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Choice or Competition: Does Integration Benefit Everyone?

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CHOICE OR COMPETITION: DOES INTEGRATION BENEFIT EVERYONE?

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Abstract. Matching markets are often fragmented, organized at a small local level. While integration of matching markets may lead to welfare gains by expanding choice, it may also harm some market participants by increasing competition for the same resources. We show that every "good" mechanism fails the monotonicity requirement that no individuals be hurt by integration. Then we provide characterization results that identify conditions under which monotonicity becomes compatible with other desirable properties of matching mechanisms.

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1. Introduction

Allocation of resources such as daycare slots, school seats and vaccines are often conducted at small local levels. For example, in Tokyo, daycare slots and elementary school seats are allocated within each of the 23 small districts that partitions the city.[1] Major cities in China such as Tianjin and Shanghai have an admission system for kindergartens where the cities are divided into small districts and a child in a given district can only be assigned to a school in the district. During the Covid-19 pandemic, Japanese government adopted the policy to first distribute vaccines to each municipality, such as each of the 23 small districts in Tokyo, which was then responsible for distributing the allotted vaccine to their residents. In the assignment of children into foster homes in Los Angeles County, CA, the assignment is conducted at an inefficiently fragmented level of regional offices (Robinson-Cortes, 2019).[2] Facing such *fragmentation* of the markets, one could hope for a welfare gain by the *integration* of the regions. What are the barriers against integration?

Integration entails two opposing effects for the individuals seeking for resources. On the one hand, it increases the *choice* because the resources in the integrated region become available. On the other hand, it increases the *competition* because the resources that were originally exclusive to the individuals in a given region become available to more individuals. The objective of this paper is to understand this tradeoff and characterize when the first effect dominates; that is, individuals become better off by integration.

For this purpose, we consider a two-sided matching model, where we refer to agents in the two sides as students and schools.[3] A "region structure" partitions the set of students and schools, and we examine how a change in the region structure affects student welfare. In particular, we ask if a mechanism in consideration is *monotone*, meaning that integration always weakly improves student welfare. Our first theorem (Theorem 1) shows that every "good" mechanism lacks monotonicity: No mechanism that is strategy-proof, Pareto efficient and individually rational is monotone. This result demonstrates that a policy-maker designing a mechanism has to admit competition to sometimes override choice if they wish to maintain strategy-proofness, Pareto efficiency and individual rationality. Or, they have to abandon at least one of these three properties to retain monotonicity. Given this impossibility, we then consider mechanisms that are Pareto efficient and individually

[1] There are some exceptions to this rule that allow for interdistrict transfers under limited scenarios, but such transfers are rarely implemented.

[2] See also Slaugh et al. (2016), who are, to our knowledge, the first to apply tools from matching theory to the problem of child adoption.

[3] See Gale and Shapley (1962), Roth (1984), Roth and Peranson (1999), and Abdulkadiroğlu and Sönmez (2003), among many others, for seminal work in two-sided matching markets.

rational (while possibly being non-strategy-proof). We show that there exist monotone mechanisms that satisfy those properties if and only if the set of allowed region structures has a type of a hierarchical structure (Theorem 2). This result demonstrates that there is a limit to monotonicity even when the requirement for strategy-proofness is lifted and completely characterizes such a limit.

Intuitively, the effect of competition is present when schools in a given region prefer the students in other regions than the students in its own region. We investigate the validity of this intuition by considering the celebrated deferred acceptance (DA) mechanism. We show that the DA mechanism (applied to each region) is monotone if and only if the school preferences favor local students (Proposition 1). However, we also prove that such a characterization does not hold for other well-known mechanisms such as the top trading cycles (TTC) mechanism (Shapley and Scarf, 1974), showing that they are not monotone even if school preferences favor locals. Our analysis demonstrates that integration improves welfare for every student in some practical scenarios under the DA mechanism while the same cannot be said for other well-known mechanisms. Hence, integration may face less public opposition under the DA mechanism than under those mechanisms.

