UTMD Working Paper
The University of Tokyo
Market Design Center

UTMD-075

Stable matching under inconsistent choice functions

Keisuke Bando
Keio University

Toshiyuki Hirai
Hosei University

Kenzo Imamura
University of Tokyo

February 13, 2025

Stable matching under inconsistent choice functions*

Keisuke Bando† Toshiyuki Hirai‡ Kenzo Imamura§

February 13, 2025

Abstract

We study stable matching in the many-to-one matching model. Substitutability, together with consistency, is known to guarantee the existence of a stable matching. We first observe that in certain applications a stable matching still exists even in the absence of consistency. We then introduce a weaker condition, *monotonicity*, and show that the combination of substitutability and monotonicity ensures the existence of a stable matching. Consistency is the rationalization axiom introduced in the choice theory literature. Our result suggests that rationalization is not necessarily required in stable matching theory. Furthermore, we analyze a stable and strategy-proof mechanism, focusing on the cumulative offer process, which is widely used in both theory and practice. We derive a necessary condition for the cumulative offer process to be stable and strategy-proof under substitutable and weakly monotonic choice functions for any proposal order, provided that there are sufficiently many doctors. This condition is stringent, highlighting that the addition of strategy-proofness imposes significant restrictions. We apply our conditions to real-life applications such as daycare allocation and college admissions.

1 Introduction

Matching theory is a central field in market design and has a lot of applications, such as hospital-resident matching, school choice, and labor markets.[1] The core concept of matching theory is *stable matching*, which requires no pair prefers each other to their current partners. Stable matching has several desirable properties. First, it ensures fairness. In the context of school choice, stability respects priority and eliminates justified envy (Abdulkadiroğlu and Sönmez, 2003). Second, stability contributes to robustness.

*This work was partially supported by JSPS KAKENHI Grant Numbers 21H04979, 22K01396, 23K01312, 23K12443, and JST ERATO Grant Number JPMJER2301, Japan.

†Department of Industrial and Systems Engineering, Faculty of Science and Technology, Keio University, email: bando@ae.keio.ac.jp.

‡Department of Economics, Hosei University, e-mail: hirai@hosei.ac.jp.

§Department of Economics, University of Tokyo, email: imamurak@e.u-tokyo.ac.jp.

[1] Applications include new employee assignment (Kojima and Odahara, 2022), internal matching markets within firms (Cowgill, 2018), and teacher assignment (Combe et al., 2022).

Institutions using stable matching are less likely to incentivize off-market transactions, making them more sustainable in the long run (Roth, 1991).

However, a stable matching does not always exist in the model of many-to-one matching: a stable matching may fail to exist under some choice functions of multi-unit demand agents or hospitals. To guarantee the existence of stable matchings, it is necessary to impose conditions on the choice functions. The two standard conditions are substitutability and consistency (Roth, 1984; Aygün and Sönmez, 2013). Substitutability excludes complementarity among unit demand agents or doctors. Consistency requires that removing a rejected doctor does not affect the set of chosen doctors. It also relates to rationalization, where a choice function can be interpreted as maximizing a hospital's utility.

In this paper, we focus on consistency under the assumption of substitutability. We begin by discussing the necessity for consistency through two examples. The first example, a simplified version of one found in Aygün and Sönmez (2012), illustrates that without consistency, there may be no stable matchings.

Example 1. Suppose that there are three doctors d_1, d_2, d_3 and one hospital h. Each doctor prefers h to her outside option. A choice function C_h is given as follows:

$$\begin{aligned} C_h(\{d_1, d_2, d_3\}) &= \emptyset, \\ C_h(\{d_1, d_2\}) &= \{d_1\}, \\ C_h(\{d_2, d_3\}) &= \{d_2\}, \\ C_h(\{d_1, d_3\}) &= \{d_3\}, \end{aligned}$$

and $C_h(\{d_i\}) = \{d_i\}$ for $i = 1, 2, 3$.

A choice function C_h violates consistency: $C_h(\{d_1, d_2, d_3\}) = \emptyset$ and $C_h(\{d_1, d_2\}) = \{d_1\}$. In addition, there is no stable matching: while an empty matching where all doctors are unmatched is clearly unstable, a matching where a doctor d_i is matched is blocked by d_{i-1} where $d_0 \equiv d_3$.

We then consider an example that is slightly different from Example 1.

Example 2. Suppose that there are three doctors d_1, d_2, d_3 and one hospital h. Each doctor prefers h to her outside option. A choice function C_h is given as follows:

$$\begin{aligned} C_h(\{d_1, d_2, d_3\}) &= \emptyset, \\ C_h(\{d_1, d_2\}) &= \{d_1, d_2\}, \\ C_h(\{d_2, d_3\}) &= \{d_2, d_3\}, \\ C_h(\{d_1, d_3\}) &= \{d_1, d_3\}, \end{aligned}$$

and $C_h(\{d_i\}) = \{d_i\}$ for $i = 1, 2, 3$.

In Example 2, the choice function also violates consistency. However, a stable matching exists: the matching where d_1 and d_2 are matched is stable since $d_3 \notin C_h(\{d_1, d_2, d_3\})$.

These examples lead to the following question: what types of consistency violations do not prevent the existence of a stable matching?

This paper introduces a condition that is weaker than consistency. The key concept is what we call the *maximal choice correspondence*. The maximal choice set refers to a set of doctors in the available set satisfying the following two properties: (i) the hospital is willing to choose every doctor in the set and (ii) the hospital is not willing to add any doctor outside the set. In Example 2, $\{\{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_3\}\}$ is the maximal choice sets at $\{d_1, d_2, d_3\}$ as, for example, $C_h(\{d_1, d_2\}) = \{d_1, d_2\}$ and $d_3 \notin C_h(\{d_1, d_2, d_3\})$.

Since there can be multiple maximal sets, it forms a correspondence. In Example 1, a maximal choice set does not exist at $\{d_1, d_2, d_3\}$. A crucial property is that constructing a new substitutable choice function as a selection from the maximal choice correspondence ensures that it satisfies consistency (Lemma 1). We show that if such selections exist, then stable matchings exist even under the inconsistent choice functions. However, there may be no substitutable selection (see Example 3). To address this, we introduce a condition called *monotonicity*. We show that under substitutability and monotonicity, stable matchings exist (Theorem 1).

Our result offers two contributions. First, it suggests that rationalization is not necessarily required in stable matching theory, thereby broadening the applicability of the theory. Inconsistent choice functions often appear in real-life applications, such as daycare allocation and college admissions. Even in such cases, we can apply stable matching theory. Second, we demonstrate that choice correspondences can be powerful tools in stable matching theory. By introducing the maximal choice correspondence, stable matchings can be analyzed even with inconsistent choice functions. Furthermore, our condition, monotonicity, builds on the condition for choice correspondence introduced by Sotomayor (1999).

We further investigate stable and strategy-proof (SP) mechanisms. In particular, we focus on the *doctor-proposing cumulative offer process* (COP), which is standard in both theory and applications. Under certain assumptions, we derive a necessary condition for COP to be both stable and strategy-proof (for any proposal order): the existence of a substitutable selection (Theorem 3). This condition is stringent because we also provide a more general condition for the existence of a stable matching (Theorem 2), which is based on conditions in choice correspondences. We employ the choice correspondence framework to address cases where a substitutable selection is not available. Therefore, the necessity of a substitutable selection highlights that the addition of strategy-proofness imposes significant restrictions.

Theoretically, we identify *order-independence* as a crucial property for COP to be both stable and strategy-proof. Order-independence means that the outcome of COP does not depend on the order in which doctors make their proposals. It is known that the conjunction of substitutability and consistency guarantees that COP is order-independent (Hirata and Kasuya, 2014). However, without consistency, this property does not hold. First, we observe that doctors can exploit order-dependence to manipulate COP (see Example 5). Then, we generalize this observation assuming that there are sufficiently many doctors (Theorem 3).

We apply our results to real-life applications. First, we consider *daycare allocation* (Kamada and Kojima, 2024). In this context, the matching market is subject to a constraint called a *general upper-bounds*, which differs from the usual capacity constraint. Under such constraints, choice functions are inconsistent, and stable matchings fail to exist. Therefore, we investigate which types of constraints can guarantee the existence of stable matchings. According to our findings, if the constraints form a *matroid*, stable matchings exist. Furthermore, under matroid constraints, there also exists a stable and SP mechanism. We also show that matroid constraints are necessary to obtain these positive results. Next, we consider *college admissions.* In theory, we often assume a strict priority ranking; however, in practice many applicants obtain identical scores, meaning that they are tied. To address this issue, tie-breaking is commonly employed to produce a strict priority ranking. In contrast, tie-breaking is not used due to fairness concerns in several countries such as Hungary (Biró and Kiselgof, 2015). The choice functions used in Hungary violate consistency. Our findings connect these distinct choice functions—one that uses tie-breaking and one that does not. In addition, our results imply that stable matchings exist, and a stable and strategy-proof mechanism also exists in college admissions in Hungary.

1.1 Related Literature

This paper contributes to the literature on stable matching theory. Aygün and Sönmez (2013) point out that substitutability alone is not sufficient to guarantee the existence of a stable matching, but they also show that adding consistency ensures its existence.[2] We show that while the conjunction of substitutability and consistency is sufficient, this is not necessary. Fleiner and Jankó (2014) propose alternative concepts of stability in situations where a choice function violates consistency. In this paper, we adopt the standard definition of stable matching. Our paper is also closely related to Caspari and Khanna (ming). They provide conditions for the existence of stable matchings in settings where doctor preferences are not assumed to be consistent. In contrast, we analyze situations in which choice functions of hospitals are not required to satisfy consistency. These studies complement one another.

Consistency is an axiom related to rationalization. Under the assumption of substitutability, Yang (2020) shows that consistency is equivalent to rationalizability. Consistency and substitutability are originally introduced in social choice theory.[3] Plott (1973) introduces the concept of path-independence, and Aizerman and Malishevski (1981) later show that the conjunction of consistency and substitutability is equivalent to path-independence.[4] Our results suggest that rationalization is not necessarily required in stable matching theory.

This paper employs techniques from the literature on stable matching with choice correspondences. Sotomayor (1999) extends the notion of substitutability for choice functions to choice correspondences. Erdil and Kumano (2019) as well as Che et al. (2019), analyze stable matchings with choice correspondences using this notion. We

[2]Zhang (2016) extends consistency to matching with contracts. In many-to-many matching with contracts, Bando et al. (2021) demonstrate the existence of stable matchings by employing the concepts of substitutability, the law of aggregate demand, and this extended notion of consistency.

[3]See Moulin (1985) for a reference.

[4]In matching theory, Blair (1988) shows the equivalence between these concepts.

apply the conditions proposed by Sotomayor (1999) to a maximal choice correspondence constructed from an inconsistent choice function. Furthermore, we introduce a condition—weak monotonicity—that further refines the notion of substitutable correspondences. We also develop a technique to analyze stable and strategy-proof mechanisms under correspondences. Thus, our results not only demonstrate the usefulness of techniques in choice correspondences but also suggest that the methods developed in this paper could potentially be applied to stable matching with choice correspondences.

This paper has implications for several applications discussed in the literature. Kamada and Kojima (2024) consider choice functions subject to constraints called general upper-bounds to analyze daycare allocation. A similar constraint arises in refugee resettlement (Delacrétaz et al., 2023). In college admissions, some countries do not use tie-breaking (Biró and Kiselgof, 2015; Rios et al., 2021). In dynamic matching markets, there may be a constraint requiring a hospital to continue hiring a doctor matched in a previous period (Bando and Kawasaki, 2024). In all these applications, the choice function satisfies substitutability but may violate consistency. By employing our results, we can analyze the existence of stable matchings as well as the design of stable and strategy-proof mechanisms in each scenario.

The remainder of this paper is organized as follows: Section 2 introduces the model. Section 3 provides the definition of maximal choice correspondence and presents conditions for the existence of a stable matching. Section 4 analyzes strategic issues of stable mechanisms. Section 5 provides applications. Section 6 concludes the study. All omitted proofs are in Appendix.

2 Model

We consider the model of many-to-one matching. Let $\mathcal{D}$ be a finite set of doctors and $\mathcal{H}$ be a finite set of hospitals where $\mathcal{D} \cap \mathcal{H} = \emptyset$. Each doctor $d \in \mathcal{D}$ has strict preferences $\succ_d$ over $\mathcal{H} \cup \{\emptyset\}$, where $\emptyset$ means being unmatched (or an outside option). Each hospital $h \in \mathcal{H}$ has a *choice function* $C_h : 2^{\mathcal{D}} \to 2^{\mathcal{D}}$ such that $C_h(D) \subseteq D$ for all $D \subseteq \mathcal{D}$. We define a rejection function by $R_h(D) = D \setminus C_h(D)$ for all $D \subseteq \mathcal{D}$.

A *matching* is a function $\mu : \mathcal{D} \to \mathcal{H} \cup \{\emptyset\}$. In words, $\mu(d)$ represents a hospital assigned to d at μ where $\mu(d) = \emptyset$ means that d is unmatched. Similarly, $\mu^{-1}(h)$ represents a set of doctors assigned to h at μ. We often write $\mu(h)$ instead of $\mu^{-1}(h)$. A matching μ is *individually rational* if $\mu(d) \succeq_d \emptyset$ for all $d \in \mathcal{D}$ and $C_h(\mu(h)) = \mu(h)$ for all $h \in \mathcal{H}$. A matching μ is *blocked* if there exists a hospital $h \in \mathcal{H}$ and a non-empty set of doctors $D \subseteq \mathcal{D} \setminus \mu(h)$ such that $D \subseteq C_h(\mu(h) \cup D)$ and $h \succ_d \mu(d)$ for all $d \in D$. We say that a matching μ is *stable* if it is individually rational and not blocked.

It is well known that a stable matching exists under *substitutability* and *consistency* defined as follows (Roth, 1984; Hatfield and Milgrom, 2005; Aygün and Sönmez, 2013).

Definition 1. Let $h \in \mathcal{H}$.

- C_h satisfies consistency if for any $D, D' \subseteq \mathcal{D}$, $C_h(D) \subseteq D' \subseteq D$ implies $C_h(D) = C_h(D')$.[5]

[5]Consistency is also known as "irrelevance of rejected contracts" in the literature of matching with

- C_h satisfies substitutability if for any $D, D' \subseteq \mathcal{D}$ with $D \subseteq D'$, $R_h(D) \subseteq R_h(D')$.

It is known that substitutability and consistency are equivalent to another single axiom called *path-independence*: a choice function C_h is path-independent if for every $D, D' \subseteq \mathcal{D}$, $C_h(D \cup D') = C_h(C_h(D) \cup C_h(D'))$ (Plott, 1973; Aizerman and Malishevski, 1981; Blair, 1988).

3 Stable Matching

In this section, we analyze the existence of a stable matching. We focus on weakening consistency under the assumption of substitutability. To deal with inconsistency, we introduce concepts related to a choice correspondence. Our main sufficient condition builds on the property for a choice correspondence proposed by Sotomayor (1999).

Motivated by the inconsistent choice functions in Examples 1 and 2, we introduce a concept.

Definition 2. For every $D \subseteq \mathcal{D}$, a set of doctors $D' \subseteq D$ is a maximal choice set in D to h if $C_h(D') = D'$ and $d \notin C_h(D' \cup \{d\})$ for all $d \in D \setminus D'$.

Intutively, $d \in D \setminus D'$ cannot claim to D' since $d \notin C_h(D' \cup \{d\})$ even though her claim may change the choice (i.e., $C_h(D' \cup \{d\}) \neq D'$). Note that when C_h satisfies consistency, $C_h(D)$ is always a maximal choice set to h for all $D \subseteq \mathcal{D}$. In general, a maximal choice set does not exist under substitutability. The choice function given in Example 1 illustrates this fact. We say that C_h is *regular* if there is a maximal choice set to h for any $D \subseteq \mathcal{D}$. The choice function given in Example 2 is regular: for $\{d_1, d_2, d_3\}$, maximal choice sets are $\{d_1, d_2\}, \{d_1, d_3\}$, and $\{d_2, d_3\}$.

Based on these concepts, we introduce a *maximal choice correspondence*. Since there would be multiple maximal choice sets as Example 2, we focus on choice correspondence.

Definition 3. A correspondence $\mathcal{C}_h : 2^{\mathcal{D}} \rightrightarrows 2^{\mathcal{D}}$ is maximal choice correspondence for a choice function C_h if

$$\mathcal{C}_h(D) = \{D' \subseteq D : D' \text{ is a maximal choice set at } D\}$$

for every $D \subseteq \mathcal{D}$.

For a choice function C_h, we define a *maximal rejection correspondence* by

$$\mathcal{R}_h(D) = \{D' : D' = D \setminus D'' \text{ for some } D'' \in \mathcal{C}_h(D)\}$$

for every $D \subseteq \mathcal{D}$. Note that $\mathcal{C}_h(D) = \mathcal{R}_h(D) = \emptyset$ when there exists no maximal choice set in D.

A key property of a maximal choice correspondence is that if we can construct a new substitutable choice function from this correspondence, it also satisfies consistency. Thus, if we can obtain substitutable selections for all hospitals, then a stable matching exists under these constructed choice functions. We show that this stable matching is

contracts (Aygün and Sönmez, 2013).

also stable under the original choice functions. Now we formally introduce concepts and results.

A choice function $\tilde{C}_h$ is a *selection* from a maximal choice correspondence $\mathcal{C}_h$ if $\tilde{C}_h(D) \in \mathcal{C}_h(D)$ for every $D \subseteq \mathcal{D}$. We say that C_h *has a substitutable selection* if there exists a selection from $\mathcal{C}_h$ that satisfies substitutability. In general, substitutable selection is not uniquely determined: in Example 2, any choice function $\tilde{C}_h$ such that $\tilde{C}_h(\{d_1, d_2, d_3\}) \in \{\{d_1, d_2\}, \{d_1, d_3\}, \{d_1, d_3\}\}$ and $\tilde{C}_h(D) = D$ for any other D is a substitutable selection from $\mathcal{C}_h$. We show that any substitutable selection satisfies consistency.

Lemma 1. Suppose that C_h satisfies substitutability and has a substitutable selection $\tilde{C}_h$. Then, $\tilde{C}_h$ satisfies consistency. Therefore, any substitutable selection from a maximal choice correspondence satisfies path-independence.

Proof. Fix an arbitrary $D \subseteq \mathcal{D}$ and $d \in D \setminus \tilde{C}_h(D)$. Since $\tilde{C}_h$ is a substitutable selection, $\tilde{C}_h(D) \cap (D \setminus \{d\}) \subseteq \tilde{C}_h(D \setminus \{d\})$. By $d \notin \tilde{C}_h(D)$ and $\tilde{C}_h(D) \subseteq D$, $\tilde{C}_h(D) = \tilde{C}_h(D) \cap (D \setminus \{d\}) \subseteq \tilde{C}_h(D \setminus \{d\})$.

Suppose that $\tilde{C}_h(D) \subsetneq \tilde{C}_h(D \setminus \{d\})$. Since $\tilde{C}_h(D \setminus \{d\})$ is a maximal choice set in $D \setminus \{d\}$, $C_h(\tilde{C}_h(D \setminus \{d\})) = \tilde{C}_h(D \setminus \{d\})$. By substitutability of C_h, $d' \in C_h(\tilde{C}_h(D) \cup \{d'\})$ for all $d' \in \tilde{C}_h(D \setminus \{d\}) \setminus \tilde{C}_h(D)$. This contradicts that $\tilde{C}_h(D)$ is a maximal choice set in D by $d' \in D$. Hence, $\tilde{C}_h(D) = \tilde{C}_h(D \setminus \{d\})$. □

Our first existence result uses the substitutable selection property. This result explains why a stable matching exists in Example 2: the choice function violates consistency but has a substitutable selection.

Proposition 1. Suppose that C_h satisfies substitutability $\tilde{C}_h$ for every $h \in \mathcal{H}$. If C_h has a substitutable selection for every $h \in \mathcal{H}$, then a stable matching exists.

