UTMD Working Paper
The University of Tokyo
Market Design Center

UTMD-073

# Double Stability in Two-Sided Matching: Priority and Preference in Harmony

Yuanju Fang
Seigakuin University

Yosuke Yasuda
Osaka University

January 1, 2025

# Double Stability in Two-Sided Matching: Priority and Preference in Harmony*

Yuanju Fang† and Yosuke Yasuda‡

January 1, 2025

**Abstract**

In two-sided matching problems, there can be ambiguity regarding whether institutions such as schools and daycares should be treated as agents who can make decisions, objects to be assigned, or both. To address this, we consider an extended college admissions model that incorporates a common priority order, often determined by external criteria such as exam scores or institutional rules, alongside the preferences of students and colleges. We define a matching as *double stable* if it satisfies priority stability and preference stability simultaneously. Our main finding establishes that a double stable matching exists if and only if the resulting outcome of the serial dictatorship mechanism coincides with that of the student-proposing deferred acceptance mechanism.

**JEL classification:** C78, D47.

**Key words:** matching, common priority, double stability.

## 1 Introduction

Matching markets have been extensively studied in the literature for decades. Since the seminal contribution by Gale and Shapley [5], the **college admissions model** has been used to analyze

*An earlier version of this paper was circulated under the title "Matching with Subjective and Objective Evaluations". We would like to thank Georgy Artemov, Shingo Ishiguro, Yuichiro Kamada, Shigehiro Serizawa, and seminar participants at CTWZ (Contract Theory Workshop on Zoom), EEA-ESEM 2024 (Rotterdam), MES-E/SE 2024 (Ho Chi Minh City), JEA 2024 (Fukuoka), and NOVA for their helpful comments. All remaining errors are ours. Yasuda gratefully acknowledges research support from Grant-in-Aid for Scientific Research (KAKENHI), 18KK0342.

†Political Science and Economics Department, Seigakuin University, 1-1 Tosaki, Ageo, Saitama, 362-8585 Japan. E-mail: fangyuanju90@gmail.com.

‡Corresponding author: Graduate School of Economics, Osaka University, 1-7 Machikaneyama, Toyonaka, Osaka 560-0043 Japan. E-mail: yosuke.yasuda@gmail.com

two-sided matching markets, such as assigning students to colleges and residents to hospitals. Besides this model, another well-known model is the **school choice model** (Abdulkadiroğlu and Sönmez [1]), which is adopted to analyze one-sided matching markets, including assigning students to schools and tenants to houses. The key difference between these two models is "the role of participants." In the college admissions model, participants on each side are *agents* who can make decisions, whereas in the school choice model, participants on one side do not reflect preferences and they are *objects* to be assigned.

In many real-life matching markets, identifying the role of participants is straightforward. For example, in the American medical labor market (Roth [7]), hospitals act as agents because they have *preferences* over residents. Conversely, in Boston's school choice market (Abdulkadiroğlu and Sönmez [1]), schools function as objects, and external criteria, like siblings and walk zones, determine schools' *priority orders* for students. However, in other markets, it can be challenging to identify participants as either agents or objects. This ambiguity gives rise to the presence of common priority orders, often determined by exam scores or institutional rules. These priority orders coexist with the preferences of colleges/schools, influencing the admission process in some student assignment markets.

An example is the college admissions in China.[1] Each student is required to take an exam and submit a preference list containing at most thirty universities. The government then uses these preferences to send student applications to universities, employing a method called the serial dictatorship (SD) mechanism with respect to 105% of university seats.[2] After receiving these applications, universities are responsible for placing students into specific departments. This task is crucial because different students contribute in different ways to the university's growth and development. Therefore, each university has its preferences for selecting students. Since the number of students admitted cannot exceed the actual quota, some students will not be accepted. Once the admission decisions are made, the main process concludes, and a supplementary process begins for those students who were not accepted.

Another example is the public high school choice in Osaka, Japan. Each student takes an exam and submits a preference list containing only one school. The government then sends

[1]The descriptions of the following two examples are based on Articles 6 and 7 of Beijing's University Recruitment Regulation and Article I.3 of Osaka's Public High School Recruitment in 2023, respectively.

