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Abstract

This study addresses the house allocation problem in which each agent initially owns an object and some objects are social endowments. By using the concept of structure of ownership rights, David Gale's Top Trading Cycles (TTC) rule is extended to this problem (Pycia and Ünver, 2017). However, some TTC rules associated with a given structure of ownership rights violate *endowments-swapping-proofness*, which requires that no pair of agents can benefit from swapping their endowments before operating a given rule. Therefore, we identify a necessary and sufficient condition for a structure of ownership rights under which the associated TTC rule is *endowments-swapping-proof*. Based on this result, we characterize a subclass of TTC rules and provide new insights into the kidney exchange problem.

Keywords: top trading cycles rule; endowments-swapping-proofness; house allocation; kidney exchange.

JEL codes: C78; D47.

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1 Introduction

We study the house allocation problem with private and social endowments, where some houses (indivisible objects) are privately owned by agents and others are owned by no agents (i.e., the unowned houses are social endowments). We assume that each agent initially owns one house and has strict preferences over all houses. The mechanism designer must determine a matching between agents and houses, provided that each agent is assigned exactly one house. Typical real-life examples of this problem are on-campus housing (Abdulkadiroğlu and Sönmez, 1999) and kidney exchange (Roth, Sönmez, and Ünver, 2004).[1] The housing market (Shapley and Scarf, 1974) is a special case of our problem in which there are no social endowments.[2]

We focus on the possibility that a pair of agents benefits from swapping their endowments before the operation of a given rule. In kidney exchange, for example, this possibility exists when a pair of patients swap their donors by exploiting legal loopholes (i.e., fake marriage and fake adoption) to obtain higher-quality kidneys. *Endowments-swapping-proofness* requires that a rule be immune to such pairwise manipulation via endowments.[3]

David Gale's Top Trading Cycles (TTC) rule is of central importance for the study of the housing market, and it is well known that this rule is *endowments-swapping-proof* (Moulin, 1995; Fujinaka and Wakayama, 2018). Following Pycia and Ünver (2017), Gale's TTC rule is extended to our house allocation problem using the concept of "structure of ownership rights," which specifies the ownership rights of unmatched houses.[4,5] In contrast to the housing market, we find that some "extended" TTC rules violate *endowments-swapping-proofness*. More specifically, there is a structure of ownership rights such that the associated TTC rule violates *endowments-swapping-proofness*. Based on this fact, we identify a necessary and sufficient condition for a structure of ownership rights such that the associated TTC rule satisfies *endowments-swapping-proofness*.

[1] In the kidney exchange model, the social endowments correspond to "Good Samaritan donors" (Sönmez and Ünver, 2006).

[2] See, for example, Sönmez and Ünver (2024a,b) for extensive surveys of matching models without monetary transfers.

[3] *Endowments-swapping-proofness* pertains to pairwise manipulation. Moulin (1995) introduced the property pertaining to manipulation by a coalition of any size in the housing markets.

[4] We assume that a structure of ownership rights grants each agent the initial ownership rights of his endowment.

[5] The class of those extended TTC rules is equivalent to that of hierarchical exchange rules introduced by Pápai (2000).

We show that an independence condition called *restricted independence* is a necessary and sufficient condition of a structure of ownership rights for *endowments-swapping-proofness* of the associated TTC rules (Theorem 1). This condition states that even if a pair of agents swap their endowments before the implementation of a given rule, the ownership right of a house that they do not own remains unchanged.

In our problem, Pycia and Ünver (2017) has characterized the set of TTC rules associated with a given structure of ownership rights by *efficiency* (no chosen matching can be changed such that no agent is worse off and some agent is better off), *individual rationality* (no agent is worse off than he is with his endowment), and *group strategy-proofness* (no group of agents can benefit by jointly misrepresenting their preferences). In conjunction with this characterization, our result leads to a new characterization of the subclass of TTC rules (Corollary 1): A rule satisfies *endowments-swapping-proofness* in addition to the three axioms if and only if it is a TTC rule associated with a *restricted independent* structure of ownership rights.

Our result has an interesting implication for the kidney exchange model (Roth, Sönmez, and Ünver, 2004), which is an important special case of our problem. Roth, Sönmez, and Ünver (2004) propose six types of the Top Trading Cycles and Chains (TTCC) rules, each of which is an extension of Gale's TTC rule to accommodate the kidney exchange model. Moreover, they show that the "TTCC rule with chain selection rule *e*" is the only *efficient* and *strategy-proof* rule among these six TTCC rules. This TTCC rule is equivalent to the TTC rule associated with a structure of ownership rights (Krishna and Wang, 2007), and this structure satisfies *restricted independence*. Thus, our result shows that this desirable TTCC rule also satisfies *endowments-swapping-proofness*.

Finally, we extend our analysis to two more general models and provide new insights. The first one is the house allocation problem with existing tenants (Abdulkadiroğlu and Sönmez, 1999), where some agents do not initially own a house. The second problem is that some agents initially own multiple houses, assuming single-unit demand.

Related literature For the housing market, several characterizations of Gale's TTC rule have been proposed (for example, see Ma (1994), Takamiya (2001), Miyagawa (2002), Hashimoto and Saito (2015), and Ekici (2024)). The characterization of the TTC rule in terms of *endowments-swapping-proofness* was first presented by Fujinaka and Wakayama (2018). Since then, an increasing number of studies have explored *endowments-swapping-proofness*. Chen and Zhao (2021) provide another characterization of the TTC rule based on *endowments-swapping-proofness*. Tamura (2023) shows that Fujinaka and Wakayama's (2018) characterization still holds on the domain of single-dipped preferences. Fujinaka and Wakayama (2024) show that, once exchange constraints are imposed, no rule satisfies *individual rationality* and *endowments-swapping-proofness*. For the multi-type housing market, Feng (2023) proposes extensions of *endowments-swapping-proofness* and characterizes TTC rules accommodating the market based on the extended axioms.[6] Notably, these studies assume no social endowments. To the best of our knowledge, our study is the first to explore the implications of *endowments-swapping-proofness* for the house allocation problem with private and social endowments.

Pycia and Ünver (2017) introduce TTC rules associated with a given structure of ownership rights in the house allocation problem with existing tenants.[7] For this problem, a subclass of TTC rules has been extensively analyzed: Abdulkadiroğlu and Sönmez (1999) propose "You Request My House—I Get Your Turn" (YRMH-IGYT) rules; Karakaya, Klaus, and Schlegel (2019) propose the TTC rules associated with "ownership-adapted" priority structures.[8] As mentioned above, our result can be extended to this general problem. The structures of ownership rights defining these TTC rules satisfy *restricted independence*. Thus, we find that both the YRMH-IGYT rules and the TTC rules associated with ownership-adapted priority structures satisfy *endowments-swapping-proofness*. See Section 5 for more details.

Organization of the paper The rest of the paper is organized as follows. Section 2 describes our model and introduces *endowments-swapping-proofness*. Section 3 defines TTC rules using a structure of ownership rights. Section 4 explains the motivation to introduce *restricted independence* and presents our main results.

[6] In the multi-type housing market, there are multiple types of indivisible objects, and assume that each agent initially owns one object of each type and receives exactly one object of each type.

[7] Pycia and Ünver (2017) also introduce the class of Trading Cycles (TC) rules, which includes that of TTC rules. Each TC rule is based on a structure of control rights in which an agent can be not only the owner of a house but also its broker. They characterize the class of TC rules in terms of *efficiency* and *group strategy-proofness*.

[8] Sönmez and Ünver (2010) characterize YRMH-IGYT rules by means of *efficiency*, *individual rationality*, *strategy-proofness*, *weak neutrality*, and *consistency*. The class of the TTC rules associated with ownership-adapted priority structures includes YRMH-IGYT rules.

In Section 4, we also apply our results to the kidney exchange problem. Section 5 discusses how our main result can be extended to more general problems. Section 6 concludes by mentioning remaining issues. Proofs omitted from the main text are relegated to appendices.

2 Preliminaries

Let I and H denote the set of **agents** and **houses**, respectively. Suppose $2 \leq |I| \leq |H| < +\infty$.[9] Each agent $i \in I$ has a **strict** preference relation $\succ_i$ over H. We denote the induced weak preference relation of $\succ_i$ by $\succsim_i$. That is, for each $\{h, h'\} \subset H$, if $h \succsim_i h'$, then either $h \succ_i h'$ or $h = h'$. Let $\mathscr{P}$ be the set of strict preferences over H. A **preference profile** is a profile $(\succ_i)_{i \in I} \in \mathscr{P}^I$. For each $i \in I$, each $\succ_i \in \mathscr{P}$, and each $h \in H$, let $U^+(\succ_i, h) = \{h' \in H \colon h' \succ_i h\}$ be the **strict upper contour set** of h according to $\succ_i$.

A **matching** is a function $x \colon I \to H$ such that for each $h \in H$, $|x^{-1}(h)| \leq 1$. We write x_i for $x(i)$, where x_i represents the house that agent i receives under x. Let X be the set of all matchings. We denote a **private endowment** by $\omega = (\omega_i)_{i \in I} \in X$, where ω_i represents the house initially owned by agent i. Note that each agent owns one house as his endowment.[10] For each $\omega \in X$, let

$$H_0^\omega = \{h \in H \colon \forall\, i \in I,\ h \neq \omega_i\}$$

be the set of unowned houses (or **social endowments**) at ω.

An **economy** $e = (\succ, \omega)$ is a pair of a preference profile $\succ \in \mathscr{P}^I$ and a private endowment $\omega \in X$. Let $\mathscr{E} = \mathscr{P}^I \times X$ be the set of economies. For each $\omega \in X$ and each $\{i, j\} \subset I$, let $\omega^{i,j} \in X$ be such that $\omega_i^{i,j} = \omega_j$, $\omega_j^{i,j} = \omega_i$, and for each $k \in I \setminus \{i, j\}$, $\omega_k^{i,j} = \omega_k$. For each $e = (\succ, \omega) \in \mathscr{E}$, let $e^{i,j} = (\succ, \omega^{i,j})$.

A rule is a function $f \colon \mathscr{E} \to X$ that maps each economy $e \in \mathscr{E}$ to a matching $f(e) \in X$. We denote the house that agent i receives at e by $f_i(e)$.

We focus on a rule that satisfies the following property.

***Endowments-swapping-proofness*:** There exist no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subseteq I$ such that $f_i(e^{i,j}) \succ_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

[9] For any finite set A, $|A|$ denotes the cardinality of A.

[10] We will discuss the implications of relaxing this assumption in Section 5.

Endowments-swapping-proofness formalized by Fujinaka and Wakayama (2018) requires that no pair of agents benefits from swapping their endowments before operating the given rule.

3 Top trading cycles

If $|I| = |H|$, there are no social endowments. This case is called the "housing market," following Shapley and Scarf (1974). The most prominent rule for this market is Gale's TTC rule because it satisfies a list of desirable axioms, including *endowments-swapping-proofness* (Moulin, 1995; Fujinaka and Wakayama, 2018). This section points out that some of the "extended" TTC rules violate *endowments-swapping-proofness* in our problem.

3.1 Structure of ownership rights

We first introduce some notation. For each $I' \subseteq I$ with $I' \neq \emptyset$, a **submatching** on I' is a function $\sigma\colon I' \to H$ such that for each $h \in H$, $|\sigma^{-1}(h)| \leq 1$. We use the notation $\sigma^{\emptyset}$ to represent the situation in which all agents and all houses are unmatched. For convenience, we consider $\sigma^{\emptyset}$ as a submatching. Let $\mathcal{S}$ be the set of all submatchings. Note that $X \subset \mathcal{S}$ and $\sigma^{\emptyset} \in \mathcal{S}$. Let $\mathring{\mathcal{S}} = \mathcal{S} \setminus X$ be the set of submatchings except for matchings. For each $\sigma \in \mathcal{S}$, let $I_\sigma \subseteq I$ and $H_\sigma \subseteq H$ be the set of matched agents and matched houses under σ, respectively; that is, for each $\sigma \in \mathcal{S}$ with $\sigma\colon I' \to H$, let

$$I_\sigma = I' \quad \text{and} \quad H_\sigma = \left\{h \in H \colon \exists\, i \in I',\, h = \sigma(i)\right\}.$$

For convenience, let $I_{\sigma^{\emptyset}} = H_{\sigma^{\emptyset}} = \emptyset$. Moreover, let $\overline{I_\sigma} = I \setminus I_\sigma$ and $\overline{H_\sigma} = H \setminus H_\sigma$ denote the set of unmatched agents and unmatched houses under σ, respectively. We often regard $\sigma \in \mathcal{S}$ as a set of matched agent-house pairs; that is, for each $\sigma \in \mathcal{S}$ with $\sigma\colon I' \to H$, let

$$\sigma = \{(i,h) \in I \times H \colon i \in I' \text{ and } h = \sigma(i)\} \quad \text{and} \quad \sigma^{\emptyset} = \emptyset.$$

A **structure of ownership rights** is a collection of functions

$$\left\{o_\sigma^\omega \colon \overline{H_\sigma} \to \overline{I_\sigma}\right\}_{(\omega,\sigma) \in X \times \mathring{\mathcal{S}}}.$$

We write the structure of ownership rights as $\mathcal{O}$. Given a private endowment $\omega \in X$ and a submatching $\sigma \in \mathring{\mathcal{S}}$, $o^{\omega}_{\sigma}(h) = i$ denotes that unmatched house $h \in \overline{H_\sigma}$ is owned by unmatched agent $i \in \overline{I_\sigma}$.

We impose the following two assumptions on $\mathcal{O}$:

(O1) For each $\omega \in X$ and each $i \in I$, $o^{\omega}_{\sigma^{\varnothing}}(\omega_i) = i$.