Proof. Let $\tilde{C}_h$ be a substitutable selection from the maximal correspondence for each $h \in \mathcal{H}$. Since $\tilde{C}_h$ satisfies substitutability and consistency for all $h \in \mathcal{H}$ from Lemma 1, there exists a stable matching μ for the problem where each hospital h has the choice function $\tilde{C}_h$. It is straightforward to see that μ is individually rational for the original problem. Suppose that μ is blocked. Then, there exist $h \in \mathcal{H}$ and nonempty $D \subseteq \mathcal{D} \setminus \mu(h)$ such that $D \subseteq C_h(\mu(h) \cup D)$ and $h \succ_d \mu(d)$ for all $d \in D$. Let $d \in D$. Stability of μ (under $(\tilde{C}_h)_{h \in \mathcal{H}}$) implies $d \notin \tilde{C}_h(\mu(h) \cup \{d\})$. By consistency of $\tilde{C}_h$, $\tilde{C}_h(\mu(h) \cup \{d\}) = \mu(h)$. Thus, $\mu(h)$ is a maximal choice set in $\mu(h) \cup \{d\}$. On the other hand, substitutability of C_h implies $d \in C_h(\mu(h) \cup \{d\})$, a contradiction. Therefore, μ is stable. □

We should note that neither doctor-optimal nor hospital-optimal stable matching may exist even assuming substitutability (Example 2). This contrasts with the setting with substitutable and consistent choice functions (Roth, 1984; Blair, 1988).

Although a substitutable selection guarantees the existence of a stable matching under inconsistent choice functions, it does not always exist. The following example illustrates this point.

Table 1: choice, maximal choice, and maximal rejection

D	$C_h(D)$	$\mathcal{C}_h(D)$	$\mathcal{R}_h(D)$
$\{d_1, d_2, d_3, d_4\}$	$\emptyset$	$\{d_1, d_2, d_4\}, \{d_2, d_4, d_3\}$	$\{d_1\}, \{d_3\}$
$\{d_1, d_2, d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_2\}$	$\{d_3\}$
$\{d_1, d_2, d_4\}$	$\{d_1, d_2, d_4\}$	$\{d_1, d_2, d_4\}$	$\emptyset$
$\{d_1, d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_1\}$
$\{d_2, d_3, d_4\}$	$\{d_2, d_3, d_4\}$	$\{d_2, d_3, d_4\}$	$\emptyset$

Example 3. Let $h \in \mathcal{H}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$. We assume that $C_h(D) = D$ for all $D \subseteq \mathcal{D}$ with $|D| \leq 2$. For D with $|D| \geq 3$, the choice function C_h, the maximal choice correspondence $\mathcal{C}_h$, and the maximal rejection correspondence $\mathcal{R}_h$ are given in Table 1.

There is no substitutable selection from $\mathcal{C}_h$. To see this, suppose that there exists a substitutable selection $\bar{C}_h$ from $\mathcal{C}_h$. Let $\bar{R}_h$ denote the rejection function of $\bar{C}_h$. Then, $\bar{R}_h(\{d_1, d_2, d_3\}) = \{d_3\}$ and $\bar{R}_h(\{d_1, d_3, d_4\}) = \{d_1\}$. By substitutability of $\bar{C}_h$, we have $\{d_1, d_3\} \subseteq \bar{R}_h(\{d_1, d_2, d_3, d_4\})$. However, $\bar{C}_h(\{d_1, d_2, d_3, d_4\}) = \{d_1, d_2, d_4\}$ or $\{d_2, d_3, d_4\}$ imply $\{d_1, d_3\} \not\subseteq \bar{R}_h(\{d_1, d_2, d_3, d_4\})$, a contradiction.

To deal with the non-existence of substitutable selection, we introduce a condition on maximal choice correspondence which is based on that in the matching under choice correspondence (Sotomayor, 1999).

Definition 4. A choice function C_h satisfies monotonicity if for any $D \subseteq \mathcal{D}$, $d \in \mathcal{D} \setminus D$, and $X \in \mathcal{R}_h(D)$, there exists $X' \in \mathcal{R}_h(D \cup \{d\})$ with $X \subseteq X'$.

Note that monotonicity is a stronger condition than regularity since $\emptyset$ is a maximal choice set in $\emptyset$, namely, $\mathcal{C}_h(\emptyset) = \{\emptyset\}$. The choice function in Example 3 satisfies monotonicity.

Now we provide our main result.

Theorem 1. If a choice function C_h has no substitutable selection but satisfies monotonicity for every hospital $h \in \mathcal{H}$, then a stable matching exists.

Unlike the proof of Proposition 1, we directly show the existence of a stable matching by using the doctor-proposing COP under monotonicity. A formal definition of COP is given in Section 4.1.

Theorem 1 and Proposition 1 offer two contributions. First, they suggest that rationalization is not necessarily required in stable matching theory, thereby broadening the applicability of the theory. As we will see Section 5, inconsistent choice functions often appear in real-life applications, such as daycare allocation (Kamada and Kojima, 2024) and college admissions in Hungary (Biró and Kiselgof, 2015). Even in such cases, we can apply stable matching theory. Second, we demonstrate that choice correspondences can be powerful tools in stable matching theory. By introducing the maximal choice correspondence, stable matchings can be analyzed even with inconsistent choice functions. In particular, substitutable selection from maximal choice correspondence restores consistency. While a substitutable selection may not exist, our condition, monotonicity, building on the condition for choice correspondence introduced by Sotomayor (1999), can guarantee the existence of stable matching. A choice function that has no substitutable selection but satisfies monotonicity arises in applications such as dynamic matching (Bando and Kawasaki, 2024). We will discuss it in Section 5.

3.1 Additional Conditions

We introduce two additional conditions for the existence of a stable matching. First, by observing that the substitutable selection property and monotonicity are independent, we introduce a condition that includes both. Next, we introduce another sufficient condition which is also based on Sotomayor (1999).

First, we introduce a general condition that includes both the substitutable selection property and monotonicity. As seen in Example 3, monotonicity does not imply the substitutable selection property. The following example illustrates that the substitutable selection property does not imply monotonicity. Therefore, both are independent.

Example 4. Let $h \in \mathcal{H}$ and $\mathcal{D} = \{d_1, d_2, d_3\}$. We assume that $C_h(\{d\}) = \{d\}$ for all $d \in \mathcal{D}$. For D with $|D| \geq 2$, the choice function C_h, the maximal choice correspondence $\mathcal{C}_h$, and the maximal rejection correspondence $\mathcal{R}_h$ are given in Table 2.

Table 2: choice, maximal choice, and maximal rejection

D	$C_h(D)$	$\mathcal{C}_h(D)$	$\mathcal{R}_h(D)$
$\{d_1, d_2, d_3\}$	$\emptyset$	$\{d_2, d_3\}$	$\{d_1\}$
$\{d_1, d_2\}$	$\emptyset$	$\{d_1\}, \{d_2\}$	$\{d_1\}, \{d_2\}$
$\{d_1, d_3\}$	$\{d_3\}$	$\{d_3\}$	$\{d_1\}$
$\{d_2, d_3\}$	$\{d_2, d_3\}$	$\{d_2, d_3\}$	$\emptyset$

Then, C_h does not satisfy monotonicity since $\{d_2\} \in \mathcal{R}_h(\{d_1, d_2\})$ and $\mathcal{R}_h(\{d_1, d_2, d_3\}) = \{\{d_1\}\}$. On the other hand, C_h has a substitutable selection $\tilde{C}_h$ such that

$$\tilde{C}_h(\{d_1, d_2, d_3\}) = \{d_2, d_3\}, \tilde{C}_h(\{d_1, d_2\}) = \{d_2\}, \tilde{C}_h(\{d_1, d_3\}) = \{d_3\}. \tilde{C}_h(\{d_2, d_3\}) = \{d_2, d_3\}.$$

To introduce a general condition, we need some additional notations. Let $p(\mathcal{D})$ be the set of finite sequences of distinct doctors; that is, $p(\mathcal{D}) = \{(d_1, \cdots, d_n) \mid n \geq 1, d_i \in \mathcal{D} \text{ for all } i \text{ and } d_i \neq d_j \text{ for all } i \neq j\}$. For each $\mathbf{d} \in p(\mathcal{D})$, let $\rho(\mathbf{d})$ be the set of doctors that appear in $\mathbf{d}$, that is, $\rho(\mathbf{d}) = \{d_1, \cdots, d_n\}$ where $\mathbf{d} = (d_1, \cdots, d_n)$. We often denote $\mathcal{C}_h(\rho(\mathbf{d}))$ and $\mathcal{R}_h(\rho(\mathbf{d}))$ by $\mathcal{C}_h(\mathbf{d})$ and $\mathcal{R}_h(\mathbf{d})$, respectively, for notational simplicity.

Definition 5. A choice function C_h satisfies weak monotonicity if there exists a function $r_h : p(\mathcal{D}) \to 2^{\mathcal{D}}$ such that for any $\mathbf{d} \in p(\mathcal{D})$, (i) $r_h(\mathbf{d}) \in \mathcal{R}_h(\mathbf{d})$ and (ii) $r_h(\mathbf{d}) \subseteq r_h(\mathbf{d}, d)$ for any $d \in \mathcal{D} \setminus \rho(\mathbf{d})$.

In the above definition, the function r_h is called the *rejection function over sequences* (induced from C_h). This function naturally defines a *choice function over sequences* by $c_h(\mathbf{d}) = \rho(\mathbf{d}) \setminus r_h(\mathbf{d})$. Note that C_h has substitutable selection if and only if it induces a rejection function r_h over sequences that is *order-independent*, that is, $r_h(\mathbf{d}) = r_h(\mathbf{d}')$ for any $\mathbf{d}, \mathbf{d}' \in p(\mathcal{D})$ with $\rho(\mathbf{d}) = \rho(\mathbf{d}')$. In general, the rejection/choice function may not be order-independent.

We show that weak monotonicity is sufficient for the existence of a stable matching. As same as the proof of Theorem 1, the doctor-proposing COP finds a stable matching where the formal definition is given in Section 4.1. The proof is given in Appendix.

Theorem 2. If C_h satisfies substitutability and weak monotonicity for every $h \in \mathcal{H}$, then a stable matching exists.

Next, we introduce another condition for the existence of a stable matching. Sotomayor (1999) generalizes substitutability for a choice function to a choice correspondence. The generalization consists of two conditions. Our monotonicity builds on one of the conditions. Using the other condition, we provide another condition for the existence.

Definition 6. A choice function C_h satisfies lower monotonicity if for any $D \subseteq \mathcal{D}$, $D' \in \mathcal{C}_h(D)$, and $d \in D'$, there exists $D'' \in \mathcal{C}_h(D \setminus \{d\})$ such that $D' \setminus \{d\} \subseteq D''$.

Note that the maximal choice correspondence $\mathcal{C}_h$ satisfies consistency; that is, for any $D \subseteq \mathcal{D}$, $D' \in \mathcal{C}_h(D)$ and $d \in D \setminus D'$, we have $D' \in \mathcal{C}_h(D \setminus \{d\})$. Then, in terms of the rejection function, lower monotonicity condition is written as follows: for any $D \subseteq \mathcal{D}$ and any $d \in D$, we have that for any $D' \in \mathcal{R}_h(D)$, there exists $D'' \in \mathcal{R}_h(D \setminus \{d\})$ such that $D'' \subseteq D'$. We also note that lower monotonicity is independent of weak monotonicity. We provide an example in Appendix.

We show that lower monotonicity guarantees the existence of a stable matching. Of theoretical interest, it is worth noting that we use the hospital-proposing COP instead of the doctor-proposing COP to find a stable matching. The proof is given in Appendix.

Proposition 2. If C_h satisfies substitutability, regularity, and lower monotonicity for every $h \in \mathcal{H}$, then a stable matching exists.

Remark 1. Weak monotonicity generalizes both monotonicity and the substitutable selection property. We prove this fact in Appendix. Moreover, there is a weak monotonic choice function that satisfies neither monotonicity nor the substitutable selection property.

Remark 2. Verifying whether weak monotonicity holds is difficult, as it requires demonstrating the existence of a monotone rejection function over sequences. To address this problem, we provide an alternative formulation of weak monotonicity, called *essential monotonicity*, in Appendix. This condition is defined directly in terms of the maximal rejection correspondence, rather than via a function over sequences, making it easier to verify. These facts are further illustrated in Appendix.

Finally, we discuss a necessary condition for the existence of a stable matching. One approach to establishing necessity is to provide a general condition that incorporates both weak monotonicity and lower monotonicity and guarantees the existence of a stable matching. However, the existence under each of these conditions is proven in a different manner. Under weak monotonicity, we use the doctor-proposing COP to obtain a stable matching. In contrast, under lower monotonicity, we use the hospital-proposing COP. Therefore, it is not straightforward to consider a general condition that incorporates both.

Although it is difficult to derive necessary conditions in the general case, we can do so under a special case—namely, when the maximal choice correspondence always outputs a singleton. In that case, we show that weak monotonicity becomes a necessary condition. While a singleton-valued maximal choice correspondence may seem restrictive, it does arise in real-life applications, such as daycare allocation, which will be discussed in Section 5.

4 Stable and Strategy-proof Mechanism

In this section, we analyze an incentive property (strategy-proofness) under stable mechanisms. First, we provide a sufficient condition for the existence of a stable and strategy-proof mechanism. Next, we discuss a necessary condition for a particular class of stable mechanisms, COP. Finally, we examine the nonexistence of any stable and strategy-proof mechanism.

Throughout this section, we assume that only workers are strategic. Therefore, a *mechanism* is defined as a function φ that selects a matching $\varphi(\succ)$ for each preference profile of doctors $\succ = (\succ_d)_{d\in\mathcal{D}}$. We denote by $\varphi(\succ)_d$ a hospital assigned to a doctor $d \in \mathcal{D}$ in $\varphi(\succ)$ where $\varphi(\succ)_d = \emptyset$ when d is unmatched at $\varphi(\succ)$. Given $C = (C_h)_{h\in\mathcal{H}}$, a mechanism φ is *stable* if it selects a stable matching $\varphi(\succ)$ at $(\succ, C)$ for each $\succ$. A mechanism φ is *strategy-proof (SP)* if $\varphi(\succ)_d \succeq_d \varphi(\succ'_d, \succ_{-d})_d$ for all $d \in \mathcal{D}$, $\succ'_d$ and $\succ$.

It is well-known that there exists a stable and SP mechanism when each hospital h has a choice function C_h satisfying substitutability and the *law of aggregate demand* (LAD), namely, $|C_h(D)| \leq |C_h(D')|$ for any $D, D' \subseteq \mathcal{D}$ with $D \subseteq D'$ (Hatfield and Milgrom, 2005).[6] This result directly implies the following proposition.

Proposition 3. Suppose that C_h has a substitutable selection from $\mathcal{C}_h$ that satisfies LAD for every $h \in H$. Then, there exists a stable and SP mechanism.

Even this basic fact distinguishes our results from those obtained under consistency. First, under substitutability and consistency, LAD for choice functions is known to be a necessary and sufficient condition for the existence of stable and SP mechanisms (Hatfield and Milgrom, 2005). However, this necessity does not hold without consistency. In Example 2, the choice function satisfies substitutability but violates LAD. By Proposition 3, as long as each hospital employs such choice functions, a stable and SP mechanism exists.[7] Next, it is known that there exists at most one stable and SP mechanism under consistency (Hirata and Kasuya, 2017).[8] Without consistency, however, there may be multiple stable and SP mechanisms. The choice function given in Example 2 illustrates this fact: every substitutable selection satisfies LAD and thus yields a stable and SP mechanism. These observations reveal that the existing results depend crucially on the assumption of consistency.

In Section 3, we demonstrated that a stable mechanism exists under conditions that are more general than the substitutable selection property. It is natural to ask whether there are more general conditions that guarantee the existence of a stable and SP mechanism. In the following sections, we show that the answer is negative. Therefore, the substitutable selection property is crucial for the existence of the desired mechanism.

---

[6]Note that the conjunction of substitutability and LAD implies consistency.

[7]Note that Proposition 3 does not require LAD for choice functions; instead, it requires LAD for a substitutable selection from a maximal choice correspondence.

[8]The cumulative offer process is the unique stable and SP mechanism under substitutability and LAD (Sakai, 2011; Hirata and Kasuya, 2017; Hatfield et al., 2021).

4.1 Cumulative Offer Process

We define a doctor-proposing cumulative offer process (COP). Throughout this section, we assume that C_h satisfies substitutability and weak monotonicity for all $h \in \mathcal{H}$ defined in Section 3.1. By definition, there exists a rejection function r_h over sequences for each $h \in H$. In COP, an unmatched doctor proposes to the best hospital among those who have not rejected her yet, and then the hospital tentatively accepts doctors by the choice function over sequences. The output of this process may depend on a proposal order since the choice function over sequences may be order-dependent. Therefore, we need to specify a proposal order for COP to be a well-defined mechanism. Here, we define a proposal order as a linear order $>$ over $\mathcal{D} \times \mathcal{H}$. This order is used to determine a proposal when there are multiple possible proposals. Formally, COP is defined as follows for any given $((r_h)_{h\in\mathcal{H}}, >)$.

- For each $h \in \mathcal{H}$, $\mathbf{d}_h(k)$ denotes a sequence of doctors who have proposed to h by step k where $\mathbf{d}_h(0) \equiv \emptyset$. For each $d \in D$, $R_d(k)$ denotes the set of hospitals that have rejected d by step k where $R_d(0) \equiv \emptyset$. Moreover, μ_k denotes a tentative matching at step k where $\mu_0(h) = \emptyset$ for each $h \in \mathcal{H}$. These are revised as follows.

- Step $k \geq 1$: The set of possible proposals is defined by

$$P_k = \{(d, h) \in \mathcal{D} \times \mathcal{H} \mid \mu_{k-1}(d) = \emptyset \text{ and } h = \max_{\succ_d} \mathcal{H} \setminus R_d(k-1)\},$$

  where for any $H \subseteq \mathcal{H}$, $\max_{\succ_d} H$ is the (unique) element in $H \cup \{\emptyset\}$ such that $\max_{\succ_d} H \succeq_d h'$ for all $h' \in H \cup \{\emptyset\}$. When $P_k = \emptyset$, the algorithm terminates at this step and outputs μ_{k-1}. Otherwise, let $(\hat{d}, \hat{h}) = \max_{>} P_k$. Then, $\hat{d}$ proposes to $\hat{h}$ with setting $\mathbf{d}_{\hat{h}}(k) = (\mathbf{d}_{\hat{h}}(k-1), \hat{d})$ and $\mathbf{d}_{h'}(k) = \mathbf{d}_{h'}(k-1)$ for all $h' \neq \hat{h}$. For each $d \in D$, define $R_d(k) = \{h \in \mathcal{H} \mid d \in r_h(\mathbf{d}(k))\}$. A tentative matching at step k is defined by $\mu_k(h) = c_h(\mathbf{d}_h(k))$ for each $h \in \mathcal{H}$. Go to Step $k+1$.

We call the above algorithm COP w.r.t. $((r_h)_{h\in\mathcal{H}}, >)$. The monotonicity of r_h guarantees that this procedure terminates in a finite step and produces a matching. Moreover, the definition of the maximal choice set guarantees that the final matching is stable. A proof is given in Appendix.

The output of COP depends on a proposal order when the rejection function over sequences is order-independent. More importantly, COP may not be SP for some proposal order in such a case. The following example illustrates this fact.

Example 5. Let $\mathcal{H} = \{h\}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$. We assume that h has the same choice function C_h as defined in Example 3. It is straightforward to see that C_h induces a rejection function r_h over sequences such that

$$r_h(d_1, d_2, d_3, d_4) = \{d_3\} \text{ and } r_h(d_2, d_3, d_4, d_1) = \{d_1\}. \tag{1}$$

Clearly, r_h is not order-independent. Let $>_1$ and $>_2$ be proposal orders such that

$$>_1\colon (d_1, h), (d_2, h), (d_3, h), (d_4, h) \text{ and } >_2\colon (d_2, h), (d_3, h), (d_4, h), (d_1, h).$$

Then, COP w.r.t. $(r_h, >_1)$ outputs the matching μ with $\mu(h) = \{d_1, d_2, d_4\}$ while COP w.r.t. $(r_h, >_2)$ outputs the matching ν with $\nu(h) = \{d_2, d_3, d_4\}$ when $h \succ_{d_i} \emptyset$ for all $i = 1, 2, 3, 4$. Therefore, the output of COP depends on a proposal order.

