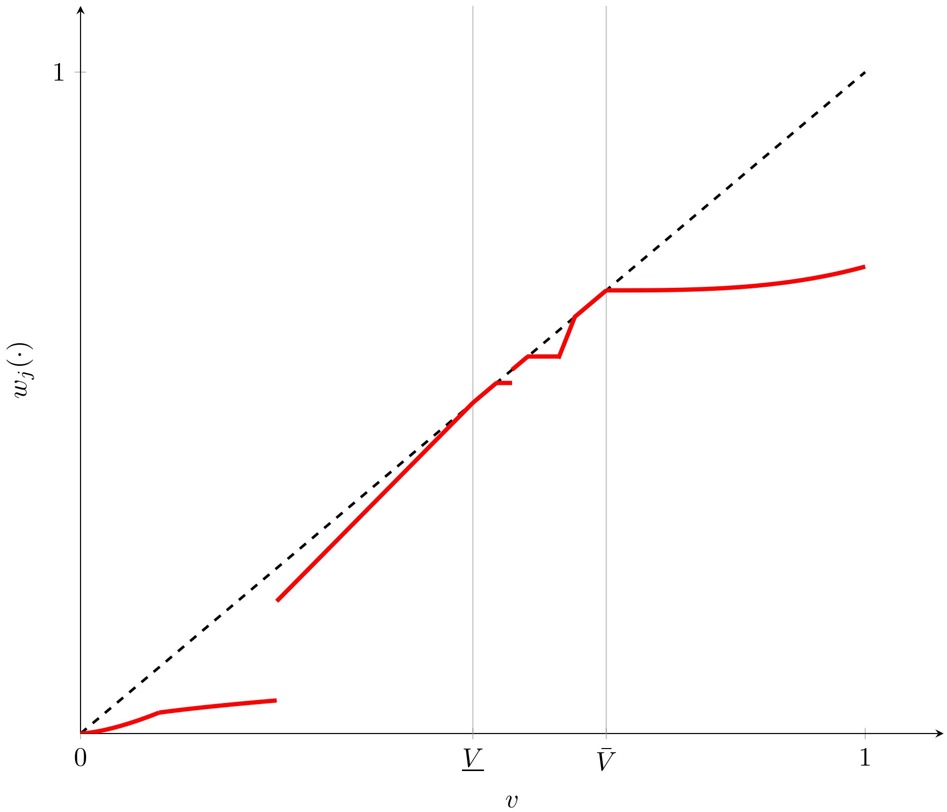

Figure 1: Wage function described in Remark 1

from "poaching" from unemployment all available workers with productivity $v \in I$ and paying them a common wage of no more than $\underline{v}$. Our argument, including the proof that an interval I with the desired property exists, is formalized below.

Proof. First, note that there exists $\varepsilon > 0$ and V' with positive measure such that $f_1(v) + f_2(v) < f(v) - \varepsilon$ and $w_j(v) = v$ for all $v \in V'$.[4] Now, arbitrarily fix $p < 1$ such that $\bar{f}(1-p) \leq p\varepsilon$ and $\frac{1}{2}p^2\varepsilon > \bar{f}(1-p)$ (note that those inequalities are satisfied by any sufficiently large $p < 1$). By Halmos (1974, Theorem A, Page 68), there exists an interval $I := [\underline{v}, \bar{v}] \subseteq [0,1]$ such that $\mu(V' \cap I) > p\mu(I)$, where $\mu(\cdot)$ is the Lebesgue measure. Then $\tilde{A}_j$ where for all v:

[4] The proof for this claim is as follows: Suppose for contradiction that for each ε, any set of productivities such that $f_1(v) + f_2(v) < f(v) - \varepsilon$ and $w_j(v) = v$ has zero measure. Then, for each $n = 1,2,\ldots$ define the set $V_n := \{v \in V | f(v) - f_1(v) - f_2(v) > \frac{1}{n} \text{ and } w_j(v) = v\}$. Then, by assumption, $V_1, V_2, \ldots$ is an increasing sequence of sets and $\bigcup_n V_n = V^* := \{v \in V | f(v) - f_1(v) - f_2(v) > 0 \text{ and } w_j(v) = v\}$. Therefore, by countable additivity of the Lebesgue measure, we have $\mu(V^*) = \lim_n \mu(V_n) = 0$, which contradicts the assumption that V has positive measure and the fact that $V \subseteq V^*$.

$$\tilde{f}_j(v) := \begin{cases} f(v) - f_j(v) - f_{-j}(v) & \text{if } v \in I, \\ f_j(v) & \text{otherwise.} \end{cases} \qquad \tilde{w}_j(v) := \begin{cases} 0 & \text{if } \tilde{f}_j(v) = 0, \\ \sup_{v' < \underline{v}} w_j(v') & \text{if } v \in I \text{ and } \tilde{f}_j(v) > 0, \\ w_j(v) & \text{otherwise.} \end{cases}$$

blocks A. To see this, note Condition 4 of the definition of block is satisfied for all $v \in I$ (firm j fires all existing workers in set I and hires unemployed workers of the same productivity), and Condition 2 of the definition of block is satisfied for all $v \notin I$ (no worker receives a wage cut and no workers are poached from firm $-j$). By construction, $\tilde{w}_j(v)$ satisfies our monotonicity condition. Therefore, it remains only to show that j's profit is higher under $\tilde{A}_j$.

