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Abstract

As a generalization of ordinal concavity we introduce a new notion of discrete concavity called *tandem concavity* defined for a function over the subsets of a finite set E endowed with an ordered partition (E_1, E_2). Every function expressed as a lexicographic composition of two ordinally concave functions satisfies tandem concavity. We apply tandem concavity to the rationalization of choice rules in stable matching problems. We show that tandem concavity rationalizes a wider class of choice rules than ordinal concavity.

Keywords: Discrete convexity, ordinal concavity, tandem concavity, stable matchings

1. Introduction

Consider a finite set E and a function $u : 2^E \to \mathbb{R}$. There are several notions of discrete concavity for this function, such as *M*$^\natural$*-concavity* [11] and *semi-strict quasi M*$^\natural$*-concavity* [5], also known as *ordinal concavity* [7, 16]. In some economic applications, E is endowed with an ordered partition (E_1, E_2) and the function u has a property that depends on the partition. For example, in a job-matching context, a firm separates the set of workers E into two types, skilled workers E_1 and unskilled workers E_2, and its profit function u depends on the composition of the two types of workers (see [4, 8]). Similarly, in dynamic matching problems, agents separate the whole set of contracts E into the set E_1 of contracts signed in period 1 and the set E_2 signed in period 2 (see [2]).

In the present note, we investigate concavity of a function defined over the subsets of a partitioned set. For analytical simplicity, we consider a partition consisting of two parts, but it is straightforward to generalize our results to the case with multiple parts (see Remark 4 in Section 3.1). We introduce a new notion called *tandem concavity* as a generalization of ordinal concavity. For two functions $u_1, u_2 : 2^E \to \mathbb{R}$ that satisfy ordinal concavity, we apply an operation called *lexicpgraphic composition* of the two functions and show that the resulting function satisfies tandem concavity under the unique maximizer condition (to be precisely defined in Section 3.2). We also apply the new notion to stable matching problems. Recent studies [2, 4, 8] show that a

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stable matching exists if every agent's choice rule satisfies a certain form of substitutability between contracts in a partitioned set. We show that a choice rule satisfies the substitutability if and only if it is *rationalized* by a tandem-concave function, i.e., the outcome of the choice rule is supported as the maximizers of a tandem-concave function. This result implies that, if every agent in a matching market has a tandem-concave utility function, then a stable matching exists.

The present note is organized as follows. Section 2 introduces some notation used here and describes a definition of ordinal concavity. In Section 3 we define tandem concavity and examine its relationship to ordinal concavity via lexicographic composition. Section 4 applies tandem concavity to the problem of rationalizing choice rules in stable matching problems.

2. Preliminaries

We introduce notation following [6]. Note in particular that $\emptyset$ denotes the empty set as usual while it also means a symbol that does not belong to the underlying set E. For any $X \in 2^E$ let $X + x = X \cup \{x\}$ for all $x \in E \setminus X$ and $X - x = X \setminus \{x\}$ for all $x \in X$. Also for $x = \emptyset$ let $X \pm x = X$.

Ordinal concavity is defined as follows.

Definition 2.1 (**Ordinal Concavity**): *A function* $u : 2^E \to \mathbb{R}$ *satisfies* ordinal concavity *if for every* $X, X' \in 2^E$ *the following statement holds*:
For every $x \in X \setminus X'$ *there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

(i) $u(X) < u(X - x + x')$, *or*

(ii) $u(X') < u(X' - x' + x)$, *or*

(iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

Remark 1: Ordinal concavity was originally called *semi-strict quasi M$^\natural$-concavity* in [5] (also see [12, 13] and [3]). The name of ordinal concavity was used by [7, 16] in the economics literature, which we adopt in the present note. □

Remark 2: Ikebe and Tamura [9] introduce discrete concavity called *twisted M$^\natural$-concavity* defined for functions over the subsets of a partitioned set (also see [15]). This notion captures concavity of utility functions in markets with money transfers. Meanwhile, our tandem concavity to be precisely defined below captures concavity of utility functions in markets *without* money transfers (see Section 4). □

3. Tandem concavity

Throughout this section, we fix an ordered partition (E_1, E_2) of E.

3.1. Definition of tandem concavity

We focus on the following new notion of discrete concavity.

Definition 3.1 (**Tandem Concavity**): *A function* $u : 2^E \to \mathbb{R}$ *satisfies* tandem concavity with respect to (E_1, E_2) *if the following three statements hold*:

(a1) *For every* $X, X' \in 2^E$ *and every* $x \in (X \setminus X') \cap E_1$ *there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

- (i) $u(X) < u(X - x + x')$, *or*
- (ii) $u(X') < u(X' - x' + x)$, *or*
- (iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

(a2) *For every* $X, X' \in 2^E$ *with* $X \cap E_1 = X' \cap E_1$ *and every* $x \in X \setminus X'$ *there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

- (i) $u(X) < u(X - x + x')$, *or*
- (ii) $u(X') < u(X' - x' + x)$, *or*
- (iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

(b) *For every* $X, X' \in 2^E$, *If* $u(X \cap E_1) > u(X' \cap E_1)$, *then* $u(X) > u(X')$.

Remark 3: Here we allow that E_1 (resp. E_2) is the empty set. If so, then (a1) (resp. (a2)) becomes null, while (a2) (resp. (a1)) becomes equivalent to the condition for ordinal concavity, and (b) always holds. Hence in this case tandem concavity means ordinal concavity. □

Remark 4: Consider $X, X' \in 2^E$. Condition (a1) requires the same condition as ordinal concavity for $x \in X \setminus X'$ with $x \in E_1$. Condition (a2) requires the same condition for $x \in E_2$ while assuming that X and X' have the same intersection with E_1. Condition (b) intuitively states that elements from E_1 are more important for achieving a higher function value than those from E_2.