(O2) For each $\omega \in X$, each $\{\sigma, \sigma'\} \subset \mathring{\mathcal{S}}$ with $\sigma \subset \sigma'$, each $i \in \overline{I_{\sigma'}}$, and each $h \in \overline{H_{\sigma'}}$,

$$o^{\omega}_{\sigma}(h) = i \implies o^{\omega}_{\sigma'}(h) = i.$$

(O1) means that each agent is granted the initial ownership right of his endowment. (O2) means that if house h is owned by agent i at a submatching and both h and i are unmatched at a larger submatching, then h is still owned by i at the larger submatching.[11]

Remark 1. By (O1) and (O2), for each $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$ and each $h \in \overline{H_\sigma}$, if there is $i \in \overline{I_\sigma}$ with $h = \omega_i$, then $o^{\omega}_{\sigma}(h = \omega_i) = i$. That is, if house h and its initial owner are unmatched at σ, then $h(= \omega_i)$ continues to be owned by the initial owner at σ. $\Diamond$

Remark 2. Our definition of a structure of ownership rights differs from Pycia and Ünver's (2017) definition in that it depends on a private endowment as well as on the submatching already formed. This specification allows us to consider the situations in which a pair of agents swap their endowments. $\Diamond$

3.2 TTC rules

Given a structure of ownership rights $\mathcal{O}$, a **TTC rule associated with $\mathcal{O}$** is a rule $TTC^{\mathcal{O}} \colon \mathscr{E} \to X$ that selects a matching obtained via the following algorithm:

TTC ALGORITHM WITH $\mathcal{O}$. Let $e = (\succ, \omega) \in \mathscr{E}$. Let $\mathbb{N}$ be the set of natural numbers.

- For each $r \in \mathbb{N}$, let $\sigma[e, r] \in \mathcal{S}$ be the submatching formalized in Round r.[12] Let $\sigma[e, 0] = \sigma^{\varnothing}$. If $\sigma[e, r] \in \mathring{\mathcal{S}}$, then the algorithm proceeds with Round $r + 1$; otherwise, the algorithm terminates, and then $TTC^{\mathcal{O}}(e) = \sigma[e, r]$.

[11] (O2) is referred to as the "consistency" of a structure of ownership rights in Pycia and Ünver (2017).

[12] Formally, it should be $\sigma^{\mathcal{O}}[e, r]$. When defining notations related to $\mathcal{O}$, we will sometimes omit $\mathcal{O}$ for simplicity if it does not cause any confusion.

- Round $r \in \mathbb{N}$ is defined as follows: The set of remaining agents and houses in Round r is $\overline{I_{\sigma[e,r-1]}} \cup \overline{H_{\sigma[e,r-1]}}$. Each $i \in \overline{I_{\sigma[e,r-1]}}$ points to his best house among $\overline{H_{\sigma[e,r-1]}}$ according to $\succ_i$. Each $h \in \overline{H_{\sigma[e,r-1]}}$ points to $o^{\omega}_{\sigma[e,r-1]}(h) \in \overline{I_{\sigma[e,r-1]}}$. Given that the number of agents and houses is finite, there is a sequence of houses and agents $C = (h_1(= h_{N+1}), i_1, \ldots, h_N, i_N)$, called a **cycle**, such that for each $n \in \{1, 2, \ldots, N\}$, h_n points to i_n and i_n points to h_{n+1}. Each agent in each cycle is assigned the house he points to.

- The submatching $\sigma[e, r]$ is the union of $\sigma[e, r-1]$ and the set of agent-house pairs matched in Round r.

Here, we introduce some notation related to the TTC algorithm with $\mathcal{O}$. For each $e \in \mathscr{E}$, each $r \in \mathbb{N}$, and each $\{m, m'\} \subset I \cup H$, we write $m \overset{(e,r)}{\to} m'$ to represent that m points to m' in Round r of the TTC algorithm with $\mathcal{O}$ at e. For convenience, we often regard cycle $C = (h_1, i_1, \ldots, h_N, i_N)$ as $C = \{h_1, i_1, \ldots, h_N, i_N\}$. Thus, for example, $m \in C$ or $\{m, m'\} \subset C$ means that $m \in \{h_1, i_1, \ldots, h_N, i_N\}$ or $\{m, m'\} \subset \{h_1, i_1, \ldots, h_N, i_N\}$, respectively. For each $e \in \mathscr{E}$ and each $r \in \mathbb{N}$, let $\mathbb{C}^{\mathcal{O}}(e, r)$ be the set of cycles in Round r of the TTC algorithm with $\mathcal{O}$ at e. Hence, if $C = (h_1(= h_{N+1}), i_1, \ldots, h_N, i_N) \in \mathbb{C}^{\mathcal{O}}(e, r)$, then for each $n \in \{1, 2, \ldots, N\}$, each $i_n \in \overline{I_{\sigma[e,r-1]}}$, and each $h_n \in \overline{H_{\sigma[e,r-1]}}$,

- $o^{\omega}_{\sigma[e,r-1]}(h_n) = i_n$;
- for each $h \in \overline{H_{\sigma[e,r-1]}} \setminus \{h_{n+1}\}$, $h_{n+1} \succ_{i_n} h$;
- $TTC^{\mathcal{O}}_{i_n}(e) = h_{n+1}$.

Let $I(e, r)$ (resp. $H(e, r)$) be the set of agents (resp. houses) that belong to a cycle in Round r of the TTC algorithm with $\mathcal{O}$ at e; that is,

$$I(e, r) = I_{\sigma[e,r]} \setminus I_{\sigma[e,r-1]} = \bigcup_{C \in \mathbb{C}^{\mathcal{O}}(e,r)} \{C \cap I\},$$
$$H(e, r) = H_{\sigma[e,r]} \setminus H_{\sigma[e,r-1]} = \bigcup_{C \in \mathbb{C}^{\mathcal{O}}(e,r)} \{C \cap H\}.$$

Thus, if $i \in I(e, r)$ (resp. $h \in H(e, r)$), then agent i is matched with a house (resp. house h is matched with an agent) in Round r of the TTC algorithm with $\mathcal{O}$ at e.

Finally, we mention two features of the TTC rules. The first states that if an unmatched agent owns an unmatched house in some round of the TTC algorithm, the house is not matched earlier than the agent. The second, which follows from the first, states that the initial endowment of an agent is not matched earlier than the agent.

Fact 1. *Let $e = (\succ, \omega) \in \mathscr{E}$, $r \in \mathbb{N}$, $i \in \overline{I_{\sigma[e,r-1]}}$, and $h \in \overline{H_{\sigma[e,r-1]}}$ with $o^{\omega}_{\sigma[e,r-1]}(h) = i$. Let $(r_i, r_h) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $h \in H(e, r_h)$. Then, $r_i \leq r_h$.*

Proof. Let $r' = \min\{r_i, r_h\}$. Note that $i \in \overline{I_{\sigma[e,r'-1]}}$ and $h \in \overline{H_{\sigma[e,r'-1]}}$ and by $r \leq r'$, $\sigma[e, r-1] \subset \sigma[e, r'-1]$. Then, by (O2), $o^{\omega}_{\sigma[e,r-1]}(h) = i$ implies $o^{\omega}_{\sigma[e,r'-1]}(h) = i$. Thus, $h \overset{(e,r')}{\rightarrow} i$. Suppose on the contrary that $r' = r_h < r_i$. Let $C' \in \mathbb{C}^{\mathcal{O}}(e, r')$ with $h \in C'$. Then, by $h \overset{(e,r'=r_h)}{\rightarrow} i$, $i \in C' \in \mathbb{C}^{\mathcal{O}}(e, r')$. However, by $r' = r_h < r_i$, $i \notin C' \in \mathbb{C}^{\mathcal{O}}(e, r')$, which is a contradiction. $\square$

Fact 2. *Let $e = (\succ, \omega) \in \mathscr{E}$ and $i \in I$. Let $(r_i, r_h) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $\omega_i \in H(e, r_{\omega_i})$. Then, $r_i \leq r_{\omega_i}$.*

Proof. Note that $i \in \overline{I_{\sigma[e,0]}}$, $\omega_i \in \overline{H_{\sigma[e,0]}}$, and by (O1), $o^{\omega}_{\sigma[e,0]}(\omega_i) = i$. It follows from Fact 1 that $r_i \leq r_{\omega_i}$. $\square$

3.3 Non-endowments-swapping-proof TTC rules

As noted above, Gale's TTC rule satisfies *endowments-swapping-proofness* in the housing market. However, there exists a structure of ownership rights $\mathcal{O}$ such that $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness* in our problem. The following example illustrates this fact.

Example 1. Let $I = \{1, 2, 3\}$ and $H = \{h_1, h_2, h_3, h_4\}$. Let $e = (\succ, \omega) \in \mathscr{E}$ be such that $\omega = (h_1, h_2, h_3)$ and

$\succ_1$	$\succ_2$	$\succ_3$
h_4	h_1	h_4
h_1	h_2	h_3
$\vdots$	$\vdots$	$\vdots$

Then, $H^{\omega}_0 = \{h_4\}$. We consider a structure of ownership rights $\mathcal{O}$ such that $o^{\omega}_{\sigma^{\varnothing}}(h_4) = 3$ and $o^{\omega^{1,2}}_{\sigma^{\varnothing}}(h_4) = 1$. Then, $TTC^{\mathcal{O}}(e) = (h_1, h_2, h_4)$ and $TTC^{\mathcal{O}}_1(e^{1,2}) = (h_4, h_1, h_3)$. Hence,

$$TTC^{\mathcal{O}}_1(e^{1,2}) = h_4 \succ_1 h_1 = TTC^{\mathcal{O}}_1(e);$$

$$TTC_2^{\mathcal{O}}(e^{1,2}) = h_1 \succ_2 h_2 = TTC_2^{\mathcal{O}}(e),$$

which implies that $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness*. ■

4 Results

4.1 Independence condition

Example 1 motivates us to identify a necessary and sufficient condition for a structure of ownership rights under which the associated TTC rule is *endowments-swapping-proof*. The source of the TTC rule not being *endowments-swapping-proof* in Example 1 is that when a pair $\{1,2\}$ swaps their endowments, the ownership right of h_4 changes from agent 3 to agent 1. Based on this observation, one might think that if a structure of ownership rights excludes such a "dependent" ownership, the associated TTC rule satisfies *endowments-swapping-proofness*. We formulate the independent condition as follows:

Independence: For each $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$, each $\{i, j\} \subseteq \overline{I_\sigma}$, and each $h \in \overline{H_\sigma}$ with $o_\sigma^\omega(h) \notin \{i, j\}$, $o_\sigma^{\omega^{i,j}}(h) = o_\sigma^\omega(h)$.

Independence states that when a pair of unmatched agents swap their endowments, the ownership rights remain unchanged except for the ownership rights of the houses they own.

We can see that *independence* is sufficient for *endowments-swapping-proofness* of TTC rules. However, this condition is not necessary, as shown in the following example.

Example 2. Suppose $I = \{1,2,3,4,5\}$ and $H = \{h_1, h_2, h_3, h_4, h_5, h_6\}$. Let $\hat{\omega} = (h_1, h_2, h_3, h_4, h_5)$ and $\Sigma = \{(\hat{\omega}, \sigma) \in X \times \mathring{\mathcal{S}} \colon (1, h_4) \in \sigma\}$. We now consider a **priority order** $\rhd$, a linear order over I. For each $\{i, j\} \subseteq I$ with $i \neq j$, $i \rhd j$ means that agent i has a higher priority than agent j under $\rhd$. Let $(\rhd^0, \rhd^\Sigma)$ be a pair of priority orders such that

$\rhd^0$	$\rhd^\Sigma$
1	1
4	5
5	4
2	2
3	3

We consider a structure of ownership rights $\mathcal{O}^{\Sigma}$ that satisfies the following conditions: For each $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$ and each $h \in \overline{H_\sigma}$,

(E1) if there is $j \in \overline{I_\sigma}$ with $h = \omega_j$, then $o^{\omega}_{\sigma}(h) = j$;

(E2) if there is no $j \in \overline{I_\sigma}$ with $h = \omega_j$ and $(\omega, \sigma) \notin \Sigma$, then for each $i \in \overline{I_\sigma} \setminus \{o^{\omega}_{\sigma}(h)\}$, $o^{\omega}_{\sigma}(h) \rhd^0 i$;

(E3) if there is no $j \in \overline{I_\sigma}$ with $h = \omega_j$ and $(\omega, \sigma) \in \Sigma$, then for each $i \in \overline{I_\sigma} \setminus \{o^{\omega}_{\sigma}(h)\}$, $o^{\omega}_{\sigma}(h) \rhd^{\Sigma} i$.

That is, $\mathcal{O}^{\Sigma}$ is such that given $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$ and $h \in \overline{H_\sigma}$, if the initial owner is unmatched, then he continues to own h; otherwise, and in addition, if $(\omega, \sigma) \notin \Sigma$ (reps. $(\omega, \sigma) \in \Sigma$), the unmatched agent with the highest priority under $\rhd^0$ (resp. $\rhd^{\Sigma}$) owns h.[13] The structure $\mathcal{O}^{\Sigma}$ violates *independence*. In particular, consider $(\hat{\omega}, \hat{\sigma} = \{(1, h_4)\})$, $\{2, 3\} \subset \overline{I_{\hat{\sigma}}}$, and $h_6 \in \overline{H_{\hat{\sigma}}}$. By (E2) and (E3), $o^{\hat{\omega}}_{\hat{\sigma}}(h_6) = 5$ and $o^{\hat{\omega}^{2,3}}_{\hat{\sigma}}(h_6) = 4$. This implies that $\mathcal{O}^{\Sigma}$ violates *independence*. However, as we will show later, $TTC^{\mathcal{O}^{\Sigma}}$ satisfies *endowments-swapping-proofness*. ■

In Example 2, the submatching $\hat{\sigma} = \{(1, h_4)\}$ is never formalized in any round of the TTC algorithm with $\mathcal{O}^{\Sigma}$ at $\hat{\omega}$ because $\hat{\omega}_4 = h_4$ is not matched earlier than agent 4 (see Fact 2). That is, although the requirement of *independence* is not satisfied for endowment-submatching pairs such as $(\hat{\omega}, \hat{\sigma})$, the fact does not affect whether $TTC^{\mathcal{O}^{\Sigma}}$ satisfies *endowments-swapping-proofness*. This suggests that it is sufficient to impose the independence condition only on endowment-submatching pairs in which the submatching can be formalized in some round of the TTC algorithm at the endowment. To weaken *independence* in this way, we introduce the "attainability" of a submatching σ, which indicates whether σ is formalized in a round of the TTC algorithm at a given private endowment.

Given a structure of ownership rights $\mathcal{O}$ and a private endowment $\omega \in X$, a submatching $\sigma \in \mathcal{S}$ is **attainable at ω under $\mathcal{O}$** if there is a sequence of submatchings $(\mu_v)_{v=1}^{u}$ such that

(S1) for each $v \in \{1, 2, \ldots, u\}$,

$$\mu_v = \left\{ \left(i_1^v (= i^v_{N^v+1}), \sigma(i_1^v)\right), \left(i_2^v, \sigma(i_2^v)\right), \ldots, \left(i^v_{N^v}, \sigma(i^v_{N^v})\right) \right\};$$

[13] (E1) implies that $\mathcal{O}^{\Sigma}$ satisfies (O1). The proof of the fact that $\mathcal{O}^{\Sigma}$ satisfies (O2) is given in Appendix D.