[2]The original version of the SD mechanism applies to 100% of university seats. The version implemented in China allows universities to receive more student applications than their actual quotas. A formal description of the SD mechanism and its two variants is provided in Section 2.

student applications to schools by using the SD mechanism with respect to 110% of school seats. After receiving these applications, schools first allocate 90% of their seats to the high-scoring students. Those who do not secure a seat enter a so-called "border zone". In this zone, since test scores may not fully reflect a student's abilities, schools have the flexibility to use their preferences, based on personal statements or extracurricular activities, to fill the last 10% of seats. Once the admission decisions are made, the process concludes, and those students who were not accepted need to explore the option of attending a private school.

Note that in the above examples, the assignment is determined not only by exam scores but also significantly by colleges'/schools' preferences. Historically, these markets were structured as a school choice model, where colleges/schools lacked the authority to select students. In 1984, the Chinese government changed this by allowing colleges to express their preferences.[3] Similarly, Japan has implemented a reform for school choice in 2016, where the local government in Osaka redesigned its admission system to incorporate schools' preferences.[4] These reforms have effectively changed the market structure. As a result, colleges/schools play a dual role as both agents and objects in these markets.

Following the original concept of stability, we can evaluate an assignment in two ways: either based on exam scores or on colleges' preferences. In the former, a matching is fair or **priority stable** if students with higher priority are matched with their more preferred colleges.[5] In the latter, a matching is stable or **preference stable** if any pair who prefer each other are matched together. Ideally, the government aims to design a mechanism that meets both stability criteria. However, as the following example shows, a **double stable** matching does not always exist.

**Example 1** *There are three students $s_1, s_2, s_3$, and three colleges $c_1, c_2, c_3$. Each college is assumed to have only one seat. Students' preferences, colleges' preferences, and a common priority order are given as follows.*

[3]In 1979, four university presidents in Shanghai petitioned the government for the autonomy to select their students (Xiao [11]). In response to this appeal, the government introduced a so-called "dummy quota policy", under which colleges could freely choose students from the applicants they received. See Articles 6 and 7 of National University Recruitment Regulation in 1984.

[4]In 2014, Osaka released its Public High School Recruitment Reform Guideline, which emphasizes the importance of respecting schools' preferences as a primary policy goal in the first section.

[5]The formal definitions of these concepts are provided in Section 2.

$$
\begin{array}{llllll}
P_{s_1}: & c_1, c_3, c_2, & P_{c_1}: & s_3, s_2, s_1, & \succ: s_1, s_2, s_3, \\
P_{s_2}: & c_1, c_2, c_3, & P_{c_2}: & s_3, s_2, s_1, & \\
P_{s_3}: & c_2, c_1, c_3 & P_{c_3}: & s_3, s_2, s_1. &
\end{array}
$$

*The unique priority stable matching is* $\mu = \binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$. *Since* $s_2$ *prefers* $c_1$ *to* $c_2 = \mu(s_2)$ *and* $c_1$ *prefers* $s_2$ *to* $s_1 = \mu(c_1)$*, this matching is not preference stable. Therefore, there is no double stable matching in this problem.*

The above example shows that priority stability and preference stability are incompatible. To achieve priority stability, colleges' preferences must be ignored, while ensuring preference stability requires compromising the importance of exam scores. Faced with this *dilemma*, the governments of China and Japan have attempted to balance both types of rankings in their admission processes. Therefore, double stability is only partially achieved in these markets.

In this paper, we explore the challenges of fully achieving double stability. Specifically, we aim to identify the conditions under which priority stability and preference stability can coexist. To address this, we introduce a common priority order into the college admissions model. We show that (i) both the SD mechanism and the student-proposing DA mechanism[6] produce a double stable matching whenever possible (Propositions 2 and 3), and surprisingly, (ii) the college-proposing DA mechanism may fail to produce a double stable matching even if it exists (Example 2). Based on these findings, we conclude that a double stable matching exists if, and only if, the outcome of the SD mechanism coincides with that of the student-proposing DA mechanism (Theorem 1). This is our main result, reflecting the difficulties of satisfying both stability criteria. Given that double stability is characterized by two well-known mechanisms, our result offers a straightforward method to determine if a dilemma may arise in a particular matching problem.

Beyond our analysis of double stability, we also consider an alternative notion of stability. A matching is **minimally stable** if it is not priority blocked and preference blocked by the *same* pair. Since this notion only requires the elimination of a double-blocking pair, the existence of a minimally stable matching is always guaranteed (Remark 4). Moreover, the set of minimally stable matchings is larger than the union of the priority stable matching and the preference

[6]In this paper, we extensively mention and use both the serial dictatorship (SD) mechanism and the deferred acceptance (DA) mechanism. The formal descriptions of these mechanisms are provided in Section 2.

stable matchings (Example 1[7]). Based on these observations, we might regard minimal stability as the weakest notion and suggest that any other stability notion should be laid on between minimal stability and double stability.