To see that firm j's profit increases, let $\delta := \underline{v} - \sup_{v \leq \underline{v}} w_j(v)$. j makes an additional profit of at least

$$\delta p \mu(I) \varepsilon + \frac{1}{2} \big(p\mu(I)\big)^2 \varepsilon,$$

from hiring workers from set $V' \cap I$ while firing existing workers from $V' \cap I$ causes no loss (because those workers were hired at wages equal to their productivities), and the loss from losing workers from $I \setminus V'$ is bounded from above by $\bar{f}\big[(1-p)\mu(I) \times (\delta + \mu(I))\big] = \bar{f}(1-p)\delta\mu(I) + \bar{f}(1-p)\mu(I)^2$. Because p satisfies $\bar{f}(1-p) \leq p\varepsilon$ and $\frac{1}{2}p^2\varepsilon > \bar{f}(1-p)$ by assumption, the total change in j's payoff is strictly positive, as desired. □

As previously mentioned, the remaining cases are addressed in the appendix.

Step 2: Equivalence between Core and Equilibrium Allocations

Next, we show that the set of core allocations of the cooperative game, which by Proposition 1 coincides with the set of Bertrand allocations, is equivalent to the set of equilibrium allocations of the non-cooperative game.

Proposition **2.** *An allocation A can be induced by an equilibrium of the non-cooperative game if and only if it is a core allocation of the cooperative game.*

Without the wage monotonicity constraint, it would be straightforward to establish this result, adopting the original Bertrand argument. However, the monotonicity constraint makes it harder to establish that non-Bertrand allocations cannot be induced by an equilibrium. Specifically, one challenge is that our non-cooperative game requires monotonicity of wage offers, while the cooperative game imposes monotonicity on the final wage schedule. The proof below establishes that one can construct a profitable deviation strategy based on a block in the cooperative game which satisfies the original monotonicity condition.

Proof. It is easy to see that any Bertrand allocation can be induced by an equilibrium: if both firms offer all workers wages equal to their productivity, then no deviation can increase either firm's profits, given the strategy profile of other agents, as the only way to hire additional workers is to pay them more than their productivity.

We show the complementary direction by separately considering each of the six cases presented in the proof of Proposition 1. Formally, consider any strategy profile that admits a profitable deviation σ_j by firm j, which can be constructed by an adaptation of the argument in the proofs of Propositions 7 and 8 of Gentile Passaro et al. (2024). Then, σ_j (when all others play their original strategies) induces allocation $A_j^{\sigma_j} := \big\{(f_j^{\sigma_j}(v), w_j^{\sigma_j}(v))\big\}_{v \in [0,1]}$ for firm j in which the wages of *employed* workers at firm i satisfy monotonicity, i.e., $w_j^{\sigma_j}(\cdot)$ is monotonic. Now consider an alternative deviation strategy σ_j' for firm j that coincides with σ_j except that j simply withholds making offers to workers (v,ℓ) who receive offers from firm j according to σ_j but decline it. By construction, this again induces the same allocation, i.e. $A_j^{\sigma_j'} := \{(f_j^{\sigma_j'}(v), w_j^{\sigma_j'}(v))\}_{v\in[0,1]} = A_j^{\sigma_j}$. Therefore, for any v such that $f_j^{\sigma_j}(v) = f^{\sigma_j'}(v) > 0$, $w_j^{\sigma_j'}(v) = w_j^{\sigma_j'}(v) = w_j^{\sigma_j}(v)$, where the first equality follows because no offers are rejected, and the second inequality follows because σ_j' induces the same allocation as does σ_j. Because $w_j^{\sigma_j}(\cdot)$ is monotonic, so is $w_j^{\sigma_j'}(\cdot)$. □

For clarity, we construct a deviation strategy corresponding to the case presented in Remark 1: Consider a deviation σ_j' by firm j such that it makes offers to workers in set $\Big([I^{\complement}\times[0,1]]\cap S_j\Big)\cup\Big([I\times[0,1]]\cap[S_j\cup S_{-j}]^{\complement}\Big)$, and the wage offer is given by $\tilde{w}_j(\cdot)$. This deviation clearly results in allocation $\tilde{A}_j$ for firm j, and the set of workers hired is measurable.[5] Therefore, j's profit strictly increases, as shown in the proof of Remark 1, proving the original allocation A cannot be induced by an equilibrium.

