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# Tandem concavity with application to matching

# problems

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## Tandem concavity with application to matching problems

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#### Abstract

As a generalization of ordinal concavity we introduce a new notion of discrete concavity called *tandem concavity* defined for a function over the subsets of a finite set E endowed with an ordered partition  $(E_1, E_2)$  of E. Every function expressed as a lexicographic composition of two ordinally concave functions satisfies tandem concavity. We apply tandem concavity to the rationalization of choice rules in stable matching problems. Moreover, we show that tandem concavity rationalizes a wider class of choice rules in matching markets than ordinal concavity.

Keywords: Discrete convexity, ordinal concavity, tandem concavity, stable matchings

## 1. Introduction

Consider a finite set E and a function  $u : 2^E \to \mathbb{R}$ . There are several notions of discrete concavity for this function, such as  $M^{\natural}$ -concavity [11] and semi-strict quasi  $M^{\natural}$ -concavity [5], also known as ordinal concavity [7, 16]. In some economic applications, E is endowed with an ordered partition  $(E_1, E_2)$  and the function u has a property that depends on the partition. For example, in a jobmatching context, a firm separates the set of workers E into two types, skilled workers  $E_1$  and unskilled workers  $E_2$ , and its profit function u depends on the composition of the two types of workers (see [4, 8]). Similarly, in dynamic matching problems, agents separate the whole set of contracts E into the set  $E_1$  of contracts signed in period 1 and the set  $E_2$  signed in period 2 (see [2]).

In the present note, we investigate concavity of a function defined over the subsets of a partitioned set. For analytical simplicity, we consider a partition consisting of two parts, but it is straightforward to generalize our results to the case with multiple parts (see Remark 4 in Section 3.1). We introduce a new notion called *tandem concavity* as a generalization of ordinal concavity. For two functions  $u_1, u_2 : 2^E \to \mathbb{R}$  that satisfy ordinal concavity, we apply an operation called *lexicpgraphic composition* of the two functions and show that the resulting function satisfies tandem concavity under the unique maximizer condition (to be precisely defined in Section 3.2).

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We also apply the new notion to stable matching problems. Recent studies [2, 4, 8] show that a stable matching exists if every agent's choice rule satisfies a certain form of substitutability between contracts in a partitioned set. We show that a choice rule satisfies the substitutability if and only if it is *rationalized* by a tandem-concave function, i.e., the outcome of the choice rule is supported as the maximizers of a tandem-concave function. This result implies that, if every agent in a matching market has a tandem-concave utility function, then a stable matching exists.

The present note is organized as follows. Section 2 introduces some notation used here and describes a definition of ordinal concavity. In Section 3 we define tandem concavity and examine its relationship to ordinal concavity via lexicographic composition. Section 4 applies tandem concavity to the problem of rationalizing choice rules in stable matching problems. Section 5 gives concluding remarks.

## 2. Preliminaries

We introduce notation following [6]. Note in particular that  $\emptyset$  denotes the empty set as usual while it also means a symbol that does not belong to the underlying set E. For any  $X \in 2^E$  let  $X + x = X \cup \{x\}$  for all  $x \in E \setminus X$  and  $X - x = X \setminus \{x\}$  for all  $x \in X$ . Also for  $x = \emptyset$  let  $X \pm x = X$ .

Ordinal concavity is defined as follows.

**Definition 2.1 (Ordinal Concavity):** A function  $u : 2^E \to \mathbb{R}$  satisfies ordinal concavity if for every  $X, X' \in 2^E$  the following statement holds:

For every  $x \in X \setminus X'$  there exists  $x' \in (X' \setminus X) \cup \{\emptyset\}$  such that

(i) u(X) < u(X - x + x'), or

(ii) 
$$u(X') < u(X' - x' + x)$$
, or

(iii) u(X) = u(X - x + x') and u(X') = u(X' - x' + x).

**Remark 1**: Ordinal concavity was originally called *semi-strict quasi*  $M^{\natural}$ -*concavity* in [5] (also see [12, 13] and [3]). The name of ordinal concavity was used by [7, 16] in the economics literature, which we adopt in the present note.

**Remark 2**: Ikebe and Tamura [9] introduce discrete concavity called *twisted*  $M^{\ddagger}$ -concavity defined for functions over the subsets of a partitioned set (also see [15]). This notion captures concavity of utility functions in markets with money transfers. Meanwhile, our tandem concavity to be precisely defined below captures concavity of utility functions in markets *without* money transfers (see Section 4).

## 3. Tandem concavity

Throughout this section, we fix an ordered partition  $(E_1, E_2)$  of E.

#### 3.1. Definition of tandem concavity

We focus on the following new notion of discrete concavity.

**Definition 3.1 (Tandem Concavity):** A function  $u : 2^E \to \mathbb{R}$  satisfies tandem concavity with respect to  $(E_1, E_2)$  if the following three statements hold:

- (a1) For every  $X, X' \in 2^E$  and every  $x \in (X \setminus X') \cap E_1$  there exists  $x' \in (X' \setminus X) \cup \{\emptyset\}$  such that
  - (i) u(X) < u(X x + x'), or
  - (ii) u(X') < u(X' x' + x), or
  - (iii) u(X) = u(X x + x') and u(X') = u(X' x' + x).
- (a2) For every  $X, X' \in 2^E$  with  $X \cap E_1 = X' \cap E_1$  and every  $x \in X \setminus X'$  there exists  $x' \in (X' \setminus X) \cup \{\emptyset\}$  such that
  - (i) u(X) < u(X x + x'), or
  - (ii) u(X') < u(X' x' + x), or
  - (iii) u(X) = u(X x + x') and u(X') = u(X' x' + x).
- (b) For every  $X, X' \in 2^{E}$ , If  $u(X \cap E_{1}) > u(X' \cap E_{1})$ , then u(X) > u(X').