We next show that COP may not be SP for some proposal ordering. To this end, we assume that there are an additional doctor d_5 and an additional hospital h'. Moreover, h is never willing to hire doctor d_5. Thus, the choice function of h is given by $\hat{C}_h(D) = C_h(D \setminus \{d_5\})$ for any $D \subseteq \mathcal{D}$. Clearly, $\hat{C}_h$ also induces a rejection function r_h over sequences satisfying (1). Suppose that h' has a unit demand preference $\succ_{h'}$ such that $\succ_{h'}: d_5, d_1, \emptyset$.

Let φ be COP w.r.t. $(r_h, r_{h'}, >)$ where $>$ is a proposal order such that

$$(d_1, h'') > (d_2, h'') > (d_3, h'') > (d_4, h'') > (d_5, h'')$$

for any $h'' = h, h'$. Let $\succ$ be a preference profile of doctors such that

$$\succ_{d_1}: h', h, \emptyset, \quad \succ_{d_2}: h, \emptyset, \quad \succ_{d_3}: h, \emptyset, \quad \succ_{d_4}: h, \emptyset, \quad \succ_{d_5}: h', \emptyset.$$

Then, we have $\varphi(\succ)_{d_1} = \emptyset$ while $\varphi(\succ'_{d_1}, \succ_{-d_1})_{d_1} = h$. Therefore, φ does not satisfy SP.

This example suggests that order-independence is a crucial property for COP to be SP. In fact, we can generalize this observation. The following result shows that order-independence is a necessary condition for COP to satisfy SP for any proposal order when there are sufficiently many doctors. The proof is given in Appendix.

Theorem 3. Assume that $|\mathcal{H}| \geq 3$. Suppose that $h \in \mathcal{H}$ has a choice function satisfying weak monotonicity and there are sufficiently many doctors so that $\max\{|C_h(D)||D \subseteq \mathcal{D}\} + 5 \leq |\mathcal{D}|$. Let r_h be a rejection function induced from C_h. If r_h does not satisfy order-independence, then there exist two other hospitals h', h'' and a proposal order $>$ such that

- h' and h'' have rejection functions $r_{h'}$ and $r'_{h''}$ induced from some unit demand preferences $\succ_{h'}$ and $\succ_{h''}$,
- COP w.r.t. $(r_h, r_{h'}, r_{h''}, r_{\mathcal{H}\setminus\{h,h',h''\}}, >)$ is not SP for any $r_{\mathcal{H}\setminus\{h,h',h''\}} = (r_{\bar{h}})_{\bar{h}\in\mathcal{H}\setminus\{h,h',h''\}}$.

This means that the substitutable selection property is necessary for COP to satisfy SP for any proposal order.

4.2 Nonexistence of Stable and SP Mechanism

In the previous section, we focused only on COP while there are possibly other stable mechanisms. For example, COP can be generalized by considering more complex proposal orders. In this section, we provide a negative result regarding general stable mechanisms.

The following condition is a natural extension of LAD into a maximal choice correspondence. We say that $\mathcal{C}_h$ satisfies LAD if for any $D_1, D_2 \subseteq \mathcal{D}$ with $D_1 \subseteq D_2$, we have that for any $D'_1 \in \mathcal{C}_h(D_1)$, there exists $D'_2 \in \mathcal{C}_h(D_2)$ such that $|D'_1| \leq |D'_2|$. We can strengthen LAD as follows: $\mathcal{C}_h$ satisfies *acceptance* if there exists an integer $q > 0$ such that $|A| = \min\{|D|, q\}$ for any $A \in \mathcal{C}_h(D)$ and any $D \subseteq \mathcal{D}$.

Proposition 4. There may not exist a stable and SP mechanism even if every hospital has a substitutable choice function whose maximal choice correspondence satisfies monotonicity and acceptance.

The proof is based on an example, which is presented in Appendix.

5 Applications

We present inconsistent choice functions used in some practical problems and how our results apply to them. We continue to call the agents in these applications "doctors" and "hospitals" to maintain consistency with the previous sections.

Below, we often use a responsive choice function of a hospital defined as usual. Let $h \in H$. Assume that h has a linear priority (or preference) order $\succ_h$ over $\mathcal{D} \cup \{\emptyset\}$, where $\emptyset$ denotes the outside option, and a quota $q_h > 0$. Then, choice function C_h is q_h-responsive w.r.t. $\succ_h$ if for any $D \subseteq \mathcal{D}$, $C_h(D) = \{d \in D | d \succ_h \emptyset\}$ if $|\{d \in D | d \succ_h \emptyset\}| \leq q_h$, and $C_h(D) = D' \subseteq D$ such that $|D'| = q_h$ and $d' \succ_h d''$ for all $d' \in D'$ and $d'' \in D \setminus D'$ otherwise. We call a choice function C_h responsive if it is q_h-responsive for some $q_h > 0$. It is well-known that a responsive choice function satisfies path-independence.

5.1 Daycare Allocation

In Japan, as in many other countries, daycare services for young children are allocated by the government using matching mechanisms. One complication in this allocation process is that the required teacher-to-child ratio varies with the age of the children, introducing a complex constraint that goes beyond those typically discussed in the matching literature (such as type-specific quotas (Abdulkadiroğlu and Sönmez, 2003) or common quotas (Abraham et al., 2007)). Kamada and Kojima (2024) introduced a new class of constraints known as *general upper-bounds*, designed to accommodate age-dependent capacity restrictions.

We examine a choice function proposed by Kamada and Kojima (2024). First, we observe that this choice function satisfies substitutability but violates consistency. We then analyze the maximal choice correspondence and find that, under general upper bounds, it violates our conditions. As a result, we narrow our focus to the subclass of constraints known as *matroids*, under which our conditions hold. To establish these results, we provide two key facts. First, we connect two distinct choice functions through maximal choice correspondences—one proposed by Kamada and Kojima (2024) and the other derived from a greedy algorithm studied by Fleiner (2001). Next, we exploit a special structure of this model: the maximal choice correspondence always outputs a singleton. We show that if maximal correspondences satisfy this single-valuedness property, then we can apply the results in the standard matching theory (e.g., Hatfield and Milgrom (2005)).

In a matching problem under general upper-bounds constraints, each hospital h is endowed with its (linear) priority order $\succ_h$ over doctors and a (feasibility) constraint $\mathcal{F}_h \subseteq 2^{\mathcal{D}}$, which defines admissible sets of doctors for h. Therefore, a matching μ is *feasible* if and only if $\mu(h) \in \mathcal{F}_h$ for all $h \in \mathcal{H}$. A constraint $\mathcal{F}_h$ is a *general upper-bounds* *constraint* for h if $D \in \mathcal{F}_h$ and $D' \subseteq D$ implies $D' \in \mathcal{F}_h$. A matching problem is called a *matching problem under general upper-bounds constraints* if any hospital has a general upper-bounds constraint.

Kamada and Kojima (2024) consider a generalization of the cumulative offer process in a matching problem under general upper-bounds constraints.[9] We introduce the choice function of hospital h used in their generalized cumulative offer process. Assume that a priority order $\succ_h$ and a general upper-bounds constraint $\mathcal{F}_h$ are given for h. Denote $D = \{d_1, \cdots, d_k\} \subseteq \mathcal{D}$ such that $d_1 \succ_h \cdots \succ_h d_k$. Then, the choice function is defined as

$$C_h(D) = \begin{cases} D & \text{if } D \in \mathcal{F}_h, \\ \emptyset & \text{if } \{d_1\} \notin \mathcal{F}_h, \\ \{d_1, \cdots, d_\ell\} & \\ \quad \text{such that } \{d_1, \cdots, d_\ell\} \in \mathcal{F}_h \text{ and } \{d_1, \cdots, d_{\ell+1}\} \notin \mathcal{F}_h & \text{otherwise.}^{10} \end{cases}$$

Note that $d_\ell \in C_h(D)$ if and only if $\{d_1, \cdots, d_\ell\} \in \mathcal{F}_h$ since $\mathcal{F}_h$ is a general upper-bounds constraint. It is easy to see that C_h is substitutable by this fact. Meanwhile, C_h may not be consistent.

Example 6. Let $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$ and $\mathcal{H} = \{h\}$. Hospital h has the following priority order over the doctors: $d_1 \succ_h d_2 \succ_h d_3 \succ_h d_4$. The constraint for h is given by $\mathcal{F}_h = \{D \subseteq \mathcal{D} |\ |D \cap \{d_1, d_3\}| \leq 1\}$. Then, $C_h(\{d_1, d_2, d_3, d_4\}) = \{d_1, d_2\}$ by $\{d_1, d_2, d_3\} \notin \mathcal{F}_h$. However, $C_h(\{d_1, d_2, d_4\}) = \{d_1, d_2, d_4\}$ by $\{d_1, d_2, d_4\} \cap \{d_1, d_3\} = \{d_1\}$. This argument shows that $\{d_1, d_2, d_4\}$ is a maximal choice set in $\mathcal{D}$. Moreover, it can be confirmed that $\{d_1, d_2, d_4\}$ is the unique maximal choice set in $\mathcal{D}$. Thus, $\mathcal{C}_h(\{d_1, d_2, d_3, d_4\}) = \{\{d_1, d_2, d_4\}\}$.

Let $\mathcal{C}_h$ be the maximal choice correspondence for C_h. For each $D \subseteq \mathcal{D}$, we show that $\mathcal{C}_h(D)$ is a singleton consisting of a set of doctors chosen by another well-investigated choice function in a matching problem under general upper-bounds constraints. Let $\hat{C}_h$ be the greedy choice function defined as follows: let $D = \{d_1, \cdots, d_k\} \subseteq \mathcal{D}$ such that $d_1 \succ_h \cdots \succ_h d_k$. Define $\hat{C}_h^0(D) = \emptyset$. For each $\ell = 1, \cdots, k$, define

$$\hat{C}_h^\ell(D) = \begin{cases} \hat{C}_h^{\ell-1}(D) \cup \{d_\ell\} & \text{if } \hat{C}_h^{\ell-1}(D) \cup \{d_\ell\} \in \mathcal{F}_h \\ \hat{C}_h^{\ell-1}(D) & \text{otherwise.} \end{cases}$$

Finally, define $\hat{C}_h(D) = \hat{C}_h^k(D)$. Unlike C_h, the greedy choice function $\hat{C}_h$ always satisfies consistency while it may violate substitutability.

In Example 6, it is easy to verify $\hat{C}_h(\{d_1, d_2, d_3, d_4\}) = \{d_1, d_2, d_4\}$, which coincides the unique maximal choice set in $\mathcal{D}$. The following result shows that this observation holds in general.

[9] Their main algorithm is called a *cutoff adjustment process*. They show that both algorithms produce the same outcome.

[10] As Kamada and Kojima (2024) note, such ℓ uniquely exists since $\mathcal{F}_h$ is a general upper-bounds constraint.

Proposition 5. Let $h \in \mathcal{H}$. Assume that $\mathcal{F}_h$ is a general upper-bounds constraint. Then, $\mathcal{C}_h(D) = \{\hat{C}_h(D)\}$ for all $D \subseteq \mathcal{D}$.

This result clarifies the relationship between the choice function by Kamada and Kojima (2024) and the greedy choice function. In particular, it implies that C_h has a substitutable selection if and only if $\hat{C}_h$ satisfies substitutability.

In general, when the maximal choice correspondence of each hospital's choice function is singleton-valued, the stable matching problem with inconsistent choice functions can be reduced to the one with consistent choice functions. The following proposition shows this fact. We emphasize that this result applies to any matching problem with singleton-valued maximal choice correspondences, not just to those under general upper-bounds constraints.

Proposition 6. Assume that for all $h \in \mathcal{H}$, C_h satisfies substitutability and has a singleton-valued maximal choice correspondence such that $\mathcal{C}_h(D) = \{\hat{C}_h(D)\}$ for all $D \subseteq \mathcal{D}$. The following facts hold.

(a) $\hat{C}_h$ satisfies consitency for all $h \in \mathcal{H}$.[11]

(b) A matching μ is stable at $(\succ, (C_h)_{h\in\mathcal{H}})$ if and only if it is stable at $(\succ, \hat{C}_h)_{h\in\mathcal{H}})$.

Proof. (a) Let $D \subseteq \mathcal{D}$. Suppose that $d \in D$ and $d \notin \hat{C}_h(D)$. By definition, $\hat{C}_h(D)$ is also a maximal choice set in $D \setminus \{d\}$. Since C_h has a singleton-valued maximal choice correspondence, $\bar{C}_h(D)$ is the unique maximal choice set in $D \setminus \{d\}$. Thus, $\hat{C}_h(D \setminus \{d\}) = \hat{C}_h(D)$,

(b) Clearly, $C_h(\mu(h)) = \mu(h)$ if and only if $\hat{C}_h(\mu(h)) = \mu(h)$, for any matching μ and all $h \in \mathcal{H}$. Thus, a matching μ is individually rational at $(\succ, (C_h)_{h\in\mathcal{H}})$ if and only if μ is individually rational at $(\succ, (\hat{C}_h)_{h\in\mathcal{H}})$.

Suppose that an individually rational matching μ is blocked at $(\succ, (C_h)_{h\in\mathcal{H}})$; that is, there exists h and nonempty $D \subseteq \mathcal{D} \setminus \mu(h)$ such that $D \subseteq C_h(\mu(h) \cup D)$. We can show $d \in \hat{C}_h(\mu(h) \cup \{d\})$ by the same argument as Lemma 1 since $\hat{C}_h$ satisfies consistency.

We show the converse. Suppose that an individually rational μ is blocked at $(\succ, (\hat{C}_h)_{h\in\mathcal{H}})$; that is, there exists h and nonempty $D \subseteq \mathcal{D}\setminus\mu(h)$ such that $D \subseteq \hat{C}_h(\mu(h) \cup D)$ and $h \succ_d \mu(d)$ for all $d \in D$. Suppose that $d \notin C_h(\mu(h) \cup \{d\})$ for all $d \in D$. By $C_h(\mu(h)) = \mu(h)$, this implies that $\mu(h)$ is the unique maximal choice set in $\mu(h) \cup D$, contradicting that $D \subseteq \hat{C}_h(\mu(h) \cup D)$ is the unique maximal choice set in $\mu(h) \cup D$. Therefore, $d \in C_h(\mu(h) \cup \{d\})$ for some $d \in D$. This implies that μ is blocked via h and $\{d\}$ at $(\succ, (C_h)_{h\in\mathcal{H}})$. □

Proposition 6 enable us to apply all results in the standard matching theory that assumes consistency such as Hatfield and Milgrom (2005) when each hospital's choice function has a singleton-valued maximal choice correspondence. For example, the substitutable selection property, which is equivalent to weak monotonicity in this setting, is a necessary condition for the existence of a stable matching in the maximal domain sense as mentioned at the end of Subsection 3.1.

[11] Note that Proposition 6 (a) does not require substitutability of $\hat{C}_h$. Thus, it is independent of Lemma 1.

We now provide the implications for daycare allocation problems. The above results allow us to focus on the greedy choice functions rather than on the original choice functions. The properties of greedy choice functions have been extensively investigated. In particular, it has been shown that the greedy choice function has nice properties when $(\mathcal{D}, \mathcal{F}_h)$ forms a matroid.

Definition 7. A pair of the set of doctors and a constraint $(\mathcal{D}, \mathcal{F}_h)$ is a matroid if (i) $\emptyset \in \mathcal{F}_h$, (ii) $D \in \mathcal{F}_h$ and $D' \subseteq D$ imply $D' \in \mathcal{F}_h$, and (iii) for any $D, D' \in \mathcal{F}_h$ with $|D| > |D'|$, $D' \cup \{d\} \in \mathcal{F}_h$ for some $d \in D \setminus D'$.

The greedy choice function $\hat{C}_h$ satisfies path-independence and LAD for any priority order if $(\mathcal{D}, \mathcal{F}_h)$ is a matroid (Fleiner, 2001; Yokoi, 2019). Therefore, a stable and strategy-proof mechanism exists under the greedy choice functions when the constraints are matroids, which in turn implies that the desired mechanism exists for the daycare allocation problem if the constraints are matroids.

We can also establish the necessary condition on the constraint structure in daycare allocation problems. Hafalir et al. (2022) show that a matroid structure of $(\mathcal{D}, \mathcal{F}_h)$ is necessary for path-independence of the greedy choice function $\hat{C}_h$. Namely, if $(\mathcal{D}, \mathcal{F}_h)$ is not a matroid, then $\hat{C}_h$ is not path-independent for some priority order. Since greedy choice functions satisfy consistency, we can apply the maximal domain result of Hatfield and Milgrom (2005): a stable matching under the greedy choice functions may not exist when a constraint is not matroid. By Propositions 5 and 6 (b), we find that a matroid structure is necessary (and sufficient) for the existence of a stable matching in the daycare allocation problem; in other words, the existence result cannot be extended to a larger class of constraints, such as general upper-bounds.

Remark 3. In Example 6, COP defined in Section 4.1 (for any proposal order) generates a matching μ^* such that $\mu^*(h) = \{d_1, d_2, d_4\}$, which is stable. Meanwhile, the generalized cumulative offer process by Kamada and Kojima (2024) generates a matching μ^{**} such that $\mu^{**}(h) = \{d_1, d_2\}$.[12] Therefore, these two algorithms generate different matchings even if the constraint for each hospital is a matroid.

5.2 College Admissions

College admission problems are prominent applications of matching theory (Balinski and Sönmez, 1999). In these problems, the choice function reflects the college's admission process. In some countries, applicants are admitted based on a priority ranking derived from scores such as those on entrance exams. In theory, we often assume a strict priority ranking; however, in practice many applicants obtain identical scores, meaning that they are tied. This can create difficulties in satisfying capacity constraints (or quotas). To address this issue, the choice function must specify how ties are resolved. In both theory

[12] Their algorithm always generates a matching called the doctor (student)-optimal fair matching if every hospital has a general upper-bounds constraint defined as follows. At a matching μ, doctor d has justified envy toward doctor d' if $\mu(d') \succ_d \mu(d)$ and $d \succ_{\mu(d')} d'$. A matching is fair if it is feasible and no doctor has justified envy toward any doctor. A matching μ is called the doctor-optimal fair matching if μ is fair and there is no fair matching μ' such that $\mu'(d) \succeq_d \mu(d)$ for all $d \in \mathcal{D}$.

and practice, tie-breaking is commonly employed to produce a strict priority ranking: Biró and Kiselgof (2015) report that college admission systems in Ireland and Turkey break ties according to specific rules such as using birthdates. In contrast, the Hungarian college admission system treats tied applicants equally. Intuitively, the college sets a cutoff score to ensure that the total number of admitted students does not exceed the quota. Thus, all students with a score lower than the cutoff are rejected. This implies that a student might be rejected even when the college has available capacity, and it is known that this choice function fails to satisfy consistency.[13]

Our findings connect these choice functions that adopt different treatments of tied applicants. We show that the choice function used in the Hungarian college admission system satisfies the substitutable selection property. Furthermore, the substitutable selection coincides with that obtained via tie-breaking, as used in college admissions in other countries.

Let $h \in \mathcal{H}$. Assume that h has a complete, transitive, and reflexive binary relation $\succsim_h$ over $\mathcal{D}$. We call this binary relation a weak priority ordering of h. We denote $\succ_h$ and $\sim_h$ the antisymmetric part and the symmetric part of $\succsim_h$, respectively. Let $\mathcal{I} = \{I_1, \cdots, I_k\}$ be a partition of $\mathcal{D}$ based on $\succsim_h$, that is, $d \sim_h d'$ for all $d, d' \in I_\ell$ for all $\ell = 1, \cdots, k$.[14] Each element of the partition $\mathcal{I}$ is called an indifference class. Without loss of generality, we assume that $d \succ_h d'$ for all $d \in I_\ell$ and $d' \in I_{\ell'}$ with $\ell < \ell'$. Assume that h also has a quota $q_h > 0$.

We introduce the unreceptive choice function that represents the Hungarian college admission system, which is formulated by Imamura and Tomoeda (2023). We say that choice function C_h of h is q_h-unreceptive w.r.t. $\succsim_h$ if

$$C_h(D) = \begin{cases} D & \text{if } |D| \leq q_h, \\ (I_1 \cup \cdots \cup I_\ell) \cap D & \\ \qquad \text{such that } |(I_1 \cup \cdots \cup I_\ell) \cap D| \leq q_h & \text{otherwise} \\ \qquad\qquad \text{and } |(I_1 \cup \cdots \cup I_{\ell+1}) \cap D| > q_h & \end{cases}$$

for all $D \subseteq \mathcal{D}$. We say that a choice function is unreceptive if it is q-unreceptive for some integer $q > 0$.