Recently, Chen et al. [3] and Miyazaki and Okamoto [6] studied an extended matching model where participants on each side have multiple preference rankings. They analyzed the difficulties of finding stable matchings by discussing a wide range of ranking structures, such as how many choices can be listed in each ranking or how many rankings participants can have. In contrast, our paper focuses on a specific ranking structure that reflects many real-life scenarios, incorporating both a common priority order and colleges' preferences over students. In Fang and Yasuda [4], we use a similar framework to examine the performance of different mechanisms, including the mechanism currently practiced in China and its improved alternatives.[8]

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 provides the main result and several associated results. In Section 4, we consider two possible extensions: minimal stability and college-specific priority orders. Section 5 concludes.

## 2 College Admissions Problem with Common Priority

In this paper, we examine an extended college admissions model in which students are ranked by two distinct types of rankings. We call this model a college admissions problem with common priority, or simply a **problem**, which is denoted by $G$ and consists of the following elements.

1. a set of students, $S = \{s_1, \cdots, s_n\}$,
2. a set of colleges, $C = \{c_1, \cdots, c_m\}$,
3. a capacity vector, $q = (q_{c_1}, \cdots, q_{c_m})$,
4. a list of strict student preferences, $P_S = (P_{s_1}, \cdots, P_{s_n})$,
5. a list of strict college preferences, $P_C = (P_{c_1}, \cdots, P_{c_m})$,
6. a strict priority order over students, $\succ$.

Each student $s \in S$ has a **strict preference** relation $P_s$ over colleges and her outside option $\emptyset$. The notation $cP_sc'$ means that $s$ prefers $c$ to $c'$. Let $R_s$ denote the weak preference relation

[7] We will revisit this example in Section 4.

[8] In Fang and Yasuda [4], motivated by the college admissions in China, we examine the dummy quota policy, which sets an upper bound on the number of applications each college can receive. This key exogenous constraint cannot be ignored, as it significantly complicates the mechanism design issues.

induced by $P_s$, i.e., $cR_sc'$ if and only if $cP_sc'$ or $c = c'$. The **common priority order** $\succ$ is also assumed to be strict, which can be based on some exogenous factors such as exam scores or institutional rules. We use the notation $s \succ s'$ to denote that $s$ has a higher priority than $s'$.

Each college $c \in C$ has a **strict preference** relation $P_c$ over the set of subsets of students. The notation $S'P_cS''$ means that $c$ would like to admit a group of students $S'$ than another group $S''$. Let $R_c$ denote the weak preference relation induced by $P_c$. We say that $P_c$ is **responsive** (Roth [8]) if for any $S' \subset S$ with $|S'| < q_c$, and $s, s' \in S \setminus S'$, (i) $S' \cup sP_cS' \cup s'$ if and only if $sP_cs'$, and (ii) $S' \cup sP_cS'$ if and only if $sP_c\emptyset$. Throughout the paper, we assume that colleges' preferences are responsive.

Now, we introduce several definitions that are needed for our analysis. A **matching** is a set valued function $\mu : S \cup C \rightrightarrows 2^{S \cup C}$ such that (i) $\mu(s) \subset C \cup \emptyset, |\mu(s)| = 1$ for any $s \in S$, (ii) $\mu(c) \subset S, |\mu(c)| \leq q_c$ for any $c \in C$, and (iii) $c \in \mu(s)$ if and only if $s \in \mu(c)$ for any $s \in S$ and $c \in C$. A matching $\mu$ is **individually rational** for students if for any $s \in S$, $\mu(s)R_s\emptyset$, and for colleges if for any $c \in C$, (i) $|\mu(c)| \leq q_c$ and (ii) for any $s \in \mu(c)$, $sR_c\emptyset$. A matching $\mu$ is **priority blocked** by a pair $(s, c)$ if (i) $cP_s\mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $s \succ \emptyset$, or (b) for some $s' \in \mu(c)$, $s \succ s'$; such a pair is called a priority blocking pair. The concept of "preference blocked" is defined analogously; A matching $\mu$ is **preference blocked** by a pair $(s, c)$ if (i) $cP_s\mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $sP_c\emptyset$, or (b) for some $s' \in \mu(c)$, $sP_cs'$; such a pair is called a preference blocking pair.

Using these blocking conditions, we define three essential stability notions.