This paper belongs to the literature in matching with constraints. Research in this literature include Abdulkadiroğlu (2005), Ergin and Sönmez (2006), Abraham, Irving and Manlove (2007), Biro et al. (2010), Hafalir, Yenmez and Yildirim (2013), Westkamp (2013), Goto et al. (2014), Kamada and Kojima (2015, 2017, 2018, 2023*b*), Kojima, Tamura and Yokoo (2018), Aygün and Turhan (2020) and Pathak et al. (2021). The main departure of the present paper is that we consider integration of multiple markets, while those earlier contributions treat the relevant market as given.

We note that Kamada and Kojima (2023*a*) also consider integration of multiple regions in a matching problem between students and schools and provide an approach complementary to the present paper. Specifically, their paper studies "partial integration" of regions, in the sense that the produced matching must satisfy the balancedness constraint: for each region, the total number of residents of other regions matched to schools in it must be equal to the total number of its residents matched to a school outside of the region.[4] That is, the paper takes as given the constraint that "full integration" of multiple regions is infeasible. The present paper, in contrast, studies whether, and to what extent, a full integration of regions is desirable. Their paper and ours are complementary in this sense.

[4]Hafalir, Kojima and Yenmez (2022) introduced a balancedness constraint in the context of interdistrict school choice. A balancedness constraint across individual institutions was introduced by Dur and Ünver (2019).

Benefit of integration as well as its possible cost has been a central issue in international economics for at least two centuries. Ricardo (1821) famously argued that opening up countries for international trade will benefit all countries through specialization and access to goods from abroad, broadening choice. Stolper and Samuelson (1941) offered a model in which, although trade improves overall welfare of a country, some sectors may be made worse off through competition. Our paper can be thought of as identifying and analyzing analogous forces of choice and competition in the context of matching problems.

2. Model

2.1. **Preliminary Definitions.** Let there be a finite set of students I and a finite set of schools S. Each student i has a strict preference relation $\succ_i$ over the set of schools and being unmatched (being unmatched is denoted by $\emptyset$). For any $s, s' \in S \cup \{\emptyset\}$, we write $s \succeq_i s'$ if and only if $s \succ_i s'$ or $s = s'$.

Each school $s \in S$ is endowed with a strict preference relation $\succ_s$ over the set of subsets of students (we use $\emptyset$ to denote the empty set with a slight abuse of notation). For any $I', I'' \subseteq I$, we write $I' \succeq_s I''$ if and only if $I' \succ_s I''$ or $I' = I''$. We denote by $\succ = (\succ_a)_{a \in I \cup S}$ the preference profile of all students and schools. For any $i, i' \in I \cup \{\emptyset\}$, we write $i \succeq_s i'$ if and only if $i \succ_s i'$ or $i = i'$.[5]

For each $s \in S$, fix a positive integer q_s. We assume that preference relation $\succ_s$ is **responsive with capacity** q_s (Roth, 1985), that is,

(1) For any $I' \subseteq I$ with $|I'| \leq q_s$, $i \in I \setminus I'$ and $i' \in I'$, $(I' \cup i) \setminus i' \succeq_s I'$ if and only if $i \succeq_s i'$, and
(2) For any $I' \subseteq I$ with $|I'| \leq q_s$ and $i' \in I'$, $I' \succeq_s I' \setminus i'$ if and only if $i' \succeq_s \emptyset$.
(3) $\emptyset \succ_s I'$ for any $I' \subseteq I$ with $|I'| > q_s$.

In words, we assume that the ranking of a student (or keeping a position vacant) is independent of her peers, and any set of students exceeding its capacity is unacceptable.