Imamura and Tomoeda (2023) show that an unreceptive choice function satisfies substitutability. Meanwhile, they also show that an unreceptive choice function may not satisfy consistency by the following example.

Example 7. Let $\mathcal{D} = \{d_1, d_2, d_3\}$. Suppose that the weak priority order of h is given as $d_1 \succ_h d_2 \sim_h d_3$. Thus, $\mathcal{I} = \{I_1, I_2\}$ where $I_1 = \{d_1\}$ and $I_2 = \{d_2, d_3\}$. Let C_h be a 2-unreceptive choice function. Then, $C_h(\{d_1, d_2, d_3\}) = \{d_1\}$ while $C_h(\{d_1, d_2\}) = \{d_1, d_2\}$. Hence C_h does not satisfy consistency.

Note that $\mathcal{C}_h(\{d_1, d_2, d_3\}) = \{\{d_1, d_2\}, \{d_1, d_3\}\}$.

[13] Rios et al. (2021) report that the college admission system in Chile also treats tied applicants equally, but in a different manner. This choice function, however, satisfies consistency.

[14] For a given set A, we say that a family $\mathcal{B}$ of nonempty subsets of A is a partition of A if $\bigcup_{B \in \mathcal{B}} B = A$ and for any $B, B' \in \mathcal{B}$, $B \neq B'$ implies $B \cap B' = \emptyset$.

For a given weak priority ordering $\succsim_h$, we say that $\hat{\succ}_h$ is a tie-breaking priority ordering of $\succsim_h$ if $\hat{\succ}_h$ is a linear rdering and $d \succ_h d'$ implies $d \hat{\succ}_h d'$ for all $d, d' \in \mathcal{D}$. Then, we can construct a q_h-responsive choice function $\hat{C}_h$ w.r.t. the tie-breaking priority ordering $\hat{\succ}_h$ and a given quota q_h. We call this choice function $\hat{C}_h$ a tie-breaking choice function of C_h.

Proposition 7. Let C_h be a q_h-unreceptive choice function w.r.t. $\succsim_h$, where $q_h > 0$ is a given integer, and $\mathcal{C}_h$ be the maximal choice correspondence for C_h. Let $\hat{C}_h$ be a tie-breaking choice function of C_h. Then, $\hat{C}_h$ is a substitutable selection of $\mathcal{C}_h$.

Proof. It suffices to show that $\hat{C}_h$ is a selection from $\mathcal{C}_h$ since $\hat{C}_h$ is a responsive choice function and a responsive choice function is substitutable. Let $\hat{\succ}_h$ be a tie-breaking priority ordering of $\succ_h$ such that $\hat{C}_h$ is responsive w.r.t. $\hat{\succ}_h$.

Fix an arbitrary $D \subseteq \mathcal{D}$. If $|D| \leq q_h$, then it is easy to see that $C_h(D) = \hat{C}_h(D) = D$ and D is a maximal choice set in D. Thus, assume that $|D| > q_h$. Denote $I^* = C_h(D)$ and $D^* = \hat{C}_h(D)$. By $|D^*| = q_h$, $C_h(D^*) = D^*$. Fix an arbitrary $d \in D \backslash D^*$. Then, $d' \hat{\succ}_h d$ for any $d' \in D^*$. By the definition of the tie-breaking priority, $d' \succsim_h d$ for any $d' \in D^*$. Let $I_\ell \in \mathcal{I}$ be the indifference class such that $d \in I_\ell$. Note that $|D^* \cup \{d\}| = q_h + 1$. Then, $C_h(D^* \cup \{d\}) = D^*$ if $D^* \cap I_\ell = \emptyset$ because $d' \succ_h d$ for any $d' \in D^*$ in this case. Meanwhile, $C_h(D^* \cup \{d\}) = D^* \setminus I_\ell$ if $D^* \cap I_\ell \neq \emptyset$ by $|D^* \setminus I_\ell| < q_h < |D^* \cup I_\ell|$ and $D^* \setminus I_\ell = D^* \cap (I_1 \cup \cdots \cup I_{\ell-1})$. In either case, $d \notin C_h(D^* \cup \{d\})$. Hence D^* is a maximal choice set in D. $\square$

Proposition 7 connects these choice functions that adopt different treatments of tied applicants. In particular, the tie-breaking–based choice functions used in theory and practice can be related to the one employed in Hungary through a maximal choice correspondence.

Since a tie-breaking choice function is responsive, it satisfies LAD. Therefore, we obtain the following corollary from Proposition 3.

Corollary 1. There exists a stable and strategy-proof mechanism in a college admission problem with unreceptive choice functions.

5.3 Dynamic Matching

Real-life matching markets are frequently dynamic, requiring match decisions to be made across multiple time periods. For instance, in Japanese hospitals, a system known as rotation assigns medical residents to various departments over successive periods. More broadly, job rotation practices are common in many workplaces. In this setting, it is natural that when an agent chooses a partner in the current period, her decision usually depends on past matchings that cannot be altered. Bando and Kawasaki (2024) propose a choice function that illustrates the choice of an agent in such a situation.

We observe that this choice function satisfies substitutability but not consistency. Next, we demonstrate that it satisfies monotonicity, which guarantees the existence of a stable matching. However, this choice function does not necessarily have a substitutable selection. Thus, by Theorem 3, we cannot expect the COP to be strategy-proof for any proposal order, provided that there are sufficiently many doctors.

Let $\bar{C}_h$ be a choice function satisfying substitutability and consistency. Therefore, $\bar{C}_h$ satisfies path-independence. Fix any $D^p \subseteq \mathcal{D}$ with $\bar{C}_h(D^p) = D^p$. Suppose that h has been matched to the doctors in D^p and is required to continue hiring them at the current period. Then, the *period-wise choice function* $\bar{C}_h(D|D^p)$ that represents the choice of h from each $D \subseteq \mathcal{D} \setminus D^p$ at the current period as follows:

$$\bar{C}_h(D|D^p) = \begin{cases} \bar{C}_h(D^p \cup D) \setminus D^p & \text{if } D^p \subseteq \bar{C}_h(D^p \cup D) \\ \emptyset & \text{otherwise .} \end{cases}$$

Note that the second case means that there exists $d \in D^p$ such that $d \notin \bar{C}_h(D^p \cup D)$. That is, h cannot continue hiring the worker d that h had hired at the past period. We assume that such a case is infeasible and $\emptyset$ is assigned.

Let $C_h(D) = C_h(D|D^p)$ for all $D \subseteq \mathcal{D} \setminus D^p$ for simplicity. We refer to C_h as the period-wise choice function of h where R_h denotes the rejection function of C_h.

Proposition 8. C_h satisfies substitutability.

Meanwhile, C_h may not be consistent.[15]

Example 8. Let $\mathcal{D} = \{d_1, d_2, d_3\}$. Let $\bar{C}_h$ be a responsive choice function w.r.t. a preference order $d_3 \succ_h d_2 \succ_h d_1 \succ_h \emptyset$ and a quota $q_h = 2$. Let $C_h(D) = \bar{C}_h(D|\{d_1\})$ for each $D \subseteq \{d_2, d_3\}$. Then, $C_h(\{d_2, d_3\}) = \emptyset$ by $\{d_1\} \nsubseteq \{d_2, d_3\} = \bar{C}_h(\{d_1, d_2, d_3\})$ whereas $C_h(\{d_2\}) = \{d_2\}$ by $\{d_1\} \subseteq \{d_1, d_2\} = \bar{C}_h(\{d_1, d_2\})$. Hence, C_h does not satisfy consistency. Note that $\mathcal{C}_h(\{d_2, d_3\}) = \{\{d_2\}, \{d_3\}\}$.

Although C_h may be inconsistent, it satisfies monotonicity. Thus, a stable matching exists by Theorem 1 when every hospital has a period-wise choice function.

Proposition 9. C_h satisfies monotonicity.

Therefore, a period-wise choice function satisfies weak monotonicity. Meanwhile, it may not have a substitutable selection as mentioned in Section 3. By Theorem 3, COP may not be SP for some proposal order under such a period-wise choice function when there are sufficiently many doctors. We illustrate this fact with the following example.

Example 9. Let $\mathcal{D} = \{d_0, d_1, d_2, d_3, d_4\}$. Let C'_h be a choice function over $\{d_1, d_2, d_3, d_4\}$ induced from the following strict preference ordering:

$$\succ'_d: \{d_1, d_2\}, \{d_1, d_3\}, \{d_2, d_4\}, \{d_3, d_4\}, \{d_1\}, \{d_2\}, \{d_3\}, \{d_4\}.$$

Then, C'_h satisfies substitutability. Define a choice function $\bar{C}_h$ over $\mathcal{D}$ by

$$\bar{C}_h(D) = \begin{cases} \{d_1, d_2\} & \text{if } \{d_1, d_2\} \subseteq D \\ C'_h(D \setminus \{d_0\}) \cup (D \cap \{d_0\}) & \text{if } \{d_1, d_2\} \nsubseteq D, \end{cases}$$

[15] The example is essentially the same as one raised in Bando and Kawasaki (2024). We show it for completeness.

for any $D \subseteq \mathcal{D}$. Then, $\bar{C}_h$ satisfies consistency and substitutability. Note that d_0 is chosen whenever $d_0 \in D$ and $\{d_1, d_2\} \nsubseteq D$. Consider the choice function C_h defined on $\{d_1, d_2, d_3, d_4\}$ such that $C_h(D) = \bar{C}_h(D|\{d_0\})$ for all $D \subseteq \{d_1, d_2, d_3, d_4\}$. Let $\mathcal{C}_h$ be the maximal choice correspondence for C_h. As proved by Proposition 9, C_h satisfies monotonicity. However, there is even no substitutable selection from $\mathcal{C}_h$.

Suppose that there exists a selection $\tilde{C}_h$ from $\mathcal{C}_h$ that satisfies substitutability. We are noting that $\mathcal{C}_h(\{d_2, d_3\}) = \{\{d_2\}\}$ and $\mathcal{C}_h(\{d_1, d_2, d_3\}) = \{\{d_1, d_3\}, \{d_2\}\}$. Thus, $\tilde{C}_h(\{d_2, d_3\}) = \{d_2\}$ and $\tilde{C}_h(\{d_1, d_2, d_3\}) = \{d_1, d_3\}$ or $\{d_2\}$. If $\tilde{C}_h(\{d_1, d_2, d_3\}) = \{d_1, d_3\}$, then $\tilde{C}_h$ violates substitutability. Thus, we must have $\tilde{C}_h(\{d_1, d_2, d_3\}) = \{d_2\}$. Similarly, we have $\tilde{C}_h(\{d_2, d_1, d_4\}) = \{d_1\}$ by $\mathcal{C}_h(\{d_1, d_4\}) = \{\{d_1\}\}$ and $\mathcal{C}_h(\{d_2, d_1, d_4\}) = \{\{d_2, d_4\}, \{d_1\}\}$. Therefore, $\tilde{R}_h(\{d_1, d_2, d_3\}) = \{d_1, d_3\}$ and $\tilde{R}_h(\{d_2, d_1, d_4\}) = \{d_2, d_4\}$. By substitutability, we must have $\tilde{R}_h(\{d_1, d_2, d_3, d_4\}) = \{d_1, d_2, d_3, d_4\}$ and hence $\tilde{C}_h(\{d_1, d_2, d_3, d_4\}) = \emptyset$. However, $\emptyset \notin \mathcal{C}_h(\{d_1, d_2, d_3, d_4\})$ by $d_3 \in C_h(\{d_3\})$, a contradiction.

We assume that there are sufficiently many other doctors that does not affect the choice of h and at least two hospitals h' and h'' apart from h. In this case, Theorem 3 implies that we can construct unit demand preferences of h' and h'' so that COP is not SP for some proposal order.

5.4 Aggregation of Choice Functions

In collective decision-making, it is necessary to aggregate the various criteria held by individual members. For example, when a firm decides whether to hire a worker, the decision is made by a group rather than by a single individual. Similarly, in college admissions, an admissions office is responsible for the decision. We model this situation as one in which individual choice functions are aggregated to form a new choice function.

Let $H \subseteq \mathcal{H}$ and let C_h be the choice function for each $h \in H$. We consider an aggregated choice function C_H. There are two criteria to aggregate individual choice functions:

1. The first criterion is permissive: for all $D \subseteq \mathcal{D}$ and every $h \in H$, $C_h(D) \subseteq C_H(D)$. Since this criterion is permissive, we consider the *minimally permissive aggregated choice function* given by
$$C_H(D) = \bigcup_{h \in H} C_h(D).$$
That is, an element is chosen if *at least one* individual selects it. For example, in a firm, hiring new staff might require approval from at least one department.

2. The second criterion is strict: for all $D \subseteq \mathcal{D}$ and every $h \in H$, $C_H(D) \subseteq C_h(D)$. Since this criterion is strict, we consider the *maximally strict aggregated choice function* given by
$$C_H(D) = \bigcap_{h \in H} C_h(D).$$
That is, an element is chosen only if *all* individuals select it. For example, in a firm, hiring new staff might require approval from all relevant departments.

Suppose that each individual choice function is path-independent. Then, it is known that the minimally permissive aggregated choice function $C_H = \bigcup_{h\in H} C_h$ is path-independent (Aizerman and Malishevski, 1981). However, the maximally strict aggregated choice function $C_H = \bigcap_{h\in H} C_h$, while satisfying substitutability, violates consistency. The following example illustrates this point.

Example 10. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $\mathcal{H} = \{h_1, h_2, h_3\}$. Suppose that the choice function of each hospital h is induced from the following preference orderings over the sets of doctors.

$$\begin{aligned}
\succ_{h_1}&: \{d_1, d_2\}, \{d_2, d_3\}, \{d_1\}, \{d_2\}, \{d_3\}, \emptyset\\
\succ_{h_2}&: \{d_2, d_3\}, \{d_3, d_1\}, \{d_2\}, \{d_3\}, \{d_1\}, \emptyset\\
\succ_{h_3}&: \{d_3, d_1\}, \{d_1, d_2\}, \{d_3\}, \{d_1\}, \{d_2\}, \emptyset.
\end{aligned}$$

Then, $C_{\mathcal{H}}(\{d_1, d_2, d_3\}) = \emptyset$, $C_{\mathcal{H}}(\{d_1, d_2\}) = \{d_2\}, C_{\mathcal{H}}(\{d_2, d_3\}) = \{d_3\}, C_{\mathcal{H}}(\{d_3, d_1\}) = \{d_1\}$, and $C_{\mathcal{H}}(\{d\}) = \{d\}$ for any $d = d_1, d_2, d_3$, which is the choice function considered in Example 1. Hence, $C_{\mathcal{H}}$ does not satisfy regularity as we have already mentioned that $\mathcal{C}^{\mathcal{H}}(\{d_1, d_2, d_3\}) = \emptyset$.

Note that the choice function $C_{\mathcal{H}}$ appears in Example 1. Therefore, no stable matching exists. This raises the question: under what conditions does the aggregation guarantee the existence of a stable matching? The following example suggests that responsiveness is key.

Example 11. Let $\mathcal{D} = \{d_1, d_2, d_3\}$ and $H = \{h_1, h_2, h_3\}$. Each hospital has a responsive choice function w.r.t. each of the following preference orderings and quotas $q_{h_1} = q_{h_2} = q_{h_3} = 2$.

$$\succ_{h_1}:\ d_1, d_2, d_3, \emptyset;\ \succ_{h_2}:\ d_2, d_3, d_1, \emptyset;\ \succ_{h_3}:\ d_3, d_1, d_2, \emptyset.$$

Then, $C_{\mathcal{H}}(\{d_1, d_2, d_3\}) = \emptyset$ and $C_{\mathcal{H}}(D) = D$ for all $D \subsetneq \mathcal{D}$, which is the choice function considered in Example 2, which is inconsistent.

Note that the choice function $C_{\mathcal{H}}$ appears in Example 2. Therefore, a stable matching exists. We can generalize this observation.

Proposition 10. *Assume that C_h is a q_h-responsive choice function w.r.t. $\succ_h$ for all $h \in H$. Let $\mathcal{C}_H$ be the maximal choice correspondence of C_H. Define a function $\bar{C} : 2^{\mathcal{D}} \to 2^{\mathcal{D}}$ as follows: let $\bar{h} = \arg\min_{h\in H} q_h$. For each $D \subseteq \mathcal{D}$, define $a(D) = \{d \in D|\ d \succ_h \emptyset \text{ for all } h \in H\}$ and $\bar{C}(D) = C_{\bar{h}}(a(D))$. Then, $\bar{C}$ is a substitutable selection from $\mathcal{C}_H$.*

Note that $a(D) \subseteq a(D')$ if $D \subseteq D'$. Then, $\bar{C}$ satisfies LAD since a responsive choice function satisfies LAD. Therefore, we can apply Proposition 3.

Corollary 2. *A stable and strategy-proof mechanism exists for a many-to-one matching problem where every hospital has a maximally strict aggregated choice function of responsive choice functions.*

6 Concluding Remarks

This paper considered a many-to-one matching problem where the hospitals' choice functions do not necessarily satisfy consistency. We proposed several weaker consistency conditions: monotonicity, the substitutable selection property, and weak monotonicity. We demonstrated that a stable matching exists if the choice function of every hospital satisfies weak monotonicity, the weakest of the three conditions. However, we also observe that this is insufficient to guarantee the existence of a stable and strategy-proof mechanism. For that, it is necessary for each hospital's choice function to satisfy the substitutable selection property. Additionally, we considered several applications in the literature and examined when our results are applicable.

There are two directions for future research. The first is to investigate the necessary conditions for the existence of stable matchings under inconsistent choice functions. The second is to consider the model of matching with contracts, in which weaker conditions for substitutability have been proposed (for example, unilateral and bilateral substitutes (Hatfield and Kojima, 2010)). Analyzing inconsistency under these weaker versions of substitutes will be reserved for future research.

References

Atila Abdulkadiroğlu and Tayfun Sönmez. 2003. School choice: A mechanism design approach. *American Economic Review* 93, 3 (2003), 729–747.

David J Abraham, Robert W Irving, and David F Manlove. 2007. Two algorithms for the student-project allocation problem. *Journal of Discrete Algorithms* 5, 1 (2007), 73–90.

Mark Aizerman and Andrew Malishevski. 1981. General theory of best variants choice: Some aspects. *IEEE Trans. Automat. Control* 26, 5 (1981), 1030–1040.

Orhan Aygün and Tayfun Sönmez. 2012. Matching with contracts: The critical role of irrelevance of rejected contracts. *Working Paper* (2012).

Orhan Aygün and Tayfun Sönmez. 2013. Matching with contracts: Comment. *American Economic Review* 103, 5 (2013), 2050–51.

Michel Balinski and Tayfun Sönmez. 1999. A tale of two mechanisms: student placement. *Journal of Economic Theory* 84, 1 (1999), 73–94.

Keisuke Bando, Toshiyuki Hirai, and Jun Zhang. 2021. Substitutes and stability for many-to-many matching with contracts. *Games and Economic Behavior* 129 (2021), 503–512.

Keisuke Bando and Ryo Kawasaki. 2024. Stability and substitutability in multi-period matching markets. *Games and Economic Behavior* 147 (2024), 533–553.

Péter Biró and Sofya Kiselgof. 2015. College admissions with stable score-limits. *Central European Journal of Operations Research* 23, 4 (2015), 727–741.

Charles Blair. 1988. The lattice structure of the set of stable matchings with multiple partners. *Mathematics of Operations Research* 13, 4 (1988), 619–628.

Gian Caspari and Manshu Khanna. forthcoming. Nonstandard choice in matching markets. *International Economic Review* (forthcoming).

Yeon-Koo Che, Jinwoo Kim, and Fuhito Kojima. 2019. Weak monotone comparative statics. *arXiv preprint arXiv:1911.06442* (2019).

Julien Combe, Olivier Tercieux, and Camille Terrier. 2022. The design of teacher assignment: Theory and evidence. *The Review of Economic Studies* 89, 6 (2022), 3154–3222.

Bo Cowgill. 2018. *Matching markets for googlers.* Harvard Business Review Press (China Case Studies).