**Remark 3**: Here we allow that  $E_1$  (resp.  $E_2$ ) is the empty set. If so, then (a1) (resp. (a2)) becomes null, while (a2) (resp. (a1)) becomes equivalent to the condition for ordinal concavity, and (b) always holds. Hence in this case tandem concavity means ordinal concavity.

**Remark 4**: Consider  $X, X' \in 2^E$ . Condition (a1) requires the same condition as ordinal concavity for  $x \in X \setminus X'$  with  $x \in E_1$ . Condition (a2) requires the same condition for  $x \in E_2$  while assuming that X and X' have the same intersection with  $E_1$ . Condition (b) intuitively states that elements from  $E_1$  are more important for achieving a higher function value than those from  $E_2$ .

The above interpretation leads us to the following generalization of (a1) and (a2) for an ordered partition  $(E_1, \ldots, E_m)$  with  $m \ge 2$ : we require ordinal concavity for  $X, X' \in 2^E$  and  $x \in X \setminus X'$  with  $x \in E_i$  while assuming that X and X' have the same intersection with  $E_k$  for all k < i, where  $i \in \{1, \ldots, m\}$ . A similar generalization can be made for (b) as well.

A formal definition of *generalized tandem concavity* is given as follows.

**Definition 3.2 (Generalized Tandem Concavity):** A function  $u : 2^E \to \mathbb{R}$  satisfies generalized tandem concavity with respect to  $(E_1, \ldots, E_m)$  if the following two statements hold (with  $E_0$  being the empty set):

(a)<sub>g</sub> For every  $i \in \{1, ..., m\}$ , every  $X, X' \in 2^E$  with  $X \cap E_k = X' \cap E_k$  ( $\forall k \in \{0, ..., i-1\}$ ), and every  $x \in (X \setminus X') \cap E_i$  there exists  $x' \in (X' \setminus X) \cup \{\emptyset\}$  such that

(i)<sub>g</sub> 
$$u(X) < u(X - x + x')$$
, or  
(ii)<sub>g</sub>  $u(X') < u(X' - x' + x)$ , or  
(iii)<sub>g</sub>  $u(X) = u(X - x + x')$  and  $u(X') = u(X' - x' + x)$ .

(b)<sub>g</sub> For every  $X, X' \in 2^E$ , if there exists  $i \in \{1, \ldots, m\}$  such that

$$u\left(X\cap\left(\cup_{k=1}^{i}E_{k}\right)\right)>u\left(X'\cap\left(\cup_{k=1}^{i}E_{k}\right)\right),$$

*then*, u(X) > u(X').

If m = 2, then Condition  $(a)_g$  for i = 1 (resp. i = 2) is equivalent to Condition (a1) (resp. (a2)) in Definition 3.1, and Condition  $(b)_g$  is equivalent to Condition (b) in Definition 3.1. The results in this note can be restated in obvious ways in terms of an ordered partition  $(E_1, \ldots, E_m)$  of E.

#### 3.2. Lexicographic composition

We introduce an operation called *lexicographic composition*, which clarifies the connection between ordinal concavity and tandem concavity. For any nonempty  $X \subseteq E$  define  $u_X : 2^X \to \mathbb{R}$ by

$$u_X(Z) = u(Z) \qquad (\forall Z \in 2^X).$$

We call  $u_X$  the *restriction* of u on X. Also for any  $X \subset E$  define  $u^X : 2^{E \setminus X} \to \mathbb{R}$  by

$$u^X(Y) = u(Y \cup X) - u(X) \quad (\forall Y \subseteq E \setminus X).$$

We call  $u^X$  the *fixing* of u by X.

We say that  $u: 2^E \to \mathbb{R}$  satisfies the *unique-maximizer condition* (UM) if the following condition holds:

(UM) For every  $X \in 2^E$  there uniquely exists a maximizer of  $\max\{u(Y) \mid Y \subseteq X\}$ .

Let us consider the lexicographical order  $\leq_{\ell}$  on  $\mathbb{R}^2$  defined by  $(a, b) <_{\ell} (c, d) \iff$  (i) a < c or (ii) a = c and b < d, for all  $a, b, c, d \in \mathbb{R}$ . Let  $(\mathbb{R}^2)_{\ell}$  be the set  $\mathbb{R}^2$  endowed with the lexicographical order  $\leq_{\ell}$ .

Let  $u_1: 2^E \to \mathbb{R}$  be an ordinally concave function with the unique maximizer condition (UM) and  $u_2: 2^E \to \mathbb{R}$  be an ordinally concave function. Now we define the *lexicographic composition*  $u_1 \bullet u_2$  of  $u_1$  and  $u_2$  as follows. For any  $X \in 2^E$  define  $X^*$  to be the (unique) maximizer of the restriction of  $u_1$  on  $E_1$ . Moreover, let  $Y_{X^*}^*$  be a maximizer of the restriction, of the fixing  $u_2^{X^*}$  by  $X^*$ , to  $X \cap E_2$ . Here note that  $Y_{X^*}^*$  may not be unique but the value of  $u_2(Y_{X^*}^* \cup X^*)$  is, with respect to the given X. Then define  $(u_1 \bullet u_2): 2^E \to (\mathbb{R}^2)_\ell$  by

$$(u_1 \bullet u_2)(X) = (u_1(X^*), u_2(Y^*_{X^*} \cup X^*)) \quad (X \in 2^E).$$

For any  $X \in 2^E$  we write  $\hat{u}_1(X)$  and  $\hat{u}_2(X)$  to denote  $u_1(X^*)$  and  $u_2(Y^*_{X^*} \cup X^*)$ , respectively.