The above interpretation leads us to the following generalization of (a1) and (a2) for an ordered partition $(E_1, \ldots, E_m)$ with $m \geq 2$: we require ordinal concavity for $X, X' \in 2^E$ and $x \in X \setminus X'$ with $x \in E_i$ while assuming that X and X' have the same intersection with E_k for all $k < i$, where $i \in \{1, \ldots, m\}$. A similar generalization can be made for (b) as well. □

A formal definition of *generalized tandem concavity* is given as follows.

Definition 3.2 (**Generalized Tandem Concavity**): *A function* $u : 2^E \to \mathbb{R}$ *satisfies* generalized tandem concavity with respect to $(E_1, \ldots, E_m)$ *if the following two statements hold* (*with* E_0 *being the empty set*):

(a)$_g$ *For every* $i \in \{1, \ldots, m\}$, *every* $X, X' \in 2^E$ *with* $X \cap E_k = X' \cap E_k$ $(\forall k \in \{0, \ldots, i-1\})$, *and every* $x \in (X \setminus X') \cap E_i$ *there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that* (i)$_g$ $u(X) < u(X - x + x')$, *or*

(ii)$_g$ $u(X') < u(X' - x' + x)$, *or*

(iii)$_g$ $u(X) = u(X - x + x')$ *and* $u(X') = u(X' - x' + x)$.

(b)$_g$ *For every* $X, X' \in 2^E$, *if there exists* $i \in \{1, \dots, m\}$ *such that*

$$u\Big(X \cap \big(\cup_{k=1}^{i} E_k\big)\Big) > u\Big(X' \cap \big(\cup_{k=1}^{i} E_k\big)\Big),$$

then, $u(X) > u(X')$.

If $m = 2$, then Condition (a)$_g$ for $i = 1$ (resp. $i = 2$) is equivalent to Condition (a1) (resp. (a2)) in Definition 3.1, and Condition (b)$_g$ is equivalent to Condition (b) in Definition 3.1. The results in this note can be restated in obvious ways in terms of an ordered partition $(E_1, \dots, E_m)$ of E.

3.2. Lexicographic composition

We introduce an operation called *lexicographic composition*, which clarifies the connection between ordinal concavity and tandem concavity. For any nonempty $X \subseteq E$ define $u_X : 2^X \to \mathbb{R}$ by

$$u_X(Z) = u(Z) \qquad (\forall Z \in 2^X).$$

We call u_X the *restriction* of u on X. Also for any $X \subset E$ define $u^X : 2^{E \setminus X} \to \mathbb{R}$ by

$$u^X(Y) = u(Y \cup X) - u(X) \quad (\forall Y \subseteq E \setminus X).$$

We call u^X the *fixing* of u by X.

We say that $u : 2^E \to \mathbb{R}$ satisfies the *unique-maximizer condition* **(UM)** if the following condition holds:

(UM) For every $X \in 2^E$ there uniquely exists a maximizer of $\max\{u(Y) \mid Y \subseteq X\}$.

Let us consider the lexicographical order $\leq_\ell$ on $\mathbb{R}^2$ defined by $(a, b) <_\ell (c, d) \iff$ (i) $a < c$ or (ii) $a = c$ and $b < d$, for all $a, b, c, d \in \mathbb{R}$. Let $(\mathbb{R}^2)_\ell$ be the set $\mathbb{R}^2$ endowed with the lexicographical order $\leq_\ell$.

Let $u_1 : 2^E \to \mathbb{R}$ be an ordinally concave function with the unique maximizer condition (UM) and $u_2 : 2^E \to \mathbb{R}$ be an ordinally concave function. Now we define the *lexicographic composition* $u_1 \bullet u_2$ of u_1 and u_2 as follows. For any $X \in 2^E$ define X^* to be the (unique) maximizer of the restriction of u_1 on $X \cap E_1$. Moreover, let $Y^*_{X^*}$ be a maximizer of the restriction, of the fixing $u_2^{X^*}$ by X^*, to $X \cap E_2$. Here note that $Y^*_{X^*}$ may not be unique but the value of $u_2\left(Y^*_{X^*} \cup X^*\right)$ is, with respect to the given X. Then define $(u_1 \bullet u_2) : 2^E \to (\mathbb{R}^2)_\ell$ by

$$(u_1 \bullet u_2)(X) = \left(u_1\left(X^*\right), u_2\left(Y^*_{X^*} \cup X^*\right)\right) \quad \left(X \in 2^E\right).$$

For any $X \in 2^E$ we write $\hat{u}_1(X)$ and $\hat{u}_2(X)$ to denote $u_1\left(X^*\right)$ and $u_2\left(Y^*_{X^*} \cup X^*\right)$, respectively.

To interpret the lexicographic composition, consider an agent who maximizes two utility functions u_1 and u_2 in stages: given $X \in 2^E$, she first maximizes u_1 among the subsets of $X \cap E_1$, with the maximizer denoted X^*, and then maximizes u_2 among the subsets of $X \cap E_2$ while fixing the choice X^* from $X \cap E_1$. The function value $(u_1 \bullet u_2)(X)$ represents her maximized utilities (called *indirect utilities* in economics) for the first and second stages.

Remark 5: Fujishige et al. [6] defined an operation of lexicographic composition for two functions in a different way. $\Box$

We say that $(u_1 \bullet u_2) : 2^E \to (\mathbb{R}^2)_\ell$ satisfies tandem concavity if it satisfies conditions (a1), (a2) and (b) in Definition 3.1 with $<$ replaced by $<_\ell$.

Proposition 3.3: *If $u_1 : 2^E \to \mathbb{R}$ is an ordinally concave function that satisfies the unique-maximizer condition* **(UM)** *and $u_2 : 2^E \to \mathbb{R}$ is ordinally concave, then the lexicographic composition $u_1 \bullet u_2$ is tandem concave.*

(Proof) We first prove that $u_1 \bullet u_2$ satisfies (a1). Consider $X, X' \in 2^E$ and $x \in (X \setminus X') \cap E_1$. We consider two cases.