David Delacrétaz, Scott Duke Kominers, and Alexander Teytelboym. 2023. Matching mechanisms for refugee resettlement. *American Economic Review* 113, 10 (2023), 2689–2717.

Aytek Erdil and Taro Kumano. 2019. Efficiency and stability under substitutable priorities with ties. *Journal of Economic Theory* 184 (2019), 104950.

Tamás Fleiner. 2001. A matroid generalization of the stable matching polytope. In *International Conference on Integer Programming and Combinatorial Optimization.* Springer, 105–114.

Tamás Fleiner and Zsuzsanna Jankó. 2014. Choice function-based two-sided markets: stability, lattice property, path independence and algorithms. *Algorithms* 7, 1 (2014), 32–59.

Isa E Hafalir, Fuhito Kojima, M Bumin Yenmez, and Koji Yokote. 2022. Design on matroids: Diversity vs. meritocracy. *arXiv preprint arXiv:2301.00237* (2022).

John William Hatfield and Fuhito Kojima. 2010. Substitutes and stability for matching with contracts. *Journal of Economic Theory* 145, 5 (2010), 1704–1723.

John William Hatfield, Scott Duke Kominers, and Alexander Westkamp. 2021. Stability, strategy-proofness, and cumulative offer mechanisms. *Review of Economic Studies* 88, 3 (2021), 1457–1502.

John William Hatfield and Paul R Milgrom. 2005. Matching with contracts. *American Economic Review* 95, 4 (2005), 913–935.

Daisuke Hirata and Yusuke Kasuya. 2014. Cumulative offer process is order-independent. *Economics Letters* 124, 1 (2014), 37–40.

Daisuke Hirata and Yusuke Kasuya. 2017. On stable and strategy-proof rules in matching markets with contracts. *Journal of Economic Theory* 168 (2017), 27–43.

Kenzo Imamura and Kentaro Tomoeda. 2023. Tie-breaking or not: A choice function approach. *Working Paper* (2023).

Yuichiro Kamada and Fuhito Kojima. 2024. Fair matching under constraints: Theory and applications. *The Review of Economic Studies* 91, 2 (2024), 1162–1199.

Fuhito Kojima and Hiroaki Odahara. 2022. Toward market design in practice: a progress report. *The Japanese Economic Review* (2022), 1–18.

Herve Moulin. 1985. Choice functions over a finite set: a summary. *Social Choice and Welfare* 2, 2 (1985), 147–160.

Charles R Plott. 1973. Path independence, rationality, and social choice. *Econometrica: Journal of the Econometric Society* (1973), 1075–1091.

Ignacio Rios, Tomas Larroucau, Giorgiogiulio Parra, and Roberto Cominetti. 2021. Improving the Chilean college admissions system. *Operations Research* 69, 4 (2021), 1186–1205.

Alvin E Roth. 1984. Stability and polarization of interests in job matching. *Econometrica: Journal of the Econometric Society* (1984), 47–57.

Alvin E Roth. 1991. A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom. *The American Economic Review* (1991), 415–440.

Toyotaka Sakai. 2011. A note on strategy-proofness from the doctor side in matching with contracts. *Review of Economic Design* 15 (2011), 337–342.

Marilda Sotomayor. 1999. Three remarks on the many-to-many stable matching problem. *Mathematical Social Sciences* 38, 1 (1999), 55–70.

Yi-You Yang. 2020. Rationalizable choice functions. *Games and Economic Behavior* 123 (2020), 120–126.

Yu Yokoi. 2019. Matroidal choice functions. *SIAM Journal on Discrete Mathematics* 33, 3 (2019), 1712–1724.

Jun Zhang. 2016. On sufficient conditions for the existence of stable matchings with contracts. *Economics Letters* 145 (2016), 230–234.

A Omitted arguments in Section 3.1

We introduce the concept of essential monotonicity and show the equivalence between weak monotonicity and essential monotonicity. We then show that weak monotonicity includes both substitutable selection and monotonicity. Moreover, we provide an example that illustrates weak monotonicity strictly includes both conditions,

To define essential monotonicity, we inductively define *essential rejection set* as follows.

- We say that any $D \in \mathcal{R}_h(\mathcal{D})$ is an essential rejection set at $\mathcal{D}$.
- Let $0 < k \leq |\mathcal{D}|$. Suppose that an essential rejection set at D is defined for any set $D \subseteq \mathcal{D}$ with $|D| \geq k$. For any $D \subseteq \mathcal{D}$ with $|D| \leq k-1$, we say that $D' \in \mathcal{R}_h(D)$ is an essential rejection set at D if for any $d \in \mathcal{D} \setminus D$, there exists an essential rejection set D'' at $D \cup \{d\}$ such that $D' \subseteq D''$.

Essential monotonicity is defined as follows.

Definition 8. We say that C_h satisfies essential monotonicity if there exists an essential rejection set at D for any $D \subseteq \mathcal{D}$.

Essential monotonicity is defined without a function over sequences unlike the definition of weak monotonicity while the two concepts are equivalent.

Lemma 2. C_h satisfies weak monotonicity if and only if it satisfies essential monotonicity.

Proof. We first show the "only if" part. Suppose that there exists a function $r_h : p(\mathcal{D}) \to 2^{\mathcal{D}}$ satisfying the two conditions. We inductively show that for any $D \subseteq \mathcal{D}$ and any $\mathbf{d} \in p(\mathcal{D})$ with $\rho(\mathbf{d}) = D$, $r_h(\mathbf{d})$ is an essential rejection set at D. The statement clearly holds when $D = \mathcal{D}$. Fix any $0 < k \leq |\mathcal{D}|$. Suppose that the statement is true for any $D \subseteq \mathcal{D}$ with $|D| \geq k$. Pick any $D \subseteq \mathcal{D}$ with $|D| = k-1$ and any $\mathbf{d} \in p(\mathcal{D})$ with $\rho(\mathbf{d}) = D$. Suppose that $r_h(\mathbf{d})$ is not an essential rejection set at D. Then, there exists $d \in \mathcal{D} \setminus D$ such that there exists no essential rejection set D' at $D \cup \{d\}$ such that $r_h(\mathbf{d}) \subseteq D'$. By the induction hypothesis, $r_h(\mathbf{d}, d)$ is an essential rejection set at $D \cup \{d\}$. Thus, $r_h(\mathbf{d}) \nsubseteq r_h(\mathbf{d}, d)$ holds, a contradiction.

We next show the "if" part. Suppose that C_h satisfies essential monotonicity. For each $n \geq 1$, let $p_n(\mathcal{D}) = \{\mathbf{d} \in p(\mathcal{D}) \mid |\rho(\mathbf{d})| \leq n\}$ be the set of sequences in $p(\mathcal{D})$ whose length is equal to or less than n. We inductively show that for each $n \geq 1$, there exists a function $r_h^n : p_n(\mathcal{D}) \to 2^{\mathcal{D}}$ such that for any $\mathbf{d} \in p_n(\mathcal{D})$, (i) $r_h^n(\mathbf{d})$ is an essentially rejection set at $\rho(\mathbf{d})$ and (ii) $r_h^n(\mathbf{d}) \subseteq r_h^n(\mathbf{d}, f)$ for any $f \in \mathcal{D} \setminus \rho(\mathbf{d})$ if $|\rho(\mathbf{d})| < n$. The claim clearly holds for $n = 1$ by essential monotonicity. Suppose that the claim is true for $n-1 (\geq 1)$ and pick any function $r_h^{n-1} : p_{n-1}(\mathcal{D}) \to 2^{\mathcal{D}}$ satisfying (i) and (ii). Define $r_h^n(\mathbf{d}) = r_h^{n-1}(\mathbf{d})$ for any $\mathbf{d} \in p_{n-1}(\mathcal{D})$. Pick any $\mathbf{d} \in p_n(\mathcal{D}) \setminus p_{n-1}(\mathcal{D})$. We denote $\mathbf{d}$ by $(d_1, \cdots, d_n)$. Since $r_h^{n-1}(d_1, \cdots, d_{n-1})$ is an essential rejection set at $\{d_1, \cdots, d_{n-1}\}$, there exists an essential rejection set R' at $\{d_1, \cdots, d_n\}$ such that $r_h^{n-1}(d_1, \cdots, d_{n-1}) \subseteq R'$. Define $r_h^n(\mathbf{d}) = R'$. Then, r_h^n satisfies (i) and (ii) in the claim. □

We now show that weak monotonicity includes both the substitutable selection property and monotonicity.

Proposition 11. Let $h \in \mathcal{H}$. If C_h satisfies either monotonicity or the substitutable selection property, then C_h satisfies weak monotonicity.

Proof. First, suppose that C_h satisfies monotonicity. Then, any $D' \in \mathcal{R}_h(D)$ is an essential rejection set at D for any $D \subseteq \mathcal{D}$. Thus, it satisfies weak monotonicity by Lemma 2.

Next, suppose that C_h has substitutable selection $\bar{C}_h$. Let $\bar{R}_h$ be the rejection function of $\bar{C}_h$. For each $\mathbf{d} \in p(\mathcal{D})$, define $r_h(\mathbf{d}) = \bar{R}^h(\mathbf{d})$. Then, r_h satisfies (i) and (ii) in Definition 5. Thus, it satisfies weak monotonicity. $\square$

The following example shows that the class of choice functions satisfying weak monotonicity is strictly larger than those satisfying monotonicity or the substitutable selection property.

Example 12. *Let $h \in \mathcal{H}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$. We assume that $C_h(D) = D$ for all $D \subseteq \mathcal{D}$ with $|D| \leq 2$. For $D \subseteq \mathcal{D}$ with $|D| \geq 3$, $C_h(D)$, $\mathcal{C}_h(D)$, $\mathcal{R}_h(D)$, and $\mathcal{E}_h(D)$ are given in Table 4 where $\mathcal{E}_h(D)$ denotes the set of all essential rejection sets at D.*

Table 3: choice and maximal choice (rejection) and essential rejection

Y	$C_h(D)$	$\mathcal{C}_h(D)$	$\mathcal{R}_h(D)$	$\mathcal{E}_h(D)$
$\{d_1, d_2, d_3, d_4\}$	$\emptyset$	$\{d_3, d_4\}, \{d_1, d_2, d_4\}$	$\{d_1, d_2\}, \{d_3\}$	$\{d_1, d_2\}, \{d_3\}$
$\{d_1, d_2, d_3\}$	$\{d_1, d_2\}$	$\{d_1, d_2\}$	$\{d_3\}$	$\{d_3\}$
$\{d_1, d_2, d_4\}$	$\{d_1, d_2, d_4\}$	$\{d_1, d_2, d_4\}$	$\emptyset$	$\emptyset$
$\{d_1, d_3, d_4\}$	$\{d_3\}$	$\{d_1, d_3\}, \{d_3, d_4\}$	$\{d_4\}, \{d_1\}$	$\{d_1\}$
$\{d_2, d_3, d_4\}$	$\emptyset$	$\{d_2, d_3\}, \{d_2, d_4\}, \{d_3, d_4\}$	$\{d_4\}, \{d_3\}, \{d_2\}$	$\{d_3\}, \{d_2\}$

Then, it can be easily confirmed that C_h satisfies substitutability and essential monotonicity (weak monotonicity). On the other hand, C_h does not satisfy monotonicity since $\{d_4\} \in \mathcal{R}_h(\{d_1, d_3, d_4\})$ but $d_4 \notin D$ for any $D \in \mathcal{R}_h(\{d_1, d_2, d_3, d_4\})$.

Moreover, C_h does not satisfy the substitutable selection property. To see this, suppose that there exists a substitutable selection $\bar{C}_h$ from $\mathcal{C}_h$. Then, $\bar{C}_h(\{d_1, d_2, d_3\}) = \{d_1, d_2\}$. Thus, $\bar{C}_h(\{d_1, d_2, d_3, d_4\}) = \{d_1, d_2, d_4\}$ by substitutability of $\bar{C}_h$. Note that $\bar{C}_h(\{d_1, d_3, d_4\}) = \{d_1, d_3\}$ or $\{d_3, d_4\}$. But either case violates substitutability of $\bar{C}_h$, a contradiction.

We next show that weak monotonicity and lower monotonicity is logically independent. The choice function defined in Table 5 will show that weak monotonicity does not imply lower monotonicity. The following example shows that lower monotonicity does not imply weak monotonicity.

Example 13. Let $h \in \mathcal{H}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$. We assume that $C_h(\{d\}) = \{d\}$ for all $d \in \mathcal{D}$. For $D \subseteq \mathcal{D}$ with $|D| \geq 2$, $C_h(D)$, $\mathcal{C}_h(D)$, $\mathcal{R}_h(D)$, and $\mathcal{E}_h(D)$ are given in Table 4 where $\mathcal{E}_h(D)$ denotes the set of all essential rejection sets at D.

In the above table, "None" indicates that there exists no essential rejection set at $\{d_1, d_2\}$. Thus, C_h does not satisfy weak monotonicity by Lemma 2. On the other hand, it can be confirmed that C_h satisfies lower monotonicity. Therefore, weak monotonicity does not imply lower monotonicity.

Table 4: choice and maximal choice (rejection) and essential rejection

Y	$C_h(D)$	$\mathcal{C}_h(D)$	$\mathcal{R}_h(D)$	$\mathcal{E}_h(D)$
$\{d_1, d_2, d_3, d_4\}$	$\emptyset$	$\{d_3, d_4\}$	$\{d_1, d_2\}$	$\{d_1, d_2\}$
$\{d_1, d_2, d_3\}$	$\emptyset$	$\{d_2, d_3\}$	$\{d_1\}$	$\{d_1\}$
$\{d_1, d_2, d_4\}$	$\emptyset$	$\{d_1, d_4\}$	$\{d_2\}$	$\{d_2\}$
$\{d_1, d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_1\}$	$\{d_1\}$
$\{d_2, d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_2\}$	$\{d_2\}$
$\{d_1, d_2\}$	$\emptyset$	$\{d_1\}, \{d_2\}$	$\{d_1\}, \{d_2\}$	None
$\{d_1, d_3\}$	$\{d_3\}$	$\{d_3\}$	$\{d_1\}$	$\{d_1\}$
$\{d_1, d_4\}$	$\{d_1, d_4\}$	$\{d_1, d_4\}$	$\emptyset$	$\emptyset$
$\{d_2, d_3\}$	$\{d_2, d_3\}$	$\{d_2, d_3\}$	$\emptyset$	$\emptyset$
$\{d_2, d_4\}$	$\{d_4\}$	$\{d_4\}$	$\{d_2\}$	$\{d_2\}$
$\{d_3, d_4\}$	$\{d_3, d_4\}$	$\{d_3, d_4\}$	$\emptyset$	$\emptyset$

B Omitted Proofs

Proofs of Theorems 1 and 2

We assume that C_h satisfied lower monotonicity for each $h \in \mathcal{H}$. We show Theorem 2. Then, it implies Theorem 1. Suppose that C_h satisfies weak monotonicity for all $h \in H$. We use COP w.r.t. $((r_h)_{h\in\mathcal{H}}, >)$ where $>$ is an arbitrary proposal order, which was formally defined in Subsection 4.1.

We first show that COP terminates in a finite step. By monotonicity of r_h, we have $\mathcal{H} \setminus R_d(k) \supseteq \mathcal{H} \setminus R_d(k+1)$ for each step k and $d \in \mathcal{D}$. Moreover, for each step k, either $\{d \in \mathcal{D} \mid \mu_{k-1}(d) = \emptyset\} \supsetneq \{d \in \mathcal{D} \mid \mu_k(d) = \emptyset\}$ or $\mathcal{H} \setminus R_d(k) \supsetneq \mathcal{H} \setminus R_d(k+1)$ for some $d \in \mathcal{D}$ holds by definition. This implies that COP terminates in a finite step because there are finite doctors and hospitals. We assume that COP terminates at step $k^* + 1$ ($k^* \geq 0$).

We next show that μ_k is a matching for each step $k = 1, \cdots, k^*$. Pick any $k = 1, \cdots, k^*$. Suppose that μ_k is not a matching. Then, there exists $d' \in \mu_k(c_{h_1}(\mathbf{d}_{h_1}(k))) \cap \mu_k(c_{h_2}(\mathbf{d}_{h_2}(k))) \neq \emptyset$ for some $h_1, h_2 \in \mathcal{H}$ with $h_1 \neq h_2$. Without loss of generality, we assume that $h_1 \succ_{d'} h_2$. By $d' \in c_{h_2}(\mathbf{d}_{h_2}(k)))$, d' has proposed to h_2 by step k. Thus, d' has been rejected by h_1 by step $k-1$. Thus, $d' \in r_{h_1}(\mathbf{d}_{h_1}(k-1))$. By monotonicity, $d' \in r_{h_1}(\mathbf{d}_{h_1}(k))$, a contradiction.

We finally show that the final matching μ_{k^*} is stable. Clearly, $\mu_{k^*}(d) \succeq \emptyset$ for all $d \in \mathcal{D}$. We also have $C_h(\mu_{k^*}(h)) = \mu_{k^*}(h)$ for all $h \in \mathcal{H}$ since $\mu_{k^*}(h) = c_h(\mathbf{d}_h(k^*))$ is a maximal choice set in $\rho(\mathbf{d}_h(k^*))$. Thus, μ_{k^*} is individually rational. Suppose that μ_{k^*} is blocked. Then, there exist $h \in \mathcal{H}$ and nonempty $D \subseteq \mathcal{D} \setminus \mu_{k^*}(h)$ such that $D \subseteq C_h(\mu_{k^*}(h) \cup D)$ and $h \succ_d \mu_{k^*}(d)$ for all $d \in D$. Pick any $\hat{d} \in D$. By substitutability of C_h, we have $\hat{d} \in C_h(\mu_{k^*}(h) \cup \{\hat{d}\})$. Moreover, $\hat{d}$ has proposed to h by $h \succ_{\hat{d}} \mu_{k^*}(\hat{d})$. Thus, $\hat{d} \in \rho(\mathbf{d}_h(k^*))$ holds, contradicting $\mu_{k^*}(h) = c_h(\mathbf{d}_h(k^*))$ is a maximal choice set at $\rho(\mathbf{d}_h(k^*))$.

Proof of Proposition 2

The proof is done by a hospital-proposing DA algorithm defined as follows.

- For each $h \in \mathcal{H}$, $A_h(k)$ denotes the set of all doctors who have rejected h by step k where $A_h(0) \equiv \emptyset$. For each $d \in \mathcal{D}$, $A_d(k)$ denotes the set of hospitals that have proposed to d by step k where $A_d(0) \equiv \emptyset$. Moreover, μ_k denotes a tentative matching at step k where $\mu_0(h) = \emptyset$ for all $h \in \mathcal{H}$. These sets are revised by the following procedure.

- Step $k \geq 1$: If $\mu_{k-1}(h) \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-1))$ for all $h \in \mathcal{H}$, then the algorithm terminates at this step and outputs μ_{k-1}. Otherwise, for each $h \in \mathcal{H}$, pick any $D^h \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-1))$ such that $\mu_{k-1}(h) \subseteq D^h$. Define $A_d(k) = A_d(k-1) \cup \{h \in \mathcal{H} \mid d \in D^h\}$ for all $d \in \mathcal{D}$. Let $\mu_k(d) = \max_{\succ_d} A_d(k)$ for all $d \in \mathcal{D}$ and $A_h(k) = \{d \in \mathcal{D} \mid h \in A_d(k) \setminus \mu_k(d)\}$ for all $h \in \mathcal{H}$. Proceed to the next step.

The following claim guarantees that the above procedure is well-defined.

Claim 1. Suppose that the hospital-proposing DA algorithm proceeds to step k and does not terminate at step k.

(a) For each $h \in \mathcal{H}$, there exists $D^h \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-1))$ such that $\mu_{k-1}(h) \subseteq D^h$.

(b) $A_d(k-1) \subsetneq A_d(k)$ for some $d \in \mathcal{D}$.

Proof. When $k = 1$, the statement holds by the regularity of the choice functions. Thus, we assume that $k \geq 2$.