To interpret the lexicographic composition, consider an agent who maximizes two utility functions  $u_1$  and  $u_2$  in stages: given  $X \in 2^E$ , she first maximizes  $u_1$  among the subsets of  $X \cap E_1$ , with the maximizer denoted  $X^*$ , and then maximizes  $u_2$  among the subsets of  $X \cap E_2$  while fixing the choice  $X^*$  from  $X \cap E_1$ . The function value  $(u_1 \bullet u_2)(X)$  represents her maximized utilities (called *indirect utilities* in economics) for the first and second stages.

**Remark 5**: Fujishige et al. [6] defined an operation of lexicographic composition for two functions in a different way.  $\Box$ 

We say that  $(u_1 \bullet u_2) : 2^E \to (\mathbb{R}^2)_\ell$  satisfies tandem concavity if it satisfies conditions (a1), (a2) and (b) in Definition 3.1 with < replaced by  $<_\ell$ .

**Proposition 3.3**: If  $u_1 : 2^E \to \mathbb{R}$  is an ordinally concave function that satisfies the uniquemaximizer condition (**UM**) and  $u_2 : 2^E \to \mathbb{R}$  is ordinally concave, then the lexicographic composition  $u_1 \bullet u_2$  is tandem concave.

(Proof) We first prove that  $u_1 \bullet u_2$  satisfies (a1). Consider  $X, X' \in 2^E$  and  $x \in (X \setminus X') \cap E_1$ . We consider two cases.

**Case 1:** Suppose  $x \notin X^*$ . Then, because of the definitions of  $X^*$  and  $\hat{u}_1$  we have

$$\hat{u}_1(X) = \hat{u}_1(X - x),$$
(3.1)

$$\hat{u}_1(X') \le \hat{u}_1(X'+x).$$
 (3.2)

If the weak inequality of (3.2) holds with strict inequality, then (a1)(ii) holds for  $x' = \emptyset$ . Suppose that the weak inequality of (3.2) holds with equality. By the unique-maximizer condition (**UM**) of  $u_1$ , we have  $(X')^* = (X' + x)^*$ . By this equation and  $x \in E_1$ , we have  $Y^*_{(X')^*} = Y^*_{(X'+x)^*}$ , which implies  $\hat{u}_2(X') = \hat{u}_2(X' + x)$ . Similarly, by (3.1) and (**UM**), we have  $X^* = (X - x)^*$  and  $Y^*_{X^*} = Y^*_{(X-x)^*}$ , which implies  $\hat{u}_2(X) = \hat{u}_2(X - x)$ . Therefore, (a1)(iii) holds for  $x' = \emptyset$ .

**Case 2:** Suppose  $x \in X^*$ . For  $X^*$ ,  $(X')^*$  and  $x \in X^* \setminus (X')^*$ , the ordinal concavity of  $u_1$  implies that there exists  $\hat{x}' \in ((X')^* \setminus X^*) \cup \{\emptyset\}$  such that

(i)<sub>1</sub>  $u_1(X^*) < u_1(X^* - x + \hat{x}')$ , or

(ii)<sub>1</sub> 
$$u_1((X')^*) < u_1((X')^* - \hat{x}' + x)$$
, or

(iii)<sub>1</sub> 
$$u_1(X^*) = u_1(X^* - x + \hat{x}')$$
 and  $u_1((X')^*) = u_1((X')^* - \hat{x}' + x)$ .

**Case 2(i)**: Suppose  $\hat{x}' \notin X$ . If (i)<sub>1</sub> holds, then (a1)(i) holds for  $x' = \hat{x}'$ . If (ii)<sub>1</sub> holds, then (a1)(ii) holds for  $x' = \hat{x}'$ . Suppose that (iii)<sub>1</sub> holds. By the latter equality,  $(X')^*$  and  $(X')^* - \hat{x}' + x$  attain the same value of  $u_1$ . Since these two subsets are included in  $(X'+x) \cap E_1$ , the unique-maximizer condition (**UM**) of  $u_1$  implies that there is a subset of  $(X'+x) \cap E_1$  that attains a higher value of  $u_1$  than the two subsets, i.e.,  $\hat{u}_1(X'+x) > \hat{u}_1(X')$ . Therefore, (a1)(ii) holds for  $x' = \emptyset$ .

**Case 2(ii)**: Suppose  $\hat{x}' \in X$ . If (i)<sub>1</sub> holds, then we obtain a contradiction to the definition of  $X^*$  that maximizes  $u_1$  among all subsets of  $X \cap E_1$ . Similarly, if (iii)<sub>1</sub> holds, then the former equality of (iii)<sub>1</sub> implies that  $X^*$  and  $X^* - x + x'$  attain the maximum of  $u_1$  among all subsets of  $X \cap E_1$ , contradicting the unique-maximizer condition (**UM**) of  $u_1$ . The remaining possibility is that (ii)<sub>1</sub> holds. Since  $(X')^* - \hat{x}' + x \subseteq X' + x$ , (a1)(ii) holds for  $x' = \emptyset$ .

Next, we prove that  $u_1 \bullet u_2$  satisfies (a2). Consider  $X, X' \in 2^E$  with  $X \cap E_1 = X' \cap E_1$  and  $x \in X \setminus X'$ . Note that  $x \in E_2$ . Since  $X \cap E_1 = X' \cap E_1$  and because of the unique-maximizer condition (UM) of  $u_1$ , we have  $X^* = (X')^*$ . We choose  $Y^*_{X^*}$  and  $Y^*_{(X')^*}$  in the following way: first, choose  $Y^*_{(X')^*}$  as an arbitrary maximizer of the restriction of  $u_2^{(X')^*}$  to  $X' \cap E_2$ . Then, choose  $Y^*_{X^*}$  as a maximizer of the restriction of  $u_2^{X^*}$  to  $X \cap E_2$  in such a way that the following (\*) holds:

(\*)  $Y_{X^*}^*$  attains the minimum of  $|Y_{X^*}^* \Delta Y_{(X')^*}^*|$ .

We consider two cases.