**Definition 1** *A matching $\mu$ is fair or* ***priority stable*** *if it is individually rational for students, and there is no priority blocking pair. A matching $\mu$ is stable or* ***preference stable*** *if it is individually rational for both students and colleges, and there is no preference blocking pair. A matching $\mu$ is said to be* ***double stable*** *if it is priority stable and preference stable.*

In other words, a matching $\mu$ is double stable if it satisfies the following three properties: (i) individually rational for both students and colleges, (ii) not priority blocked by any pair, and (iii) not preference blocked by any pair. Additionally, for a given matching $\mu'$, a pair $(s, c)$ is defined as a **double-blocking pair** if $\mu'$ is both priority blocked and preference blocked by the pair $(s, c)$. Apparently, there exists no double-blocking pair for a double stable matching.

A **mechanism** $\phi$ is a procedure that selects a matching for each problem. Let $\phi(G)$ denote the matching selected by mechanism $\phi$. In this paper, we focus on two specific mechanisms: the serial dictatorship (SD) mechanism and the deferred acceptance (DA) mechanism. Below we describe each mechanism in detail. The first one is the SD mechanism, in which the assignment is based on a common priority order and students' preferences.[9]

**The SD mechanism**

Step 1: The student with the highest priority is assigned to her top choice.

Step $t \geq 2$: The student with the $t$-th highest priority is assigned to her top choice among all colleges except the ones whose quotas have been filled. Note that the mechanism is terminated when all students have chosen a college or all colleges have filled their quotas.

The second one is the (student-proposing) DA mechanism, in which the assignment is based on colleges' preferences and students' preferences.[10]

**The student-proposing DA mechanism**

Step 1: Each student proposes to her top choice. Each college (i) considers its applicants in this step; (ii) tentatively accepts those applicants up to its quota, one at a time, following its preferences; and (iii) rejects the remaining applicants.

Step $t \geq 2$: Each student that has been rejected in the previous step proposes to her next choice. Each college (i) considers its applicants in this step and all tentatively matched applicants in the previous step; (ii) tentatively accepts those applicants up to its quota, one at a time, following its preferences; and (iii) rejects the remaining applicants. Note that the mechanism is terminated when no student's proposal is rejected.

Recall that, in our model, students are ranked by two distinct types of rankings. If only one type of rankings affects assignments, i.e., either common priority or colleges' preferences are completely irrelevant, then such a mechanism could be considered extreme. Conversely, if both types of rankings are utilized to determine assignments, the mechanism would be moderate. We formally define these two concepts as follows.

[9]The SD mechanism described here is the one with respect to 100% of colleges' quotas. The SD mechanism with respect to 105% or 110% of colleges' quotas is defined in the same way by slightly increasing colleges' quotas.

[10]The college-proposing DA mechanism can be defined in almost the same way by swapping the roles of colleges and students, hence we do not describe it here.

**Definition 2** *A mechanism $\phi$ is called* **extreme** *if for any problem $G = (P_S, P_C, \succ)$, either (i) $\phi(P_S, P'_C, \succ) = \phi(P_S, P_C, \succ)$ for any $P'_C$ or (ii) $\phi(P_S, P_C, \succ') = \phi(P_S, P_C, \succ)$ for any $\succ'$, where $P'_C$ is an arbitrary list of colleges' preferences and $\succ'$ is an arbitrary common priority order. A mechanism $\phi$ is* **moderate** *if it is not extreme.*

Most related papers in the matching literature assume that participants on each side have only one type of ranking over those on the other side. Consequently, nearly all known mechanisms are extreme by their definitions. For example, the SD mechanism is extreme because it disregards colleges' preferences. Similarly, both the student-proposing and college-proposing DA mechanisms are extreme because common priority plays no role in the admission process.[11]

# 3 Characterizations of Double Stable Matching

In this section, we investigate the conditions that allow for the coexistence of priority stability and preference stability. As discussed in Section 1, double stability is an attractive and natural stability notion. However, its existence is not always guaranteed.

**Proposition 1** *There may exist no double stable matching.*

Given this non-existence result, a natural next question is whether there exists a mechanism that satisfies double stability whenever possible. The answer is affirmative, and there are at least two mechanisms that fulfill this purpose. The first one is the SD mechanism.