Case 1: Suppose $x \notin X^*$. Then, because of the definitions of X^* and $\hat{u}_1$ we have

$$\hat{u}_1(X) = \hat{u}_1(X - x), \tag{3.1}$$

$$\hat{u}_1(X') \leq \hat{u}_1(X' + x). \tag{3.2}$$

If the weak inequality of (3.2) holds with strict inequality, then (a1)(ii) holds for $x' = \emptyset$. Suppose that the weak inequality of (3.2) holds with equality. By the unique-maximizer condition **(UM)** of u_1, we have $(X')^* = (X' + x)^*$. By this equation and $x \in E_1$, we have $Y^*_{(X')^*} = Y^*_{(X'+x)^*}$, which implies $\hat{u}_2(X') = \hat{u}_2(X' + x)$. Similarly, by (3.1) and **(UM)**, we have $X^* = (X - x)^*$ and $Y^*_{X^*} = Y^*_{(X-x)^*}$, which implies $\hat{u}_2(X) = \hat{u}_2(X - x)$. Therefore, (a1)(iii) holds for $x' = \emptyset$.

Case 2: Suppose $x \in X^*$. For X^*, $(X')^*$ and $x \in X^* \setminus (X')^*$, the ordinal concavity of u_1 implies that there exists $\hat{x}' \in ((X')^* \setminus X^*) \cup \{\emptyset\}$ such that

(i)$_1$ $u_1(X^*) < u_1(X^* - x + \hat{x}')$, or

(ii)$_1$ $u_1((X')^*) < u_1((X')^* - \hat{x}' + x)$, or

(iii)$_1$ $u_1(X^*) = u_1(X^* - x + \hat{x}')$ and $u_1((X')^*) = u_1((X')^* - \hat{x}' + x)$.

Case 2(i): Suppose $\hat{x}' \notin X$. If (i)$_1$ holds, then (a1)(i) holds for $x' = \hat{x}'$. If (ii)$_1$ holds, then (a1)(ii) holds for $x' = \hat{x}'$. Suppose that (iii)$_1$ holds. By the latter equality, $(X')^*$ and $(X')^* - \hat{x}' + x$ attain the same value of u_1. Since these two subsets are included in $(X' + x) \cap E_1$, the unique-maximizer condition **(UM)** of u_1 implies that there is a subset of $(X' + x) \cap E_1$ that attains a higher value of u_1 than the two subsets, i.e., $\hat{u}_1(X' + x) > \hat{u}_1(X')$. Therefore, (a1)(ii) holds for $x' = \emptyset$.

Case 2(ii): Suppose $\hat{x}' \in X$. If (i)$_1$ holds, then we obtain a contradiction to the definition of X^* that maximizes u_1 among all subsets of $X \cap E_1$. Similarly, if (iii)$_1$ holds, then the former equality of (iii)$_1$ implies that X^* and $X^* - x + x'$ attain the maximum of u_1 among all subsets of $X \cap E_1$, contradicting the unique-maximizer condition **(UM)** of u_1. The remaining possibility is that (ii)$_1$ holds. Since $(X')^* - \hat{x}' + x \subseteq X' + x$, (a1)(ii) holds for $x' = \emptyset$.

Next, we prove that $u_1 \bullet u_2$ satisfies (a2). Consider $X, X' \in 2^E$ with $X \cap E_1 = X' \cap E_1$ and $x \in X \setminus X'$. Note that $x \in E_2$. Since $X \cap E_1 = X' \cap E_1$ and because of the unique-maximizer condition **(UM)** of u_1, we have $X^* = (X')^*$. We choose $Y^*_{X^*}$ and $Y^*_{(X')^*}$ in the following way: first, choose $Y^*_{(X')^*}$ as an arbitrary maximizer of the restriction of $u_2^{(X')^*}$ to $X' \cap E_2$. Then, choose $Y^*_{X^*}$ as a maximizer of the restriction of $u_2^{X^*}$ to $X \cap E_2$ in such a way that the following $(*)$ holds:

$(*)$ $Y^*_{X^*}$ attains the minimum of $|Y^*_{X^*} \Delta Y^*_{(X')^*}|$.

We consider two cases.

Case 1′: Suppose $x \notin Y^*_{X^*}$. Then,

$$\hat{u}_2(X) = \hat{u}_2(X - x), \tag{3.3}$$

$$\hat{u}_2(X') \leq \hat{u}_2(X' + x). \tag{3.4}$$

If the weak inequality of (3.4) holds with strict inequality, then by $(X')^* = (X' + x)^*$ and $\hat{u}_1(X') = \hat{u}_1(X' + x)$ (which follows from $x \in E_2$), (a2)(ii) holds for $x' = \emptyset$. Similarly, if the weak inequality of (3.4) holds with equality, then together with (3.3), it implies that (a2)(iii) holds for $x' = \emptyset$.

Case 2′: Suppose $x \in Y^*_{X^*}$. Then, For $Y^*_{X^*} \cup X^*$, $Y^*_{(X')^*} \cup (X')^*$ and $x \in (Y^*_{X^*} \cup X^*) \setminus (Y^*_{(X')^*} \cup (X')^*)$, the ordinal concavity of u_2 implies that there exists $\tilde{x}' \in \big((Y^*_{(X')^*} \cup (X')^*) \setminus (Y^*_{X^*} \cup X^*)\big) \cup \{\emptyset\}$ such that

(i)$_2$ $u_2(Y^*_{X^*} \cup X^*) < u_2\big((Y^*_{X^*} - x + \tilde{x}') \cup X^*\big)$, or

(ii)$_2$ $u_2\big(Y^*_{(X')^*} \cup (X')^*\big) < u_2\big((Y^*_{(X')^*} - \tilde{x}' + x) \cup (X')^*\big)$, or

(iii)$_2$ $u_2(Y^*_{X^*} \cup X^*) = u_2\big((Y^*_{X^*} - x + \tilde{x}') \cup X^*\big)$ and
$u_2\big(Y^*_{(X')^*} \cup (X')^*\big) = u_2\big((Y^*_{(X')^*} - \tilde{x}' + x) \cup (X')^*\big)$.