**Case 1':** Suppose  $x \notin Y_{X^*}^*$ . Then,

$$\hat{u}_2(X) = \hat{u}_2(X - x),$$
(3.3)

$$\hat{u}_2(X') \le \hat{u}_2(X'+x).$$
 (3.4)

If the weak inequality of (3.4) holds with strict inequality, then by  $(X')^* = (X' + x)^*$  and  $\hat{u}_1(X') = \hat{u}_1(X' + x)$  (which follows from  $x \in E_2$ ), (a2)(ii) holds for  $x' = \emptyset$ . Similarly, if the weak inequality of (3.4) holds with equality, then together with (3.3), it implies that (a2)(iii) holds for  $x' = \emptyset$ .

**Case 2':** Suppose  $x \in Y_{X^*}^*$ . Then, For  $Y_{X^*}^* \cup X^*$ ,  $Y_{(X')^*}^* \cup (X')^*$  and  $x \in (Y_{X^*}^* \cup X^*) \setminus (Y_{(X')^*}^* \cup (X')^*)$ , the ordinal concavity of  $u_2$  implies that there exists  $\tilde{x}' \in ((Y_{(X')^*}^* \cup (X')^*) \setminus (Y_{X^*}^* \cup X^*)) \cup \{\emptyset\}$  such that

(i)<sub>2</sub> 
$$u_2(Y_{X^*}^* \cup X^*) < u_2((Y_{X^*}^* - x + \tilde{x}') \cup X^*)$$
, or

(ii)<sub>2</sub> 
$$u_2(Y^*_{(X')^*} \cup (X')^*) < u_2((Y^*_{(X')^*} - \tilde{x}' + x) \cup (X')^*)$$
, or

(iii)<sub>2</sub>  $u_2(Y^*_{X^*} \cup X^*) = u_2((Y^*_{X^*} - x + \tilde{x}') \cup X^*)$  and  $u_2(Y^*_{(X')^*} \cup (X')^*) = u_2((Y^*_{(X')^*} - \tilde{x}' + x) \cup (X')^*).$ 

**Case 2'(i)**: Suppose  $\tilde{x}' \notin X$ . If (i)<sub>2</sub> holds, then by  $X^* = (X - x + \tilde{x}')^*$  and  $\hat{u}_1(X) = \hat{u}_1(X - x + \tilde{x}')$  (which follows from  $x, \tilde{x}' \in E_2$ ), (a2)(i) holds for  $x' = \tilde{x}'$ . Similarly, if (ii)<sub>2</sub> holds, then (a2)(ii) holds for  $x' = \tilde{x}'$ , and if (iii)<sub>2</sub> holds, then (a2)(iii) holds for  $x' = \tilde{x}'$ .

**Case 2'(ii)**: Suppose  $\tilde{x}' \in X$ . If (i)<sub>2</sub> holds, then we obtain a contradiction to  $Y_{X^*}^*$  maximizing the restriction of  $u_2^{X^*}$  to  $X \cap E_2$ . If (iii)<sub>2</sub> holds, then the former equality of (iii)<sub>2</sub> implies that  $Y_{X^*}^*$  and  $Y_{X^*}^* - x + \tilde{x}'$  attain the maximum of the restriction of  $u_2^{X^*}$  to  $X \cap E_2$ . Since  $Y_{X^*}^* - x + \tilde{x}' \subseteq X \cap E_2$ , we obtain a contradiction to the choice of  $Y_{X^*}^*$  (recall (\*)). The remaining possibility is that (ii)<sub>2</sub> holds. Note that  $Y_{(X')^*}^* - \tilde{x}' + x \subseteq (X' + x) \cap E_2$ . Together with  $(X')^* = (X' - \tilde{x}' + x)^*$  and  $\hat{u}_1(X') = \hat{u}_1(X' - \tilde{x}' + x)$  (which follows from  $x, \tilde{x}' \in E_2$ ), (ii)<sub>2</sub> implies that (a2)(ii) holds for  $x' = \emptyset$ .

Finally, we prove that  $u_1 \bullet u_2$  satisfies (b). Consider  $X, X' \in 2^E$ . Suppose that

$$(u_1 \bullet u_2)(X \cap E_1) > (u_1 \bullet u_2)(X' \cap E_1).$$

Since  $(X \cap E_1) \cap E_2 = (X' \cap E_1) \cap E_2 = \emptyset$ , we have  $\hat{u}_2(X \cap E_1) = \hat{u}_2(X' \cap E_1)$ . Therefore, the above displayed inequality holds only if  $\hat{u}_1(X \cap E_1) > \hat{u}_1(X' \cap E_1)$ , which implies  $\hat{u}_1(X) > \hat{u}_1(X')$ . Hence,  $(u_1 \bullet u_2)(X) > (u_1 \bullet u_2)(X')$  holds.

## 4. Application to stable matching problems

Consider a two-sided matching problem between agents and institutions, e.g., workers and firms or students and schools. Fix an institution in the market and let E denote the set of contracts that the institution chooses from. For example, in a job-matching market, E is the set of job candidates for the firm. The institution's *choice rule* is a function  $C : 2^E \to 2^E$  such that  $C(X) \subseteq X$  for all  $X \in 2^E$ . Given  $X \in 2^E$ , C(X) specifies the set of contracts that the institution chooses from X.

The following is a standard condition of a choice rule.

**Definition 4.1 (Irrelevance of Rejected Contracts** [1]): A choice rule  $C : 2^E \to 2^E$  satisfies irrelevance of rejected alternatives contracts if for every  $X \in 2^E$  and  $x \in X$ , it holds that

$$x \notin C(X) \Longrightarrow C(X-x) = C(X).$$

Since Kelso and Crawford [10] (also see [14]) it has long been recognized that *substitutability* of a choice rule is essential for the existence of a stable matching.