We first show (a). Since the algorithm does not terminate at step $k-1$, for each $h \in \mathcal{H}$, we can take $B^h \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-2))$ such that $\mu_{k-2}(h) \subseteq B^h$. By definition, $A_d(k-1) = A_d(k-2) \cup \{h \in \mathcal{H} \mid d \in B^h\}$ for all $d \in \mathcal{D}$. Fix an arbitrary $\hat{h} \in H$. By definition, we have $A_{\hat{h}}(k-2) \subseteq A_{\hat{h}}(k-1)$. By lower monotonicity, there exists $D^{\hat{h}} \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-1))$ such that $B^{\hat{h}} \setminus A_{\hat{h}}(k-1) \subseteq D^{\hat{h}}$. We show that $\mu_{k-1}(\hat{h}) \subseteq D^{\hat{h}}$. Pick any $d \in \mu_{k-1}(\hat{h})$. Then, $\hat{h} = \max_{\succ_d} A_d(k-1)$ and $d \notin A_{\hat{h}}(k-1)$. If $\hat{h} \in A_d(k-2)$, then $\hat{h} = \max_{\succ_d} A_d(k-2) = \mu_{k-2}(d)$, which implies $d \in D^{\hat{h}}$ by $d \in \mu_{k-2}(\hat{h}) \subseteq B^{\hat{h}}$ and $B^{\hat{h}} \setminus A_{\hat{h}}(k-1) \subseteq D^{\hat{h}}$. If $\hat{h} \in A_d(k-1) \setminus A_d(k-2)$, then $d \in B^{\hat{h}}$ by definition, which implies $d \in D^{\hat{h}}$ by $d \notin A_{\hat{h}}(k-1)$ and $B^{\hat{h}} \setminus A_{\hat{h}}(k-1) \subseteq D^{\hat{h}}$.

We next show (b). By (a), for each $h \in \mathcal{H}$, there exists $D^h \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k-1))$ such that $\mu_{k-1}(h) \subseteq D^h$. Since the algorithm does not terminate at step k, there exists $\tilde{h} \in H$ such that $\mu_{k-1}(\tilde{h}) \notin \mathcal{C}_{\tilde{h}}(\mathcal{D} \setminus A_{\tilde{h}}(k-1))$. Then, we must have $\mu_{k-1}(\tilde{h}) \subsetneq D^{\tilde{h}}$. Therefore, there exists $d' \in D^{\tilde{h}} \setminus \mu_{k-1}(\tilde{h})$. Suppose that $\tilde{h} \in A_{d'}(k-1)$. By $d' \notin A_{\tilde{h}}(k-1)$, we have $\tilde{h} = \mu_{k-1}(d')$, contradicting that $d' \notin \mu_{k-1}(\tilde{h})$. Thus, $\tilde{h} \notin A_{d'}(k-1)$ and $\tilde{h} \in A_{d'}(k)$ where the latter follows from $d' \in D^{\tilde{h}}$. Therefore, $A_{d'}(k-1) \subsetneq A_{d'}(k)$ holds. □

By (b) of the above claim, the algorithm terminates in a finite step $k^* + 1$ ($k^* \geq 0$) since the set of hospitals is finite. We show that μ_{k^*} is stable. Clearly, $\mu_{k^*}(d) = \max_{\succ_d} A_d(k^*) \succeq_d \emptyset$ for all $d \in \mathcal{D}$. Moreover, $C_h(\mu_{k^*}(h)) = \mu_{k^*}(h)$ for all $h \in \mathcal{H}$ since $\mu_{k*}(h) \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k^*))$. Thus, μ_{k^*} is individually rational.

Suppose that μ_{k^*} is blocked. Then, there exist $h \in \mathcal{H}$ and nonempty $D \subseteq \mathcal{D} \setminus \mu_{k^*}(h)$ such that $D \subseteq C_h(\mu_{k^*}(h) \cup D)$ and $h \succ_d \mu_{k^*}(d)$ for all $d \in D$. Pick any $\hat{d} \in D$. By substitutability of C_h, we have $\hat{d} \in C_h(\mu_{k^*}(h) \cup \{\hat{d}\})$. By $h \succ_{\hat{d}} \mu_{k^*}(\hat{d}) = \max_{\succ_{\hat{d}}} A_{\hat{d}}(k^*)$, we have $\hat{d} \in \mathcal{D} \setminus A_h(k^*)$. By the definition of maximal choice set, this implies $\hat{d} \notin C_h(\mu_{k*}(h) \cup \{\hat{d}\})$ since $\mu_{k^*}(h) \in \mathcal{C}_h(\mathcal{D} \setminus A_h(k^*))$, a contradiction.

Proof of Theorem 3

We introduce some notation used throughout this proof. For any $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ and any $d_i \in \{d_1, \cdots, d_n\}$, define $c_h(\mathbf{d})[d_i] = d_i$ if $d_i \in c_h(\mathbf{d})$ and $c_h(\mathbf{d})[d_i] = \emptyset$ if $d_i \in r_h(\mathbf{d})$. For any $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ and $i = 1, \cdots, n$, define $\mathbf{d}(i) = d_i$. For any $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ and any $i, j \in \{1, \cdots, n\}$ with $i < j$, let $\mathbf{d}[i, j]$ be a sequence constructed from $\mathbf{d}$ by moving i'th term into j's term without changing the ordering of the other terms, that is,

$$\mathbf{d}[i, j] = (d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_{j-1}, d_j, d_i, d_{j+1}, \cdots, d_n).$$

For example, when $\mathbf{d} = (d_1, d_2, d_3, d_4, d_5)$, $\mathbf{d}[1, 4] = (d_2, d_3, d_4, d_1, d_5)$. Such a move is called a *simple move*. We define $\mathbf{d}[i, j] = \mathbf{d}$ when $i = j$.

We introduce two conditions that are used in the proof.

- r_h satisfies condition A if for any sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$, there exist no $i, j \in \{1, \cdots, n\}$ such that $i < j$ and $c_h(\mathbf{d})[d_i] \neq c_h(\mathbf{d}[i, j])[d_i]$.
- r_h satisfies condition B if for any sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$, there exists no $i = 1, \cdots, n-1$ such that $c_h(\mathbf{d})[d_n] \neq c_h(\mathbf{d}[i-1, i])[d_n]$.

Lemma 3. r_h satisfies order-independence if and only if it satisfies conditions A and B.

Proof. The "only if" part clearly holds. We show the "if" part. Suppose that r_h satisfies conditions A and B while it does not satisfy order-independence. Then, there are sequences $\mathbf{d}, \mathbf{d}' \in \rho^h(X)$ with $\rho(\mathbf{d}) = \rho(\mathbf{d}')$ and $r_h(\mathbf{d}) \neq r_h(\mathbf{d}')$. We denote $\mathbf{d} = (d_1, \cdots, d_n)$.

We first transform $\mathbf{d}'$ into $\mathbf{d}$ by simple moves.

- Step 1. Set $\mathbf{y}^1 = \mathbf{d}'$.
- Step k. If $\{i = 1, \cdots, n \mid \mathbf{y}^k(i) \neq \mathbf{d}(i)\} = \emptyset$, then output $\mathbf{y}^k$. Otherwise, let $\hat{i} = \max\{i = 1, \cdots, n | \mathbf{y}^k(i) \neq \mathbf{d}(i)\}$. Then, there exists $i' < \hat{i}$ such that $\mathbf{y}^k(i') = \mathbf{d}(\hat{i})$ (by $\rho(\mathbf{y}^k) = \rho(\mathbf{d})$). Set $\mathbf{y}^{k+1} = \mathbf{y}[i', \hat{i}]$ and go to step $k + 1$.

It is straightforward to see that this procedure terminates in a finite step $k^*(\geq 2)$ with $\mathbf{y}^{k^*} = \mathbf{d}$. By $c_h(\mathbf{d}) \neq c_h(\mathbf{d}')$, we have that $c_h(\mathbf{y}^k) \neq c_h(\mathbf{y}^{k+1})$ for some $k = 1, \cdots k^* - 1$.

From the above argument, we assume, without loss of generality, that there exist $i, j \in \{1, \cdots, n\}$ such $i < j$ and $c_h(\mathbf{d}) \neq c_h(\mathbf{d}[i, j])$. We denote $\mathbf{d}' = \mathbf{d}[i, j]$. Consider $d_l \in \{d_1, \cdots, d_n\}$ such that $c_h(\mathbf{d})[d_l] \neq c_h(\mathbf{d}')[d_l]$. By condition A, we have $d_l \neq d_i$. Let $l' \in \{1, \cdots, n\}$ with $\mathbf{d}'(l') = d_l$. By condition A, we have $c_h(\mathbf{d})[d_l] = c_h(\mathbf{d}[l, n])[d_l]$ and $c_h(\mathbf{d}')[d_l] = c_h(\mathbf{d}'[l', n])[d_l]$. Note that the only difference between $\mathbf{d}[l, n]$ and $\mathbf{d}'[l', n]$ is the position of d_i. Therefore, starting from $\mathbf{d}[l, n]$, simple adjacent moves of d_i lead to $\mathbf{d}'[l', n]$. By condition B, $c_h(\mathbf{d}[l, n])[d_l] = c_h(\mathbf{d}'[l', n])[d_l]$. Thus, we have $c_h(\mathbf{d})[d_l] = c_h(\mathbf{d}')[d_l]$ from $c_h(\mathbf{d})[d_l] = c_h(\mathbf{d}[l, n])[d_l]$ and $c_h(\mathbf{d}')[d_l] = c_h(\mathbf{d}'[l', n])[d_l]$, contradicting the choice of d_l. $\square$

Lemma 4. Suppose that there are sufficiently many doctors so that $\max\{|C_h(D')||D' \subseteq \mathcal{D}\} + 2 \leq |\mathcal{D}|$. Suppose that r_h does not satisfy condition A. Then, there exist $h' \neq h$ and a proposal order $>$ such that

- h' has a rejection function $r_{h'}$ over sequences induced from a unit demand preference $\succ_{h'}$,
- *COP* w.r.t. $(r_h, r_{h'}, r_{\mathcal{H}\setminus\{h,h'\}}, >)$ is not SP for any $r_{\mathcal{H}\setminus\{h,h'\}} = (r_{\bar{h}})_{\bar{h}\in\mathcal{H}\setminus\{h,h'\}}$.

Proof. We first characterize condition A by a simpler condition.

Claim 2. r_h satisfies condition A if and only if for any sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$, there exist no $i = 1, \cdots, n-1$ such that $c_h(\mathbf{d})[d_i] \neq c_h(\mathbf{d}[i,n])[d_i]$.

Proof. The "only if" part clearly holds. We show that the "if" part. We assume that for any sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$, there exist no $i = 1, \cdots, n-1$ such that $c_h(\mathbf{d})[d_i] \neq c_h(\mathbf{d}[i,n])[d_i]$. Suppose that condition A is not satisfied. Then, we can take a sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ such that there exist i, j $(i < j < n)$ satisfying $c_h(\mathbf{d})[d_i] \neq c_h(\mathbf{d}[i,j])[d_i]$. We denote $\mathbf{d}' = \mathbf{d}[i,j]$. By assumption, $c_h(\mathbf{d})[d_i] = c_h(\mathbf{d}[i,n])[d_i]$ and $c_h(\mathbf{d}')[d_i] = c_h(\mathbf{d}'[j,n])[d_i]$. By $\mathbf{d}[i,n] = \mathbf{d}'[j,n]$, $c_h(\mathbf{d}[i,n])[d_i] = c_h(\mathbf{d}'[j,n])[d_i]$. Thus, $c_h(\mathbf{d})[d_i] = c_h(\mathbf{d}')[d_i]$, contradicting the choice of i, j. □

We now show this lemma. Suppose that r_h does not satisfy condition A. From Claim 2, we can take a sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ such that there exists $i = 1, \cdots, n-1$ satisfying $c_h(\mathbf{d})[d_i] \neq c_h(\mathbf{d}[i,n])[d_i]$.

By the assumption of $\max\{|C_h(D')||D' \subseteq \mathcal{D}\} + 2 \leq |\mathcal{D}|$, we can take $d' \in \mathcal{D}$ such that

$$d' \notin c_h(d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n) \cup \{d_i\}.$$

Recall that we are assuming that there is a hospital $h' \neq h$. There are two cases to consider.

Case 1. $c_h(\mathbf{d})[d_i] = d_i$ and $c_h(\mathbf{d}[i,n])[d_i] = \emptyset$.

We assume that h' has a unit demand preference $\succ_{h'}$: $d', d_i, \emptyset$. Consider any $r_{\bar{h}}$ for any $\bar{h} \in \mathcal{H} \setminus \{h, h'\}$. Define $\succ_{d_1}, \cdots, \succ_{d_n}$ by

- $\succ_d$: $h, h', \emptyset$ for all $d \in \{d_1, \cdots, d_n\} \setminus \{d_i\}$,
- $\succ_{d_i}$: $h', h, \emptyset$.

If $d' \in \{d_1, \cdots, d_n\}$, $\succ_{d'}$ has already been defined in the above way. If $d' \notin \{d_1, \cdots, d_n\}$, define $\succ_{d'}$ by

- $\succ_{d'}$: $h', \emptyset$.

We assume that $\succ_d$ ranks $\emptyset$ first for any $d \in \mathcal{D}\setminus(\{d_1, \cdots, d_n\} \cup \{d'\})$. By the construction of doctors' preference orders, we may ignore hospitals besides h and h'. Thus, we may assume that $\mathcal{H} = \{h, h'\}$ without changing the property of COP.

Consider a proposal order $>$ over $\mathcal{D} \times \{h, h'\}$ such that

$$(d_1, h), \cdots, (d_{i-1}, h), (d_i, h'), (d_i, h), (d_{i+1}, h), \cdots, (d_n, h), \cdots .$$

Let φ be COP w.r.t. $(r_h, r_{h'}, >)$.

We first show $\varphi(\succ)_{d_i} = \emptyset$. Let us consider COP w.r.t. $(r_h, r_{h'}, >)$. In the first n steps, $d_1, \cdots, d_{i-1}$ propose to h, d_i proposes to h', and $d_{i+1}, \cdots, d_n$ propose to h. Thus, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' \\ (d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n) & (d_i) \end{pmatrix}.$$

At this moment, d' is unmatched since $d' \notin c_h(d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n) \cup \{d_i\}$ and the definition of $>$. Thus, d' proposes to h' and d_i is rejected at some subsequent step. Then, according to $>$, d_i proposes to h and rejected because $c_h(\mathbf{d}[i,n])[d_i] = \emptyset$. This implies $\varphi(\succ)_{d_i} = \emptyset$.

We next show $\varphi(\succ'_{d_i}, \succ_{-d_i})_{d_i} = h$ where $\succ'_{d_i}$: $h, \emptyset$. Let us consider COP w.r.t. $(r_h, r_{h'}, >)$. In the first n steps, $d_1, \cdots, d_n$ propose to h;

$$\begin{pmatrix} h & h' \\ (d_1, \cdots, d_{i-1}, d_i, d_{i+1}, \cdots, d_n) & \end{pmatrix}.$$

This implies $\varphi(\succ)_{d_i} = h$ by $c_h(\mathbf{d})[d_i] = d_i$. Therefore, φ is not SP.

Case 2. $c_h(\mathbf{d})[d_i] = \emptyset$ and $c_h(\mathbf{d}[i,n])[d_i] = d_i$.

We assume that hospital h' has a unit demand preference $\succ_{h'}$: $d', d_i, \emptyset$. Consider any $r_{\bar{h}}$ for any $\bar{h} \in \mathcal{H} \setminus \{h, h'\}$. Define $\succ_{d_1}, \cdots, \succ_{d_n}$ by

- $\succ_d$: $h, h', \emptyset$ for all $d \in \{d_1, \cdots, d_n\} \setminus \{d_i\}$,
- $\succ_{d_i}$: $h, \emptyset$.

If $d' \in \{d_1, \cdots, d_n\}$, $\succ_{d'}$ has already been defined in the above way. If $d' \notin \{d_1, \cdots, d_n\}$, define $\succ_{d'}$ by

- $\succ_{d'}$: $h', \emptyset$.

We assume that $\succ_d$ ranks $\emptyset$ first for any $d \in \mathcal{D} \setminus (\{d_1, \cdots, d_n\} \cup \{d'\})$. Again, we may assume that $\mathcal{H} = \{h, h'\}$ because any other hospital is unacceptable for any doctor.

Consider the proposal ordering $>$ over $D \times \{h, h'\}$ such that

$$(d_1, h), \cdots, (d_{i-1}, h), (d_i, h'), (d_i, h), (d_{i+1}, h), \cdots, (d_n, h), \cdots.$$

Let φ be COP w.r.t. $(r_h, r_{h'}, >)$. We first show $\varphi(\succ)_{d_i} = \emptyset$. Let us consider COP w.r.t. $(r_h, r_{h'}, >)$. In the first n steps, $d_1, \cdots, d_n$ propose to h;

$$\begin{pmatrix} h & h' \\ (d_1, \cdots, d_{i-1}, d_i, d_{i+1}, \cdots, d_n) & \end{pmatrix}.$$

This implies $\varphi(\succ)_{d_i} = \emptyset$ by $c_h(\mathbf{d})[d_i] = \emptyset$.

We next show $\varphi(\succ'_{d_i}, \succ_{-d_i})_{d_i} = h$ where $\succ'_{d_i}$: $h', h, \emptyset$. Let us consider COP w.r.t. $(r_h, r_{h'}, >)$. In the first n steps, $d_1, \cdots, d_{i-1}$ propose to h, d_i proposes to h', $d_{i+1}, \cdots, d_n$ propose to h. Thus, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' \\ (d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n) & (d_i) \end{pmatrix}.$$

At this moment, d' is unmatched since $d' \notin c_h(d_1, \cdots, d_{i-1}, d_{i+1}, \cdots, d_n) \cup \{d_i\}$ and the definition of $>$. Thus, d' proposes to h' and d_i is rejected at some subsequent step. Then, according to $>$, d_i proposes to h and rejected because $c_h(\mathbf{d}[i,n])[d_i] = \emptyset$. This implies $\varphi(\succ)_{d_i} = h$. Therefore, φ is not SP. $\square$

Lemma 5. Suppose that there are sufficiently many doctors so that $\max\{|C_h(D) \mid D \subseteq \mathcal{D}\} + 5 \leq |\mathcal{D}|$. Suppose that r_h does not satisfy condition B. Then, there exist two hospitals $h', h'' \neq h$ and a proposal order $>$ such that

- h' and h'' have rejection functions $r_{h'}$ and $r_{h''}$ induced from some unit demand preferences $\succ_{h'}$ and $\succ_{h''}$,
- *COP* w.r.t. $(r_h, r_{h'}, r_{h''}, r_{\mathcal{H}\setminus\{h,h',h''\}}, >)$ is not SP for any $r_{\mathcal{H}\setminus\{h,h',h''\}} = (r_{\bar{h}})_{\bar{h}\in\mathcal{H}\setminus\{h,h',h''\}}$.

Proof. Suppose that r_h does not satisfy condition B. We can take a sequence $\mathbf{d} = (d_1, \cdots, d_n) \in p(\mathcal{D})$ such that there exists $i = 1, \cdots, n-1$ satisfying $c_h(\mathbf{d})[d_n] \neq c_h(\mathbf{d}[i-1, i])[d_n]$. Without loss of generality, we assume that $c_h(\mathbf{d})[d_n] = \emptyset$ and $c_h(\mathbf{d}[i-1, i])[d_n] = d_n$. Let $d_1, \cdots, d_n$ be doctors corresponding to $(d_1, \cdots, d_n)$.

Case 1. $i \leq n-2$.

By the assumption of $\max\{|C_h(D) \mid D \subseteq \mathcal{D}\} + 5 \leq |\mathcal{D}|$, we can take $d', d'' \in \mathcal{D}$ so that

$$d' \notin c_h(d_1, \cdots, d_{n-2}) \cup \{d_{i-1}, d_{n-1}, d_n\} \text{ and } d'' \notin c_h(d_1, \cdots, d_{n-1}) \cup \{d_i, d_n, d'\}.$$

Moreover, we can take $\hat{d}' \in \mathcal{D}$ so that

$$\hat{d}' \notin c_h((d_1, \cdots, d_{n-2})[i-1, i]) \cup \{d_{i-1}, d_{n-1}, d_n, d''\}.$$

Note that $d' \neq d''$ and $\hat{d}' \neq d''$ by definition while $d' = \hat{d}'$ is possible.