(S2) for each $\{v, v'\} \subset \{1, 2, \ldots, u\}$ with $v \neq v'$,

$$\mu_v \cap \mu_{v'} = \emptyset \quad \text{and} \quad \bigcup_{v=1}^{u} \mu_v = \sigma;$$

(S3) for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in I_{\mu_v}$,

$$o^{\omega}_{\mu^{v-1}}(\sigma(i_n^v)) = i_{n+1}^v,$$

where $\mu^0 = \emptyset = \sigma^{\emptyset}$ and $\mu^{v-1} = \bigcup_{z=1}^{v-1} \mu_z$.

We denote the set of attainable submatchings at ω under $\mathcal{O}$ by $\mathcal{M}^{\mathcal{O}}(\omega)$. Note that $\sigma^{\emptyset} \in \mathcal{M}^{\mathcal{O}}(\omega)$.[14]

The following proposition states that the attainability concept identifies the submatchings that can be formalized in a round of the TTC algorithm.

Proposition 1. *Given a structure of ownership rights $\mathcal{O}$, for each $\omega \in X$,*

$$\mathcal{M}^{\mathcal{O}}(\omega) = \left\{ \sigma \in \mathcal{S} \colon \exists \succ \in \mathscr{P}^I, \exists r \in \mathbb{N} \cup \{0\}, \sigma = \sigma[(\succ, \omega), r] \right\}.$$

Proof. See Appendix A. □

The following condition is a weaker version of *independence*, imposing the independent condition only on endowment-attainable submatching pairs.

Restricted independence: For each $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$ with $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$, each $\{i, j\} \subset \overline{I_\sigma}$, and each $h \in \overline{H_\sigma}$ with $o^{\omega}_{\sigma}(h) \notin \{i, j\}$, $o^{\omega^{i,j}}_{\sigma}(h) = o^{\omega}_{\sigma}(h)$.

4.2 Necessary and sufficient condition

The following theorem states that *restricted independency* of the structure of ownership rights is necessary and sufficient for *endowments-swapping-proofness* of the associated TTC rule.

Theorem 1. *Given a structure of ownership rights $\mathcal{O}$, the TTC rule associated with $\mathcal{O}$ satisfies endowments-swapping-proofness if and only if $\mathcal{O}$ satisfies restricted independence.*

Proof. See Appendix B. □

[14] We should consider the sequence $(\mu_1 (= \sigma^{\emptyset}))$.

Remark 3. The structure $\mathcal{O}^{\Sigma}$ in Example 2 satisfies *restricted independence* (see Appendix D). Therefore, by Theorem 1, $TTC^{\mathcal{O}^{\Sigma}}$ is *endowments-swapping-proof*. $\diamondsuit$

Remark 4. If $|I| = |H|$ (that is, the housing market), any structure of ownership rights satisfies *restricted independence* (see Appendix D). Notably, in the housing market, any TTC rule associated with a structure of ownership rights is equivalent to Gale's TTC rule. Thus, it follows from Theorem 1 that Gale's TTC rule is *endowments-swapping-proof* (Moulin, 1995; Fujinaka and Wakayama, 2018). $\diamondsuit$

Pycia and Ünver (2017) propose a characterization of TTC rules associated with a given structure of ownership rights in terms of the following three axioms: A chosen matching cannot be changed in such a way that all agents are at least as well off as they were at the matching and at least one agent is better off; no one is made worse off by participating in the rule; no group of agents can gain by jointly misrepresenting their preferences.

Efficiency: For each $e = (\succ, \omega) \in \mathscr{E}$, there is no $x \in X$ such that for each $i \in I$, $x_i \succsim_i f_i(e)$ and for some $j \in I$, $x_j \succ_j f_j(e)$.

Individual rationality: For each $e = (\succ, \omega) \in \mathscr{E}$ and each $i \in I$, $f_i(e) \succsim_i \omega_i$.

Group strategy-proofness: For each $e = (\succ, \omega) \in \mathscr{E}$, there are no $J \subseteq I$ and $e' = ((\succ'_J, \succ_{-J}), \omega) \in \mathscr{E}$ such that for each $i \in J$, $f_i(e') \succsim_i f_i(e)$ and for some $j \in J$, $f_j(e') \succ_j f_j(e)$.

Theorem 2 (Pycia and Ünver, 2017)**.** *A rule satisfies efficiency, individual rationality, and group strategy-proofness if and only if it is a TTC rule associated with a structure of ownership rights.*

As a corollary to Theorem 1 and Theorem 2, we obtain a characterization of TTC rules associated with a given *restricted independent* structure of ownership rights.

Corollary 1. *A rule satisfies efficiency, individual rationality, group strategy-proofness, and endowments-swapping-proofness if and only if it is a TTC rule associated with a restricted independent structure of ownership rights.*

4.3 Application to kidney exchange

Kidney exchange is a particularly important real-life application of our model. To apply our theorem to kidney exchange, we describe the kidney exchange model.[15] In this subsection, we assume that

$$H = \left\{h_1, h_2, \ldots, h_n, h_0^1, h_0^2, \ldots, h_0^n\right\};$$
$$\forall \{\omega, \omega'\} \subset X,\ H_0^{\omega} = H_0^{\omega'} = \left\{h_0^1, h_0^2, \ldots, h_0^n\right\}.$$

We simply write $H_0 = \{h_0^1, h_0^2, \ldots, h_0^n\}$ for H_0^{ω}. Here, each $h \in H \setminus H_0$ (resp. $h_0 \in H_0$) represents a "kidney of a living-donor paired with a patient" (resp. a "waitlist option for deceased-donor transplant"). Note that $|I| = |H_0|$ (that is, $|H| = 2|I|$). We impose the following assumptions on $\mathscr{E}$:

(KE1) For each $e = (\succ, \omega) \in \mathscr{E}$, each $i \in I$, and each $\{h_0^j, h_0^k\} \subset H_0$,

$$j < k \implies h_0^j \succ_i h_0^k.$$

(KE2) For each $e = (\succ, \omega) \in \mathscr{E}$, for each $i \in I$, and each $h \in H \setminus H_0$,

- $h \succ_i h_0^n (\in H_0) \implies [\forall h_0 \in H_0,\ h \succ_i h_0]$;
- $h_0^1 (\in H_0) \succ_i h \implies [\forall h_0 \in H_0,\ h_0 \succ_i h]$.

Let $\mathscr{E}^{\mathrm{KE}} \subset \mathscr{E}$ be a set of economies that satisfy (KE1) and (KE2). Note that for each $e \in \mathscr{E}^{\mathrm{KE}}$ and each $\{i, j\} \subset I$, $e^{i,j} \in \mathscr{E}^{\mathrm{KE}}$.

Roth, Sönmez, and Ünver (2004) generalize Gale's TTC rule to apply it to this model. Then, they advocate **Top Trading Cycles and Chains (TTCC) rules**, each of which selects a matching via an algorithm that repeatedly finds not only cycles but also "chains."[16] A **chain** is a sequence of houses and agents $(\bar{h}_1, \bar{i}_1, \bar{h}_2, \bar{i}_2, \ldots, \bar{i}_{N-1}, \bar{h}_N)$ such that for each $n \in \{1, 2, \ldots, N-1\}$, $\bar{h}_n$ points to $\bar{i}_n$ and $\bar{i}_n$ points to $\bar{h}_{n+1}$, and $\bar{h}_N \in H_0$. In the TTCC algorithm, each house $\bar{h}_0 \in H_0$ does not point to an agent; therefore, a chain can emerge. The TTCC algorithm must select

[15] The kidney exchange model was originally proposed by Roth, Sönmez, and Ünver (2004). However, the model here is based on Krishna and Wang (2007) and differs slightly from Roth, Sönmez, and Ünver's model. The additional assumptions imposed below are necessary to consider the kidney exchange model by Roth, Sönmez, and Ünver (2004) under our setting. See Krishna and Wang (2007) for a detailed explanation of the assumptions.

[16] For conciseness, we omit the detailed description of the TTCC algorithm. See Roth, Sönmez, and Ünver (2004) and Krishna and Wang (2007) for the formal definition of the TTCC rules.

one chain only if there is no cycle at some round. If multiple chains exist simultaneously, the selection of a chain (and whether it is removed or retained in the subsequent round) depends on a certain "chain selection rule."

Roth, Sönmez, and Ünver (2004) propose six types of chain selection rules and define six TTCC rules accordingly. They demonstrate that only one of these rules, called **TTCC rule with chain selection rule *e***, satisfies both *efficiency* and *strategy-proofness*. In addition, this rule is equivalent to the TTC rule associated with the following structure of ownership rights $\mathcal{O}^{\rhd}$ (Krishna and Wang, 2007):

(KE3) For each $(\omega, \sigma) \in X \times \mathring{\mathcal{S}}$ and each $h \in \overline{H_\sigma}$, if there is no $j \in \overline{I_\sigma}$ with $h = \omega_j$, then for each $i \in \overline{I_\sigma} \setminus \{o^\omega_\sigma(h)\}$, $o^\omega_\sigma(h) \rhd i$, where $\rhd$ is an arbitrary priority order.

By Remark 1, for each $h \in \overline{H_\sigma}$, if there is $j \in \overline{I_\sigma}$ with $h = \omega_j$, $o^\omega_\sigma(h) = j$. That is, $\mathcal{O}^{\rhd}$ is such that for each $h \in H$, its initial owner, if any, is ordered the highest and the others are ordered according to $\rhd$; if h is unmatched, then the highest-ranked agent among the unmatched agents owns h. This structure $\mathcal{O}^{\rhd}$ satisfies *restricted independence*.[17] Therefore, by invoking Theorem 1, we find that the TTC rule associated with $\mathcal{O}^{\rhd}$ satisfies *endowments-swapping-proofness*. That is, the TTCC rule with chain selection rule e satisfies *endowments-swapping-proofness* as well as *efficiency* and *strategy-proofness*.

5 Extensions

In this section, we extend our findings in two directions. First, we allow that some agents have no endowments.[18] We can extend our results to this problem. In particular, our proof of Theorem 1 can be applied to this extended setting because *endowments-swapping-proofness* pertains to endowments-swapping by a pair of "existing tenants." Second, we allow agents to own multiple houses initially. Our results hold for this setting as long as each agent receives one house at a matching.

[17] The kidney exchange model is considered the house allocation problem with existing tenants in which there are no "new applicants." We will discuss the general problem and show that $\mathcal{O}^{\rhd}$ satisfies *restricted independence* in Section 5.1. See Remark 5.

[18] Abdulkadiroğlu and Sönmez (1999) introduce this extended problem called the "house allocation problem with existing tenants."

5.1 House allocation with existing tenants

Let $I = I_E \cup I_N$, where I_E and I_N denote the set of "existing tenants" who initially own houses and the set of "new applicants" who initially do not own any house, respectively. Let $\mathcal{S}_{I_E}$ be the set of submatchings with $\sigma\colon I_E \to H$. Notably, $\mathcal{S}_{I_E}$ is the set of private endowments for this problem. With abuse of notation, we denote a private endowment by $\omega = (\omega_i)_{i \in I_E} \in \mathcal{S}_{I_E}$. Let $\mathscr{E}^{\mathrm{ET}} = \mathscr{P}^I \times \mathcal{S}_{I_E}$ be the set of economies.[19]

We adjust *endowments-swapping-proofness* by requiring that a rule is immune to endowments-swapping collusion by a pair of existing tenants.

Endowments-swapping-proofness: There exist no $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{ET}}$ and $\{i, j\} \subseteq I_E$ such that $f_i(e^{i,j}) \succ_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

A structure of ownership rights $\mathcal{O}$ is a collection of functions

$$\left\{o^{\omega}_{\sigma} : \overline{H_\sigma} \to \overline{I_\sigma}\right\}_{(\omega,\sigma) \in \mathcal{S}_{I_E} \times \mathring{\mathcal{S}}},$$

and (O1) and (O2) are defined as follows:

(O1) For each $\omega \in \mathcal{S}_{I_E}$ and each $i \in I_E$, $o^{\omega}_{\sigma^{\emptyset}}(\omega_i) = i$.

(O2) For each $\omega \in \mathcal{S}_{I_E}$, each $\{\sigma, \sigma'\} \subset \mathring{\mathcal{S}}$ with $\sigma \subset \sigma'$, each $i \in \overline{I_{\sigma'}}$, and each $h \in \overline{H_{\sigma'}}$,

$$o^{\omega}_{\sigma}(h) = i \implies o^{\omega}_{\sigma'}(h) = i.$$

We also adjust *restricted independence* as follows:

Restricted independence: For each $(\omega, \sigma) \in \mathcal{S}_{I_E} \times \mathring{\mathcal{S}}$ with $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$, each $\{i, j\} \subset \overline{I_\sigma} \cap I_E$, and each $h \in \overline{H_\sigma}$ with $o^{\omega}_{\sigma}(h) \notin \{i, j\}$, $o^{\omega^{i,j}}_{\sigma}(h) = o^{\omega}_{\sigma}(h)$.

We now focus on a structure of ownership rights based on priority orders. For each $h \in H$, let $\rhd_h$ be a **priority order of *h***, which is a linear order over I. Suppose that a structure of ownership rights $\mathcal{O}$ satisfies the following:

(O3) For each $(\omega, \sigma) \in \mathcal{S}_{I_E} \times \mathring{\mathcal{S}}$ and each $h \in \overline{H_\sigma}$, if there is no $j \in \overline{I_\sigma}$ with $h = \omega_j$, then for each $i \in \overline{I_\sigma} \setminus \{o^{\omega}_{\sigma}(h)\}$, $o^{\omega}_{\sigma}(h) \rhd_h i$.

[19] All the other notations, such as $TTC^{\mathcal{O}}$ and $\mathcal{M}^{\mathcal{O}}(\omega)$, are defined similarly. Thus, we omit the definitions.