**Definition 4.2 (Substitutability):** A choice rule  $C : 2^E \to 2^E$  satisfies substitutability if for every  $X \in 2^E$  and  $x, y \in X$  with  $x \neq y$ , it holds that

$$x \in C(X) \implies x \in C(X-y).$$

If the above condition fails, then there exist  $X \in 2^E$  and distinct  $x, y \in X$  such that  $x \in C(X)$  and  $x \notin C(X-y)$ . This means that x is chosen if it is coupled with y but not if y is absent, exhibiting complementarity between x and y. Substitutability rules out this type of complementarity.

We introduce an additional concept to see the connection between substitutability and discrete concavity.

**Definition 4.3 (Rationalization):** A function  $u : 2^E \to \mathbb{R}$  rationalizes a choice rule  $C : 2^E \to 2^E$ if for every  $X \in 2^E$ , it holds that

$$u(C(X)) > u(X') \qquad (\forall X' \subseteq X, \, X' \neq C(X)).$$

If u rationalizes C, then given  $X \in 2^E$ , C chooses the unique maximizer of u among all subsets of X. Rationalization is a fundamental step in the economic analysis to convert choice behaviors into utility-maximization problems.

**Theorem 4.4** ([16, Theorem 1']): A choice rule  $C : 2^E \to 2^E$  satisfies irrelevance of rejected contracts and substitutability if and only if there exists an ordinally concave function  $u : 2^E \to \mathbb{R}$ that rationalizes C. Moreover, in the only-if part, u can be constructed so that, for every  $X, X' \in 2^E$  and  $x \in X \setminus X'$ , there exists  $x' \in (X' \setminus X) \cup \{\emptyset\}$  that satisfies Condition (i) or (ii) of ordinal concavity (i.e., the possibility of Condition (iii) can be eliminated).

Institutions in the real world often have choice rules that violate substitutability due to complementarity between contracts. Recent studies [2, 4, 8] found weaker variants of substitutability that accommodate a specific form of complementarity and also guarantee the existence of a stable matching. We investigate the implication of weakening substitutability for the rationalizing function.

Following Bando and Kawasaki [2], we divide the set E of contracts into the set  $E_1$  of period-1 contracts and the set  $E_2$  of period-2 contracts.

**Definition 4.5 (Period-wise Substitutability** [2]): A choice rule  $C : 2^E \to 2^E$  satisfies periodwise substitutability if for every  $X \in 2^E$ ,  $i \in \{1, 2\}$ , and  $x, y \in X \cap E_i$  with  $x \neq y$ ,

$$x \in C(X) \implies x \in C(X-y).$$

This condition requires substitutability between contracts in the same period. Therefore, it allows complementarity between contracts in different periods. For example, suppose that a firm considers hiring worker  $x \in E_2$  in period 2 only if it hires worker  $y \in E_1$  in period 1. Then,  $C(\{x, y\}) = \{x, y\}$  and  $C(\{x\}) = \emptyset$ . For  $X = \{x, y\}$ , substitutability is violated, while periodwise substitutability is satisfied because x and y are contracts in different periods.

**Remark 6**: Existing studies [2, 4, 8] offer detailed accounts of real-life examples of choice rules that violate substitutability but satisfy the weaker variant of substitutability. The above example of a firm hiring workers is taken from Huang [8], who introduces a condition called *unidirectional substitutes and complements*. Dur et al. [4] introduce a condition called *partitionability*. As noted by [2], each of these conditions is equivalent to the conjunction of period-wise substitutability and future invariance defined below.

**Definition 4.6 (Future Invariance** [2]): A choice rule  $C : 2^E \to 2^E$  satisfies future invariance if for every  $X \in 2^E$  and  $x \in X \cap E_2$ ,

$$C(X) \cap E_1 = C(X - x) \cap E_1.$$

This condition states that period-2 contracts do not affect the choice of period-1 contracts.

Our main theorem states that a choice rule satisfies the above conditions if and only if it is rationalized by a tandem-concave function.

**Theorem 4.7**: A choice rule  $C : 2^E \to 2^E$  satisfies irrelevance of rejected contracts, periodwise substitutability, and future invariance if and only if there exists a tandem-concave function  $u : 2^E \to \mathbb{R}$  that rationalizes C.

(Proof) The *if part*: Suppose that there exists a tandem-concave function  $u: 2^E \to \mathbb{R}$  that rationalizes C.

It is clear that if a choice rule is rationalized by some function, then C satisfies *irrelevance of rejected contracts*.

We prove that C satisfies *Future Invariance*. Consider  $X \in 2^E$  and  $x \in X \cap E_2$ . Our goal is to prove that

$$C(X) \cap E_1 = C(X - x) \cap E_1.$$

Since  $C(X - x) \subseteq X - x \subseteq X$  and u rationalizes C, we have

$$u(C(X)) \ge u(C(X-x)),$$

with equality holding only if C(X) = C(X - x). Combining this inequality with the contraposition of Condition (b) of tandem concavity for  $X \leftarrow C(X - x)$  and  $X' \leftarrow C(X)$ , we obtain

$$u(C(X) \cap E_1) \ge u(C(X - x) \cap E_1).$$
 (4.1)

If (4.1) holds with strict inequality, then by Condition (b) of tandem concavity for  $X \leftarrow C(X) \cap E_1$ and  $X' \leftarrow C(X - x)$ , we have  $u(C(X) \cap E_1) > u(C(X - x))$ . Since  $C(X) \cap E_1 \subseteq X - x$ (which follows from  $x \in E_2$ ), we obtain a contradiction to the fact that C(X - x) maximizes uamong all subsets of X - x. Therefore, (4.1) holds with equality. Combining this equation with the fact that  $C(X \cap E_1)$  maximizes u among all subsets of  $X \cap E_1$ , we have

$$u(C(X \cap E_1)) \ge u(C(X) \cap E_1) = u(C(X - x) \cap E_1).$$
(4.2)