Student i is said to be **acceptable** to school s if $i \succ_s \emptyset$ (and unacceptable otherwise). Similarly, s is acceptable to i if $s \succ_i \emptyset$. It will turn out that only rankings of acceptable partners matter for our analysis, so we often write only acceptable partners to denote preferences and priorities. For example,

$$\succ_i: s, s'$$

[5] We denote singleton set $\{x\}$ by x when there is no confusion.

means that school s is the most preferred, s' is the second most preferred, and s and s' are the only acceptable schools under preferences $\succ_i$ of student i. We also use analogous expressions for school preferences.

A **matching** μ is a mapping that satisfies (i) $\mu_i \in S \cup \{\emptyset\}$ for all $i \in I$, (ii) $\mu_s \subseteq I$ for all $s \in S$, and (iii) for any $i \in I$ and $s \in S$, $\mu_i = s$ if and only if $i \in \mu_s$. That is, a matching simply specifies which student is assigned to which school (if any).

A matching is **individually rational** if $\mu_a \succeq_a \emptyset$ for every $a \in I \cup S$.

2.2. **Regions.** Fix a **base of regions**, which is a partition R^0 of $I \cup S$. A region structure R is a partition of $I \cup S$ such that each $r \in R$ is of the form $r = r^1 \cup \cdots \cup r^k$ with $r^1, \ldots, r^k \in R^0$. An element $r \in R$ is called a region. Note that each s belongs to a single $r \in R$ and each i is a resident of a single $r \in R$. To simplify the exposition of some results, we hereafter assume that $|r \cap I| \geq 2$ and $|r \cap S| \geq 1$ hold for each $r \in R^0$. We denote by $\mathcal{Q}$ a nonempty subset of the set of all region structures.

We call tuple $(I, S, \mathcal{Q})$ an **environment**.

A matching μ is **feasible under** R if, for all $r \in R$ and $i \in r \cap I$, we have $\mu_i \in r \cup \{\emptyset\}$. A matching μ is **Pareto efficient under** R if (i) it is feasible under R and (ii) there exists no other matching μ' that is feasible under R and satisfies $\mu'_a \succeq_a \mu_a$ for every $a \in I \cup S$.

Given a matching μ, a pair $(i, s) \in I \times S$ is called a **blocking pair** if $s \succ_i \mu_i$ and there is $I' \subseteq \mu_s \cup \{i\}$ such that $I' \succ_s \mu_s$. A matching μ is **stable under** R if (i) it is feasible under R, (ii) $s = \mu_i$ implies $s \succeq_i \emptyset$ and $i \in \mu_s$ implies $i \succeq_s \emptyset$, and (iii) it does not have any blocking pair (i, s) such that there exists $r \in R$ with $i, s \in r$. Gale and Shapley (1962) imply that there is a unique stable matching μ^* under R such that for every stable matching μ under R and every $i \in I$, we have $\mu^*_i \succeq \mu_i$. Call it a **student-optimal stable matching** (or, **SOSM**) **under** R.

A **mechanism** φ is a function from the set of preference profile-region structure pairs to the set of feasible matchings. That is, $\varphi(\succ, R)$ is a feasible matching under R.

Mechanism φ is **strategy-proof** if

$$\varphi_i(\succ, R) \succeq_i \varphi_i(\succ'_i, \succ_{-i}, R),$$

for every region structure $R \in \mathcal{Q}$, preference profile $\succ$, $i \in I$, and student preferences $\succ'_i$.[6]

Mechanism φ is **individually rational** if $\varphi(\succ, R)$ is individually rational for all $\succ$ and $R \in \mathcal{Q}$. Similarly, φ is **Pareto efficient** if $\varphi(\succ, R)$ is Pareto efficient under R for all $\succ$ and $R \in \mathcal{Q}$.

[6]We note that the definition requires reporting true preferences be a best reply for students only.

We say that φ is the **deferred acceptance mechanism** (or, the **DA mechanism**) if, for any input $(\succ, R)$, the matching $\varphi(\succ, R)$ is the SOSM under R given preference profile $\succ$.

3. Limits of Monotone Mechanisms

We are now ready to introduce the key concept of this paper, monotonicity.