**Proposition 2** *The SD mechanism produces a unique priority stable matching, hence it always eliminates a double-blocking pair.*[12] *Moreover, the SD mechanism produces a double stable matching whenever it exists.*

**Proof.** Under the SD mechanism, a student with a higher priority always faces a (weakly) larger set of available colleges than a student with lower priority does, hence no student constitutes a priority blocking pair. To establish the uniqueness, suppose on the contrary that there are two distinct priority stable matchings, say $\mu$ and $\mu'$. Let $s$ be a student with the highest priority

[11] In Fang and Yasuda [4], motivated by the fact that the government in China has abandoned the use of extreme mechanisms, we analyze moderate mechanisms, including the current mechanism and its improved alternatives.

[12] Note that a unique priority stable matching can also be derived by the student proposing DA mechanism under which each college's preferences are replaced by the common priority order. See Balinski and Sönmez [2].

who is assigned to different colleges between $\mu$ and $\mu'$. Assume, without loss of generality, $\mu(s) P_s \mu'(s)$. Since each priority stable matching is individually rational for students, $\mu(s) \neq \emptyset$ must hold, i.e., $\mu(s)$ must be a college. By our presumption, under the matching $\mu'$, college $\mu(s)$ either (i) has an empty seat or (ii) is assigned to some student who has a lower priority than $s$. In either case, $\mu'$ is priority blocked by a pair of $s$ and $\mu(s)$. A contradiction.

The last part is immediate from the definition of double stability and the uniqueness of the priority stable matching. ■

Since every double stable matching must be priority stable and the priority stable matching is always unique in our model, there cannot exist more than one double stable matchings. Thus, Proposition 2 implies the following.

**Remark 1** *There exists at most one double stable matching.*

While the SD mechanism may look attractive, it is extreme and cannot take any college's preferences into account. Therefore, the uniqueness of the priority stable matching (Proposition 2) implies that, if a mechanism respects colleges' preferences even a little, it has to give up priority stability. That is, incorporating colleges' preferences is *never* compatible with priority stability, which is formally expressed as follows.

**Remark 2** *The SD mechanism is a unique priority stable mechanism. Thus, there exists no mechanism which is both moderate and priority stable.*

The second mechanism that fulfills our purpose, i.e., achieving double stability whenever it is possible, is the student-proposing DA mechanism. This mechanism ignores priority stability but always achieves preference stability.

**Proposition 3** *The DA mechanisms, both student-proposing and college-proposing, produce a preference stable matching, hence they always eliminate a double-blocking pair. Moreover, the student-proposing DA mechanism produces a double stable matching whenever it exists.*

For the proof of Proposition 3, we use the following well-known properties of the priority stable matching and the preference stable matching. Lemma 1 is derived by Balinski and Sönmez [2] and Lemma 2 is established by Gale and Shapley [5], hence we omit the proofs.

**Lemma 1** *Every priority stable matching is Pareto efficient for students.*[13]

**Lemma 2** *There always exist (possibly multiple) preference stable matchings. Among them, one outcome, called the student optimal stable matching (SOSM), Pareto dominates (based on students' preferences) all other stable outcomes. The SOSM is derived by the student-proposing DA mechanism.*

**Proof of Proposition 3.** For any college admissions problem (with responsive preferences), it is widely known that the DA mechanism always produces a preference stable matching (See Gale and Shapley [5] or Roth and Sotomayor [9]).

To prove the second part, suppose that the student-proposing DA mechanism produces a matching $\hat{\mu}$, which is different from the double stable matching $\mu^*$. Note that $\mu^*$ should satisfy both priority stability and preference stability. Lemma 2 implies that $\mu^*$ is Pareto dominated by $\hat{\mu}$, hence $\mu^*$ is not a Pareto efficient matching for students. This contradicts Lemma 1. ■

When there exists a double stable matching, the student-proposing DA mechanism can find it by Proposition 3. One may wonder if the college-proposing DA mechanism also has the same property. However, the following example shows that the answer is negative.

**Example 2** *Let $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2, c_3\}$, and $q = (1, 1, 1)$. Students' preferences, colleges' preferences, and a common priority order are given as follows.*

$$\begin{array}{llllll} P_{s_1}: & c_1, c_2, c_3, & P_{c_1}: & s_1, s_2, s_3, & \succ: & s_1, s_2, s_3, \\ P_{s_2}: & c_1, c_2, c_3, & P_{c_2}: & s_1, s_3, s_2, & & \\ P_{s_3}: & c_1, c_3, c_2, & P_{c_3}: & s_1, s_2, s_3. & & \end{array}$$