4 Discussion

We show that multi-product "menu" pricing under a monotonicity constraint leads to the familiar Bertrand outcome that competition erodes firm profits. As described in the Introduction, monotone pricing is a relevant constraint in certain markets affected by adverse selection, moral hazard, and legal constraints. Therefore, our paper supports the common, often unmodeled, assumption that prices must equal marginal cost (or marginal productivity in labor-market settings) under competition.

Our finding is robust to a number of model alterations:

- Our model allows firms to restrict the set of worker types who receive wage offers, and to "ration" wage offers to different types. The proof of Proposition 1 obtains if we assume firms are required to offer all workers contracts, implying that our result holds if firms can only compete over wages.
- Proposition 1 shows that our main result also holds in a cooperative game, suggesting that details of offer timing do not drive our finding.
- The central contribution of Proposition 2 is to show that we can obtain any allocation with monotonic *accepted* wages via a strategy that makes monotonic wage *offers*. Because only workers who receive wage offers are eligible to work at that firm, if a firm's wage offers are monotonic, so too are the accepted offers. Therefore, our main result holds if we instead require wage monotonicity among employed workers.
- Our result extends in the usual way when there are more than two firms.

Overall, our main result—together with its robustness to various modeling specifications—suggests that the conclusion of the original Bertrand model holds in a wide variety of environments. We therefore view this paper as providing justification for the pervasive use of the

[5] To see that the set of workers hired is measurable, note that I is measurable by assumption, and taking a countable number of unions, intersections, and complements of measurable sets results in a measurable set.

Bertrand prediction as a building block for richer models designed to answer new economic questions.

Appendix: Proof of Proposition 1

Proof. Consider any Bertrand allocation $A=\left\{(f_i(v),w_i(v))\right\}_{v\in[0,1],i=1,2}$. Then the following hold for almost all $v\in[0,1]$:

B1 $f_1(v)+f_2(v)=f(v)$, and

B2 for all $i\in\{1,2\}$, $w_i(v)=v$ if $f_i(v)>0$.

We establish the desired result through two lemmas regarding these enumerated conditions.

Lemma **1.** *Any allocation A satisfying B1 and B2 is a core allocation.*

Proof of Lemma 1. Suppose not for the sake of contradiction. Then there are a firm j and a distinct allocation (for firm j) $\tilde{A}_j := \left\{(\tilde{f}_j(v),\tilde{w}_j(v))\right\}_{v\in[0,1]}$ that blocks A. In order for $\tilde{A}_j$ to block A it must be that $\pi_j^{\tilde{A}_j} > \pi_j^{A_j}$. However,

$$\pi_j^{A_j} = \int_0^1 \left[v - w_j(v)\right] f_j(v)dv = 0 \geq \int_0^1 \left[v - \tilde{w}_j(v)\right] \tilde{f}_j(v)dv = \pi_j^{\tilde{A}_j}.$$

The second equality follows because, by the construction of A, either $f_j(v)=0$ or $w_j(v)=v$ for almost all v, therefore, the integrand almost always equals zero. The inequality follows because of the following exhaustive cases for almost all v, corresponding, respectively, to Conditions 1-4 of the definition of block:

- Suppose $\tilde{w}_j(v) \geq w_j(v)$ and $\tilde{w}_j(v) > w_{-j}(v)$, then it must be that $\tilde{w}_j(v) \geq v$ since $\max\{w_j(v),w_{-j}(v)\}=v$, which makes the integrand weakly negative,
- Suppose $\tilde{w}_j(v) \geq w_j(v)$ and $\tilde{f}_j(v) + f_{-j}(v) \leq f(v)$. If $\tilde{f}_j(v) = 0$ then the integrand is weakly negative. If $\tilde{f}_j(v)>0$ then it must be that $f_{-j}(v)<f(v)$, and by the construction of A that $f_j(v)+f_{-j}(v) = f(v)$, it must be that $f_j(v) > 0$. Therefore, it must be that $w_j(v)=v$, and the requirement that $\tilde{w}_j(v)\geq w_j(v)$ makes the integrand weakly negative.
- Suppose $\tilde{w}_j(v) > w_{-j}(v)$ and $\tilde{f}_j(v) + f_j(v) \leq f(v)$. If $\tilde{f}_j(v) = 0$ then the integrand is weakly negative. If $\tilde{f}_j(v) > 0$ then it must be that $f_j(v) < f(v)$, and by the construction of A that $f_j(v)+f_{-j}(v) = f(v)$, it must be that $f_{-j}(v) > 0$. Therefore, it must be that $w_{-j}(v)=v$, and the requirement that $\tilde{w}_j(v)>w_{-j}(v)$ makes the integrand strictly negative.
- Suppose $\tilde{f}_j(v)+f_j(v)+f_{-j}(v) \leq f(v)$ then it must be that $\tilde{f}_j(v)=0$ since by the construction of A it is the case that $f_j(v)+f_{-j}(v)=f(v)$. Therefore, the integrand is weakly negative.