Recall that we are assuming that there are two hospitals h' and h'' apart from h. We assume that h' has a unit demand preference $\succ_{h'}$ such that

- d' is ranked first,
- $d_{n-1} \succ_{h'} d_n \succ_{h'} d_{i-1} \succ_{h'} \emptyset$,
- $d \succ_{h'} d_{n-1}$ for all $d \in \{d_1, \cdots, d_{n-2}, \hat{d}'\} \setminus \{d_{i-1}, d'\}$.

Hospital h'' has a unit demand preference $\succ_{h''}$: $d'', d_n, d_i, \emptyset$. Consider any $r_{\bar{h}}$ for any $\bar{h} \in \mathcal{H} \setminus \{h, h', h''\}$.

Define $\succ_{d_1}, \cdots, \succ_{d_n}$ by

- $\succ_d$: $h, h', h'', \emptyset$ for all $d \in \{d_1, \cdots, d_{i-2}\}$,
- $\succ_{d_{i-1}}$: $h', h, h'', \emptyset$,
- $\succ_{d_i}$: $h'', h, h', \emptyset$,
- $\succ_d$: $h, h', h'', \emptyset$ for all $d \in \{d_{i+1}, \cdots, d_{n-2}\}$,

- $\succ_{d_{n-1}}$: $h', h, h'', \emptyset$,
- $\succ_{d_n}$: $h', h'', h, \emptyset$.

If $d', d'' \in \{d_1, \cdots, d_n\}$, $\succ_{d'}$ and $\succ_{d''}$ have already been defined in the above way. If $d', d'' \notin \{d_1, \cdots, d_n\}$, define $\succ_{d'}$ and $\succ_{d''}$ by

- $\succ_{d'}$: $h', \emptyset$,
- $\succ_{d''}$: $h'', \emptyset$.

If $\hat{d}' = d'$ or $\hat{d}' \in \{d_1, \cdots, d_n\}$, $\succ_{\hat{d}'}$ has already been defined. If $\hat{d}' \neq d'$ and $\hat{d}' \notin \{d_1, \cdots, d_n\}$, define

- $\succ_{\hat{d}'}$: $h', \emptyset$.

We assume that $\succ_d$ ranks $\emptyset$ first for any $d \in \mathcal{D} \setminus (\{d_1, \cdots, d_n\} \cup \{d', d'', \hat{d}'\})$. By the construction of doctors' preference orders, we may ignore hospitals besides h, h', and h''. Thus, we may assume that $\mathcal{H} = \{h, h', h''\}$ without changing the property of COP.

Consider a proposal ordering $>$ over $\mathcal{D} \times \{h, h', h''\}$ such that

$$(d_1, h), \cdots, (d_{i-2}, h), (d_n, h'), (d_n, h''), (d_n, h), (d_{i-1}, h'), (d_{i-1}, h), (d_i, h''), (d_i, h),$$
$$(d_{n-1}, h'), (d_{n-1}, h), (d_{i+1}, h), \cdots, (d_{n-2}, h), \cdots\cdots, (d'', h'').$$

Note that (d'', h'') is ranked at the bottom.

Let φ be COP w.r.t. $(r_h, r_{h'}, \succ_{h''}, >)$. The following two claims show that φ is not SP.

Claim 3. $\varphi(\succ)_{d_n} = \emptyset$.

Proof. In the following argument, we omit any step at which h' and h'' are proposed by unacceptable doctors to them.

In the first $i-1$ steps, $d_1, \cdots, d_{i-2}$ propose to h and d_n proposes to h'. Then, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}) & (d_n) & \end{pmatrix}.$$

In the next three steps, d_{i-1} proposes to h' while rejected, d_{i-1} proposes to h, and d_i proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_{i-1}) & (d_n, \cancel{d_{i-1}}) & (d_i) \end{pmatrix}.$$

In the next three steps, d_{n-1} proposes to h' while d_n is rejected, d_n proposes to h'' while d_i is rejected, and d_i proposes to h. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_{i-1}, d_i) & (\cancel{d_n}, \cancel{d_{i-1}}, d_{n-1}) & (\cancel{d_i}, d_n) \end{pmatrix}.$$

In the subsequent steps, $d_{i+1}, \cdots, d_{n-2}$ propose to h;

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_{i-1}, d_i, d_{i+1}, \cdots, d_{n-2}) & (\cancel{d_n}, \cancel{d_{i-1}}, d_{n-1}) & (\cancel{d_i}, d_n) \end{pmatrix}.$$

At this moment, d' is unmatched since $d' \notin c_h(d_1, \cdots, d_{n-2}) \cup \{d_{n-1}, d_n\}$ and the definition of $>$. Note that $d' \neq d_{i-1}, d''$ and $(d', h') > (d'', h'')$ by definition. This implies that d' proposes to h' while accepted in some subsequent step. Moreover, this happens before d'' proposes to h''. Therefore, d_{n-1} is rejected by h' and proposes to h before d'' proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_{i-1}, d_i, d_{i+1}, \cdots, d_{n-2}, d_{n-1}) & (\cancel{d_n}, \cancel{d_{i-1}}, \cancel{d_{n-1}}, \cdots, d') & (\cancel{d_i}, d_n) \end{pmatrix}.$$

At this moment, d'' is unmatched since $d'' \notin c_h(d_1, \cdots, d_{n-1}) \cup \{d', d_n\}$ and the definition of $>$. Note that $d'' \neq d_i$ and d'' is never accepted by h' since d' is ranked first at $\succ_{h'}$. Therefore, d'' proposes h'' in some subsequent step by the construction of $>$. Then, d_n is rejected since d'' is ranked first at $\succ_{h''}$. This implies $\varphi(\succ)_{d_n} = \emptyset$ by $d_n \notin c_h(d_1, \cdots, d_n)$. $\square$

Claim 4. $\varphi(\succ'_{d_n}, \succ_{-d_n})_{d_n} \neq \emptyset$ where $\succ'_{d_n}: h'', h', h, \emptyset$.

Proof. In the following argument, we omit any step at which h' and h'' are proposed by unacceptable doctors to them.

In the first $i-1$ steps, $d_1, \cdots, d_{i-2}$ propose to h and d_n proposes to h''. Then, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}) & & (d_n) \end{pmatrix}.$$

In the next three steps, d_{i-1} proposes to h', d_i proposes to h'' while rejected, and d_i proposes to h;

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_i) & (d_{i-1}) & (d_n, \cancel{d_i}) \end{pmatrix}.$$

In the next two steps, d_{n-1} proposes to h' while d_{i-1} is rejected, d_{i-1} proposes to h. Thus, we have

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_i, d_{i-1}) & (\cancel{d_{i-1}}, d_{n-1}) & (d_n, \cancel{d_i}) \end{pmatrix}.$$

In the subsequent steps, $d_{i+1}, \cdots, d_{n-2}$ propose to h;

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_i, d_{i-1}, d_{i+1}, \cdots, d_{n-2}) & (\cancel{d_{i-1}}, d_{n-1}) & (d_n, \cancel{d_i}) \end{pmatrix}.$$

At this moment, $\hat{d}'$ is unmatched since $\hat{d}' \notin c_h((d_1, \cdots, d_{n-2})[i-1, i]) \cup \{d_{i-1}, d_{n-1}, d_n, d''\}$ and the definition of $>$. Note that $\hat{d}' \neq d_{i-1}, d''$. This implies that at least one doctor in $\{d_1, \cdots, d_{n-2}, \hat{d}', d'\} \setminus \{d_{i-1}\}$, who is ranked higher than d_{n-1} at $\succ_{h'}$, proposes to h' in some subsequent step. Moreover, this happens before d'' proposes to h'' since (d'', h'') is ranked at the bottom in $>$. Therefore, d_{n-1} is rejected by h' and proposes to h before d'' proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{i-2}, d_i, d_{i-1}, d_{i+1}, \cdots, d_{n-2}, d_{n-1}) & (\cancel{d_{i-1}}, \cancel{d_{n-1}}, \cdots, \hat{d}', \cdots) & (\cancel{d_i}, d_n) \end{pmatrix}.$$

If d_n is not rejected by h'' in any subsequent step, $\varphi(\succ)_{d_n} = h''$. Otherwise, $\varphi(\succ)_{d_n} = h$ by $d_n \in c_h((d_1, \cdots, d_n)[i-1, i])$. Hence $\varphi(\succ)_{d_n} \neq \emptyset$. □

Case 2. $i = n - 1$.

By the assumption of $\max\{|C_h(D) \mid D \subseteq \mathcal{D}\} + 5 \leq |\mathcal{D}|$, we can take $d', d'' \in \mathcal{D}$ such that

$$d' \notin c_h(d_1, \cdots, d_{n-2}) \cup \{d_{n-2}, d_{n-1}, d_n\} \text{ and } d'' \notin c_h(d_1, \cdots, d_{n-1}) \cup \{d_{n-1}, d_n, d'\}.$$

Moreover, we can take $\hat{d}' \in \mathcal{D}$ such that

$$\hat{d}' \notin c_h(d_1, \cdots, d_{n-3}, d_{n-1}) \cup \{d_{n-2}, d_n, d''\}.$$

Note that $d' \neq d''$ and $\hat{d}' \neq d''$ by definition while $d' = \hat{d}'$ is possible.

Recall that we are assuming that there are two hospitals h' and h'' apart from h. We assume that h' has a unit demand preference $\succ_{h'}$ such that

- d' is ranked first,
- $d_n \succ_{h'} d_{n-2} \succ_{h'} \emptyset$,
- $d \succ_{h'} d_n$ for all $d \in \{d_1, \cdots, d_{n-1}, \hat{d}'\} \setminus \{d', d_{n-2}\}$.

Hospital h'' has a unit demand preference $\succ_{h''}$: $d'', d_n, d_{n-1}, \emptyset$. Consider any $r_{\bar{h}}$ for any $\bar{h} \in \mathcal{H} \setminus \{h, h', h''\}$.

Define $\succ_{d_1}, \cdots, \succ_{d_n}$ by

- $\succ_d$: $h, h', h'', \emptyset$ for all $d \in \{d_1, \cdots, d_{n-3}\}$,
- $\succ_{d_{n-2}}$: $h', h, h'', \emptyset$,
- $\succ_{d_{n-1}}$: $h'', h, h', \emptyset$,
- $\succ_{d_n}$: $h', h'', h, \emptyset$.

If $d', d'' \in \{d_1, \cdots, d_n\}$, $\succ_{d'}$ and $\succ_{d''}$ have already been defined in the above way. If $d', d'' \notin \{d_1, \cdots, d_n\}$, define $\succ_{d'}$ and $\succ_{d''}$ by

- $\succ_{d'}$: $h', \emptyset$,
- $\succ_{d''}$: $h'', \emptyset$.

If $\hat{d}' = d'$ or $\hat{d}' \in \{d_1, \cdots, d_n\}$, $\succ_{\hat{d}'}$ has already been defined. If $\hat{d}' \neq d'$ and $\hat{d}' \notin \{d_1, \cdots, d_n\}$, define

- $\succ_{\hat{d}'}$: $h', \emptyset$.

We assume that $\succ_d$ ranks $\emptyset$ first for any $d \in \mathcal{D} \setminus (\{d_1, \cdots, d_n\} \cup \{d', d'', \hat{d}'\})$. Again, we may assume that $\mathcal{H} = \{h, h', h''\}$ without changing any property of COP since any other hospital is unacceptable for any doctors.

Consider the proposal ordering $>$ over $\mathcal{D} \times \{h, h', h''\}$ such that

$$(d_1, h), \cdots, (d_{n-3}, h), (d_n, h'), (d_n, h''), (d_n, h), (d_{n-2}, h'), (d_{n-2}, h),$$
$$(d_{n-1}, h''), (d_{n-1}, h), \cdots\cdots, (d'', h'').$$

Note that (d'', h'') is ranked at the bottom.

Let φ be COP w.r.t. $(r_h, r_{h'}, r_{h''}, >)$. The following two claims show that φ is not SP.

Claim 5. $\varphi(\succ)_{d_n} = \emptyset$.

Proof. In the following argument, we omit any step at which h' and h'' are proposed by unacceptable doctors to them.

In the first $n-2$ steps, $d_1, \cdots, d_{n-3}$ propose to h and d_n proposes to h'. Then, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}) & (d_n) & \end{pmatrix}.$$

In the next three steps, d_{n-2} proposes to h' while rejected, d_{n-2} proposes to h, and d_{n-1} proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}, d_{n-2}) & (d_n, \cancel{d_{n-2}}) & (d_{n-1}) \end{pmatrix}.$$

At this moment, d' is unmatched since $d' \notin c_h(d_1, \cdots, d_{n-2}) \cup \{d_{n-2}, d_{n-1}, d_n\}$ and the definition of $>$. Note that $d' \neq d_{n-2}, d''$ and $(d', h') > (d'', h'')$ by definition. This implies that d' proposes to h' while accepted in some subsequent step. Moreover, this happens before d'' proposes to h''. Therefore, d_n is rejected by h' and proposes to h'' (while d_{n-1} is rejected) before d'' proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}, d_{n-2}) & (\cancel{d_n}, \cancel{d_{n-2}}, \cdots, d') & (\cancel{d_{n-1}}, d_n) \end{pmatrix}.$$

In the next step, d_{n-1} proposes to h,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}, d_{n-2}, d_{n-1}) & (\cancel{d_n}, \cancel{d_{n-2}}, \cdots, d') & (\cancel{d_{n-1}}, d_n) \end{pmatrix}.$$

At this moment, d'' is unmatched since $d'' \notin c_h(d_1, \cdots, d_{n-1}) \cup \{d_{n-1}, d_n, d'\}$ and the definition of $>$. Note that $d'' \neq d_{n-1}$ and d'' is never accepted by h' since d' is ranked first at $\succ_{h'}$. Therefore, d'' proposes h'' in some subsequent step by the construction of $\succ$. Then, d_n is rejected since d'' is ranked first at $\succ_{h''}$. This implies $\varphi(\succ)_{d_n} = \emptyset$ by $d_n \notin c_h(d_1, \cdots, d_n)$.

$\square$

Claim 6. $\varphi(\succ'_{d_n}, \succ_{-d_n})_{d_n} \neq \emptyset$ where $\succ'_{d_n}: h'', h', h, \emptyset$.

Proof. In the following argument, we omit any step at which h' and h'' are proposed by unacceptable doctors to them.

In the first $n-2$ steps, $d_1, \cdots, d_{n-3}$ propose to h and d_n proposes to h''. Then, the sequence of proposals for each hospital is given by

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}) & & (d_n) \end{pmatrix}.$$

In the next three steps, d_{n-2} proposes to h', d_{n-1} proposes to h'' while rejected, and d_{n-1} proposes to h. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}, d_{n-1}) & (d_{n-2}) & (d_n, \cancel{d_{n-1}}) \end{pmatrix}.$$

At this moment, $\hat{d}'$ is unmatched since $\hat{d}' \notin c_h(d_1, \cdots, d_{n-3}, d_{n-1}) \cup \{d_{n-2}, d_n, d''\}$ and the definition of $>$. Note that $\hat{d}' \neq d''$. This implies that at least one doctor in $\{d_1, \cdots, d_{n-1}, \hat{d}', d'\} \setminus \{d_{n-2}\}$, who is ranked higher than d_{n-2} at $\succ_{h'}$, proposes to h' in some subsequent step. Moreover, this happens before d'' proposes to h'' since (d'', h'') is ranked at the bottom in $>$. Therefore, d_{n-2} is rejected by h' and proposes to h before d'' proposes to h''. Thus,

$$\begin{pmatrix} h & h' & h'' \\ (d_1, \cdots, d_{n-3}, d_{n-1}, d_{n-2}) & (\cancel{d_{n-2}}, \cdots, \hat{d}', \cdots) & (d_n, \cancel{d_{n-1}}) \end{pmatrix}.$$

If d_n is not rejected by h'' in any subsequent step, $\varphi(\succ)_{d_n} = h''$. Otherwise, $\varphi(\succ)_{d_n} = h$ by $d_n \in c_h((d_1, \cdots, d_n)[i-1, i])$. Hence $\varphi(\succ'_{d_n}, \succ_{-d_n})_{d_n} \neq \emptyset$. □

Therefore, d_n can be better off by misreporting her preference order in both Case 1 (Claims 3 and 4) and Case 2 (Claims 5 and 6). □

We now show Theorem 3. Suppose that r_h is not order-independent. Thus, r_h does not satisfy conditon A or condition B from Lemma 3. In either case, we have the desired result from Lemmas 4 and 5.

Proof of Proposition 4

Let $\mathcal{H} = \{h_1, h_2\}$ and $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$. We assume that $C_{h_1}(D) = D$ for all $D \subseteq \mathcal{D}$ with $|D| \leq 2$. For D with $|D| \geq 3$, C_{h_1}, $\mathcal{C}_{h_1}$, and $\mathcal{R}_{h_1}$ are given in Table 5.

Then, C_{h_1} satisfies substitutability, monotonicity, and acceptance.[16] On the other hand, C_{h_1} does not have a substitutable selection. To see this, suppose that there exists a substitutable selection $\bar{C}_{h_1}$ from $\mathcal{C}_{h_1}$. Then, $\bar{C}_{h_1}(\{d_1, d_2, d_3\}) = \{d_1, d_3\}$ and $\bar{C}_h(\{d_1, d_2, d_4\}) = \{d_2, d_4\}$. By substitutability of $\bar{C}_{h_1}$, we have $\bar{C}_{h_1}(\{d_1, d_2, d_3, d_4\}) =$

[16]Note that C_{h_1} does not satisfy lower monotonicity since $\{d_2, d_4\} \in \mathcal{C}_{h_1}(\mathcal{D})$ but $C_{h_1}(\mathcal{D} \setminus \{d_4\}) = \{\{d_1, d_3\}\}$. Therefore, monotonicity does not imply lower monotonicity.

Table 5: choice and maximal choice (rejection)

D	$C_{h_1}(D)$	$\mathcal{C}_{h_1}(D)$	$\mathcal{R}_{h_1}(D)$
$\{d_1, d_2, d_3, d_4\}$	$\emptyset$	$\{d_1, d_3\}, \{d_2, d_4\}$	$\{d_2, d_4\}, \{d_1, d_3\}$
$\{d_1, d_2, d_3\}$	$\{d_1, d_3\}$	$\{d_1, d_3\}$	$\{d_2\}$
$\{d_1, d_2, d_4\}$	$\{d_2, d_4\}$	$\{d_2, d_4\}$	$\{d_1\}$
$\{d_1, d_3, d_4\}$	$\{d_1, d_3\}$	$\{d_1, d_3\}$	$\{d_4\}$
$\{d_2, d_3, d_4\}$	$\{d_2, d_4\}$	$\{d_2, d_4\}$	$\{d_3\}$

$\{d_1, d_3\}$ and $\bar{C}_{h_1}(\{d_1, d_2, d_3.d_4\}) = \{d_2, d_4\}$, a contradiction. Let C_{h_2} be the choice function induced from the unit demand preferences given by

$$\succ_{h_2}: d_3, d_4, d_1, d_2, \emptyset.$$

We denote $C = (C_{h_1}, c_{h_2})$.

Suppose that there exists a stable and SP mechanism φ in this example for contradiction. Let $\succ^1$ be the preference profile of doctors defined by

$$\succ_{d_1}: h_2, h_1, \emptyset, \quad \succ_{d_2}: h_2, h_1, \emptyset \quad \succ_{d_3}: h_1, h_2, \emptyset \quad \succ_{d_4}: h_1, h_2, \emptyset.$$

At $(\succ^1, C)$, the set of all stable matchings consists of

$$\mu_1 = \begin{pmatrix} h_1 & h_2 & \emptyset \\ d_1, d_3 & d_4 & d_2 \end{pmatrix} \text{ and } \mu_2 = \begin{pmatrix} h_1 & h_2 & \emptyset \\ d_2, d_4 & d_3 & d_1 \end{pmatrix}.$$

Case 1. $\varphi(\succ^1) = \mu_1$.