Case 2′(i): Suppose $\tilde{x}' \notin X$. If (i)$_2$ holds, then by $X^* = (X - x + \tilde{x}')^*$ and $\hat{u}_1(X) = \hat{u}_1(X - x + \tilde{x}')$ (which follows from $x, \tilde{x}' \in E_2$), (a2)(i) holds for $x' = \tilde{x}'$. Similarly, if (ii)$_2$ holds, then (a2)(ii) holds for $x' = \tilde{x}'$, and if (iii)$_2$ holds, then (a2)(iii) holds for $x' = \tilde{x}'$.

Case 2′(ii): Suppose $\tilde{x}' \in X$. If (i)$_2$ holds, then we obtain a contradiction to $Y^*_{X^*}$ maximizing the restriction of $u_2^{X^*}$ to $X \cap E_2$. If (iii)$_2$ holds, then the former equality of (iii)$_2$ implies that $Y^*_{X^*}$ and $Y^*_{X^*} - x + \tilde{x}'$ attain the maximum of the restriction of $u_2^{X^*}$ to $X \cap E_2$. Since $Y^*_{X^*} - x + \tilde{x}' \subseteq X \cap E_2$, we obtain a contradiction to the choice of $Y^*_{X^*}$ (recall $(*)$). The remaining possibility is that (ii)$_2$ holds. Note that $Y^*_{(X')^*} - \tilde{x}' + x \subseteq (X' + x) \cap E_2$. Together with $(X')^* = (X' - \tilde{x}' + x)^*$ and $\hat{u}_1(X') = \hat{u}_1(X' - \tilde{x}' + x)$ (which follows from $x, \tilde{x}' \in E_2$), (ii)$_2$ implies that (a2)(ii) holds for $x' = \emptyset$.

Finally, we prove that $u_1 \bullet u_2$ satisfies (b). Consider $X, X' \in 2^E$. Suppose that

$$(u_1 \bullet u_2)(X \cap E_1) > (u_1 \bullet u_2)(X' \cap E_1).$$

Since $(X \cap E_1) \cap E_2 = (X' \cap E_1) \cap E_2 = \emptyset$, we have $\hat{u}_2(X \cap E_1) = \hat{u}_2(X' \cap E_1)$. Therefore, the above displayed inequality holds only if $\hat{u}_1(X \cap E_1) > \hat{u}_1(X' \cap E_1)$, which implies $\hat{u}_1(X) > \hat{u}_1(X')$. Hence, $(u_1 \bullet u_2)(X) > (u_1 \bullet u_2)(X')$ holds. □

4. Application to stable matching problems

In this section, we present an application of tandem concavity to matching problems considered in recent works by Huang [8], Dur et al. [4], and Bando and Kawasaki [2]. We first explain their main results and contributions. To guarantee the existence of stable matching, existing studies typically assume that the agents' choice rules satisfy a condition called *substitutability*. Roughly speaking, this condition requires agents to view matching partners as substitutes, thus eliminating any forms of complementarity; a formal definition is given in Definition 4.2 below. However, there are real-life situations in which substitutability is violated. To take an example from Dur-Morrill-Phan [4], consider a hospital that wants to hire a new nurse only if it can hire a new doctor. Then, doctors are complementary to nurses, and hence the hospital's choice rules violates substitutability (see Example 1 below). The papers [2, 4, 8] show that the existence of stable matching is guaranteed even under a weaker variant of substitutability that alllows complementarity across different groups of agents. This result deepens our understanding about the essential condition for the existence of stable matching as well as expands the applicability of matching theory. We show that the weaker variant of substitutability is *rationalized*, in a sense specified later in Definition 4.3, by tandem concave utility functions.

Consider a two-sided matching problem between individuals and institutions, e.g., workers and firms or students and schools. Fix an institution in the market and let E denote the set of contracts that the institution chooses from. For example, in a job-matching market, E is the set of job candidates for the firm. The institution's *choice rule* is a function $C : 2^E \to 2^E$ such that $C(X) \subseteq X$ for all $X \in 2^E$. Given $X \in 2^E$, $C(X)$ specifies the set of contracts that the institution chooses from X.

The following is a standard condition of a choice rule.

Definition 4.1 (**Irrelevance of Rejected Contracts** [1]): *A choice rule $C : 2^E \to 2^E$ satisfies* irrelevance of rejected alternatives contracts *if for every $X \in 2^E$ and $x \in X$, it holds that*

$$x \notin C(X) \implies C(X - x) = C(X).$$

Since Kelso and Crawford [10] (also see [14]) it has long been recognized that *substitutability* of a choice rule is essential for the existence of stable matching.

Definition 4.2 (**Substitutability**): *A choice rule $C : 2^E \to 2^E$ satisfies* substitutability *if for every $X \in 2^E$ and $x, y \in X$ with $x \neq y$, it holds that*

$$x \in C(X) \implies x \in C(X - y).$$

If the above condition fails, then there exist $X \in 2^E$ and distinct $x, y \in X$ such that $x \in C(X)$ and $x \notin C(X - y)$. This means that x is chosen if it is coupled with y but not if y is absent, exhibiting complementarity between x and y. Substitutability rules out this type of complementarity.

We introduce an additional concept to see the connection between substitutability and discrete concavity.

Definition 4.3 (**Rationalization**): *A function* $u : 2^E \to \mathbb{R}$ rationalizes a choice rule $C : 2^E \to 2^E$ *if for every* $X \in 2^E$*, it holds that*

$$u(C(X)) > u(X') \qquad (\forall X' \subseteq X,\ X' \neq C(X)).$$

If u rationalizes C, then given $X \in 2^E$, C chooses the unique maximizer of u among all subsets of X. Rationalization is a fundamental step in the economic analysis to convert choice problems into utility-maximization problems.