$\pi_j^{A_j} \geq \pi_j^{\tilde{A}_j}$ contradicts the premise that $\pi_j^{\tilde{A}_j} > \pi_j^{A_j}$. Therefore, A is a core allocation.

□

Lemma **2.** *There exist no core allocations which do not satisfy both B1 and B2.*

Proof. Suppose for contradiction that there is a core allocation $A=\{(f_i(v),w_i(v))\}_{v\in[0,1],i=1,2}$ such that there exists a set V with positive (Lebesgue) measure where either B1 or B2 fails for all $v \in V$. We proceed by considering six exhaustive cases: By countable additivity of measures, the set of productivities that fails one of B1 or B2 has positive measure if and only if at least one of the sets in the following six cases has a positive measure.

First, suppose there exist a firm j and a subset of productivities $V \subset [0,1]$ with positive measure such that $w_j(v)>v$ for all $v\in V$. Then $\tilde{A}_j$ where for all v:

$$\tilde{f}_j(v):=\begin{cases} f_j(v) & \text{if } v\notin V, \\ 0 & \text{if } v\in V. \end{cases} \qquad \tilde{w}_j(v):=\begin{cases} w_j(v) & \text{if } v\notin V, \\ 0 & \text{if } v\in V. \end{cases}$$

blocks A as j's profit increases and Condition 4 of the definition of block is satisfied for all $v\in V$ (i.e. the workers in V are fired) and for all $v\in[0,1]\setminus V$ Condition 2 of the definition of block is satisfied (i.e. there is no change in the hiring or wages of workers in $[0,1]\setminus V$). By construction, $\tilde{w}_j(v)$ satisfies our monotonicity condition.

Therefore, we proceed with the assumption that for each firm j, $w_j(v)\le v$ for almost all v. Second, suppose there exist a firm j and a subset of productivities V with positive measure such that $f_1(v)+f_2(v)<f(v)$ and $w_j(v)<v$ for all $v\in V$. Then $\tilde{A}_j$ where for all v:

$$\tilde{f}_j(v):=f(v)-f_{-j}(v). \qquad \tilde{w}_j(v):=\begin{cases} 0 & \text{if } \tilde{f}_j(v)=0, \\ \sup\limits_{v'\le v} w_j(v') & \text{otherwise.} \end{cases}$$

blocks A as firm j's profit increases as some previously unemployed workers are hired at a wage strictly less than their productivity while all existing workers at j continue to be employed at the same wage as before, and Condition 2 of the definition of block is satisfied for all $v\in[0,1]$ (i.e. no worker receives a wage cut and no workers are poached from firm $-j$). By construction, $\tilde{w}_j(v)$ satisfies monotonicity.

Third, there exists a firm j and a subset of productivities V with positive measure such that $f_1(v)+f_2(v)<f(v)$ and $w_j(v)=v$ for all $v\in V$. This case has been addressed in Remark 1.

The previous two cases exhaust the possibility of a core allocation in which $f_1(v)+f_2(v)<f(v)$ for any subset of productivities with positive measure. Therefore, we proceed with the assumption that $f_1(v)+f_2(v)=f(v)$ for almost all v.

Fourth, suppose that there exist j and a set V of productivities with positive measure such that $w_j(v)<v$ and $f_j(v)=f(v)$ for all $v\in V$. Then, there exists $\varepsilon\in(0,1)$ and V' with positive measure such that $w_j(v)<v-\varepsilon$ and $f_j(v)=f(v)$ for all $v\in V'$.[6] For any $p<1$, by Halmos (1974, Theorem A, Page 68), there exists an interval $I^p:=[\underline{v}^p,\bar{v}^p]\subseteq[0,1]$ such that $\mu(V'\cap I^p)>p\mu(I^p)$, where $\mu(\cdot)$ is the Lebesgue measure. Consider the following cases.