Definition 1. A mechanism φ is **monotone** if, for all $R, R' \in \mathcal{Q}$, $r \in R$, $r' \in R'$ such that $r \subseteq r'$, $i \in r \cap I$, and $\succ$, we have $\varphi_i(\succ, R') \succeq_i \varphi_i(\succ, R)$.

In words, monotonicity requires that all students be made weakly better off when regions expand. Note that we do not require schools be weakly better off. Our negative results (such as Theorem 1) clearly hold under a stronger requirement that all students and schools be made better off as a result of expansion.

Definition 2. We say that $\mathcal{Q}$ **admits a merger** if there exist $R, R' \in \mathcal{Q}$, distinct $r_1, r_2 \in R$ and $r' \in R'$ such that $r_1 \cup r_2 \subseteq r'$.

We regard admitting a merger as a minimal requirement. The condition is satisfied if, for instance, $\mathcal{Q}$ includes the base of regions or the grand region structure (i.e., the partition consisting of a single cell) and contains at least two region structures.

Theorem 1. *Fix an environment $(I, S, \mathcal{Q})$ such that $\mathcal{Q}$ admits a merger. There exists no monotone mechanism that is strategy-proof, Pareto efficient, and individually rational.*

Proof. Consider a monotone mechanism φ that is Pareto efficient and individually rational. We will show that φ is not strategy-proof.

Because $\mathcal{Q}$ admits a merger, there exist $R, R' \in \mathcal{Q}$ with the following property: there exist distinct $r_1, r_2 \in R$ and $r' \in R'$ such that $r_1 \cup r_2 \subseteq r'$. Fix such (R, R', r_1, r_2, r') arbitrarily.

Let $\{s_1, i_1, i_1'\} \subseteq r_1$ and $\{s_2, i_2\} \subseteq r_2$: Such schools and students exist because regions are constructed from a base of regions. Consider a preference profile such that:

$$\begin{aligned} &\succ_{i_1}: s_2, s_1, \quad &&\succ_{s_1}: i_2, i_1, i_1', \\ &\succ_{i_1'}: s_2, s_1, \quad &&\succ_{s_2}: i_1, i_1', i_2, \\ &\succ_{i_2}: s_1, s_2, && \end{aligned}$$

and the capacities of s_1 and s_2 are both one, while all other schools and students prefer $\emptyset$ the most.

By feasibility and the fact that $i_1, i'_1 \notin r_2$, we have $\varphi_{i_1}(\succ, R) \neq s_2$ and $\varphi_{i'_1}(\succ, R) \neq s_2$. Similarly, we have $\varphi_{i_2}(\succ, R) \neq s_1$. These facts and the Pareto efficiency of φ imply $\varphi_{i_2}(\succ, R) = s_2$ and either $\varphi_{i_1}(\succ, R) = s_1$ or $\varphi_{i'_1}(\succ, R) = s_1$. Assume $\varphi_{i_1}(\succ, R) = s_1$—the proof for the case with $\varphi_{i'_1}(\succ, R) = s_1$ is symmetric.

Consider R'. Because of the monotonicity of φ and $r_1 \cup r_2 \subseteq r'$, it must be that $\varphi_{i_1}(\succ, R') \succeq_{i_1} \varphi_{i_1}(\succ, R) = s_1$ and $\varphi_{i_2}(\succ, R') \succeq_{i_2} \varphi_{i_2}(\succ, R) = s_2$. This and the Pareto efficiency of φ imply $\varphi_{i_1}(\succ, R') = s_2$ and $\varphi_{i_2}(\succ, R') = s_1$.

Now, consider another preference relation $\succ'_{i_1}$ of i_1 such that

$$\succ'_{i_1}: s_2,$$

and let $\succ' := (\succ'_{i_1}, \succ_{-i_1})$. Then, by the individual rationality of φ, we have $\varphi_{i_1}(\succ', R) = \emptyset$. This and Pareto efficiency of φ imply that $\varphi_{i'_1}(\succ', R) = s_1$ and $\varphi_{i_2}(\succ', R) = s_2$.