Recall again Remark 1: By (O1) and (O2), for each $h \in \overline{H_\sigma}$, if there is $i \in \overline{I_\sigma}$ with $h = \omega_i$, $o^\omega_\sigma(h) = i$. Therefore, $\mathcal{O}$ that satisfies (O3) is such that for each $h \in H$, its initial owner, if any, is ordered the highest and the others are ordered according to $\rhd_h$; if h is unmatched, then the highest-ranked agent among the unmatched agents owns h. Notably, the ordering excluding the initial owners may vary from house to house. The following proposition states that (O3) is a stronger condition than *restricted independence*.

Proposition 2. *If a structure of ownership rights satisfies (O3), it satisfies restricted independence.*

Proof. See Appendix C. □

It is noteworthy that the structure of ownership rights that satisfies (O3) is equivalent to the "ownership-adapted priority structure" introduced by Karakaya, Klaus, and Schlegel (2019). Theorem 1 and Proposition 2 together imply that any TTC rule associated with an ownership-adapted priority structure also satisfies *endowments-swapping-proofness*.

Remark 5. Suppose that a structure of ownership rights $\mathcal{O}$ satisfies (O3) for a profile of priority orders, $(\rhd_h)_{h \in H}$, such that for each $\{h', h''\} \subset H$, $\rhd_{h'} = \rhd_{h''}$. Then, the TTC rule associated with $\mathcal{O}$ is equivalent to the YRMH-IGYT rule (Abdulkadiroğlu and Sönmez, 1999).[20] Thus, any YRMH-IGYT rule also satisfies *endowments-swapping-proofness*. Additionally, if $I_N = \emptyset$, then this structure $\mathcal{O}$ is equivalent to $\mathcal{O}^\rhd$ in the kidney exchange model of Section 4.3. Therefore, by Proposition 2, $\mathcal{O}^\rhd$ satisfies *restricted independence*. ◊

Remark 6. Karakaya, Klaus, and Schlegel (2019) introduce an "acyclicity" condition and characterize the TTC rules associated with ownership-adapted acyclic priority structures by means of *efficiency*, *individual rationality*, *strategy-proofness*, *consistency*, and *reallocation-proofness* (or *non-bossiness*).[21] Notably, according to Proposition 2, an ownership-adapted priority structure satisfies *restricted independence*, regardless of whether the priority structure is acyclic. Recall that Pycia and

[20] See, for example, Sönmez and Ünver (2010) for a formal description of the YRMH-IGYT rule.

[21] The "acyclicity" condition states that rankings of agents do not reverse substantially between priority orders of two houses. Also, *consistency* requires that the removal of a set of agents and houses does not affect the matching of the remaining agents and houses; *reallocation-proofness* requires that no pair of agents can gain by misrepresenting their preferences and swapping their matched houses within the pair; and *non-bossiness* requires that no one can affect other agents' matchings without changing his matching. See Karakaya, Klaus, and Schlegel (2019) for the formal definitions of these properties.

Ünver (2017) characterize the TTC rules associated with a given structure of ownership rights. Note also that Theorem 1 and Corollary 1 hold in this setting because the proof of Theorem 1 can be applied. By comparing the characterizations in the two prior studies with our own, we see that the class of rules characterized by Corollary 1 lies between the class characterized by Karakaya, Klaus, and Schlegel (2019) and the class characterized by Pycia and Ünver (2017). $\Diamond$

5.2 Multiple endowments

We introduce some definitions for the multiple endowment case, which differ slightly from those for the single endowment case.[22] With abuse of notation, for each $i \in I$, let $\omega_i \subset H$ denote the set of i's endowments. Suppose that for each $i \in I$, $\omega_i \neq \emptyset$, for each $\{i, i'\} \subset I$ with $i \neq i'$, $\omega_i \cap \omega_{i'} = \emptyset$, and $\bigcup_{i \in I} \omega_i \subset H$. Let $\omega = (\omega_i)_{i \in I}$ be a private endowment and Ω be the set of private endowments. An economy is defined as $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{ME}} = \mathscr{P}^I \times \Omega$. For each $\omega \in \Omega$ and each $\{i, j\} \subset I$, let

$$\Omega^{i,j}(\omega) = \left\{ \omega' \in \Omega \colon \begin{array}{l} \omega'_i \cup \omega'_j = \omega_i \cup \omega_j; \\ \forall k \in I \setminus \{i, j\},\ \omega'_k = \omega_k \end{array} \right\}.$$

Endowments-swapping-proofness is extended for the multiple endowments case as follows:

***Endowments-swapping-proofness*:** There exist no $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{ME}}$, $\{i, j\} \subset I$, and $\omega' \in \Omega^{i,j}(\omega)$ such that $f_i(\succ, \omega') \succ_i f_i(e)$ and $f_j(\succ, \omega') \succ_j f_j(e)$.

A structure of ownership rights is denoted by

$$\{o^{\omega}_{\sigma} : \overline{H_\sigma} \to \overline{I_\sigma}\}_{(\omega,\sigma) \in \Omega \times \mathring{\mathcal{S}}},$$

and (O1) is defined as follows:

(O1) For each $\omega \in \Omega$, each $i \in I$, and each $h \in \omega_i$, $o^{\omega}_{\sigma^{\emptyset}}(h) = i$.

Restricted independence is also extended for this case as follows:

***Restricted independence*:** For each $(\omega, \sigma) \in \Omega \times \mathring{\mathcal{S}}$ with $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$, each $\{i, j\} \subset \overline{I_\sigma}$, each $h \in \overline{H_\sigma}$ with $o^{\omega}_{\sigma}(h) \notin \{i, j\}$, and each $\omega' \in \Omega^{i,j}(\omega)$, $o^{\omega'}_{\sigma}(h) = o^{\omega}_{\sigma}(h)$.

[22] All the other notations are defined similarly. Thus, we omit the definitions.

Applying the proof of Theorem 1 to the multiple endowments case under the single-unit demand assumption is straightforward with these modifications.

6 Concluding remarks

We conclude with a list of the remaining questions and a proposal for future research directions.

6.1 Characterization without efficiency

It is of interest to identify the rules that satisfy *individual rationality*, *group strategy-proofness*, and *endowments-swapping-proofness*. In the housing market, Fujinaka and Wakayama (2018) characterize Gale's TTC rule based on these three axioms. Feng (2023) extends this characterization to the multi-type housing market. Building on these studies, a natural question arises: which rules satisfy these axioms in our problem? In our problem, dropping *efficiency* from Corollary 1 can enlarge the class of rules. For instance, consider the following rule $f^{\star}$: for each $e = (\succ, \omega) \in \mathscr{E}$, $f^{\star}(e)$ selects the matching obtained by Gale's TTC algorithm, assuming that the set of available houses is restricted to $H \setminus H_0^{\omega}$. Importantly, this rule $f^{\star}$ satisfies the three axioms, although it is not associated with a structure of ownership rights. It remains an open question to characterize the class of rules obtained by dropping *efficiency* from Corollary 1.

6.2 General priority structure

In Section 5.1, we assume that a priority order of a house is independent of endowments. Without this assumption, some ownership-adopted priority structures may violate *restricted independence* even if the agents are ranked according to the priority order. Given $\omega \in X$ and $h \in H$, let $\rhd_h^{\omega}$ be a **priority order for h at ω**. Consider a structure of ownership rights $\mathcal{O}$ that satisfies the following: for a private endowment $\omega \in X$, a house $h \in H_0^{\omega}$, and a pair of agents $\{i, j\} \subset I$,

- for each $k \in I \setminus \{i, j\}$, $k \rhd_h^{\omega} i \rhd_h^{\omega} j$ and $i \rhd_h^{\omega^{i,j}} j \rhd_h^{\omega^{i,j}} k$;
- for each $(\omega', \sigma) \in \{\omega, \omega^{i,j}\} \times \mathring{\mathcal{S}}$ with $h \in \overline{H_\sigma}$ and each $k \in \overline{I_\sigma} \setminus \{o_\sigma^{\omega'}(h)\}$, $o_\sigma^{\omega'}(h) \rhd_h^{\omega'} k$.

This structure $\mathcal{O}$ violates *restricted independence* because $\sigma^{\emptyset} \in \mathcal{M}^{\mathcal{O}}(\omega)$, $o^{\omega}_{\sigma^{\emptyset}}(h) \notin \{i, j\}$, and $o^{\omega^{i,j}}_{\sigma^{\emptyset}}(h) = i$. This implies that $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness*. That is, the TTC rule associated with an ownership-adapted priority structure may violate *endowments-swapping-proofness* if the priority order of a house depends on endowments. It is an open question to identify the class of profiles of such general priority orders such that the TTC rules associated with the priority orders satisfy *endowments-swapping-proofness*.

6.3 Multi-demands

Atlamaz and Klaus (2007) investigate the object reallocation problem in which each agent has multi-demands and multiple endowments.[23] They show that on the domain of separable preferences, no rule satisfies *efficiency*, *individual rationality*, and a weaker property than *endowments-swapping-proofness*, called *transfer-proofness* (no pair of agents can benefit from transferring part of one agent's endowments to the other agent before the operation of a given rule).[24] Thus, this incompatibility holds with *endowments-swapping-proofness* in the multi-demands setting. One way to avoid this negative result is to restrict the domain of preferences. For the multi-demands model, several "generalized" TTC rules have been proposed in previous studies (Pápai, 2003; Biró, Klijn, and Pápai, 2022; Altuntaş, Phan, and Tamura, 2023; Feng, 2023). It remains for future research to investigate whether these generalized TTC rules satisfy *endowments-swapping-proofness* on a restricted domain.

A Appendix: Proof of Proposition 1

Let $\omega \in X$ and

$$\mathcal{S}^*(\omega) = \left\{ \sigma \in \mathcal{S} \colon \exists \succ \in \mathscr{P}^I, \exists r \in \mathbb{N} \cup \{0\}, \sigma = \sigma[(\succ, \omega), r] \right\}.$$

[23] Atlamaz and Klaus (2007) consider the problem in which an agent's endowment can be empty and there can be social endowments. Therefore, their problem is considered a generalization of house allocation with existing tenants.

[24] *Transfer-proofness* is weaker than *endowments-swapping-proofness* as defined for the multiple endowments case in Section 5.2. This is because *endowments-swapping-proofness* pertains to all forms of manipulation involving the swapping of endowments between agents, including cases in which one agent transfers only part of his endowments to another. Also note that *transfer-proofness* discussed by Atlamaz and Klaus (2007) is stronger than the one proposed here because it allows the receiver in the manipulating pair to be indifferent after the rule is operated. Nevertheless, their proof shows that the incompatibility persists even under this weaker condition.

We prove $\mathcal{M}^{\mathcal{O}}(\omega) = \mathcal{S}^*(\omega)$ in two steps.

Step 1: $\boldsymbol{\mathcal{M}^{\mathcal{O}}(\omega) \subseteq \mathcal{S}^*(\omega)}$. Let $\rho \in \mathcal{M}^{\mathcal{O}}(\omega)$. If $\rho = \sigma^{\emptyset}$, then $\sigma^{\emptyset} = \sigma[(\succ, \omega), 0]$ for each $\succ \in \mathscr{P}^I$. Thus, we below consider the case $\rho \neq \sigma^{\emptyset}$. Then, there exists a sequence of submatchings $(\mu_v)_{v=1}^{u}$ that satisfies (S1) to (S3). Note that by (S1), for each $v \in \{1, 2, \ldots, u\}$,

$$\mu_v = \left\{ (i_1^v (= i_{N^v+1}^v), \rho(i_1^v)), (i_2^v, \rho(i_2^v)), \ldots, (i_{N^v}^v, \rho(i_{N^v}^v)) \right\} \subset \rho.$$

For each $v \in \{1, 2, \ldots, u\}$, let $P^v = \left\{ \rho(i_1^1), \rho(i_1^2), \ldots, \rho(i_1^v) \right\}$. We proceed in three steps.

▶ **Substep 1-1: Constructing the economy.** Let $\succ' \in \mathscr{P}^I$ be such that

(i) for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in I_{\mu^v}$,

- for each $m \in \{1, 2, \ldots, v-2\}$, $\rho(i_1^m) \succ'_{i_n^v} \rho(i_1^{m+1})$;
- $U^+(\succ'_{i_n^v}, \rho(i_n^v)) = P^{v-1}$;

(ii) for each $k \in \overline{I_\rho}$,

- for each $m \in \{1, 2, \ldots, u-1\}$, $\rho(i_1^m) \succ'_k \rho(i_1^{m+1})$;
- $U^+(\succ'_k, \rho(i_1^u)) = P^{u-1}$.

This preference profile $\succ'$ can be described as follows:

$$\begin{array}{ccccccc}
\succ'_{i_n^1 \in I_{\mu_1}} & \succ'_{i_n^2 \in I_{\mu_1}} & \cdots & \succ'_{i_n^v \in I_{\mu_v}} & \cdots & \succ'_{i_n^u \in I_{\mu_u}} & \succ'_{k \in \overline{I_\rho}} \\
\hline
\rho(i_n^1) & \rho(i_1^1) & \cdots & \rho(i_1^1) & \cdots & \rho(i_1^1) & \rho(i_1^1) \\
\vdots & \rho(i_n^2) & & \rho(i_1^2) & \cdots & \rho(i_1^2) & \rho(i_1^2) \\
 & \vdots & & \vdots & & \vdots & \vdots \\
 & & & \rho(i_1^{v-1}) & \cdots & \rho(i_1^{v-1}) & \rho(i_1^{v-1}) \\
 & & & \rho(i_n^v) & & \vdots & \vdots \\
 & & & \vdots & & \rho(i_1^{u-1}) & \rho(i_1^{u-1}) \\
 & & & & & \rho(i_n^u) & \rho(i_1^u) \\
 & & & & & \vdots & \vdots
\end{array}$$

Let $e' = (\succ', \omega) \in \mathscr{E}$. Note that $\sigma[e', 0] = \mu^0 = \emptyset = \sigma^{\emptyset}$.

▶ **Substep 1-2: For each $\boldsymbol{v \in \{1, 2, \ldots, u\}}$, $\boldsymbol{\sigma[e', v] = \mu^v}$.** Let $v \in \{1, 2, \ldots, u\}$. Suppose that for each $z \in \{0, 1, \ldots, v-1\}$, $\sigma[e', z] = \mu^z$. Then, by the definition of $\succ'$ (Substep 1-1),

- for each $k \in \overline{I_{\sigma[e',v-1]}} \setminus I_{\mu_v} = \overline{I_{\mu^{v-1}}} \setminus I_{\mu_v}$, $k \overset{(e',v)}{\to} \rho(i_1^v)$;
- for each $i_n^v \in I_{\mu_v}$, $i_n^v \overset{(e',v)}{\to} \rho(i_n^v)$.