1. Suppose that there is no $V^p\subseteq[\bar{v}^p,1]$ with positive measure such that $w_{-j}(v)<w_j(\bar{v}^p)$ and $f_{-j}(v)>0$ for all $v\in V^p$ and for all p sufficiently close to 1. Let $w^p:=\sup_{v<\underline{v}^p}w_{-j}(v)$. Then for a constant $\varepsilon'\in(0,\varepsilon)$, $\tilde{A}_{-j}$ where for all v:

[6] The proof is analogous to that in Footnote 4.

$$\tilde{f}_{-j}(v) := \begin{cases} f(v) & \text{if } v \in I^p, \\ f_{-j}(v) & \text{otherwise.} \end{cases} \qquad \tilde{w}_{-j}(v) := \begin{cases} 0 & \text{if } \tilde{f}_{-j}(v) = 0, \\ \max\{w^p, w_j(v) + \varepsilon'\} & \text{if } v \in I^p, \\ \max\{\sup_{v \le \bar{v}^p} w_j(v) + \varepsilon', w_{-j}(v)\} & \text{if } v \in [\bar{v}^p, 1] \text{ and } \tilde{f}_{-j}(v) > 0, \\ w_{-j}(v) & \text{otherwise.} \end{cases}$$

blocks A for sufficiently small ε' for the following reasons: First, for all $v \in I^p$, Condition 4 of the definition of block is satisfied, and second, for all $v \notin I^p$, Condition 2 of the definition of block is satisfied. Note also that $\tilde{w}_j(v)$ satisfies monotonicity by construction. It therefore remains only to show that $\tilde{A}_{-j}$ increases the profit of firm $-j$.

We proceed by showing that firm $-j$'s "gain" from poaching workers in $I^p \cap V'$ exceeds the "loss" of at most p fraction of workers over interval I^p for sufficiently large p. After doing so, we show that the loss in profit resulting from the increased wage to workers with productivities $v \ge \bar{v}^p$ is arbitrarily small, thus completing the argument that $-j$'s profit increases.

The loss from losing existing workers in I^p is upper bounded by

$$\bar{f}(1-p)(\bar{v}^p - \underline{v}^p)(\bar{v}^p - w^p). \tag{1}$$

This follows because, in the worst case, there are at most p fraction of workers in I^p who are lost by firm $-j$, with this p fraction loaded into the rightmost part of I^p.

The gain from poaching workers in $I^p \cap V'$ is at least

$$\underline{f} \int_{\underline{v}^p}^{\bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p)} \min\{v - w^p, \varepsilon - \varepsilon'\} dv.$$

Let $v^p := \max\big\{\min\big\{w^p + \varepsilon - \varepsilon', \bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p)\big\}, \underline{v}^p\big\}$. Then we can rewrite the lower bound on the gain as

$$\underline{f} \int_{\underline{v}^p}^{v^p} (v - w^p) dv + \underline{f} \int_{v^p}^{\bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p)} (\varepsilon - \varepsilon') dv.$$

We can rewrite this as:

$$(v^p - \underline{v}^p)(\underline{v}^p - w^p)\underline{f} + \frac{1}{2}(v^p - \underline{v}^p)^2 \underline{f} + (\varepsilon - \varepsilon')\big[\bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p) - v^p\big]\underline{f},$$

which, because $\varepsilon - \varepsilon' \in (0,1)$ and all bracketed terms are non-negative, is no smaller than

$$\begin{aligned} &(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\big[(v^p - \underline{v}^p)(\underline{v}^p - w^p) + (v^p - \underline{v}^p)^2 + \big[\bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p) - v^p\big]\big] \\ =&(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\big[(v^p - \underline{v}^p)(v^p - w^p) + \big[\bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p) - v^p\big]\big] \end{aligned} \tag{2}$$

and therefore, a lower bound on "the net gain," i.e. (2)-(1), equals

$$(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}\big[(v^p-\underline{v}^p)(v^p-w^p)+\big[\bar{v}^p-(1-p)\big(\bar{v}^p-\underline{v}^p\big)-v^p\big]\big]-\bar{f}(1-p)(\bar{v}^p-\underline{v}^p)(\bar{v}^p-w^p). \tag{3}$$

We now consider (3) in light of the three possible values v^p can take:

First, suppose that $v^p=w^p+\varepsilon-\varepsilon'$. (3) is proportional to

$$\frac{(v^p-\underline{v}^p)(v^p-w^p)+\bar{v}^p-v^p}{(1-p)(\bar{v}^p-\underline{v}^p)}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}},$$

and

$$\begin{aligned}\frac{(v^p-\underline{v}^p)(v^p-w^p)+\bar{v}^p-v^p}{(1-p)(\bar{v}^p-\underline{v}^p)}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}&=\frac{(v^p-\underline{v}^p)(v^p-w^p)+\underline{v}^p-v^p}{(1-p)(\bar{v}^p-\underline{v}^p)}+\frac{1}{1-p}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}\\&=\frac{(v^p-\underline{v}^p)[(v^p-w^p)-1]}{(1-p)(\bar{v}^p-\underline{v}^p)}+\frac{1}{1-p}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}\\&\geq\frac{(v^p-w^p)-1}{(1-p)}+\frac{1}{1-p}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}\\&=\frac{v^p-w^p}{1-p}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}\\&=\frac{\varepsilon-\varepsilon'}{1-p}-1-\frac{\bar{f}(\bar{v}^p-w^p)}{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}},\end{aligned}$$