Let $\succ^2$ be the preference profile of doctors defined by

$$\succ_{d_1}: h_2, h_1, \emptyset, \quad \succ'_{d_2}: h_1, \emptyset \quad \succ_{d_3}: h_1, h_2, \emptyset \quad \succ_{d_4}: h_1, h_2, \emptyset.$$

At $(\succ^2, C)$, the set of all stable matchings is still $\{\mu_1, \mu_2\}$. By SP, we must have $\varphi(\succ^2) = \mu_1$. Let $\succ^3$ be the preference profile of doctors defined by

$$\succ_{d_1}: h_2, h_1, \emptyset, \quad \succ'_{d_2}: h_1, \emptyset \quad \succ'_{d_3}: h_1, \emptyset \quad \succ_{d_4}: h_1, h_2, \emptyset.$$

At $(\succ^3, C)$, the set of all stable matchings is $\{\mu_1, \mu_3\}$ where

$$\mu_3 = \begin{pmatrix} h_1 & h_2 & \emptyset \\ d_2, d_4 & d_1 & d_3 \end{pmatrix}.$$

By SP, we must have $\varphi(\succ^3) = \mu_1$. Let $\succ^4$ be the preference profile of doctors defined by

$$\succ'_{d_1}: h_2, \emptyset, \quad \succ'_{d_2}: h_1, \emptyset \quad \succ'_{d_3}: h_1, \emptyset \quad \succ_{d_4}: h_1, h_2, \emptyset.$$

At $(\succ^4, C)$, μ_3 is the unique stable matching. Thus, $\varphi(\succ^4)_{d_1} = h_2 \succ_{d_1} h_1 = \varphi(\succ^3)_{d_1}$, contradicting the assumption that φ satisfies SP.

Case 2. $\varphi(\succ^1) = \mu_2$.

We can show that φ does not satisfy SP by a similar argument as in Case 1 as follows. Let $\hat{\succ}^2$ be the preference profile of doctors defined by

$$\succ'_{d_1}: h_1, \emptyset, \quad \succ_{d_2}: h_2, h_1, \emptyset \quad \succ_{d_3}: h_1, h_2, \emptyset \quad \succ_{d_4}: h_1, h_2, \emptyset.$$

At $(\hat{\succ}^2, C)$, the set of all stable matchings is still $\{\mu_1, \mu_2\}$. By SP, we must have $\varphi(\hat{\succ}^2) = \mu_2$. Let $\hat{\succ}^3$ be the preference profile of doctors defined by

$$\succ'_{d_1}: h_1, \emptyset, \quad \succ_{d_2}: h_2, h_1, \emptyset \quad \succ_{d_3}: h_1, h_2, \emptyset \quad \succ'_{d_4}: h_1, \emptyset.$$

At $(\hat{\succ}^3, C)$, the set of all stable matchings is $\{\mu_2, \mu_4\}$ where

$$\mu_4 = \begin{pmatrix} h_1 & h_2 & \emptyset \\ d_1, d_3 & d_2 & d_4 \end{pmatrix}.$$

By SP, we must have $\varphi(\hat{\succ}^3) = \mu_2$. Let $\hat{\succ}^4$ be the preference profile of doctors defined by

$$\succ'_{d_1}: h_1, \emptyset, \quad \succ'_{d_2}: h_2, \ \emptyset \quad \succ_{d_3}: h_1, h_2, \emptyset \quad \succ'_{d_4}: h_1, \emptyset.$$

At $(\hat{\succ}^4, C)$, μ_4 is the unique stable matching. Thus, $\varphi(\hat{\succ}^4)_{d_2} = h_2 \succ_{d_2} h_1 = \varphi(\hat{\succ}^3)_{d_2}$, contradicting the assumption that φ satisfies SP.

Proof of Proposition 5

Fix an arbitrary h. Let $\mathcal{F}_h$ be a general upper-bounds constraint for h, and $\succ_h$ be a priority order over $\mathcal{D}$ of h. Fix an arbitrary $D \subseteq \mathcal{D}$. Denote $D = \{d_1, \cdots, d_k\}$ with $d_1 \succ_h \cdots \succ_h d_k$.

We begin by showing that $\hat{C}_h(D) \in \mathcal{C}_h(D)$. Denote $\hat{C}_h(D) = \{d'_1, \cdots, d'_{k'}\}$ with $d'_1 \succ_h \cdots \succ_h d'_{k'}$. By definition, $\hat{C}_h(D) \in \mathcal{F}_h$. Therefore, $\hat{C}_h(D) = C_h(\hat{C}_h(D))$.

Fix an arbitrary $d_\ell \in D \setminus \hat{C}_h(D)$. Then, $\hat{C}_h^{\ell-1}(D) \cup \{d_\ell\} \notin \mathcal{F}_h$. By the construction of $\hat{C}_h$, we have

$$\underbrace{d'_1 \succ_h \cdots \succ_h d'_{\ell'}}_{\text{Doctors in } \hat{C}_h^{\ell-1}(D)} \succ_h d_\ell \succ_h \underbrace{d'_{\ell'+1} \succ_h \cdots \succ_h d'_{k'}}_{\text{Doctors in } \hat{C}_h(D) \setminus \hat{C}_h^{\ell-1}(D)}.$$

Then, by the definition of C_h and $\hat{C}_h^{\ell-1}(D) \in \mathcal{F}_h$, $\hat{C}_h^{\ell-1}(D) \cup \{d_\ell\} \notin \mathcal{F}_h$ implies that $C_h(\hat{C}_h(D) \cup \{d_\ell\}) = \hat{C}_h^{\ell-1}(D)$. Thus, $d_\ell \notin C_h(\hat{C}_h(D) \cup \{d_\ell\})$. Hence, $\hat{C}_h(D) \in \mathcal{C}_h(D)$.

We turn to showing that $\hat{C}_h(D)$ is the unique element in $\mathcal{C}_h(D)$. Fix an arbitrary $D' \in \mathcal{C}_h(D)$. Note that $D' \in \mathcal{F}_h$.

Claim 7. $D' \cap \{d_1\} = \hat{C}_h^1(D)$.

Proof. It suffices to show that $d_1 \in D'$ if and only if $d_1 \in \hat{C}_h^1(D)$. First, assume that $d_1 \in D'$. We have $C_h(D') = D'$ by $D' \in \mathcal{C}_h(D)$. Thus, $D' \in \mathcal{F}_h$. Since $\mathcal{F}_h$ is a general upper-bounds constraint, $\{d_1\} \in \mathcal{F}_h$. Thus, $\hat{C}_h^1(D) = \{d_1\}$.

Next, assume that $d_1 \notin D'$. Then, $\{d_1\} \notin \mathcal{F}_h$; otherwise, $\{d_1\} \subseteq C_h(D' \cup \{d_1\})$ by definition, contradicting that $D' \in \mathcal{C}_h(D)$. By $\hat{C}_h^0(D) \cup \{d_1\} = \{d_1\} \notin \mathcal{F}_h$, $d_1 \notin \hat{C}_h^1(D)$. $\square$

Claim 8. Let $\ell = 2, \cdots, k$. Assume that $D' \cap \{d_1, \cdots, d_{\ell-1}\} = \hat{C}_h^{\ell-1}(D)$. Then, $D' \cap \{d_1, \cdots, d_\ell\} = \hat{C}_h^{\ell}(D)$.

Proof. By the definition of $\hat{C}_h^{\ell}$ and $\hat{C}_h^{\ell-1}(D) = D' \cap \{d_1, \cdots, d_{\ell-1}\}$,

$$\hat{C}_h^{\ell}(D) = \begin{cases} \hat{C}_h^{\ell-1}(D) \cup \{d_\ell\} & \text{if } (D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \in \mathcal{F}_h, \\ \hat{C}_h^{\ell-1}(D) & \text{otherwise.} \end{cases}$$

Thus, it suffices to show that $d_\ell \in D'$ if and only if $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \in \mathcal{F}_h$.

Assume that $d_\ell \in D'$. Then, $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \subseteq D'$. Since $D' \in \mathcal{F}_h$ and $\mathcal{F}_h$ is a general upper-bounds constraint, $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \in \mathcal{F}_h$.

Assume that $d_\ell \notin D'$. Suppose that $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \in \mathcal{F}_h$. Then, $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \subseteq C_h(D' \cup \{d_\ell\})$ by

$$\underbrace{d_1' \succ_h \cdots \succ_h d_{\ell'}'}_{\text{Doctors in } D' \cap \{d_1, \cdots, d_{\ell-1}\} = \hat{C}_h^{\ell-1}(D)} \succ_h d_\ell \succ_h \underbrace{d_1'' \succ_h \cdots \succ_h d_{k''}''}_{\text{Doctors in } D' \setminus \{d_1, \cdots, d_{\ell-1}\}}.$$

Thus, $d_\ell \in C_h(D' \cup \{d_\ell\})$. This contradicts that $D' \in \mathcal{C}_h(D)$. Thus, $(D' \cap \{d_1, \cdots, d_{\ell-1}\}) \cup \{d_\ell\} \notin \mathcal{F}_h$. □

By $D' \subseteq D = \{d_1, \cdots, d_k\}$ and Claims 7, 8, $D' = D' \cap \{d_1, \cdots, d_k\} = \hat{C}_h^k(D) = \hat{C}_h(D)$. ■

Proof of Proposition 7

It suffices to show that $\hat{C}_h$ is a selection from $\mathcal{C}_h$ since $\hat{C}_h$ is a responsive choice function and a responsive choice function is substitutable. Let $\hat{\succ}_h$ be a tie-breaking priority ordering of $\succ_h$ such that $\hat{C}_h$ is responsive w.r.t. $\hat{\succ}_h$.

Fix an arbitrary $D \subseteq \mathcal{D}$. If $|D| \leq q_h$, then it is easy to see that $C_h(D) = \hat{C}_h(D) = D$ and D is a maximal choice set in D. Thus, assume that $|D| > q_h$. Denote $I^* = C_h(D)$ and $D^* = \hat{C}_h(D)$. By $|D^*| = q_h$, $C_h(D^*) = D^*$. Fix an arbitrary $d \in D \setminus D^*$. Then, $d' \hat{\succ}_h d$ for any $d' \in D^*$. By the definition of the tie-breaking priority, $d' \succsim_h d$ for any $d' \in D^*$. Let $I_\ell \in \mathcal{I}$ be the indifference class such that $d \in I_\ell$. Note that $|D^* \cup \{d\}| = q_h + 1$. Then, $C_h(D^* \cup \{d\}) = D^*$ if $D^* \cap I_\ell = \emptyset$ because $d' \succ_h d$ for any $d' \in D^*$ in this case. Meanwhile, $C_h(D^* \cup \{d\}) = D^* \setminus I_\ell$ if $D^* \cap I_\ell \neq \emptyset$ by $|D^* \setminus I_\ell| < q_h < |D^* \cup I_\ell|$ and $D^* \setminus I_\ell = D^* \cap (I_1 \cup \cdots \cup I_{\ell-1})$. In either case, $d \notin C_h(D^* \cup \{d\})$. Hence D^* is a maximal choice set in D. ■

Proof of Proposition 8

Let $D, D' \subseteq \mathcal{D} \setminus D^p$ with $D \subseteq D'$. Pick any $d \in R_h(D)$. There are two cases to consider. We first assume that $D^p \subseteq \bar{C}_h(D^p \cup D)$. Then, $d \notin \bar{C}_h(D^p \cup D)$ by $d \in R_h(D)$. By substitutability of $\bar{C}_h$, we have $d \notin \bar{C}_h(D^p \cup D')$. This implies $d \in R_h(D')$ regardless of whether $D^p \subseteq \bar{C}_h(D^p \cup D')$ or not. We next assume that $D^p \nsubseteq \bar{C}_h(D^p \cup D)$. By substitutability of $\bar{C}_h$, we also have $D^p \nsubseteq \bar{C}_h(D^p \cup D')$ and thus $C_h(D') = \emptyset$. Therefore, $d \in D \subseteq D' = R_h(D')$. Hence $R_h(D) \subseteq R_h(D')$ holds. ■

Proof of Proposition 9

Let $D \subseteq \mathcal{D} \setminus D^p$ and $d \in (\mathcal{D} \setminus D^p) \setminus D$. Pick any $R \in \mathcal{R}_h(D)$. Note that $d \notin R$ by $R \subseteq D$. It is sufficient to show that $R \subseteq (D \cup \{d\}) \setminus D'$ for some $D' \in \mathcal{C}_h(D \cup \{d\})$. By definition, $\bar{D} = D \setminus R$ is a maximal choice set in D. Thus, $C_h(\bar{D}) = \bar{D}$ and $d' \notin C_h(\bar{D} \cup \{d'\})$ for all $d' \in D \setminus \bar{D}$. Therefore, when $d \notin C_h(\bar{D} \cup \{d\})$, $\bar{D}$ is a maximal choice set in $D \cup \{d\}$, and the proof is done. Thus, we assume that $d \in C_h(\bar{D} \cup \{d\})$. Note that $C_h(\bar{D} \cup \{d\}) \neq \emptyset$ implies that $D^p \subseteq \bar{C}_h(D^p \cup \bar{D} \cup \{d\})$. Therefore, the assumption that $d \in C_h(\bar{D} \cup \{d\})$ holds if and only if $d \in \bar{C}_h(D^p \cup \bar{D} \cup \{d\})$ and $D^p \subseteq C_h(D^p \cup \bar{D} \cup \{d\})$ hold. Note also that $\bar{C}_h(D^p \cup \bar{D} \cup \{d\}) = D^p \cup C_h(\bar{D} \cup \{d\})$.

Let $D^* = C_h(\bar{D} \cup \{d\})$. Then, $R \cap D^* = \emptyset$ by $D^* \subseteq \bar{D} \cup \{d\}$, $R \cap \bar{D} = \emptyset$, and $d \notin R$. It follows that $R \subseteq (D \cup \{d\}) \setminus D^*$ from $R \subseteq D$. It remains to show that D^* is a maximal choice set in $D \cup \{d\}$. We first show that $C_h(D^*) = D^*$. Suppose that $C_h(D^*) \subsetneq D^*$. Then, there exists some $d' \in D^* = C_h(\bar{D} \cup \{d\})$ such that $d' \notin C_h(D^*)$. By substitutability of C_h (Proposition 8) and $D^* \subseteq \bar{D} \cup \{d\}$, we have $d' \notin C_h(\bar{D} \cup \{d\}) = D^*$, contradicting the choice of d'. Hence $C_h(D^*) = D^*$. Pick any $\bar{d} \in D \cup \{d\} \setminus D^*$. Note that $\bar{d} \in D$ by $d \in C_h(\bar{D} \cup \{d\}) = D^*$. We prove that $\bar{d} \notin C_h(D^* \cup \{\bar{d}\})$ by distinguishing two cases.

Case 1. $\bar{d} \in \bar{D}$.

By $\bar{d} \notin D^* = C_h(\bar{D} \cup \{d\})$, $\bar{d} \notin \bar{C}_h(D^p \cup \bar{D} \cup \{d\})$. By consistency of $\bar{C}_h$, $\bar{d} \notin \bar{C}_h(\bar{C}_h(D^p \cup \bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup C_h(\bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup D^* \cup \{d\})$. Thus, $\bar{d} \notin C_h(D^* \cup \{\bar{d}\})$.

Case 2. $\bar{d} \notin \bar{D}$.

We have $\bar{d} \notin C_h(\bar{D} \cup \{\bar{d}\})$ since $\bar{D}$ is a maximal choice set in D and $\bar{d} \in D$. We further consider the following two subcases. First, assume that $D^p \nsubseteq \bar{C}_h(D^p \cup \bar{D} \cup \{\bar{d}\})$. Then, $D^p \nsubseteq \bar{C}_h(D^p \cup \bar{D} \cup \{d, \bar{d}\})$ by substitutability of $\bar{C}_h$. By path-independence of $\bar{C}_h$, $D^p \nsubseteq \bar{C}_h(\bar{C}_h(D^p \cup \bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup C_h(\bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup D^* \cup \{\bar{d}\})$, which implies $\bar{d} \notin C_h(D^* \cup \{\bar{d}\}) = \emptyset$. Next, assume that $D^p \subseteq \bar{C}_h(D^p \cup \bar{D} \cup \{\bar{d}\})$. Then, $\bar{d} \notin \bar{C}_h(D^p \cup \bar{D} \cup \{\bar{d}\})$ by $\bar{d} \notin C_h(\bar{D} \cup \{\bar{d}\})$. By substitutability of $\bar{C}_h$, $\bar{d} \notin \bar{C}_h(D^p \cup \bar{D} \cup \{d, \bar{d}\})$. By path-independence of $\bar{C}_h$, $\bar{d} \notin \bar{C}_h(\bar{C}_h(D^p \cup \bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup C_h(\bar{D} \cup \{d\}) \cup \{\bar{d}\}) = \bar{C}_h(D^p \cup D^* \cup \{\bar{d}\})$. Thus, $\bar{d} \notin C_h(D^* \cup \{\bar{d}\})$ holds regardless of whether $D^p \subseteq \bar{C}_h(D^p \cup D^* \cup \{\bar{d}\})$ or not.

Since $\bar{d} \notin \hat{C}_h(D^* \cup \{\bar{d}\})$ holds for both cases, D^* is a maximal choice set in $D \cup \{d\}$. Hence $\hat{C}_h$ satisfies monotonicity. ■

Proof of Proposition 10

We first show that $\bar{C}(D)$ is a maximal choice set in any $D \subseteq \mathcal{D}$. Fix an arbitrary $D \subseteq \mathcal{D}$.

First, assume that $|a(D)| \leq q_{\bar{h}}$. Note that $|a(D)| \leq q_h$ for all $h \in H$. Then, $a(D) = C_h(a(D))$ for all $h \in H$. Thus, $C_H(a(D)) = a(D)$. Note that $\bar{C}(D) = a(D)$ in this case. For any $d' \in D \setminus a(D)$, $d' \notin C_{h'}(a(D) \cup \{d'\})$ for some $h' \in H$ such that d' is unacceptable for h'. It follows that $d' \notin C_H(a(D) \cup \{d'\})$ for any $d' \in D \setminus a(D)$. Hence $\bar{C}(D) = a(D) \in \mathcal{C}_H(D)$.

Next, assume that $|a(D)| > q_{\bar{h}}$. Denote $\bar{D} = \bar{C}(D) = C_{\bar{h}}(a(D))$. By the choice of $\bar{h}$ and $|a(D)| > q_{\bar{h}}$, we have that $|\bar{D}| = q_{\bar{h}} \leq q_h$ for all $h \in H$. By $\bar{D} \subseteq a(D)$, $C_h(\bar{D}) = \bar{D}$ for all $h \in H$. Hence $C_H(\bar{D}) = \bar{D}$.

Fix an arbitrary $d' \in D \setminus \bar{D}$. If $d' \notin a(D)$, then $d' \notin C_h(\bar{D} \cup \{d\})$ for some $h \in H$ such that d' is unacceptable for h. Thus, $d' \notin C_H(\bar{D} \cup \{d'\})$ if $d' \notin a(D)$. Therefore, assume that $d' \in a(D)$. Since $\bar{D} = C_{\bar{h}}(a(D))$ and $d' \in a(D) \setminus \bar{D}$, we have that $\bar{d} \succ_{\bar{h}} d'$ for all $\bar{d} \in \bar{D}$. Since $|\bar{D}| = q_{\bar{h}}$, $\bar{D} = C_{\bar{h}}(\bar{D} \cup \{d'\})$. Therefore, $d' \notin C_H(\bar{D} \cup \{d'\})$. Thus, $\bar{D} \in \mathcal{C}_H(D)$. Hence $\bar{C}$ is a selection from $\mathcal{C}^{\bar{h}}$.

It remains to show that $\bar{C}$ is substitutable. To show this, we use the well-known fact that a responsive choice function is substitutable. Fix an arbitrary $\hat{d} \in \mathcal{D} \setminus D$. If $\hat{d} \notin a(D \cup \{\hat{d}\})$, then $a(D) = a(D \cup \{\hat{d}\})$. Thus, $\bar{C}(D \cup \{\hat{d}\}) \cap D = C_{\bar{h}}(a(D \cup \{\hat{d}\}) \cap D = C_{\bar{h}}(a(D)) \cap D = \bar{C}(D) \cap D = \bar{C}(D)$ if $\hat{d} \notin a(D \cup \{\hat{d}\})$. Therefore, assume that $\hat{d} \in a(D \cup \{\hat{d}\})$. By $a(D \cup \{\hat{d}\}) = a(D) \cup \{\hat{d}\}$ and a responsive choice function is substitutable, $\bar{C}(D \cup \{\hat{d}\}) \cap D = C_{\bar{h}}(a(D \cup \{\hat{d}\})) \cap D \subseteq C_{\bar{h}}(a(D)) = \bar{C}(D)$. Thus, $\bar{C}$ is substitutable. ■