In addition, by (S3), for each $i_n^v \in I_{\mu_v}$, $o^{\omega}_{\sigma[e',v-1]}(\rho(i_n^v)) = o^{\omega}_{\mu^{v-1}}(\rho(i_n^v)) = i_{n+1}^v$. These imply

$$\mathbb{C}^{\mathcal{O}}(e',v) = \left\{(\rho(i_{N^v}^v), i_1^v, \rho(i_1^v), i_2^v, \dots, \rho(i_{N^v-1}^v), i_{N^v}^v)\right\} \quad \text{and} \quad \sigma[e',v] = \mu^v.$$

▶ **Substep 1-3: Concluding.** By Substep 1-2 and (S2), $\sigma[e',u] = \bigcup_{v=1}^{u} \mu_v = \rho$, which implies $\rho \in S^*(\omega)$.

Step 2: $\boldsymbol{\mathcal{S}^*(\omega) \subseteq \mathcal{M}^{\mathcal{O}}(\omega)}$. Let $\sigma \in \mathcal{S}^*(\omega)$. Then,

$$\exists \succ \in \mathscr{P}^I, \exists r \in \mathbb{N} \cup \{0\}, \sigma = \sigma[(\succ, \omega), r].$$

Let $e = (\succ, \omega)$. If $r = 0$, then it is obvious that $\sigma[e,0] = \sigma^{\emptyset} \in \mathcal{M}^{\mathcal{O}}(\omega)$. Thus, we assume $r > 0$. Let $u = \left|\bigcup_{t=1}^{r} \mathbb{C}^{\mathcal{O}}(e,t)\right|$. Then, we can order the cycles in $\bigcup_{t=1}^{r} \mathbb{C}^{\mathcal{O}}(e,t)$ as $(C_v)_{v=1}^{u}$, where for each $\{v, v'\} \subset \{1, 2, \dots, u\}$,

$$[C_v \in \mathbb{C}^{\mathcal{O}}(e,s), C_{v'} \in \mathbb{C}^{\mathcal{O}}(e,s'), \text{and } s < s'] \implies v < v'.$$

In addition, for each $v \in \{1, 2, \dots, u\}$, we can regard

$$C_v = (h_1^v, i_1^v, h_2^v, i_2^v, \dots, h_{N^v}^v, i_{N^v}^v)$$

as a submatching $\mu_v \in \mathcal{S}$ such that

$$\mu_v = \left\{(i_1^v(= i_{N^v+1}^v), h_2^v(= \sigma[e,r](i_1^v)), \dots, (i_{N^v}^v, h_1^v(= \sigma[e,r](i_{N^v}^v)))\right\}.$$

The sequence of submatchings $(\mu_v)_{v=1}^{u}$ satisfies both (S1) and (S2). Therefore, we below show that (S3) holds. Pick any $v \in \{1, 2, \dots, u\}$ and any $i_n^v \in I_{\mu_v}$. Let $C_v \in \mathbb{C}^{\mathcal{O}}(e,s)$ and $v^* \in \{0, 1, \dots, u\}$ be such that

$$v^* = \begin{cases} 0 & \text{if } s = 1 \\ \max\left\{v' \in \{1, 2, \dots, u\} : C_{v'} \in \mathbb{C}^{\mathcal{O}}(e, s-1)\right\} & \text{if } s \geq 2. \end{cases}$$

Note that $\mu^{v^*} = \sigma[e, s-1] \subset \mu^{v-1}$. Then, $i_n^v \in \overline{I_{\mu^{v-1}}}$, $h_{n+1}^v \in \overline{H_{\mu^{v-1}}}$, and by

$C_v \in \mathbb{C}^{\mathcal{O}}(e,s)$,

$$o^{\omega}_{\sigma[e,s-1]}(\sigma[e,r](i^v_n)) = o^{\omega}_{\sigma[e,s-1]}(h^v_{n+1}) = i^v_{n+1},$$

which together with (O2) implies $o^{\omega}_{\mu^{v-1}}(\sigma[e,r](i^v_n)) = i^v_{n+1}$. Hence, we have $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$. $\square$

B Appendix: Proof of Theorem 1

B.1 The if part

We begin by proving an important lemma for the proof of the if part. This lemma states that the swapping of endowments between a pair of agents does not affect the cycles formed in the round prior to the round in which one of the two agents was removed from the TTC algorithm at the original economy, nor does it change the timing of the formation of the cycles.

Lemma 1. *Suppose that a structure of ownership rights $\mathcal{O}$ satisfies restricted independence. Let $e = (\succ, \omega) \in \mathscr{E}$ and $\{i,j\} \subset I$ with $i \neq j$. Let $(r_i, r_j) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. Then, for each $t \in \{1, 2, \ldots, \min\{r_i, r_j\} - 1\}$,*

$$\mathbb{C}^{\mathcal{O}}(e,t) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j},t).$$

Proof. Let $r = \min\{r_i, r_j\}$ and $t \in \{1, 2, \ldots, r-1\}$. Suppose that

$$\forall s \in \{1,2,\ldots,t-1\},\ \mathbb{C}^{\mathcal{O}}(e,s) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j},s). \tag{1}$$

We now show $\mathbb{C}^{\mathcal{O}}(e,t) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j},t)$. Let

$$C^* = (h_1(= h_{N+1}), i_1, h_2, i_2, \ldots, h_N, i_N) \in \mathbb{C}^{\mathcal{O}}(e,t).$$

Let $\{i_n, h_n\} \subset C^*$. Then,

$$TTC^{\mathcal{O}}_{i_n}(e) = h_{n+1} \quad \text{and} \quad o^{\omega}_{\sigma[e,t-1]}(h_n) = i_n. \tag{2}$$

By $i_n \in I(e,t)$ and $t \leq r-1$,

$$i_n \notin \{i,j\}. \tag{3}$$

Then, $h_n \notin \{\omega_i, \omega_j\}$; otherwise, by $\{i,j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,t-1]}}$ and (2), $o^{\omega}_{\sigma[e,t-1]}(h_n) = i_n \in \{i,j\}$, which is a contradiction to (3). By (2) and (O2), there is $s_n \in$

$\{1,2,\ldots,t\}$ such that for each $s \in \{1,2,\ldots,t\}$,

- if $s < s_n$, then $o^{\omega}_{\sigma[e,s-1]}(h_n) \neq i_n$;
- if $s_n \leq s$, then $o^{\omega}_{\sigma[e,s-1]}(h_n) = i_n$.

Let $(r'_{i_n}, r'_{h_n}, r'_{h_{n+1}}) \in \mathbb{N}^3$ be such that $i_n \in I(e^{i,j}, r'_{i_n})$, $h_n \in H(e^{i,j}, r'_{h_n})$, and $h_{n+1} \in H(e^{i,j}, r'_{h_{n+1}})$. We proceed in eight steps.

Step 1: $h_{n+1} \succsim_{i_n} TTC^{\mathcal{O}}_{i_n}(e^{i,j})$. Let $h \in U^{+}(\succ_{i_n}, h_{n+1})$. By $i_n \in I(e,t)$ and (2), $h \in H_{\sigma[e,t-1]}$. Then, there are $s' \in \{1,2,\ldots,t-1\}$ and $\ell \in I(e,s')$ with $TTC^{\mathcal{O}}_{\ell}(e) = h$. It then follows from (1) that for each $s \in \{1,2,\ldots,t-1\}$ and each $k \in I(e,s)$,

$$k \in I(e^{i,j}, s) \quad \text{and} \quad TTC^{\mathcal{O}}_{k}(e) = TTC^{\mathcal{O}}_{k}(e^{i,j}).$$

This implies $\ell \in I(e^{i,j}, s')$ and $TTC^{\mathcal{O}}_{\ell}(e^{i,j}) = TTC^{\mathcal{O}}_{\ell}(e) = h$. Hence, $TTC^{\mathcal{O}}_{i_n}(e^{i,j}) \neq h$. That is, $h_{n+1} \succsim_{i_n} TTC^{\mathcal{O}}_{i_n}(e^{i,j})$.

Step 2: $r'_{h_{n+1}} \leq r'_{i_n}$. It follows from Step 1.

Step 3: $s_n \leq r'_{h_n}$. Suppose on the contrary that $r'_{h_n} < s_n$. Let $\ell = o^{\omega}_{\sigma[e,r'_{h_n}-1]}(h_n)$. By $r'_{h_n} < s_n$, $\ell \neq i_n$. This together with $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n$ implies $\ell \in I_{\sigma[e,s_n-1]}$.[25] Then, there is $r_\ell \in \{r'_{h_n}, r'_{h_n}+1, \ldots, s_n - 1 (\leq t-1)\}$ such that $\ell \in I(e, r_\ell)$. This together with (1) implies that

$$\exists C^{\ell} \in \mathbb{C}^{\mathcal{O}}(e, r_\ell) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j}, r_\ell),\ \ell \in C^{\ell} \tag{4}$$

and $\ell \in I(e^{i,j}, r_\ell)$. By $r'_{h_n} \leq r_\ell$, it holds that $\ell \in \overline{I_{\sigma[e^{i,j}, r'_{h_n}-1]}}$. In addition, it follows from (1) and $r'_{h_n} - 1 < s_n \leq t$ that

$$\sigma[e, r'_{h_n}-1] \subseteq \sigma[e^{i,j}, r'_{h_n}-1]. \tag{5}$$

Note that by Proposition 1, $\sigma[e, r'_{h_n}-1] \in \mathcal{M}^{\mathcal{O}}(\omega)$ and by $r'_{h_n} < s_n \leq t \leq r-1$, $\{i,j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,r'_{h_n}-1]}}$. Further, $h_n \in \overline{H_{\sigma[e,r'_{h_n}-1]}}$ and $o^{\omega}_{\sigma[e,r'_{h_n}-1]}(h_n) = \ell \notin$

[25] If $\ell \notin I_{\sigma[e,s_n-1]}$, by $\sigma[e, r'_{h_n}-1] \subset \sigma[e, s_n-1]$, (O2) implies $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = \ell$, which contradicts $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n$.

$\{i,j\}$.[26] Then, *restricted independence* implies

$$o^{\omega^{i,j}}_{\sigma[e,r'_{h_n}-1]}(h_n) = o^{\omega}_{\sigma[e,r'_{h_n}-1]}(h_n) = \ell. \tag{6}$$

Then, by $h_n \in \overline{H_{\sigma[e^{i,j},r'_{h_n}-1]}}$ and $\ell \in \overline{I_{\sigma[e^{i,j},r'_{h_n}-1]}}$, (O2) together with (5) and (6) implies

$$o^{\omega^{i,j}}_{\sigma[e^{i,j},r'_{h_n}-1]}(h_n) = \ell.$$

Thus,

$$\exists C^{+\ell} \in \mathbb{C}^{\mathcal{O}}(e^{i,j}, r'_{h_n}),\ \{h_n, \ell\} \subset C^{+\ell}.$$

By $\ell \in C^{\ell} \in \mathbb{C}^{\mathcal{O}}(e^{i,j}, r_\ell)$, this together with (4) implies $h_n \in C^{+\ell} = C^{\ell} \in \mathbb{C}^{\mathcal{O}}(e, r_\ell)$, which contradicts $h_n \in C^* \in \mathbb{C}^{\mathcal{O}}(e,t)$ where $r_\ell \leq s_n - 1 < t$.

Step 4: $r'_{i_n} \leq r'_{h_n}$. Note that $h_n \in \overline{H_{\sigma[e,s_n-1]}}$ and by (2) and (3), $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n \notin \{i,j\}$. By Proposition 1, $\sigma[e, s_n - 1] \in \mathcal{M}^{\mathcal{O}}(\omega)$. Since $\{i,j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,s_n-1]}}$, *restricted independence* implies

$$o^{\omega^{i,j}}_{\sigma[e,s_n-1]}(h_n) = o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n. \tag{7}$$

It then follows from (1) and $s_n - 1 \leq t - 1$ that

$$\sigma[e, s_n - 1] \subseteq \sigma[e^{i,j}, s_n - 1]. \tag{8}$$

This together with Step 3 implies that

$$\sigma[e, s_n - 1] \overset{(8)}{\subseteq} \sigma[e^{i,j}, s_n - 1] \overset{\text{Step 3}}{\subset} \sigma[e^{i,j}, r'_{h_n} - 1]. \tag{9}$$

In addition, by $h_n \in H(e^{i,j}, r'_{h_n})$, $h_n \in \overline{H_{\sigma[e^{i,j},r'_{h_n}-1]}}$. If $i_n \in \overline{I_{\sigma[e^{i,j},r'_{h_n}-1]}}$, (O2) together with (7) and (9) implies

$$o^{\omega^{i,j}}_{\sigma[e^{i,j},r'_{h_n}-1]}(h_n) = i_n.$$

Then, by Fact 1. we have $r'_{i_n} \leq r'_{h_n}$. If $i_n \in I_{\sigma[e^{i,j},r'_{h_n}-1]}$, we have $r'_{i_n} < r'_{h_n}$.

Step 5: $r^* = r'_{i_n} = r'_{h_n}$. By Steps 2 and 4,

$$r'_{i_1} \overset{\text{Step 4}}{\leq} r'_{h_1} \overset{\text{Step 2}}{\leq} r'_{i_N} \overset{\text{Step 4}}{\leq} r'_{h_N} \overset{\text{Step 2}}{\leq} r'_{i_{N-1}} \overset{\text{Step 4}}{\leq} \dots \overset{\text{Step 2}}{\leq} r'_{i_2} \overset{\text{Step 4}}{\leq} r'_{h_2} \overset{\text{Step 2}}{\leq} r'_{i_1}.$$

[26] If $o^{\omega}_{\sigma[e,r'_{h_n}-1]}(h_n) = \ell \in \{i,j\}$, by $\{i,j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,s_n-1]}}$ and $\sigma[e, r'_{h_n} - 1] \subset \sigma[e, s_n - 1]$, (O2) implies $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = \ell \in \{i,j\}$. Since $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n$, we have $\ell = i_n \in \{i,j\}$, which contradicts (3).

Then, there is $r^* \in \mathbb{N}$ such that $r^* = r'_{i_n} = r'_{h_n}$.