Theorem 4.4 ([16, Theorem 1′]): *A choice rule* $C : 2^E \to 2^E$ *satisfies irrelevance of rejected contracts and substitutability if and only if there exists an ordinally concave function* $u : 2^E \to \mathbb{R}$ *that rationalizes* C*. Moreover, in the only-if part,* u *can be constructed so that, for every* $X, X' \in 2^E$ *and* $x \in X \setminus X'$*, there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *that satisfies Condition* (i) *or* (ii) *of ordinal concavity (i.e., the possibility of Condition* (iii) *can be eliminated).*

We now introduce a weaker variant of substitutability. Following Bando and Kawasaki [2], we present this condition in the context of multi-period matching problems. Assume that E is partitioned into two sets, E_1 and E_2, where E_1 denotes the set of period-1 contracts and E_2 denotes the set of period-2 contracts.

Definition 4.5 (**Period-wise Substitutability** [2]): *A choice rule* $C : 2^E \to 2^E$ *satisfies* period-wise substitutability *if for every* $X \in 2^E$*,* $i \in \{1, 2\}$*, and* $x, y \in X \cap E_i$ *with* $x \neq y$*,*

$$x \in C(X) \implies x \in C(X - y).$$

This condition requires substitutability between contracts in the same period. Therefore, it allows complementarity between contracts in different periods. We present an example, borrowed from Huang [8], of a choice rule that violates substitutability but satisfies period-wise substitutability.

Example 1: Consider a hospital that wants to hire a doctor x in period 1 and a nurse y in period 2. Let $E = \{x, y\}$, $E_1 = \{x\}$, and $E_2 = \{y\}$. The hospital's choice rule is given as follows:

$$C(\emptyset) = \emptyset,\ C(\{x\}) = \{x\},\ C(\{y\}) = \emptyset,\ C(\{x, y\}) = \{x, y\}. \tag{4.1}$$

The key point is that the hospital is willing to hire a nurse y in period 2 only if it can hire a doctor x in period 1. Note that $C(\{x, y\}) = \{x, y\}$ and $C(\{y\}) = \emptyset$ exhibit a violation of substitutability. Meanwhile, these choices are allowed in period-wise substitutability because x and y are contracts in different periods. □

Remark 6: Existing studies [2, 4, 8] offer detailed accounts of real-life examples of choice rules that violate substitutability but satisfy the weaker variant of substitutability. Huang [8] introduces a condition called *unidirectional substitutes and complements*. Dur et al. [4] introduce a condition called *partitionability*. As noted by [2], each of these conditions is equivalent to the conjunction of period-wise substitutability and future invariance defined below. □

Definition 4.6 (**Future Invariance** [2]): *A choice rule* $C : 2^E \to 2^E$ *satisfies* future invariance *if for every* $X \in 2^E$ *and* $x \in X \cap E_2$,

$$C(X) \cap E_1 = C(X - x) \cap E_1.$$

This condition states that period-2 contracts do not affect the choice of period-1 contracts.

If every institution has a choice rule that satisfies irrelevance of rejected contracts, period-wise substitutability, and future invariance, then stable matching always exists (see [2, Proposition 2] for details). In the context of matching with contracts over two periods, stable matching is defined in the same manner as in the static matching market. A subset of contracts $E' \subseteq E$ is stable if (i) no individual or institution can be better off on their own, and (ii) there is no blocking coalition, i.e., no group of individuals and institutions can be better off by signing alternative contracts from E. Here, being "better off" is defined in terms of individuals' preferences over contracts (which are not considered in the current paper) and institutions' choice rules. We note that members of a blocking coalition can choose contracts from either period 1 or period 2 (or both). Therefore, the distinction between periods 1 and 2 has no bearing on the definition of stability; it matters only for defining the properties of choice rules.

Our main theorem states that a choice rule satisfies the three conditions if and only if it is rationalized by a tandem-concave function.

Theorem 4.7: *A choice rule* $C : 2^E \to 2^E$ *satisfies irrelevance of rejected contracts, period-wise substitutability, and future invariance if and only if there exists a tandem-concave function* $u : 2^E \to \mathbb{R}$ *that rationalizes* C.

(Proof) *The if part*: Suppose that there exists a tandem-concave function $u : 2^E \to \mathbb{R}$ that rationalizes C.

It is clear that if a choice rule is rationalized by some function, then C satisfies *irrelevance of rejected contracts*.

We prove that C satisfies *Future Invariance*. Consider $X \in 2^E$ and $x \in X \cap E_2$. Our goal is to prove that

$$C(X) \cap E_1 = C(X - x) \cap E_1.$$

Since $C(X - x) \subseteq X - x \subseteq X$ and u rationalizes C, we have

$$u(C(X)) \geq u(C(X - x)),$$

with equality holding only if $C(X) = C(X - x)$. Combining this inequality with the contraposition of Condition (b) of tandem concavity for $X \leftarrow C(X - x)$ and $X' \leftarrow C(X)$, we obtain

$$u(C(X) \cap E_1) \geq u(C(X - x) \cap E_1). \tag{4.2}$$

If (4.2) holds with strict inequality, then by Condition (b) of tandem concavity for $X \leftarrow C(X) \cap E_1$ and $X' \leftarrow C(X - x)$, we have $u(C(X) \cap E_1) > u(C(X - x))$. Since $C(X) \cap E_1 \subseteq X - x$ (which follows from $x \in E_2$), we obtain a contradiction to the fact that $C(X - x)$ maximizes u

among all subsets of $X - x$. Therefore, (4.2) holds with equality. Combining this equation with the fact that $C(X \cap E_1)$ maximizes u among all subsets of $X \cap E_1$, we have

$$u(C(X \cap E_1)) \geq u(C(X) \cap E_1) = u(C(X - x) \cap E_1). \tag{4.3}$$

If (4.3) holds with strict inequality, we have

$$u(C(X \cap E_1)) > u(C(X) \cap E_1).$$

Since $C(X \cap E_1) = C(X \cap E_1) \cap E_1$, we have

$$u(C(X \cap E_1) \cap E_1) > u(C(X) \cap E_1).$$

By Condition (b) of tandem concavity for $X \leftarrow C(X \cap E_1)$ and $X' \leftarrow C(X)$, we have $u(C(X \cap E_1)) > u(C(X))$, a contradiction to the definition of $C(X)$. Therefore, (4.3) holds with equality. This means that all the three subsets $C(X \cap E_1)$, $C(X) \cap E_1$, and $C(X - x) \cap E_1$ attain the maximum of u among all subsets of $X \cap E_1$. By the definition of rationalization, there is a unique maximizer. Hence, all the three subsets are the same subset. In particular, $C(X) \cap E_1 = C(X - x) \cap E_1$, as desired.