*The unique double stable matching is $\mu^* = \binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$. We can calculate it by using either the SD or the student-proposing DA mechanism. However, the college-proposing DA mechanism produces a different matching $\hat{\mu} = \binom{s_1\ s_2\ s_3}{c_1\ c_3\ c_2}$.*

In the matching literature, the student-proposing DA mechanism is typically preferred to the college-proposing DA mechanism because the former is superior to the latter in students'

[13] A matching $\mu$ is Pareto dominated for students by another matching $\mu'$, if (i) for each $s \in S$, $\mu'(s) R_s \mu(s)$ and (ii) for some $s' \in S$, $\mu'(s') P_s \mu(s')$. By using this notion, we say that a matching is Pareto efficient for students if it is not Pareto dominated for students by any other matching.

welfare (see Lemma 2) and incentives (see Roth [8]).[14] We provide a different and additional rationale for preferring the student-proposing DA mechanism; it always produces a double stable matching when possible, whereas the college-proposing DA mechanism does not. Since the outcomes of these two mechanisms coincide when the preference stable matching is unique, we can also derive the following property.

**Remark 3** *If the college-proposing DA mechanism produces a double stable matching, the preference stable matching must be unique.*

We are now ready to present the main result of the paper.

**Theorem 1** *A double stable matching exists if and only if a unique priority stable matching coincides with a student optimal stable matching; that matching becomes a unique double stable matching.*

**Proof.** The "if" part is trivial by the definition of double stability. To prove the "only if" part, suppose on the contrary that a double stable matching exists when a priority stable matching and a student optimal stable matching are strictly distinct. Then, at least one of Propositions 2 and 3 must be violated. A contradiction. ■

Theorem 1 implies that a double stable matching exists if and only if the resulting outcome of the SD mechanism using the common priority order coincides with that of the student-proposing DA mechanism using the colleges' preferences. Since these two mechanisms are computationally simple, one can easily figure out if a double stable matching exists or not.[15]

Now, we consider a problem $G = (P_S, P_C, \succ)$ that has a known double stable matching $\mu^*$. To explore how stability is affected by changes, we modify this problem by replacing $\succ$ with $\succ'$, while keeping all other elements unchanged, resulting in a new problem, $G = (P_S, P_C, \succ')$. Since the only difference between $G$ and $G'$ is the common priority, the student optimal matching for $G'$ should still be $\mu^*$. This leads us to the following corollary.

[14]The student-proposing DA mechanism is strategy-proof for students, i.e., for any problem $G$, truthful reporting is a dominant strategy for each student. By contrast, the college-proposing DA mechanism does not exhibit this incentive property.

[15]Borrowing a concept from computer science, the time complexity of the SD mechanism is linear ($O(n)$) and that of the DA mechanism is quadratic ($O(n^2)$), where $n$ is a number of inputs (i.e., students in our case).

**Corollary 1** *Suppose that $G = (P_S, P_C, \succ)$ has a double stable matching. Then, a modified problem $G' = (P_S, P_C, \succ')$ also has a double stable matching if and only if the (unique) priority stable matching of $G'$ is identical to that of $G$.*

In light of Corollary 1, a common priority order that guarantees the existence of a double stable matching is essentially pinned down as unique; any significant changes to this order would eliminate the possibility of such existence. Similarly, changing colleges' preferences, rather than the priority order, yields almost the same result.

**Corollary 2** *Suppose that $G = (P_S, P_C, \succ)$ has a double stable matching. Then, a modified problem $G'' = (P''_S, P''_C, \succ)$ also has a double stable matching if and only if the (unique) student optimal stable matching of $G''$ is identical to that of $G$.*

# 4 Extensions

In this section, we consider two extensions of our model. First, we introduce a weaker version of double stability. Then, we allow college-specific priority orders and examine whether the results obtained in Section 3 still hold in this broader context.

## 4.1 Minimally Stable Matching

As discussed earlier, the notion of double stability is often too strong to guarantee its existence. To address this issue, we explore a weaker version of double stability. There are various methods to weaken this notion. For example, one could count the number of blocking pairs for each stability criterion, then call the matching stable as long as this number does not exceed some predetermined upper bound.[16] In this paper, we focus on an extremely weak version of double stability, called "minimal stability". It requires only the elimination of a double-blocking pair; any violation of a single stability criterion is allowed unlimited.

**Definition 3** *A matching $\mu$ is called **minimally stable** if there is no double-blocking pair under $\mu$, i.e., $\mu$ is not priority blocked and preference blocked by the same student-college pair.*

[16] According to this definition, our original double stability coincides with its special case in which the upper bound is set equal to 0. Clearly, choosing any positive upper bound weakens the notion of double stability.