Step 6: $C^* \in \mathbb{C}^{\mathcal{O}}(e^{i,j}, r^*)$. Step 5 implies that

$$\{i_1, i_2, \ldots, i_N\} \subset \overline{I_{\sigma[e^{i,j}, r^*-1]}} \quad \text{and} \quad \{h_1, h_2, \ldots, h_N\} \subset \overline{H_{\sigma[e^{i,j}, r^*-1]}}. \tag{10}$$

Recall that by Step 1, $h_{n+1} \succsim_{i_n} TTC^{\mathcal{O}}_{i_n}(e^{i,j})$. Hence, $TTC^{\mathcal{O}}_{i_n}(e^{i,j}) = h_{n+1} = TTC^{\mathcal{O}}_{i_n}(e)$; if $h_{n+1} \succ_{i_n} TTC^{\mathcal{O}}_{i_n}(e^{i,j})$, agent i_n does not receive his best object in $\overline{H_{\sigma[e^{i,j}, r^*-1]}}$ in Round $r^* = r'_{i_n}$ at $e^{i,j}$. Note that $h_n \in \overline{H_{\sigma[e, s_n-1]}}$ and by (2) and (3), $o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n \notin \{i, j\}$. Further, by Proposition 1, $\sigma[e, s_n - 1] \in \mathcal{M}^{\mathcal{O}}(\omega)$. Since $\{i, j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,s_n-1]}}$, *restricted independence* implies

$$o^{\omega^{i,j}}_{\sigma[e,s_n-1]}(h_n) = o^{\omega}_{\sigma[e,s_n-1]}(h_n) = i_n. \tag{11}$$

By Steps 3 and 5,

$$s_n \overset{\text{Step 3}}{\leq} r^* \overset{\text{Step 5}}{=} r'_{h_n}. \tag{12}$$

Then, it follows from (1) and $s_n - 1 \leq t - 1$ that

$$\sigma[e, s_n - 1] \subseteq \sigma[e^{i,j}, s_n - 1]. \tag{13}$$

By (12) and (13),

$$\sigma[e, s_n - 1] \overset{(13)}{\subseteq} \sigma[e^{i,j}, s_n - 1] \overset{(12)}{\subseteq} \sigma[e^{i,j}, r^* - 1]. \tag{14}$$

Then, by (10), (O2) together with (11) and (14) implies $o^{\omega^{i,j}}_{\sigma[e^{i,j},r^*-1]}(h_n) = i_n$. Hence, $C^* = (h_1, i_1, h_2, i_2, \ldots, h_N, i_N) \in \mathbb{C}^{\mathcal{O}}(e^{i,j}, r^*)$.

Step 7: $r^* = t$. Suppose on the contrary that $t \neq r^*$. There are two cases.

• **Case 7-1: $t < r^*$.** Note that $h_n \in \overline{H_{\sigma[e,t-1]}}$ and by (2) and (3), $o^{\omega}_{\sigma[e,t-1]}(h_n) = i_n \notin \{i, j\}$. Further, by Proposition 1, $\sigma[e, t-1] \in \mathcal{M}^{\mathcal{O}}(\omega)$. Since $\{i, j\} \subset \overline{I_{\sigma[e,r-1]}} \subset \overline{I_{\sigma[e,t-1]}}$, *restricted independence* implies

$$o^{\omega^{i,j}}_{\sigma[e,t-1]}(h_n) = o^{\omega}_{\sigma[e,t-1]}(h_n) = i_n. \tag{15}$$

Then, it follows from (1) that

$$\sigma[e, t - 1] \subseteq \sigma[e^{i,j}, t - 1]. \tag{16}$$

In addition, by $t < r^*$, $i_n \in \overline{I_{\sigma[e^{i,j},r^*-1]}} \subset \overline{I_{\sigma[e^{i,j},t-1]}}$ and $h_n \in \overline{H_{\sigma[e^{i,j},r^*-1]}} \subset \overline{H_{\sigma[e^{i,j},t-1]}}$. Hence, (O2) together with (15) and (16) implies

$$o^{\omega^{i,j}}_{\sigma[e^{i,j},t-1]}(h_n) = i_n.$$

Then, by $C^* \notin \mathbb{C}^{\mathcal{O}}(e^{i,j},t)$ and $C^* \in \mathbb{C}^{\mathcal{O}}(e^{i,j},r^*)$ (Step 6), there are $i_m \in C^*$ and $\bar{h} \in \overline{H_{\sigma[e^{i,j},t-1]}}$ such that $\bar{h} \succ_{i_m} h_{m+1} = TTC^{\mathcal{O}}_{i_m}(e^{i,j})$ and $i_m \overset{(e^{i,j},t)}{\to} \bar{h}$. Note that by (16),

$$\bar{h} \in \overline{H_{\sigma[e^{i,j},t-1]}} \subseteq \overline{H_{\sigma[e,t-1]}}.$$

However, by $\bar{h} \succ_{i_m} h_{m+1} = TTC^{\mathcal{O}}_{i_m}(e)$ and $i_m \in C^* \in \mathbb{C}^{\mathcal{O}}(e,t)$,

$$\bar{h} \in H_{\sigma[e,t-1]},$$

which is a contradiction.

- **Case 7-2: $r^* < t$.** Then, by Steps 3 and 5,

$$s_n \overset{\text{Step 3}}{\leq} r^* \overset{\text{Step 5}}{=} r'_{h_n} \leq t-1,$$

which implies $o^{\omega}_{\sigma[e,(t-1)-1]}(h_n) = i_n$. Then, by $C^* \notin \mathbb{C}^{\mathcal{O}}(e,t-1)$ and $C^* \in \mathbb{C}^{\mathcal{O}}(e,t)$, there are $i_m \in C^*$ and $\bar{h} \in H(e,t-1)$ such that $\bar{h} \succ_{i_m} h_{m+1} = TTC^{\mathcal{O}}_{i_m}(e)$ and $i_m \overset{(e,t-1)}{\to} \bar{h}$. Further, by (1), $\bar{h} \in H(e,t-1)$ implies

$$\bar{h} \in H(e^{i,j},t-1). \tag{17}$$

However, by Step 6, $\bar{h} \succ_{i_m} h_{m+1} = TTC^{\mathcal{O}}_{i_m}(e^{i,j})$ and $i_m \in C^* \in \mathbb{C}^{\mathcal{O}}(e^{i,j},r^*)$, which imply $\bar{h} \in H_{\sigma[e^{i,j},r^*-1]}$. Then, there is $r'_{\bar{h}} \in \{1,2,\ldots,r^*-1\}$ such that

$$\bar{h} \in H(e^{i,j},r'_{\bar{h}}).$$

Since $r'_{\bar{h}} < t-1$, this contradicts (17).

Step 8: Concluding. By Steps 6 and 7, $C^* \in \mathbb{C}^{\mathcal{O}}(e^{i,j},t)$. Hence, we have $\mathbb{C}^{\mathcal{O}}(e,t) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j},t)$. □

We are ready to prove the if part.

Proof of the if part. Suppose on the contrary that a structure of ownership rights $\mathcal{O}$ satisfies *restricted independence* but $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness*.

Then, there exist $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset I$ such that

$$TTC_i^{\mathcal{O}}(e^{i,j}) \succ_i TTC_i^{\mathcal{O}}(e) \quad \text{and} \quad TTC_j^{\mathcal{O}}(e^{i,j}) \succ_j TTC_j^{\mathcal{O}}(e). \tag{18}$$

Let $(r_i, r_j) \in \mathbb{N}^2$ be such that $i \in I(e, r_i)$ and $j \in I(e, r_j)$. Without loss of generality, we assume $r_i \leq r_j$. If $r_i = 1$, then by (18), agent i does not receive his best object in H in Round 1 at e, which is a contradiction. Thus, $r_i \geq 2$. By (18), letting $\bar{h} = TTC_i^{\mathcal{O}}(e^{i,j})$, $\bar{h} \in H_{\sigma[e, r_i - 1]}$. Then, there exist $t \in \{1, 2, \ldots, r_i - 1\}$ and $k \in I(e, t)$ such that $TTC_k^{\mathcal{O}}(e) = \bar{h}$. In addition, by Lemma 1, $\mathbb{C}^{\mathcal{O}}(e, t) \subseteq \mathbb{C}^{\mathcal{O}}(e^{i,j}, t)$, which implies $TTC_k^{\mathcal{O}}(e^{i,j}) = TTC_k^{\mathcal{O}}(e) = \bar{h}$. Since $i \neq k$, this contradicts $TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h}$. $\square$

B.2 The only if part

We prove this part by the contrapositive. Suppose that a structure of ownership rights $\mathcal{O}$ violates *restricted independence*. Then, there exist $\omega \in X$, $\rho \in \mathcal{M}^{\mathcal{O}}(\omega)$, $\{i, j\} \subset \overline{I_\rho}$, and $\bar{h} \in \overline{H_\rho}$ such that

$$o_\rho^\omega(\bar{h}) \notin \{i, j\} \quad \text{and} \quad o_\rho^{\omega^{i,j}}(\bar{h}) \neq o_\rho^\omega(\bar{h}).$$

By $\rho \in \mathcal{M}^{\mathcal{O}}(\omega)$, there exists a sequence of submatchings $(\mu_v)_{v=1}^u$ that satisfies (S1) to (S3). Without loss of generality, we assume that for each $v \in \{0, 1, \ldots, u-1\}$ and each $h \in \overline{H_{\mu^v}}$ with $o_{\mu^v}^\omega(h) \notin \{i, j\}$,

$$o_{\mu^v}^{\omega^{i,j}}(h) = o_{\mu^v}^\omega(h).\text{[27]} \tag{19}$$

Let $\ell = o_\rho^\omega(\bar{h}) \notin \{i, j\}$ and $\ell' = o_\rho^{\omega^{i,j}}(\bar{h}) (\neq \ell)$. Note that $\{\ell, \ell'\} \subset \overline{I_\rho}$. There are two cases.

• **Case 1: $\ell' \in \{i, j\}$.** Without loss of generality, we assume $\ell' (= o_\rho^{\omega^{i,j}}(\bar{h})) = i$. Note that by (S1), for each $v \in \{1, 2, \ldots, u\}$,

$$\mu_v = \left\{ (i_1^v (= i_{N^v+1}^v), \rho(i_1^v)), (i_2^v, \rho(i_2^v)), \ldots, (i_{N^v}^v, \rho(i_{N^v}^v)) \right\} \subset \rho.$$

[27] For each $v \in \{0, 1, \ldots, u\}$, $\mu^v \in \mathcal{M}^{\mathcal{O}}(\omega)$, and $\{i, j\} \subset \overline{I_\rho} = \overline{I_{\mu^u}} \subset \overline{I_{\mu^v}}$. Hence, we can consider the smallest $v' \in \{0, 1, \ldots, u\}$ such that (i) there exists $\bar{h} \in \overline{H_{\mu^{v'}}}$ such that $o_{\mu^{v'}}^\omega(\bar{h}) \notin \{i, j\}$ and $o_{\mu^{v'}}^{\omega^{i,j}}(\bar{h}) \neq o_{\mu^{v'}}^\omega(\bar{h})$, and (ii) for each $v \in \{0, 1, \ldots, v'-1\}$ and each $h \in \overline{H_{\mu^v}}$ with $o_{\mu^v}^\omega(h) \notin \{i, j\}$, $o_{\mu^v}^{\omega^{i,j}}(h) = o_{\mu^v}^\omega(h)$.

Recall that for each $v \in \{1, 2, \ldots, u\}$, let $P^v = \{\rho(i_1^1), \rho(i_1^2), \ldots, \rho(i_1^v)\}$. We proceed in four steps.

Step 1-1: Constructing the economy. Let $\succ \in \mathscr{P}^I$ be such that:

(i) for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in I_{\mu^v}$,

- for each $m \in \{1, 2, \ldots, v-2\}$, $\rho(i_1^m) \succ_{i_n^v} \rho(i_1^{m+1})$;
- $U^+(\succ_{i_n^v}, \rho(i_n^v)) = P^{v-1}$;

(ii) for each $k \in \overline{I_\rho}$ and each $m \in \{1, 2, \ldots, u-1\}$, $\rho(i_1^m) \succ_k \rho(i_1^{m+1})$;

(iii) for each $k \in \overline{I_\rho} \setminus \{i, j, \ell\}$, $U^+(\succ_k, \rho(i_1^u)) = P^{u-1}$;

(iv) $\rho(i_1^u) \succ_i \overline{h} \succ_i \omega_i$ and $U^+(\succ_i, \omega_i) = P^u \cup \{\overline{h}\}$;

(v) $\rho(i_1^u) \succ_j \omega_i \succ_j \omega_j$ and $U^+(\succ_j, \omega_j) = P^u \cup \{\omega_i\}$;

(vi) $\rho(i_1^u) \succ_\ell \overline{h}$ and $U^+(\succ_\ell, \overline{h}) = P^u$.

This preference profile $\succ$ can be described as follows:

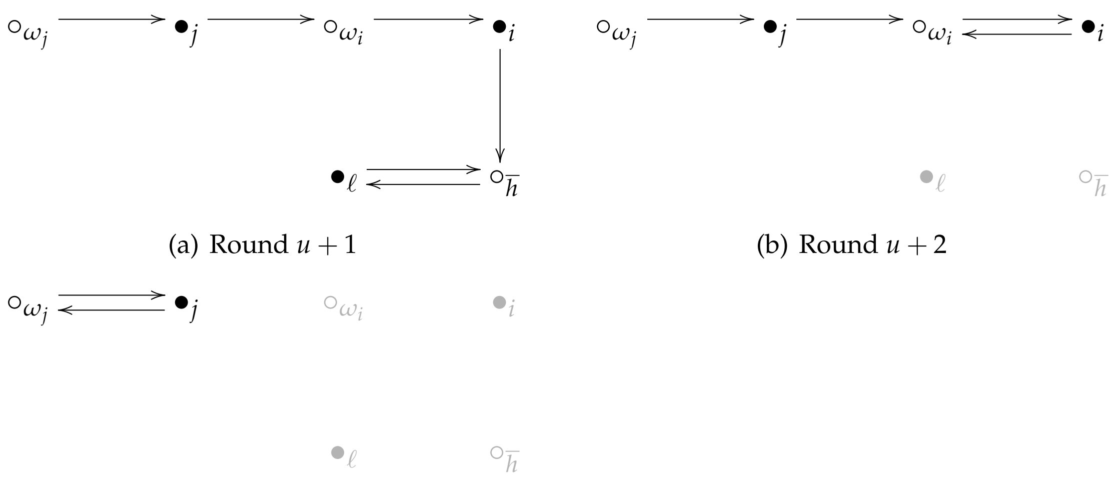
$\succ_{i_n^1 \in I_{\mu_1}}$	$\succ_{i_n^2 \in I_{\mu_1}}$	$\cdots$	$\succ_{i_n^v \in I_{\mu_v}}$	$\cdots$	$\succ_{i_n^u \in I_{\mu_u}}$	$\succ_{k \in \overline{I_\rho} \setminus \{i,j,\ell\}}$	$\succ_i$	$\succ_j$	$\succ_\ell$
$\rho(i_n^1)$	$\rho(i_1^1)$	$\cdots$	$\rho(i_1^1)$	$\cdots$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$
$\vdots$	$\rho(i_n^2)$		$\rho(i_1^2)$	$\cdots$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			$\rho(i_1^{v-1})$	$\cdots$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$
			$\rho(i_n^v)$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			$\vdots$		$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$
					$\rho(i_n^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$
					$\vdots$	$\vdots$	$\overline{h}$	ω_i	$\overline{h}$
							ω_i	ω_j	$\vdots$
							$\vdots$	$\vdots$	

Let $e = (\succ, \omega) \in \mathscr{E}$.