If (4.2) holds with strict inequality, we have

$$u(C(X \cap E_1)) > u(C(X) \cap E_1).$$

Since  $C(X \cap E_1) = C(X \cap E_1) \cap E_1$ , we have

$$u(C(X \cap E_1) \cap E_1) > u(C(X) \cap E_1).$$

By Condition (b) of tandem concavity for  $X \leftarrow C(X \cap E_1)$  and  $X' \leftarrow C(X)$ , we have  $u(C(X \cap E_1)) > u(C(X))$ , a contradiction to the definition of C(X). Therefore, (4.2) holds with equality. This means that all the three subsets  $C(X \cap E_1)$ ,  $C(X) \cap E_1$ , and  $C(X - x) \cap E_1$  attain the maximum of u among all subsets of  $X \cap E_1$ . By the definition of rationalization, there is a unique maximizer. Hence, all the three subsets are the same subset. In particular,  $C(X) \cap E_1 = C(X - x) \cap E_1$ , as desired.

We prove that C satisfies *period-wise substitutability*. Suppose, to the contrary, that the condition fails, i.e., there exist  $X \in 2^E$ ,  $i \in \{1, 2\}$ , and  $x, y \in X \cap E_i$  with  $x \neq y$  such that  $x \in C(X)$  and  $x \notin C(X-y)$ . We divide the remaining part into two parts and derive a contradiction in each case.

**Case 1:** Suppose i = 1. Consider  $C(X), C(X - y) \in 2^E$  and  $x \in C(X) \setminus C(X - y)$ . Since  $x \in E_1$ , by Condition (a1) of tandem concavity, there exists  $x' \in (C(X - y) \setminus C(X)) \cup \{\emptyset\}$  such that

(i)<sub>1</sub> 
$$u(C(X)) < u(C(X) - x + x')$$
, or

(ii)<sub>1</sub> 
$$u(C(X - y)) < u(C(X - y) - x' + x)$$
, or

(iii)<sub>1</sub> u(C(X)) = u(C(X) - x + x') and u(C(X - y)) = u(C(X - y) - x' + x).

If (i)<sub>1</sub> or the former equality of (iii)<sub>1</sub> holds, then we obtain a contradiction to the fact that C(X) uniquely maximizes u among all subsets of X. Similarly, if (ii)<sub>1</sub> or the latter equality of (iii)<sub>1</sub> holds, then we obtain a contradiction to the fact that C(X - y) uniquely maximizes u among all subsets of X - y.

**Case 2:** Suppose i = 2. Since we have already proved that C satisfies Future Invariance,  $y \in E_2$ implies  $C(X) \cap E_1 = C(X-y) \cap E_1$ . Consider  $C(X), C(X-y) \in 2^E$  and  $x \in C(X) \setminus C(X-y)$ . By Condition (a2) of tandem concavity, there exists  $x' \in (C(X-y) \setminus C(X)) \cup \{\emptyset\}$  such that

(i)<sub>2</sub> 
$$u(C(X)) < u(C(X) - x + x')$$
, or

(ii)<sub>2</sub> 
$$u(C(X - y)) < u(C(X - y) - x' + x)$$
, or

 $\text{(iii)}_2 \ u(C(X)) = u(C(X) - x + x') \ \text{and} \ u(C(X - y)) = u(C(X - y) - x' + x).$ 

If (i)<sub>2</sub> or the former equality of (iii)<sub>2</sub> holds, then we obtain a contradiction to the fact that C(X) uniquely maximizes u among all subsets of X. Similarly, if (ii)<sub>2</sub> or the latter equality of (iii)<sub>2</sub> holds, then we obtain a contradiction to the fact that C(X - y) uniquely maximizes u among all subsets of X - y.

This completes the proof of the if part.

The only-if part: Consider a choice rule  $C : 2^E \rightarrow 2^E$  that satisfies irrelevance of rejected contracts, period-wise substitutability, and future invariance.

First, define  $C_1: 2^E \to 2^E$  by

$$C_1(X) = C(X) \cap E_1 \quad (\forall X \in 2^E).$$

We show that  $C_1$  satisfies *irrelevance of rejected contracts* and *substitutability*.

Proof of C<sub>1</sub> satisfying irrelevance of rejected contracts: Consider X ∈ 2<sup>E</sup> and x ∈ X such that x ∉ C<sub>1</sub>(X). By x ∉ C<sub>1</sub>(X) = C(X) ∩ E<sub>1</sub>, we have x ∉ C(X) or x ∉ E<sub>1</sub>. If x ∉ C(X), then by the irrelevance of rejected contracts of C, we have C(X) = C(X - x). This leads to

$$C(X) \cap E_1 = C(X - x) \cap E_1.$$

If  $x \notin E_1$ , then  $x \in E_2$ . By the future invariance of C, the above equation holds. Therefore, in either case, the above equation holds. Since the left-hand side is equal to  $C_1(X)$  and the right-hand side is equal to  $C_1(X - x)$ , the desired condition follows.

• Proof of  $C_1$  satisfying substitutability: Consider  $X \in 2^E$  and  $x, y \in X$  with  $x \neq y$ . Suppose  $x \in C_1(X) = C(X) \cap E_1$ . If  $y \in E_1$ , then the period-wise substitutability of C implies  $x \in C(X - y)$ . Hence,  $x \in C(X - y) \cap E_1 = C_1(X - y)$ , as desired. If  $y \in E_2$ , then

$$C_1(X) = C(X) \cap E_1 = C(X - y) \cap E_1 = C_1(X - y),$$

where the second equality follows from the future invariance of C. By  $x \in C_1(X)$ , we have  $x \in C_1(X - y)$ , as desired.