By Propositions 2 and 3, the SD and DA mechanisms both eliminate a double-blocking pair, hence they are minimally stable mechanisms. This implies that, in contrast to double stability, the existence of minimal stability is always guaranteed.

**Remark 4** *There always exist minimally stable matchings.*

In what follows, we revisit Example 1 to illustrate that a matching, which is neither priority stable nor preference stable, can still be minimally stable.

**Example 3** *Let* $S = \{s_1, s_2, s_3\}$*,* $C = \{c_1, c_2, c_3\}$*, and* $q = (1, 1, 1)$*. Students' preferences, colleges' preferences, and a common priority order are given as follows.*

$$\begin{array}{llllll} P_{s_1}: & c_1, c_3, c_2, & P_{c_1}: & s_3, s_2, s_1, & \succ: s_1, s_2, s_3, \\ P_{s_2}: & c_1, c_2, c_3, & P_{c_2}: & s_3, s_2, s_1, & \\ P_{s_3}: & c_2, c_1, c_3 & P_{c_3}: & s_3, s_2, s_1. & \end{array}$$

*In this example, the unique priority stable matching is* $\mu^1 = \binom{s_1\ s_2\ s_3}{c_1\ c_2\ c_3}$*, and the unique preference stable matching is* $\mu^2 = \binom{s_1\ s_2\ s_3}{c_3\ c_1\ c_2}$*. Now consider a different matching* $\mu^3 = \binom{s_1\ s_2\ s_3}{c_3\ c_2\ c_1}$*. Since* $\mu^3$ *is different from* $\mu^1$ *and* $\mu^2$*, it is neither priority stable nor preference stable. However, since there is no double-blocking pair,* $\mu^3$ *is a minimally stable matching.*

Example 3 shows that the condition for minimal stability is very weak; thus, the set of minimally stable matchings is large. More precisely, this set is strictly larger than the union of the (unique) priority stable matching and preference stable matchings. This finding may broaden the scope of research in market design. For example, it would be interesting to explore a moderate mechanism that achieves minimal stability while also satisfying some additional desirable properties. We leave such investigation for future research.[17]

Additionally, one might be interested in a mathematical structure of the set of minimally stable matchings.[18] Unfortunately, this set does not form a lattice with respect to a partial order of students' preferences. To verify this property, let us first introduce the following mapping $\lambda$.

[17] In Fang and Yasuda [4], we analyze a few moderate mechanisms and show that, while our proposal satisfies minimal stability, the assignment mechanism currently used in Chinese college admission does not.

[18] In a standard one-to-one or one-to-many matching problem, it is widely known that the set of stable matchings has a lattice structure. For a detailed discussion of these lattice properties, see chapters 2 and 3 of [9].

Given any pairs of matchings $\mu$ and $\mu'$, $\lambda$ is defined as

$$\lambda(s) = \mu(s) \text{ if } \mu(s) R_s \mu'(s)$$
$$= \mu'(s) \text{ otherwise,}$$

for each student $s \in S$. In example 3, $\mu^1$ and $\mu^2$ are minimally stable matchings. By applying $\lambda$, we have

$$\lambda(s_1) = c_1, \ \lambda(s_2) = c_1, \ \lambda(s_3) = c_2,$$

which is not even a matching, as both $s_1$ and $s_2$ are assigned to the same college $c_1$ (Note that $c_1$ has a quota of 1). Thus, the following result holds.

**Remark 5** *The set of minimally stable matchings is not a lattice under students' preferences.*

### 4.2 College-specific Priority Orders

Until now, we have assumed that priority order is common across colleges. Our model, based on this assumption, includes many real-life examples, such as college admissions in China and public high school choice in Osaka, as mentioned in Section 1. However, there could be other situations in which different colleges have different priority orders over students. For example, each college may assign different weights to each subject. In such a case, priority orders are no longer common across colleges, even if students take the same exam.