Step 1-2: $TTC_i^{\mathcal{O}}(e) = \omega_i$ and $TTC_j^{\mathcal{O}}(e) = \omega_j$. By the argument similar to Step 1 in the proof of Proposition 1, we have $\sigma[e, u] = \bigcup_{v=1}^{u} \mu_v = \rho$. Then, we can easily check that

$$(\overline{h}, \ell) \in \mathbb{C}^{\mathcal{O}}(e, u+1), \quad (\omega_i, i) \in \mathbb{C}^{\mathcal{O}}(e, u+2), \quad \text{and} \quad (\omega_j, j) \in \mathbb{C}^{\mathcal{O}}(e, u+3).$$

This implies that $TTC_i^{\mathcal{O}}(e) = \omega_i$ and $TTC_j^{\mathcal{O}}(e) = \omega_j$ (Figure 1).



(a) Round $u+1$

(b) Round $u+2$

(c) Round $u+3$

Figure 1: Step 1-2.

Step 1-3: $TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h}$ and $TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i$. Note that for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in I_{\mu^v}$, $\rho(i_n^v) \in \overline{H_{\mu^{v-1}}}$. In addition, by $i_{n+1}^v \in I_\rho$ and $\{i, j\} \subset \overline{I_\rho}$,

$$o^{\omega}_{\mu^{v-1}}(\rho(i_n^v)) = i_{n+1}^v \notin \{i, j\}.$$

It thus follows from (19) that for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in I_{\mu^v}$,

$$o^{\omega^{i,j}}_{\mu^{v-1}}(\rho(i_n^v)) = o^{\omega}_{\mu^{v-1}}(\rho(i_n^v)) = i_{n+1}^v.$$

Then, by the argument similar to Step 1 in the proof of Proposition 1, we have $\sigma[e^{i,j}, u] = \bigcup_{v=1}^u \mu_v = \rho$. Then, by $o^{\omega^{i,j}}_{\sigma[e^{i,j},u]}(\bar{h}) = o^{\omega^{i,j}}_{\rho}(\bar{h}) = i$ and $o^{\omega^{i,j}}_{\sigma[e^{i,j},u]}(\omega_i) = o^{\omega^{i,j}}_{\rho}(\omega_i) = j$ (Remark 1), it holds that

$$\{(\bar{h}, i), (\omega_i, j)\} \subset \mathbb{C}^{\mathcal{O}}(e^{i,j}, u+1),$$

which implies that $TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h}$ and $TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i$ (Figure 2).

Step 1-4: Conclusion. By Steps 1-2 and 1-3,

$$TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h} \succ_i \omega_i = TTC_i^{\mathcal{O}}(e),$$
$$TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i \succ_j \omega_j = TTC_j^{\mathcal{O}}(e),$$

which imply that $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness*.

• **Case 2: $\ell' \notin \{i, j\}$.** We proceed in four steps.

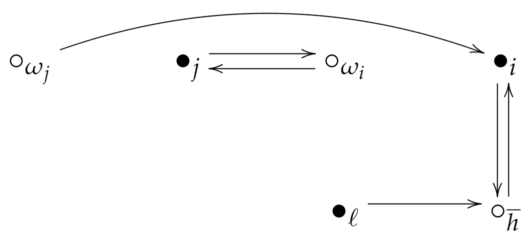


Figure 2: Step 1-3.

Step 2-1: Constructing the economy. Let $\succ \in \mathscr{P}^I$ be such that

(i) for each $v \in \{1, 2, \ldots, u\}$ and each $i_n^v \in \left\{i_1^v, i_2^v, \ldots, i_{N^v}^v\right\}$,

- for each $m \in \{1, 2, \ldots, v-2\}$, $\rho(i_1^m) \succ_{i_n^v} \rho(i_1^{m+1})$;
- $U^+(\succ_{i_n^v}, \rho(i_n^v)) = P^{v-1}$;

(ii) for each $k \in \overline{I_\rho}$ and each $m \in \{1, 2, \ldots, u-1\}$, $\rho(i_1^m) \succ_k \rho(i_1^{m+1})$;

(iii) for each $k \in \overline{I_\rho} \setminus \{i, j, \ell, \ell'\}$, $U^+(\succ_k, \rho(i_1^u)) = P^{u-1}$;

(iv) $\rho(i_1^u) \succ_i \overline{h} \succ_i \omega_i$ and $U^+(\succ_i, \omega_i) = P^u \cup \{\overline{h}\}$;

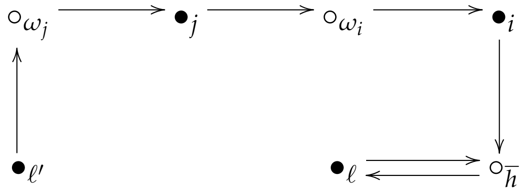
(v) $\rho(i_1^u) \succ_j \omega_i \succ_j \omega_j$ and $U^+(\succ_j, \omega_j) = P^u \cup \{\omega_i\}$;

(vi) $\rho(i_1^u) \succ_\ell \overline{h}$ and $U^+(\succ_\ell, \overline{h}) = P^u$;

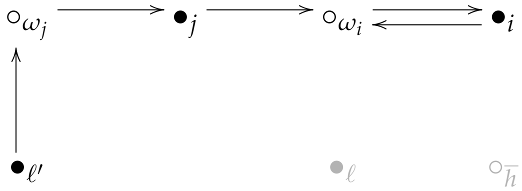
(vii) $\rho(i_1^u) \succ_{\ell'} \omega_j$ and $U^+(\succ_{\ell'}, \omega_j) = P^u$.

This preference profile $\succ$ can be described as follows:

$\succ_{i_n^1 \in I_{\mu_1}}$	$\succ_{i_n^2 \in I_{\mu_1}}$	$\cdots$	$\succ_{i_n^v \in I_{\mu_v}}$	$\cdots$	$\succ_{i_n^u \in I_{\mu_u}}$	$\succ_{k \in \overline{I_\rho} \setminus \{i,j,\ell,\ell'\}}$	$\succ_i$	$\succ_j$	$\succ_\ell$	$\succ_{\ell'}$
$\rho(i_n^1)$	$\rho(i_1^1)$	$\cdots$	$\rho(i_1^1)$	$\cdots$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$	$\rho(i_1^1)$
$\vdots$	$\rho(i_n^2)$		$\rho(i_1^2)$	$\cdots$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$	$\rho(i_1^2)$
	$\vdots$		$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			$\rho(i_1^{v-1})$	$\cdots$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$	$\rho(i_1^{v-1})$
			$\rho(i_n^v)$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
			$\vdots$		$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$	$\rho(i_1^{u-1})$
					$\rho(i_n^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$	$\rho(i_1^u)$
					$\vdots$	$\vdots$	$\bar{h}$	ω_i	$\bar{h}$	ω_j
							ω_i	ω_j	$\vdots$	$\vdots$
							$\vdots$	$\vdots$		



(a) Round $u+1$



(b) Round $u+2$

(c) Round $u+3$

Figure 3: Step 2-2.

Let $e = (\succ, \omega) \in \mathscr{E}$.

Step 2-2: $TTC_i^{\mathcal{O}}(e) = \omega_i$ and $TTC_j^{\mathcal{O}}(e) = \omega_j$. By the argument similar to Step 1 in the proof of Proposition 1, we have $\sigma[e,u] = \bigcup_{v=1}^{u} \mu_v = \rho$. Then, we can easily check that

$$(\bar{h}, \ell) \in \mathbb{C}^{\mathcal{O}}(e, u+1), \quad (\omega_i, i) \in \mathbb{C}^{\mathcal{O}}(e, u+2), \quad \text{and} \quad (\omega_j, j) \in \mathbb{C}^{\mathcal{O}}(e, u+3).$$

This implies that $TTC_i^{\mathcal{O}}(e) = \omega_i$ and $TTC_j^{\mathcal{O}}(e) = \omega_j$ (Figure 3).

Step 2-3: $TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h}$ and $TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i$. By the argument similar to Step 1-3, we have $\sigma[e^{i,j}, u] = \bigcup_{v=1}^{u} \mu_v = \rho$. Then, by $o_{\sigma[e^{i,j},u]}^{\omega^{i,j}}(\bar{h}) = o_{\rho}^{\omega^{i,j}}(\bar{h}) = \ell'$, $o_{\sigma[e^{i,j},u]}^{\omega^{i,j}}(\omega_i) = o_{\rho}^{\omega^{i,j}}(\omega_i) = j$, and $o_{\sigma[e^{i,j},u]}^{\omega^{i,j}}(\omega_j) = o_{\rho}^{\omega^{i,j}}(\omega_j) = i$ (Remark 1), it holds that

$$\{(\omega_j, i, \bar{h}, \ell'), (\omega_i, j)\} \subset \mathbb{C}^{\mathcal{O}}(e^{i,j}, u+1),$$

which implies that $TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h}$ and $TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i$ (Figure 4).

Step 2-4: Conclusion. By Steps 2-2 and 2-3,

$$TTC_i^{\mathcal{O}}(e^{i,j}) = \bar{h} \succ_i \omega_i = TTC_i^{\mathcal{O}}(e),$$
$$TTC_j^{\mathcal{O}}(e^{i,j}) = \omega_i \succ_j \omega_j = TTC_j^{\mathcal{O}}(e),$$

which imply that $TTC^{\mathcal{O}}$ violates *endowments-swapping-proofness*. $\square$

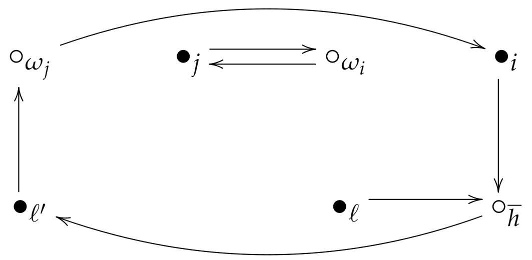


Figure 4: Step 2-3.

C Appendix: Proof of Proposition 2

Given a profile of priority orders $\rhd_H = (\rhd_h)_{h \in H}$, suppose that a structure of ownership rights $\mathcal{O}^{\rhd_H}$ satisfies (O3). We then show that $\mathcal{O}^{\rhd_H}$ satisfies *restricted independence*. Let $\omega \in \mathcal{S}_{I_E}$, $\sigma \in \mathcal{M}^{\mathcal{O}^{\rhd_H}}(\omega)$, $\{i, j\} \subset \overline{I_\sigma} \cap I_E$, and $h \in \overline{H_\sigma}$ be such that $o^\omega_\sigma(h) \notin \{i, j\}$. There are two cases.

• **Case 1: For some $k \in \overline{I_\sigma}$, $h = \omega_k$.** Then,

$$o^\omega_\sigma(h) = o^\omega_\sigma(\omega_k) \overset{\text{(O1) and (O2)}}{=} k \notin \{i, j\}, \tag{20}$$

which implies $\omega^{i,j}_k = \omega_k = h$. In addition, by $k \in \overline{I_\sigma}$ and $\omega^{i,j}_k = \omega_k = h \in \overline{H_\sigma}$,

$$o^{\omega^{i,j}}_\sigma(h) = o^{\omega^{i,j}}_\sigma(\omega_k) = o^{\omega^{i,j}}_\sigma(\omega^{i,j}_k) \overset{\text{(O1) and (O2)}}{=} k \overset{(20)}{=} o^\omega_\sigma(h).$$

• **Case 2: For each $k \in \overline{I_\sigma}$, $h \neq \omega_k$.** By (O3), for each $k \in \overline{I_\sigma} \setminus \{o^\omega_\sigma(h)\}$, $o^\omega_\sigma(h) \rhd_h k$. If there is $k \in \overline{I_\sigma}$ such that $h = \omega^{i,j}_k$, then one of the following holds:

(a) $k = i$. Then, $h = \omega^{i,j}_i = \omega_j$.

(b) $k = j$. Then, $h = \omega^{i,j}_j = \omega_i$.

(c) $k \notin \{i, j\}$. Then, $h = \omega^{i,j}_k = \omega_k$.

All cases contradict that for each $k \in \overline{I_\sigma}$, $h \neq \omega_k$. Thus, there is no $k \in \overline{I_\sigma}$ such that $h = \omega^{i,j}_k$. Then, by (O3), for each $k \in \overline{I_\sigma} \setminus \{o^{\omega^{i,j}}_\sigma(h)\}$, $o^{\omega^{i,j}}_\sigma(h) \rhd_h k$. Hence, $o^{\omega^{i,j}}_\sigma(h) = o^\omega_\sigma(h)$. $\square$

D Appendix: Omitted proofs

D.1 Omitted proof in Example 2

Here, we show that $\mathcal{O}^\Sigma$ satisfies (O2). Let $\omega \in X$, $\{\sigma, \sigma'\} \subset \mathring{\mathcal{S}}$ with $\sigma \subset \sigma'$, $i \in \overline{I_{\sigma'}}$, and $h \in \overline{H_{\sigma'}}$. Suppose $o^\omega_\sigma(h) = i$. If $h = \omega_i$, by (E1), $o^\omega_{\sigma'}(h) = i$. Thus, it suffices to consider the case in which there is no $j \in \overline{I_\sigma}$ with $h = \omega_j$. There are two cases.