where the first inequality follows because $1\geq\bar{v}^p>\underline{v}^p$ and $\bar{v}^p\geq v^p\geq\underline{v}^p\geq w^p\geq 0$ for all p which implies that $\frac{v^p-\underline{v}^p}{\bar{v}^p-\underline{v}^p}\in[0,1]$ and $v^p-w^p\leq 1$, the final equality follows because $v^p=w^p+\varepsilon-\varepsilon'$. Clearly this expression is positive for any sufficiently large $p<1$ since $\varepsilon-\varepsilon'>0$ by assumption.

Second, suppose that $v^p=\bar{v}^p-(1-p)(\bar{v}^p-\underline{v}^p)$. Then (3) is proportional to

$$\frac{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}{(1-p)}\frac{p(v^p-w^p)}{\bar{v}^p-w^p}-\bar{f}.$$

We can see that for all p

$$\begin{aligned}\frac{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}{(1-p)}\frac{p(v^p-w^p)}{\bar{v}^p-w^p}-\bar{f}&=\frac{(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}}{(1-p)}\left[p-p(1-p)\frac{\bar{v}^p-\underline{v}^p}{\bar{v}^p-w^p}\right]-\bar{f}\\&\geq(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}\left[\frac{p}{1-p}-p\right]-\bar{f}\\&=(\varepsilon-\varepsilon')\frac{1}{2}\underline{f}\frac{p^2}{1-p}-\bar{f},\end{aligned}$$

where the first equality comes from substituting in $v^p = \bar{v}^p - (1-p)(\bar{v}^p - \underline{v}^p)$, and the inequality follows because $\frac{\bar{v}^p - \underline{v}^p}{\bar{v}^p - w^p} \le 1$ for all p because $\bar{v}^p > \underline{v}^p$ and $\bar{v}^p \ge v^p \ge \underline{v}^p \ge w^p$. Clearly this expression is positive for any sufficiently large $p<1$ since $\varepsilon - \varepsilon' > 0$ by assumption.

Third, suppose that $v^p = \underline{v}^p$. Then (3) is proportional to

$$(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\frac{p}{1-p} - \bar{f}(\bar{v}^p - w^p).$$

Noting that each of the terms in parentheses is non-negative by construction and $\bar{v}^p > \underline{v}^p \ge w^p$, then the above equation is bounded below by

$$(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\frac{p}{1-p} - \bar{f}.$$

This expression is positive for any sufficiently large $p<1$ since $\varepsilon - \varepsilon' > 0$ by assumption.

Therefore, renormalizing the calculated "net gain" term from each of the three possible values v^p can take, we have shown that firm $-j$'s change in profit from workers with $v \in I^p \cap V'$ is at least

$$(1-p)(\bar{v}^p - \underline{v}^p) \times \min\left\{ (\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\left[\frac{\varepsilon - \varepsilon'}{1-p} - 1 - \frac{\bar{f}(\bar{v}^p - w^p)}{(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}}\right], (\bar{v}^p - w^p)\left[(\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\frac{p^2}{1-p} - \bar{f}\right], (\varepsilon - \varepsilon')\frac{1}{2}\underline{f}\frac{p}{1-p} - \bar{f}\right\},$$

and this expression is positive for every $\varepsilon' \in (0,\varepsilon)$ and sufficiently large $p<1$. Moreover, it can be observed by inspection that this expression is decreasing in ε'. Furthermore, the wage paid for workers in $[\bar{v}^p, 1]$ may increase at most by ε', resulting in a loss of profit from the increased wage being bounded from above by ε'. From these observations, for any sufficiently large $p<1$ and sufficiently small $\varepsilon' > 0$, firm $-j$ strictly profits with the block, i.e. $\pi_{-j}^{\tilde{A}_{-j}} > \pi_{-j}^{A_{-j}}$ as desired.