We prove that C satisfies *period-wise substitutability*. Suppose, to the contrary, that the condition fails, i.e., there exist $X \in 2^E$, $i \in \{1, 2\}$, and $x, y \in X \cap E_i$ with $x \neq y$ such that $x \in C(X)$ and $x \notin C(X - y)$. We divide the remaining part into two parts and derive a contradiction in each case.

Case 1: Suppose $i = 1$. Consider $C(X), C(X - y) \in 2^E$ and $x \in C(X) \setminus C(X - y)$. Since $x \in E_1$, by Condition (a1) of tandem concavity, there exists $x' \in (C(X - y) \setminus C(X)) \cup \{\emptyset\}$ such that

(i)$_1$ $u(C(X)) < u(C(X) - x + x')$, or

(ii)$_1$ $u(C(X - y)) < u(C(X - y) - x' + x)$, or

(iii)$_1$ $u(C(X)) = u(C(X) - x + x')$ and $u(C(X - y)) = u(C(X - y) - x' + x)$.

If (i)$_1$ or the former equality of (iii)$_1$ holds, then we obtain a contradiction to the fact that $C(X)$ uniquely maximizes u among all subsets of X. Similarly, if (ii)$_1$ or the latter equality of (iii)$_1$ holds, then we obtain a contradiction to the fact that $C(X - y)$ uniquely maximizes u among all subsets of $X - y$.

Case 2: Suppose $i = 2$. Since we have already proved that C satisfies Future Invariance, $y \in E_2$ implies $C(X) \cap E_1 = C(X - y) \cap E_1$. Consider $C(X), C(X - y) \in 2^E$ and $x \in C(X) \setminus C(X - y)$. By Condition (a2) of tandem concavity, there exists $x' \in (C(X - y) \setminus C(X)) \cup \{\emptyset\}$ such that

(i)$_2$ $u(C(X)) < u(C(X) - x + x')$, or

(ii)$_2$ $u(C(X - y)) < u(C(X - y) - x' + x)$, or

(iii)$_2$ $u(C(X)) = u(C(X) - x + x')$ and $u(C(X - y)) = u(C(X - y) - x' + x)$.

If (i)$_2$ or the former equality of (iii)$_2$ holds, then we obtain a contradiction to the fact that $C(X)$ uniquely maximizes u among all subsets of X. Similarly, if (ii)$_2$ or the latter equality of (iii)$_2$ holds, then we obtain a contradiction to the fact that $C(X-y)$ uniquely maximizes u among all subsets of $X-y$.

This completes the proof of the if part.

The only-if part: Consider a choice rule $C : 2^E \to 2^E$ that satisfies irrelevance of rejected contracts, period-wise substitutability, and future invariance.

First, define $C_1 : 2^E \to 2^E$ by

$$C_1(X) = C(X) \cap E_1 \quad (\forall X \in 2^E).$$

We show that C_1 satisfies *irrelevance of rejected contracts* and *substitutability*.

- *Proof of C_1 satisfying irrelevance of rejected contracts*: Consider $X \in 2^E$ and $x \in X$ such that $x \notin C_1(X)$. By $x \notin C_1(X) = C(X) \cap E_1$, we have $x \notin C(X)$ or $x \notin E_1$. If $x \notin C(X)$, then by the irrelevance of rejected contracts of C, we have $C(X) = C(X-x)$. This leads to

$$C(X) \cap E_1 = C(X-x) \cap E_1.$$

  If $x \notin E_1$, then $x \in E_2$. By the future invariance of C, the above equation holds. Therefore, in either case, the above equation holds. Since the left-hand side is equal to $C_1(X)$ and the right-hand side is equal to $C_1(X-x)$, the desired condition follows.

- *Proof of C_1 satisfying substitutability*: Consider $X \in 2^E$ and $x, y \in X$ with $x \neq y$. Suppose $x \in C_1(X) = C(X) \cap E_1$. If $y \in E_1$, then the period-wise substitutability of C implies $x \in C(X-y)$. Hence, $x \in C(X-y) \cap E_1 = C_1(X-y)$, as desired. If $y \in E_2$, then

$$C_1(X) = C(X) \cap E_1 = C(X-y) \cap E_1 = C_1(X-y),$$

  where the second equality follows from the future invariance of C. By $x \in C_1(X)$, we have $x \in C_1(X-y)$, as desired.

Therefore, C_1 satisfies irrelevance of rejected contracts and substitutability. By Theorem 4.4, there exists a function $u_1 : 2^E \to \mathbb{R}$ that rationalizes C_1. The proof of the theorem by [16] shows that u_1 can be constructed so that $u_1(\emptyset) = 0$.

Next, for any $X_1 \subseteq E_1$, we define $C_2^{(X_1)} : 2^{E_2} \to 2^{E_2}$ by

$$C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2 \quad (\forall X_2 \in 2^{E_2}).$$

We show that $C_2^{(X_1)}$ satisfies *irrelevance of rejected contracts* and *substitutability*.

- *Proof of $C_2^{(X_1)}$ satisfying irrelevance of rejected contracts*: Consider $X_2 \in 2^{E_2}$ and $x \in X_2 \setminus C_2^{(X_1)}(X_2)$. Since $x \notin C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2$ and $x \in X_2 \subseteq E_2$, we have

$x \notin C(X_2 \cup X_1)$. By the irrelevance of rejected contracts of C, we have $C(X_2 \cup X_1) = C((X_2 \cup X_1) - x) = C((X_2 - x) \cup X_1)$. This leads to

$$C(X_2 \cup X_1) \cap E_2 = C((X_2 - x) \cup X_1) \cap E_2.$$

Since the left-hand side is equal to $C_2^{(X_1)}(X_2)$ and the right-hand side is equal to $C_2^{(X_1)}(X_2 - x)$, the desired condition follows.