- **Case 1: $\omega \neq \hat{\omega}$ or $(1, h_4) \notin \sigma'$.** Then, $\{(\omega, \sigma), (\omega, \sigma')\} \cap \Sigma = \emptyset$. By (E2), for each $j \in \overline{I_\sigma} \setminus \{o^\omega_\sigma(h) = i\}$, $i \rhd^0 j$. By $\sigma \subset \sigma'$, $(i \in)\overline{I_{\sigma'}} \subset \overline{I_\sigma}$, which implies that for each $j \in \overline{I_{\sigma'}} \setminus \{i\}$, $i \rhd^0 j$. Hence, by (E2), $o^\omega_{\sigma'}(h) = i$.

- **Case 2: $\omega = \hat{\omega}$ and $(1, h_4) \in \sigma'$.** We proceed in two steps.

Step 2-1: $(1, h_4) \in \sigma$. Suppose on the contrary that $(1, h_4) \notin \sigma$. Then, $(\hat{\omega}, \sigma) \notin \Sigma$. If $1 \in I_\sigma$, then there is $h' \in H \setminus \{h_4\}$ with $(1, h') \in \sigma$ and thus, $\sigma \subset \sigma'$ implies $(1, h') \in \sigma'$, which contradicts $(1, h_4) \in \sigma'$. Hence, $1 \in \overline{I_\sigma}$. By (E2) and $\omega = \hat{\omega}$, for each $j \in \overline{I_\sigma} \setminus \{o^{\hat{\omega}}_\sigma(h) = i\}$, $i \rhd^0 j$. This together with $1 \in \overline{I_\sigma}$ and the definition of $\rhd^0$ implies $i = 1 \in \overline{I_{\sigma'}}$, which contradicts $(1, h_4) \in \sigma'$.

Step 2-2: Concluding. By Step 2-1, $\{(\hat{\omega}, \sigma), (\hat{\omega}, \sigma')\} \subset \Sigma$. By (E3), for each $j \in \overline{I_\sigma} \setminus \{o^{\hat{\omega}}_\sigma(h) = i\}$, $i \rhd^\Sigma j$. By $\sigma \subset \sigma'$, $(i \in)\overline{I_{\sigma'}} \subset \overline{I_\sigma}$, which implies that for each $j \in \overline{I_{\sigma'}} \setminus \{i\}$, $i \rhd^\Sigma j$. Hence, by (E3), $o^{\hat{\omega}}_{\sigma'}(h) = o^\omega_{\sigma'}(h) = i$. □

D.2 Omitted proof in Remark 3

Here, we show that $\mathcal{O}^\Sigma$ satisfies *restricted independence*. Let $\omega \in X$, $\sigma \in \mathcal{M}^{\mathcal{O}^\Sigma}(\omega)$, $\{i, j\} \subset \overline{I_\sigma}$, and $h \in \overline{H_\sigma}$ be such that $o^\omega_\sigma(h) \notin \{i, j\}$. We first consider the case in which there is $k \in \overline{I_\sigma}$ with $h = \omega_k$. Then, by (E1), $o^\omega_\sigma(h) = k \notin \{i, j\}$, which implies $\omega^{i,j}_k = \omega_k = h$. Hence, by (E1), $o^{\omega^{i,j}}_\sigma(h) = k = o^\omega_\sigma(h)$.

Next, we consider the case in which there is no $k \in \overline{I_\sigma}$ with $h = \omega_k$. Let $\ell = o^\omega_\sigma(h) \notin \{i, j\}$. By (E2) and (E3), ℓ is the agent with highest priority in $\overline{I_\sigma}$ under either $\rhd^0$ or $\rhd^\Sigma$; that is, we have either

- for each $k \in \overline{I_\sigma} \setminus \{\ell\}$, $\ell \rhd^0 k$ or
- for each $k \in \overline{I_\sigma} \setminus \{\ell\}$, $\ell \rhd^\Sigma k$.

Then, there is no $k \in \overline{I_\sigma}$ with $h = \omega^{i,j}_k$.[28] Let $\ell' = o^{\omega^{i,j}}_\sigma(h)$. By (E2) and (E3), ℓ' is the agent with highest priority in $\overline{I_\sigma}$ under either $\rhd^0$ or $\rhd^\Sigma$; that is, we have either

[28]Otherwise, one of the following holds:

- for each $k \in \overline{I_\sigma} \setminus \{\ell'\}$, $\ell' \rhd^0 k$ or
- for each $k \in \overline{I_\sigma} \setminus \{\ell'\}$, $\ell' \rhd^\Sigma k$.

There are three cases.

• **Case 1: Either $\hat{\omega} \notin \{\omega, \omega^{i,j}\}$ or $(1, h_4) \notin \sigma$.** Then, $\{(\omega, \sigma), (\omega^{i,j}, \sigma)\} \cap \Sigma = \emptyset$. By (E2), both ℓ and ℓ' are the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^0$ and thus, $\ell = \ell'$.

• **Case 2: $\omega = \hat{\omega}$ and $(1, h_4) \in \sigma$.** Then, $(\hat{\omega}, \sigma) \in \Sigma$ and $(\hat{\omega}^{i,j}, \sigma) \notin \Sigma$. By (E2) and (E3), ℓ (resp. ℓ') is the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^\Sigma$ (resp. $\rhd^0$). By $\sigma \in \mathcal{M}^{\mathcal{O}^\Sigma}(\hat{\omega})$ and $(1, h_4) \in \sigma$, there is a sequence of submatchings $(\mu_v)_{v=1}^u$ that satisfies (S1) to (S3) and in particular, for some $v \in \{1, 2, \ldots, u\}$,

$$\mu_v = \left\{ \ldots, (i_n^v = 1, \sigma(i_n^v) = h_4), (i_{n+1}^v = o_{\mu^{v-1}}^{\hat{\omega}}(h_4), \sigma(i_{n+1}^v)), \ldots \right\} \subset \sigma$$

and $\mu^{v-1} \subset \sigma$. We proceed in two steps.

Step 2-1: $\{4\} \subset I_\sigma$. Suppose on the contrary that $\{4\} \subset \overline{I_\sigma}$. By $\mu^{v-1} \subset \sigma$, $\{4\} \subset \overline{I_{\mu^{v-1}}}$. This together with $\hat{\omega}_4 = h_4 \in \overline{H_{\mu^{v-1}}}$ and (E1) implies $i_{n+1}^v = o_{\mu^{v-1}}^{\hat{\omega}}(h_4) = 4$. Thus, $\{4\} \subset I_{\mu_v} \subset I_\sigma$, which is a contradiction.

Step 2-2: Concluding. Step 2-1 together with $(1, h_4) \in \sigma$ implies $\overline{I_\sigma} \subset \{2, 3, 5\}$. Since the orders under $\rhd^0$ and $\rhd^\Sigma$ in $\{2, 3, 5\}$ are the same, ℓ is also the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^0$; that is, $\ell = \ell'$.

• **Case 3: $\omega^{i,j} = \hat{\omega}$ and $(1, h_4) \in \sigma$.** Then, $(\omega, \sigma) \notin \Sigma$ and $(\omega^{i,j} = \hat{\omega}, \sigma) \in \Sigma$. By (E2) and (E3), ℓ (resp. ℓ') is the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^0$ (resp. $\rhd^\Sigma$). We proceed in three steps.

Step 3-1: $\omega_4 = h_4$. Suppose on the contrary that $\omega_4 \neq h_4$. By $\omega_4^{i,j} = \hat{\omega}_4 = h_4$,

$$4 \in \{i, j\} \subset \overline{I_\sigma}. \tag{21}$$

This together with $(1, h_4) \in \sigma$ and the definition of $\rhd^0$ implies that agent 4 is the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^0$. Hence, by (21), $o_\sigma^\omega(h) = \ell = 4 \in \{i, j\}$, which is a contradiction to $\ell \notin \{i, j\}$.

(a) $k = i$. Then, $h = \omega_i^{i,j} = \omega_j$.

(b) $k = j$. Then, $h = \omega_j^{i,j} = \omega_i$.

(c) $k \in \overline{I_\sigma} \setminus \{i, j\}$. Then, $h = \omega_k^{i,j} = \omega_k$.

However, these contradict that there is no $k \in \overline{I_\sigma}$ with $h = \omega_k$.

Step 3-2: $\{4\} \subset I_\sigma$. By using Step 3-1, an argument similar to Case 2 shows $\{4\} \subset I_\sigma$.

Step 3-3: Concluding. Step 3-2 together with $(1, h_4) \in \sigma$ implies $\overline{I_\sigma} \subset \{2, 3, 5\}$. Since the orders under $\rhd^0$ and $\rhd^\Sigma$ in $\{2, 3, 5\}$ are the same, ℓ is also the agent with highest priority in $\overline{I_\sigma}$ under $\rhd^\Sigma$; that is, $\ell = \ell'$. □

D.3 Omitted proof in Remark 4

We prove the following proposition.

Proposition 3. *If $|I| = |H|$, then any structure of ownership rights $\mathcal{O}$ satisfies restricted independence.*

Proof. Let $\omega \in X$, $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$, $\{i, j\} \subset \overline{I_\sigma}$, and $h \in \overline{H_\sigma}$ with $o_\sigma^\omega(h) \notin \{i, j\}$. We proceed in three steps.

Step 1: $H_\sigma = \bigcup_{k \in I_\sigma} \{\omega_k\}$. By $\sigma \in \mathcal{M}^{\mathcal{O}}(\omega)$, there is a sequence of submatchings $(\mu_v)_{v=1}^{u}$ that satisfies (S1) to (S3). Note that

$$H_{\mu^0} (= H_{\sigma^\varnothing}) = \varnothing = \bigcup_{k \in I_{\mu^0} (= I_{\sigma^\varnothing} = \varnothing)} \{\omega_k\}.$$

Suppose that for each $z \in \{0, 1, \ldots, v-1\}$,

$$H_{\mu^z} = \bigcup_{k \in I_{\mu^z}} \{\omega_k\}. \tag{22}$$

Note that by (S1) and (S3),

$$\mu_v = \left\{ (i_1^v (= i_{N^v+1}^v), \sigma(i_1^v)), (i_2^v, \sigma(i_2^v)), \ldots, (i_{N^v}^v, \sigma(i_{N^v}^v)) \right\} \subset \sigma$$

and

$$\forall\, i_n^v \in I_{\mu_v},\ o_{\mu^{v-1}}^\omega(\sigma(i_n^v)) = i_{n+1}^v. \tag{23}$$

Let $\sigma(i_n^v) \in H_{\mu_v}$. By $\sigma(i_n^v) \in \overline{H_{\mu^{v-1}}}$ and (22), for each $\ell \in I_{\mu^{v-1}}$, $\sigma(i_n^v) \neq \omega_\ell$, which together with $|I| = |H|$ implies that

$$\exists\, k \in \overline{I_{\mu^{v-1}}},\ \sigma(i_n^v) = \omega_k. \tag{24}$$

By $k \in \overline{I_{\mu^{v-1}}}$ and $\omega_k = \sigma(i_n^v) \in \overline{H_{\mu^{v-1}}}$,

$$k \overset{\text{(O1) and (O2)}}{=} o^{\omega}_{\mu^{v-1}}(\omega_k) \overset{(24)}{=} o^{\omega}_{\mu^{v-1}}(\sigma(i_n^v)) \overset{(23)}{=} i_{n+1}^v,$$

that is, $\sigma(i_n^v) = \omega_k = \omega_{i_{n+1}^v}$. Hence,

$$\begin{aligned} H_{\mu^v} &= H_{\mu^{v-1}} \cup \{\sigma(i_1^v), \sigma(i_2^v), \dots, \sigma(i_{N^v}^v)\} \\ &= H_{\mu^{v-1}} \cup \left\{\omega_{i_1^v}, \omega_{i_2^v}, \dots, \omega_{i_{N^v}^v}\right\} \\ &= \bigcup_{k \in I_{\mu^{v-1}}} \{\omega_k\} \cup \bigcup_{k \in I_{\mu_v}} \{\omega_k\} \\ &= \bigcup_{k \in I_{\mu^v}} \{\omega_k\}. \end{aligned}$$

Consequentially, we have

$$H_\sigma = H_{\mu^u} = \bigcup_{k \in I_\sigma = I_{\mu^u}} \{\omega_k\}.$$

Step 2: $\overline{H_\sigma} = \bigcup_{k \in \overline{I_\sigma}} \{\omega_k\}$. Let $h' \in \overline{H_\sigma}$. By Step 1, for each $\ell \in I_\sigma$, $h' \neq \omega_\ell$, which together with $|I| = |H|$ implies that there is $k \in \overline{I_\sigma}$ such that $h' = \omega_k$. Hence,

$$\overline{H_\sigma} \subseteq \bigcup_{k \in \overline{I_\sigma}} \{\omega_k\}.$$

Conversely, let $h' \in \bigcup_{k \in \overline{I_\sigma}} \{\omega_k\}$. Because there is $k \in \overline{I_\sigma}$ such that $h' = \omega_k$,

$$h' \notin \bigcup_{k \in I_\sigma} \{\omega_k\} \overset{\text{Step 1}}{=} H_\sigma.$$

Hence,

$$\bigcup_{k \in \overline{I_\sigma}} \{\omega_k\} \subseteq \overline{H_\sigma}.$$

Step 3: Concluding. Then, by $h \in \overline{H_\sigma}$ and Step 2, there is $k \in \overline{I_\sigma}$ such that $h = \omega_k$. By $k \in \overline{I_\sigma}$ and $\omega_k = h \in \overline{H_\sigma}$,

$$o^{\omega}_{\sigma}(h) = o^{\omega}_{\sigma}(\omega_k) \overset{\text{(O1) and (O2)}}{=} k \notin \{i, j\}, \tag{25}$$

which implies that $\omega_k^{i,j} = \omega_k = h$. In addition, by $k \in \overline{I_\sigma}$ and $\omega_k^{i,j} = \omega_k = h \in \overline{H_\sigma}$,

$$o_\sigma^{\omega^{i,j}}(h) = o_\sigma^{\omega^{i,j}}(\omega_k) = o_\sigma^{\omega^{i,j}}(\omega_k^{i,j}) \overset{\text{(O1) and (O2)}}{=} k \overset{(25)}{=} o_\sigma^{\omega}(h).$$

This completes the proof. □

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