Therefore,  $C_1$  satisfies irrelevance of rejected contracts and substitutability. By Theorem 4.4, there exists a function  $u_1: 2^E \to \mathbb{R}$  that rationalizes  $C_1$ . The proof of the theorem by [16] shows that  $u_1$  can be constructed so that  $u_1(\emptyset) = 0$ .

Next, for any  $X_1 \subseteq E_1$ , we define  $C_2^{(X_1)}: 2^{E_2} \to 2^{E_2}$  by

$$C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2 \quad (\forall X_2 \in 2^{E_2}).$$

We show that  $C_2^{(X_1)}$  satisfies *irrelevance of rejected contracts* and *substitutability*.

• Proof of  $C_2^{(X_1)}$  satisfying irrelevance of rejected contracts: Consider  $X_2 \in 2^{E_2}$  and  $x \in$  $X_2 \setminus C_2^{(X_1)}(X_2)$ . Since  $x \notin C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2$  and  $x \in X_2 \subseteq E_2$ , we have  $x \notin C(X_2 \cup X_1)$ . By the irrelevance of rejected contracts of C, we have  $C(X_2 \cup X_1) =$  $C((X_2 \cup X_1) - x) = C((X_2 - x) \cup X_1)$ . This leads to

$$C(X_2 \cup X_1) \cap E_2 = C((X_2 - x) \cup X_1) \cap E_2.$$

Since the left-hand side is equal to  $C_2^{(X_1)}(X_2)$  and the right-hand side is equal to  $C_2^{(X_1)}(X_2 - C_2)$ x), the desired condition follows.

• Proof of  $C_2^{(X_1)}$  satisfying substitutability: Consider  $X_2 \in 2^{E_2}$  and  $x, y \in X_2$  with  $x \neq y$  such that  $x \in C_2^{(X_1)}(X_2) = C(X_2 \cup X_1) \cap E_2$ . Since  $x, y \in X_2 \subseteq E_2$ , the periodwise substitutability of C implies  $x \in C((X_2 \cup X_1) - y) = C((X_2 - y) \cup X_1)$ . Hence,  $x \in C((X_2 - y) \cup X_1) \cap E_2 = C_2^{(X_1)}(X_2 - y)$ , as desired.

Therefore, for any  $X_1 \subseteq E_1$ ,  $C_2^{(X_1)}$  satisfies irrelevance of rejected contracts and substitutability. By Theorem 4.4, there exists a function  $u_2^{(X_1)}: 2^{E_2} \to \mathbb{R}$  that rationalizes  $C_2^{(X_1)}$ . The proof of the theorem by [16] shows that  $u_2^{(X_1)}$  can be constructed so that  $u_2^{(X_1)}(\emptyset) = 0$ . Now, for a sufficiently large K > 0 define  $u: 2^E \to \mathbb{R}_{\geq 0}$  by

$$u(X) = Ku_1(X \cap E_1) + u_2^{(X \cap E_1)}(X \cap E_2) \quad (\forall X \in 2^E).$$

An appropriate value of K will be given below. We show that u satisfies tandem concavity.

• Proof of u satisfying Condition (b): Consider  $X, X' \in 2^E$  with  $u(X \cap E_1) > u(X' \cap E_1)$ , equivalently,  $u_1(X \cap E_1) > u_1(X' \cap E_1)$ . Then we have

$$u(X) - u(X') = \{Ku_1(X \cap E_1) + u_2^{(X \cap E_1)}(X \cap E_2)\} - \{Ku_1(X' \cap E_1) + u_2^{(X' \cap E_1)}(X' \cap E_2)\} = K\{u_1(X \cap E_1) - u_1(X' \cap E_1)\} + \{u_2^{(X \cap E_1)}(X \cap E_2) - u_2^{(X' \cap E_1)}(X' \cap E_2)\} > 0,$$

where we choose K > 0 in such a way that the above inequality holds for all  $X, X' \in 2^E$ with  $u_1(X \cap E_1) > u_1(X' \cap E_1)$ . Therefore, u(X) > u(X') holds.

• Proof of u satisfying Condition (a1): Consider  $X, X' \in 2^E$  and  $x \in (X \setminus X') \cap E_1$ . By the ordinal concavity of  $u_1$ , for  $x \in (X \cap E_1) \setminus (X' \cap E_1)$ , there exists  $x' \in ((X' \cap E_1) \setminus (X \cap E_1)) \cup \{\emptyset\}$  such that

(i)<sub>1</sub> 
$$u_1(X \cap E_1) < u_1((X \cap E_1) - x + x')$$
, or  
(ii)<sub>1</sub>  $u_1(X' \cap E_1) < u_1((X' \cap E_1) - x' + x)$ , or  
(iii)<sub>1</sub>  $u_1(X \cap E_1) = u_1((X \cap E_1) - x + x')$  and  $u_1(X' \cap E_1) = u_1((X' \cap E_1) - x' + x)$ .

As stated in the latter part of Theorem 4.4, x' is chosen so that (i)<sub>1</sub> or (ii)<sub>1</sub> holds (i.e., the possibility of (iii)<sub>1</sub> can be eliminated). If (i)<sub>1</sub> holds,

$$u(X \cap E_1) = Ku_1(X \cap E_1) < Ku_1((X \cap E_1) - x + x') = Ku_1((X - x + x') \cap E_1) = u((X - x + x') \cap E_1).$$

As we have already proved that u satisfies Condition (b), the above inequality implies u(X) < u(X - x + x'). Therefore, Condition (a1)(i) holds. Similarly, if (ii)<sub>1</sub> holds, then Condition (a1)(ii) holds.