- *Proof of $C_2^{(X_1)}$ satisfying substitutability*: Consider $X_2 \in 2^{E_2}$ and $x, y \in X_2$ with $x \neq y$ such that $x \in C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2$. Since $x, y \in X_2 \subseteq E_2$, the period-wise substitutability of C implies $x \in C((X_2 \cup X_1) - y) = C((X_2 - y) \cup X_1)$. Hence, $x \in C((X_2 - y) \cup X_1) \cap E_2 = C_2^{(X_1)}(X_2 - y)$, as desired.

Therefore, for any $X_1 \subseteq E_1$, $C_2^{(X_1)}$ satisfies irrelevance of rejected contracts and substitutability. By Theorem 4.4, there exists a function $u_2^{(X_1)} : 2^{E_2} \to \mathbb{R}$ that rationalizes $C_2^{(X_1)}$. The proof of the theorem by [16] shows that $u_2^{(X_1)}$ can be constructed so that $u_2^{(X_1)}(\emptyset) = 0$.

Now, for a sufficiently large $K > 0$ define $u : 2^E \to \mathbb{R}_{\geq 0}$ by

$$u(X) = K u_1(X \cap E_1) + u_2^{(X \cap E_1)}(X \cap E_2) \quad (\forall X \in 2^E).$$

An appropriate value of K will be given below. We show that u satisfies tandem concavity.

- *Proof of u satisfying Condition* (b): Consider $X, X' \in 2^E$ with $u(X \cap E_1) > u(X' \cap E_1)$, equivalently, $u_1(X \cap E_1) > u_1(X' \cap E_1)$. Then we have

$$\begin{aligned} & u(X) - u(X') \\ & = \big\{K u_1(X \cap E_1) + u_2^{(X \cap E_1)}(X \cap E_2)\big\} - \big\{K u_1(X' \cap E_1) + u_2^{(X' \cap E_1)}(X' \cap E_2)\big\} \\ & = K\big\{u_1(X \cap E_1) - u_1(X' \cap E_1)\big\} + \big\{u_2^{(X \cap E_1)}(X \cap E_2) - u_2^{(X' \cap E_1)}(X' \cap E_2)\big\} \\ & > 0, \end{aligned}$$

  where we choose $K > 0$ in such a way that the above inequality holds for all $X, X' \in 2^E$ with $u_1(X \cap E_1) > u_1(X' \cap E_1)$. Therefore, $u(X) > u(X')$ holds.

- *Proof of u satisfying Condition* (a1): Consider $X, X' \in 2^E$ and $x \in (X \setminus X') \cap E_1$. By the ordinal concavity of u_1, for $x \in (X \cap E_1) \setminus (X' \cap E_1)$, there exists $x' \in \big((X' \cap E_1) \setminus (X \cap E_1)\big) \cup \{\emptyset\}$ such that

  (i)$_1$ $u_1(X \cap E_1) < u_1\big((X \cap E_1) - x + x'\big)$, or

  (ii)$_1$ $u_1(X' \cap E_1) < u_1\big((X' \cap E_1) - x' + x\big)$, or

  (iii)$_1$ $u_1(X \cap E_1) = u_1\big((X \cap E_1) - x + x'\big)$ and $u_1(X' \cap E_1) = u_1\big((X' \cap E_1) - x' + x\big)$.

As stated in the latter part of Theorem 4.4, x' is chosen so that (i)$_1$ or (ii)$_1$ holds (i.e., the possibility of (iii)$_1$ can be eliminated). If (i)$_1$ holds,

$$\begin{aligned} u(X \cap E_1) &= Ku_1(X \cap E_1) \\ &< Ku_1\big((X \cap E_1) - x + x'\big) \\ &= Ku_1\big((X - x + x') \cap E_1\big) \\ &= u\big((X - x + x') \cap E_1\big). \end{aligned}$$

As we have already proved that u satisfies Condition (b), the above inequality implies $u(X) < u(X - x + x')$. Therefore, Condition (a1)(i) holds. Similarly, if (ii)$_1$ holds, then Condition (a1)(ii) holds.

- *Proof of u satisfying Condition* (a2): Consider $X, X' \in 2^E$ with $X \cap E_1 = X' \cap E_1$ and $x \in X \setminus X'$. By the ordinal concavity of $u_2^{(X \cap E_1)}$, for $x \in (X \cap E_2) \setminus (X' \cap E_2)$, there exists $x' \in \big((X' \cap E_2) \setminus (X \cap E_2)\big) \cup \{\emptyset\}$ such that

  (i)$_2$ $u_2^{(X\cap E_1)}(X \cap E_2) < u_2^{(X\cap E_1)}\big((X \cap E_2) - x + x'\big)$, or

  (ii)$_2$ $u_2^{(X\cap E_1)}(X' \cap E_2) < u_2^{(X\cap E_1)}\big((X' \cap E_2) - x' + x\big)$, or

  (iii)$_2$ $u_2^{(X\cap E_1)}(X \cap E_2) = u_2^{(X\cap E_1)}\big((X \cap E_2) - x + x'\big)$ and
  $u_2^{(X\cap E_1)}(X' \cap E_2) = u_2^{(X\cap E_1)}\big((X' \cap E_2) - x' + x\big)$.