To analyze these situations, we consider an extended version of our original model. We call this model a **college admission problem with college-specific priority** and represent it by a 6-tuple $G = (S, C, q, P_S, P_C, \succ_C)$. Elements 1 through 5 are identical to the original setting; the extended model only replaces 6 (a common priority) with 6′ (college-specific priority orders). This generalization leads us to modify the stability notions.[19]

**Definition 4** *In a college admission problem with college-specific priority, we say that a student-college pair $(s, c)$ is a* ***priority blocking pair*** *for the matching $\mu$, if (i) $cP_s\mu(s)$ and (ii) either (a) $|\mu(c)| < q_c$ and $sP_c\emptyset$, or (b) for some student $s' \in \mu(c)$, $s \succ_c s'$. A matching $\mu$ is* ***priority stable*** *if it is individually rational for students and there is no priority blocking pair. A matching $\mu$ is* ***double stable*** *if it is priority stable and preference stable.*

[19]Note that the definition of preference stability is unchanged since it is independent of priority orders.

Since colleges' preferences and college-specific priority orders are both strict rankings, there is no mathematical distinction between them. Therefore, the mathematical structure of priority stable matchings must be the same as that of preference stable matching. This implies that there are possibly multiple priority stable matchings. Consequently, the uniqueness of priority stable matching cannot be guaranteed, which implies that Proposition 2 no longer holds. Our next example shows that most of the results in Section 3 do not hold in the extended model.

**Example 4** *Let $S = \{s_1, s_2, s_3\}$, $C = \{c_1, c_2\}$, and $q = (1, 1)$. Students' preferences, colleges' preferences, and college-specific priority orders are given as follows.*

$$\begin{array}{llllll} P_{s_1}: & c_2, c_1, & P_{c_1}: & s_1, s_2, s_3, & \succ_{c_1}: & s_1, s_2, s_3, \\ P_{s_2}: & c_1, c_2, & P_{c_2}: & s_2, s_1, s_3, & \succ_{c_2}: & s_2, s_3, s_1, \\ P_{s_3}: & c_2, c_1. & & & & \end{array}$$

*There is a unique double stable matching $\mu^* = \binom{s_1\ s_2\ s_3}{c_1\ c_2\ \emptyset}$, which is different from (and also Pareto dominated for students by) a student optimal preference stable matching $\mu' = \binom{s_1\ s_2\ s_3}{c_2\ c_1\ \emptyset}$.*

In the above example, the student-proposing DA mechanism (with respect to $P_C$) produces a matching $\mu'$, which is different from the matching $\mu^*$. This implies that Proposition 3 is no longer true. The failure of Proposition 3 automatically implies that Theorem 1, our main result, does not hold either. Given these negative findings, we now understand that the common priority structure is indeed the key to rich properties derived in Section 3.

## 5 Concluding Remarks

In this paper, we study an extended college admissions model where students are ranked by two distinct types of rankings. We introduce the concept of double stability and derive some useful properties of this new stability concept. Moreover, we also provide its characterization through two existing well-known mechanisms.

While most of our findings rely on the structure of common priority and may not directly apply to more general settings, several important questions arise. For example, in the broader context, one may wonder if there exists a mechanism that produces a double stable matching whenever possible? Another natural question is how we can impose certain restrictions on the preference domain to guarantee the existence of a double stable matching? The questions raised here may be an interesting research direction in the future.

## References


[1] Abdulkadiroğlu, A. and Sönmez, T. (2003). School choice: A mechanism design approach. American Economic Review, 93: 729-747.

[2] Balinski, M. L. and Sönmez, T. (1999). A tale of two mechanisms: Student placement. Journal of Economic Theory, 84: 73-94.

[3] Chen, J., Niedermeier, R., and Skowron, P. (2018). Stable marriage with multi-modal preferences. In Proceedings of the 19th ACM conference on economics and computation, EC '18, 269–286.

[4] Fang, Y. and Yasuda, Y. (2021). Misalignment between Test Scores and Colleges' Preferences: Chinese College Admissions Reconsidered. mimeo.

[5] Gale, D. and Shapley, L. S. (1962). College admissions and the stability of marriage. American Mathematical Monthly, 69: 9-15.

[6] Miyazaki, S. and Okamoto, K. (2019). Jointly stable matchings. Journal of Combinatorial Optimization, 38: 646–665.

[7] Roth, A. E. (1984). The evolution of the labor market for medical interns and residents: a case study in game theory. Journal of Political Economy, 92: 991-1016.

[8] Roth, A. E. (1985). The college admissions problem is not equivalent to the marriage problem, Journal of Economic Theory, 36: 277-288.

[9] Roth, A. E. and Sotomayor, M. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis*. Cambridge University Press.

[10] Roth, A.E. and Peranson, E. (1999). The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design. American Economic Review, 89, 748–780.

[11] Xiao, G. (1979). Four presidents of universities in Shanghai appeal that universities should have some degree of autonomy. People's Daily.