• Proof of u satisfying Condition (a2): Consider  $X, X' \in 2^E$  with  $X \cap E_1 = X' \cap E_1$  and  $x \in X \setminus X'$ . By the ordinal concavity of  $u_2^{(X \cap E_1)}$ , for  $x \in (X \cap E_2) \setminus (X' \cap E_2)$ , there exists  $x' \in ((X' \cap E_2) \setminus (X \cap E_2)) \cup \{\emptyset\}$  such that

(i)<sub>2</sub> 
$$u_2^{(X\cap E_1)}(X\cap E_2) < u_2^{(X\cap E_1)}((X\cap E_2) - x + x')$$
, or  
(ii)<sub>2</sub>  $u_2^{(X\cap E_1)}(X'\cap E_2) < u_2^{(X\cap E_1)}((X'\cap E_2) - x' + x)$ , or  
(iii)<sub>2</sub>  $u_2^{(X\cap E_1)}(X\cap E_2) = u_2^{(X\cap E_1)}((X\cap E_2) - x + x')$  and  
 $u_2^{(X\cap E_1)}(X'\cap E_2) = u_2^{(X\cap E_1)}((X'\cap E_2) - x' + x)$ .

As stated in the latter part of Theorem 4.4, x' is chosen so that (i)<sub>2</sub> or (ii)<sub>2</sub> holds (i.e., the possibility of (iii)<sub>2</sub> can be eliminated). If (i)<sub>2</sub> holds,

$$u_2^{(X \cap E_1)}(X \cap E_2) < u_2^{(X \cap E_1)}((X \cap E_2) - x + x') = u_2^{X(X \cap E_1)}((X - x + x') \cap E_2).$$
(4.3)

This leads to

$$u(X) = Ku_1(X \cap E_1) + u_2^{(X \cap E_1)}(X \cap E_2)$$
  

$$< Ku_1(X \cap E_1) + u_2^{(X \cap E_1)}((X - x + x') \cap E_2)$$
  

$$= Ku_1((X - x + x') \cap E_1) + u_2^{((X - x + x') \cap E_1)}((X - x + x') \cap E_2)$$
  

$$= u(X - x + x'),$$

where the first inequality follows from (4.3) and the second equality follows from  $x \in E_2$ and  $x' \in E_2 \cup \{\emptyset\}$ . Therefore, Condition (a2)(i) holds. If (ii)<sub>2</sub> holds, then by  $X \cap E_1 = X' \cap E_1$ , the same argument as above establishes that Condition (a2)(ii) holds. Finally, we show that u rationalizes C. Consider  $X \in 2^E$  and  $X' \subseteq X$  with  $X' \neq C(X)$ . Our goal is to prove

$$u(C(X)) > u(X').$$
 (4.4)

Suppose  $C(X) \cap E_1 \neq X' \cap E_1$ . Then, we have  $C_1(X) = C(X) \cap E_1 \neq X' \cap E_1$ . Since  $X' \cap E_1 \subseteq X$  and  $u_1$  rationalizes  $C_1$ , we have

$$u_1(C_1(X)) = u_1(C(X) \cap E_1) > u_1(X' \cap E_1).$$

This inequality together with the definition of u implies

$$u(C(X) \cap E_1) = Ku_1(C(X) \cap E_1) > Ku_1(X' \cap E_1) = u(X' \cap E_1).$$

Since we have already proved that u satisfies Condition (b) of tandem concavity, we have (4.4), as desired.

Suppose  $C(X) \cap E_1 = X' \cap E_1$ . Then,

$$C_2^{(C(X)\cap E_1)}(X\cap E_2) = C\left((X\cap E_2)\cup (C(X)\cap E_1)\right)\cap E_2$$
  
=  $C\left(\left((X\cap E_2)\cup (C(X)\cap E_1)\cup ((X\cap E_1)\setminus C(X))\right)\cap E_2$   
=  $C\left((X\cap E_2)\cup (X\cap E_1)\right)\cap E_2$   
=  $C(X)\cap E_2,$  (4.5)

where the second equality follows from the irrelevance of rejected contracts of C (i.e., adding rejected contracts  $(X \cap E_1) \setminus C(X)$  does not change the outcome of the choice rule). Note that  $C(X) \cap E_2 \neq X' \cap E_2$  since  $C(X) \cap E_1 = X' \cap E_1, X' \subseteq X$ , and  $X' \neq C(X)$ . Therefore, it follows from (4.5) together with the fact that  $u_2^{(C(X)\cap E_1)}$  rationalizes  $C_2^{(C(X)\cap E_1)}$  that we obtain

$$u_{2}^{(C(X)\cap E_{1})}\left(C_{2}^{(C(X)\cap E_{1})}(X\cap E_{2})\right) = u_{2}^{(C(X)\cap E_{1})}(C(X)\cap E_{2}) > u_{2}^{(C(X)\cap E_{1})}(X'\cap E_{2}).$$
(4.6)

This leads to

$$u(C(X)) = Ku_1(C(X) \cap E_1) + u_2^{(C(X) \cap E_1)}(C(X) \cap E_2)$$
  
>  $Ku_1(X' \cap E_1) + u_2^{(X' \cap E_1)}(X' \cap E_2)$   
=  $u(X'),$ 

where the inequality follows from  $C(X) \cap E_1 = X' \cap E_1$  and (4.6). Therefore, (4.4) holds.  $\Box$ 

Theorem 4.7 states that, if every agent chooses contracts in such a way to maximize a tandemconcave (utility) function, then their choice rules satisfy the stated conditions. As proven by [2] (also see [4, 8]), if the agents' choice rules satisfy the stated conditions, then a stable matching exists. Therefore, tandem concavity is a meaningful assumption in stable matching problems.

## 5. Concluding Remarks

We have investigated a new concavity, called tandem concavity, for a function defined over the subsets of a partitioned set. The key idea of tandem concavity is to require ordinal concavity within each part of an ordered partition sequentially. Although mathematically simple, this generalization is meaningful in economic applications. It should be emphasized that we have shown that tandem concavity rationalizes a wider class of choice rules in matching markets than ordinal concavity.

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