  As stated in the latter part of Theorem 4.4, x' is chosen so that (i)$_2$ or (ii)$_2$ holds (i.e., the possibility of (iii)$_2$ can be eliminated). If (i)$_2$ holds,

$$u_2^{(X\cap E_1)}(X \cap E_2) < u_2^{(X\cap E_1)}\big((X \cap E_2) - x + x'\big) = u_2^{X(X\cap E_1)}\big((X - x + x') \cap E_2\big). \tag{4.4}$$

  This leads to

$$\begin{aligned} u(X) &= Ku_1(X \cap E_1) + u_2^{(X\cap E_1)}(X \cap E_2) \\ &< Ku_1(X \cap E_1) + u_2^{(X\cap E_1)}\big((X - x + x') \cap E_2\big) \\ &= Ku_1\big((X - x + x') \cap E_1\big) + u_2^{((X-x+x')\cap E_1)}\big((X - x + x') \cap E_2\big) \\ &= u(X - x + x'), \end{aligned}$$

  where the first inequality follows from (4.4) and the second equality follows from $x \in E_2$ and $x' \in E_2 \cup \{\emptyset\}$. Therefore, Condition (a2)(i) holds. If (ii)$_2$ holds, then by $X \cap E_1 = X' \cap E_1$, the same argument as above establishes that Condition (a2)(ii) holds.

Finally, we show that u rationalizes C. Consider $X \in 2^E$ and $X' \subseteq X$ with $X' \neq C(X)$. Our goal is to prove

$$u(C(X)) > u(X'). \tag{4.5}$$

Suppose $C(X) \cap E_1 \neq X' \cap E_1$. Then, we have $C_1(X) = C(X) \cap E_1 \neq X' \cap E_1$. Since $X' \cap E_1 \subseteq X$ and u_1 rationalizes C_1, we have

$$u_1(C_1(X)) = u_1(C(X) \cap E_1) > u_1(X' \cap E_1).$$

This inequality together with the definition of u implies

$$u(C(X) \cap E_1) = Ku_1(C(X) \cap E_1) > Ku_1(X' \cap E_1) = u(X' \cap E_1).$$

Since we have already proved that u satisfies Condition (b) of tandem concavity, we have (4.5), as desired.

Suppose $C(X) \cap E_1 = X' \cap E_1$. Then,

$$\begin{aligned}
C_2^{(C(X)\cap E_1)}(X \cap E_2) &= C\Big((X \cap E_2) \cup (C(X) \cap E_1)\Big) \cap E_2 \\
&= C\Big(\big((X \cap E_2) \cup (C(X) \cap E_1) \cup \big((X \cap E_1) \setminus C(X)\big)\Big) \cap E_2 \\
&= C\Big((X \cap E_2) \cup (X \cap E_1)\Big) \cap E_2 \\
&= C(X) \cap E_2,
\end{aligned} \tag{4.6}$$

where the second equality follows from the irrelevance of rejected contracts of C (i.e., adding rejected contracts $(X \cap E_1) \setminus C(X)$ does not change the outcome of the choice rule). Note that $C(X) \cap E_2 \neq X' \cap E_2$ since $C(X) \cap E_1 = X' \cap E_1$, $X' \subseteq X$, and $X' \neq C(X)$. Therefore, it follows from (4.6) together with the fact that $u_2^{(C(X)\cap E_1)}$ rationalizes $C_2^{(C(X)\cap E_1)}$ that we obtain

$$u_2^{(C(X)\cap E_1)}\big(C_2^{(C(X)\cap E_1)}(X \cap E_2)\big) = u_2^{(C(X)\cap E_1)}(C(X) \cap E_2) > u_2^{(C(X)\cap E_1)}(X' \cap E_2). \tag{4.7}$$

This leads to

$$\begin{aligned}
u(C(X)) &= Ku_1(C(X) \cap E_1) + u_2^{(C(X)\cap E_1)}(C(X) \cap E_2) \\
&> Ku_1(X' \cap E_1) + u_2^{(X'\cap E_1)}(X' \cap E_2) \\
&= u(X'),
\end{aligned}$$

where the inequality follows from $C(X) \cap E_1 = X' \cap E_1$ and (4.7). Therefore, (4.5) holds. □

Theorem 4.7 states that, if every agent chooses contracts in such a way to maximize a tandem-concave (utility) function, then their choice rules satisfy the stated conditions. We introduce a simple example of rationalization by tandem-concave functions.

Example 2: Consider the same choice rule as in Example 1. Let $E = \{x, y\}$, $E_1 = \{x\}$, $E_2 = \{y\}$, and the hospital's choice rule is given by (4.1). Recall that this choice rule violates substitutability, whereas it satisfies period-wise substitutability. One easily verifies that it also satisfies irrelevance of rejected contracts and future invariance. By Theorem 4.7, there exists a function u that rationalizes the choice rule, such as the following one:

$$u(\emptyset) = 0,\ u(\{x\}) = 1,\ u(\{y\}) = -1,\ u(\{x, y\}) = 2.$$

Note that u violates ordinal concavity because, for $\{x, y\}$, $\emptyset$, and $y \in \{x, y\} \setminus \emptyset$, we have

$$u(\{x, y\}) > u(\{x\}) \text{ and } u(\emptyset) > u(\{y\}).$$

Meanwhile, u satisfies tandem concavity. Here we demonstrate how the above violation of ordinal concavity is circumvented under tandem concavity. Since $y \notin E_1$, condition (a1) of Definition 3.1 is irrelevant. Since $\{x, y\} \cap E_1 = \{x\} \neq \emptyset = \emptyset \cap E_1$, condition (a2) is irrelevant. Therefore, nothing is required for $\{x, y\}$, $\emptyset$, and $y \in \{x, y\} \setminus \emptyset$, and hence conditions (a1) and (a2) of tandem concavity are satisfied for $\{x, y\}$, $\emptyset$, and $y \in \{x, y\} \setminus \emptyset$. □

Remark 7: We clarify our contribution in relation to recent papers partly coauthored by the present authors. Fujishige et al. [6] study various properties of ordinal concavity and its weaker variant, *ordinal weak-concavity*, including their implications for rationalized choice rules. Hafalir et al. [7] apply ordinal concavity to the analysis of choice rules with distributional objectives. Yokote et al. [16] investigate the rationalization of substitutable choice rules via ordinally concave functions. While sharing the same interest in the connection between discrete concavity and choice rules, none of these papers examine properties of choice rules that accommodate complementarity. The notion of tandem concavity is newly introduced in the current paper. □

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