2. Suppose that there exists a subset of $[0,1)$ whose supremum is 1 such that, for each p in that subset, there is a set $V^p \subseteq [\bar{v}^p, 1]$ with positive measure such that $w_{-j}(v) < w_j(\bar{v}^p)$ and $f_{-j}(v) > 0$ for all $v \in V^p$. Fix any such p and V^p. Consider $\tilde{A}_j$ where for all v:

$$\tilde{f}_j(v) := \begin{cases} f(v) & \text{if } v \in V^p, \\ f_j(v) & \text{otherwise.} \end{cases} \qquad \tilde{w}_j(v) := \begin{cases} 0 & \text{if } \tilde{f}_j(v) = 0, \\ w_j(v) & \text{otherwise.} \end{cases}$$

$\tilde{A}_j$ blocks A_j for the following reasons: Condition 1 of the definition of block is satisfied for all $v \in V^p$ since $w_{-j}(v) < w_j(\bar{v}^p) \le \tilde{w}_j(v)$ for all $v \in V^p$ by construction, and Condition 2 of the definition of block is satisfied for all $v \notin V^p$. Moreover, firm j obtains a strictly higher profit under this allocation.[7]

Fifth, suppose that there exist j and $V \subset [0,1]$ with positive measure such that $0 \le w_{-j}(v) \le w_j(v) < v$ and $f_j(v) \in (0, f(v))$ for all $v \in V$. Then, there exists $\varepsilon > 0$ and $V' \subset V$ with positive measure such that $0 \le w_{-j}(v) \le w_j(v) < v - \varepsilon$ and $f_j(v) \in (0, f(v) - \varepsilon)$ for all $v \in V'$. Then for a constant $\varepsilon' > 0$, consider $\tilde{A}_j$ where for all v:

[7] In the proposed block $\tilde{A}_j$, one could alternatively set $\tilde{w}_j(v) := w_j(v)$ for every $v \in [0,1]$, and the proof works without change.

$$\tilde{f}_j(v) := \begin{cases} f(v) & \text{if } v \in V', \\ f_j(v) & \text{otherwise.} \end{cases} \quad \tilde{w}_j(v) := \begin{cases} 0 & \text{if } \tilde{f}_j(v) = 0, \\ w_j(v) + \varepsilon' & \text{otherwise.} \end{cases}$$

$\tilde{A}_j$ blocks A_j for any sufficiently small $\varepsilon' > 0$ for the following reasons: Condition 1 of the definition of block is satisfied for all $v \in V'$ since $w_{-j}(v) \leq w_j(\bar{v}) < \tilde{w}_j(v)$ for all $v \in V'$ by construction, and Condition 2 of the definition of block is satisfied for all $v \notin V'$. To see that firm j's profit increases, first note that j benefits from hiring workers from V', which results in an additional profit of at least $(\varepsilon - \varepsilon')\varepsilon\mu(V')$. Meanwhile, j may lose from paying more for existing workers, but the associated loss is bounded from above by $\varepsilon'\beta$. Therefore, for any sufficiently small ε', firm j's profit increases, as desired. Note also that monotonicity is satisfied by $\tilde{w}_j(\cdot)$ because $w_j(\cdot)$ is monotone and ε' is a constant.

Cases 4 and 5 exhaust the possibility of a core allocation in which there exists a set V' of positive measure such that $\max\{w_1(v), w_2(v)\} < v$ for almost all $v \in V'$. Therefore, we proceed with the assumption that for almost any $v \in [0,1]$ there exists a firm j such that $w_j(v) = v$.

Sixth, suppose there exist a set V'' of positive measure and a firm j such that $0 \leq w_{-j}(v) < w_j(v) = v$ and $f_{-j}(v) \in (0, f(v))$ for all $v \in V''$. Intuitively, we proceed by showing that firm j can fire some subset of its workers who receive wages equal to productivity, and poach workers of the same productivity from firm $-j$. We proceed by constructing a set of workers with positive measure where such a maneuver is feasible.

Following earlier arguments, there exist $\delta > 0$ and a set V' with $\mu(V') > 0$ such that $0 \leq w_{-j}(v) + \delta < w_j(v) = v$ and $f_{-j}(v) \in (0, f(v))$ for all $v \in V'$.

Let $\mathrm{cl}(V')$ be the closure of V'. $\mathrm{cl}(V')$ is compact because it is a closed and bounded subset of $[0,1]$. For any $v \in [0,1]$ and $\varepsilon > 0$, define $B_\varepsilon(v) := (v - \varepsilon, v + \varepsilon) \cap [0,1]$ to be the ε-ball around v. Consider a collection of sets $\{B_\varepsilon(v')\}_{v' \in \mathrm{cl}(V')}$ where $\varepsilon < \frac{\delta}{2}$. It is obvious that $\{B_\varepsilon(v')\}_{v' \in \mathrm{cl}(V')}$ covers $\mathrm{cl}(V')$ and, because $\mathrm{cl}(V')$ is compact, there exist $v_1, v_2, ..., v_n \in \mathrm{cl}(V')$ such that $\{B_\varepsilon(v_i)\}_{i=1}^n$ covers $\mathrm{cl}(V')$, that is,

$$\bigcup_{i=1}^n B_\varepsilon(v_i) \supseteq \mathrm{cl}(V').$$

Therefore, it follows that

$$\bigcup_{i=1}^n [B_\varepsilon(v_i) \cap V'] = V'.$$

Because $\mu(V') > 0$, this implies that

$$\mu\left(\bigcup_{i=1}^n [B_\varepsilon(v_i) \cap V']\right) > 0,$$

so there exists $i \in \{1, ..., n\}$ such that $\mu(B_\varepsilon(v_i) \cap V') > 0$.

Given the conclusion of the preceding paragraph, fix $i \in \{1, ... n\}$ such that $\mu(B_\varepsilon(v_i) \cap V') > 0$. We will show that there exists $v'_i \in B_\varepsilon(v_i) \cap V'$ such that $\mu([v_i - \varepsilon, v'_i] \cap V') > 0$. To see this, suppose not for contradiction. Let $\bar{v} := \sup B_\varepsilon(v_i) \cap V'$. Take a sequence $(v^k)_{k=1}^\infty$ such that $v^k \in B_\varepsilon(v_i) \cap V'$ for each k and $\lim_{k \to \infty} v^k = \bar{v}$ (such a sequence exists by definition of $\bar{v}$.) By the assumption made for the purpose of contradiction, we have that $\mu\big([v_i - \varepsilon, v^k] \cap V'\big) = 0$ for each $k = 1, 2, ...$ Since the sets $\big([v_i - \varepsilon, v^k] \cap V'\big)_{k=1}^\infty$ form an increasing sequence of measurable sets, we have $0 = \mu\big([v_i - \varepsilon, \bar{v}] \cap V'\big) = \mu\big([v_i - \varepsilon, v_i + \varepsilon] \cap V'\big) = \mu\big(B_\varepsilon(v_i) \cap V'\big) > 0$, where the inequality is assumed at the beginning of the current paragraph. This is a contradiction.

Therefore, following the preceding paragraph, fix $v_i' \in B_\varepsilon(v_i) \cap V'$ with the property that $\mu\big([v_i - \varepsilon, v_i'] \cap V'\big) > 0$. Because $v_i' < v_i + \varepsilon$ and $\varepsilon < \frac{\delta}{2}$, we have $[v_i' - \delta, v_i'] \supseteq [v_i - \varepsilon, v_i']$. Hence, noting that $[v_i' - \delta, v_i'] \cap V'$ and $[v_i - \varepsilon, v_i'] \cap V'$ are measurable, $\mu\big([v_i' - \delta, v_i'] \cap V'\big) \geq \mu\big([v_i - \varepsilon, v_i'] \cap V'\big) > 0$.

We now show firm j can block allocation A via workers whose productivities fall in $[v_i' - \delta, v_i']$. To do so, we observe that $w_{-j}(v_i') < v_i' - \delta$ because $v_i' \in V'$. Thus, by the monotonicity of w_{-j}, $w_{-j}(v) < v_i' - \delta$ for all $v \in [v_i' - \delta, v_i']$. This implies $w_{-j}(v) < v$ for all $v \in [v_i' - \delta, v_i']$. Therefore, by the ongoing assumption (following the conclusions of Cases 4 and 5) that $\max\{w_1(v), w_2(v)\} = v$ for almost every $v \in [0,1]$, it follows that $w_j(v) = v$ for almost all $v \in [v_i' - \delta, v_i']$.

Consider $\tilde{A}_j$ where

$$\tilde{f}_j(v) := \begin{cases} f_j(v) & \text{if } v \notin [v_i' - \delta, v_i'], \\ f_{-j}(v) & \text{if } v \in [v_i' - \delta, v_i']. \end{cases} \qquad \tilde{w}_j(v) := \begin{cases} w_j(v) & \text{if } v \notin [v_i' - \delta, v_i'], \\ v_i' - \delta & \text{if } v \in [v_i' - \delta, v_i'] \text{ and } \tilde{f}_j(v) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\tilde{A}_j$ blocks A_j for the following reasons: First, it is obvious from construction that $\tilde{w}_j$ satisfies monotonicity. Condition 3 of the definition of block is satisfied for all $v \in [v_i' - \delta, v_i']$ (i.e. the workers previously employed by firm $-j$ are successfully poached and some workers are fired), and Condition 2 of the definition of block is satisfied for all $v \notin [v_i' - \delta, v_i']$ (i.e. workers in this set do not experience changes to hiring or wages). It is also the case that $\tilde{A}_j$ provides firm j with higher profit than A_j: newly poached workers from $[v_i' - \delta, v_i'] \cap V'$ (of whom there are a positive measure) are paid lower wages than their productivity in allocation $\tilde{A}_j$ while all newly-fired workers are from $[v_i' - \delta, v_i']$ and received wages equal to productivity from j in allocation A_j. This shows that A is not a core allocation.

As these six cases are exhaustive and none of them admits a core allocation, we have completed the argument that any core allocation must be a Bertrand allocation. $\square$

$\square$

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