Now, consider R' again. Because of the monotonicity of φ and $r_1 \cup r_2 \subseteq r'$, it must be that $\varphi_{i'_1}(\succ', R') \succeq_{i'_1} \varphi_{i'_1}(\succ', R) = s_1$ and $\varphi_{i_2}(\succ', R') \succeq_{i_2} \varphi_{i_2}(\succ', R) = s_2$. This and the Pareto efficiency of φ imply $\varphi_{i'_1}(\succ', R') = s_2$ and $\varphi_{i_2}(\succ', R') = s_1$. Therefore, $\varphi_{i_1}(\succ', R') = \emptyset$.

Therefore, $\varphi_{i_1}(\succ, R') = s_2 \succ'_{i_1} \emptyset = \varphi_{i_1}(\succ', R')$, showing that φ is not strategy-proof. □

This result demonstrates that every "good" mechanism lacks monotonicity. Specifically, as long as we require standard desiderata of strategy-proofness, Pareto efficiency, and individual rationality, the mechanism cannot be monotone. This result thus shows a limit to the policymakers aiming to achieve monotonicity.

We note that none of the conditions in Theorem 1 is extraneous: The DA mechanism satisfies all conditions except for monotonicity. A mechanism that always returns an empty matching, i.e., a matching in which every student is unmatched, satisfies all conditions except for Pareto efficiency. A mechanism under which every student is matched to her first choice satisfies all conditions except for individual rationality (for schools). A mechanism that satisfies all conditions except for strategy-proofness is analyzed in the next result. To do so, we begin by introducing a restriction on the region structures.

Definition 3. The region structures $\mathcal{Q}$ is **weakly hierarchical** if there exist no $R, R', R'' \in \mathcal{Q}$ such that there are $r \in R$, $r' \in R'$, and $r'' \in R''$ satisfying $r \cap r' \neq \emptyset$, $r \not\subseteq r'$, $r' \not\subseteq r$, and $r \cup r' \subseteq r''$.

Note that if $\mathcal{Q}$ satisfies the following property that we would call hierarchical, then it is also weakly hierarchical, hence the name. The property is that for all $R, R' \in \mathcal{Q}$, $r \in R$, and $r' \in R'$, we have $r \subseteq r'$, $r' \subseteq r$, or $r \cap r' = \emptyset$. For example, if integration is possible only along an existing government structure, e.g., from districts within a municipality to the entire municipality, or from municipalities within a county to the entire county, then the region structures form a hierarchy, and thus a weak hierarchy. In contrast, suppose that there are three (mutually disjoint) municipalities A, B, and C, and A could be merged only with B or only with C or with both B and C. This case gives rise to region structures that are not weakly hierarchical. We note that, while weakly hierarchical region structures do not necessarily admit a merger or vice versa, any hierarchical region structures with cardinality of at least two admit a merger.

Theorem 2. *Fix an environment $(I, S, \mathcal{Q})$. There is a monotone mechanism that is Pareto efficient and individually rational if and only if $\mathcal{Q}$ is weakly hierarchical.*

Proof. "Only if" direction:

Consider a mechanism φ that is Pareto efficient and individually rational. We will show that φ is not monotone if $\mathcal{Q}$ is not weakly hierarchical.

Suppose that $\mathcal{Q}$ is not weakly hierarchical. Then, there must exist $R, R', R'' \in \mathcal{Q}$, $r \in R$, $r' \in R'$ and $r'' \in R''$ such that $r \setminus r'$, $r' \setminus r$ and $r \cap r'$ are all nonempty and $r \cup r' \subseteq r''$.

Take such (R, R', R'', r, r', r'') and take an arbitrary $s \in r \cap r' \cap S$, $i \in (r \setminus r') \cap I$ and $i' \in (r' \setminus r) \cap I$. Such a school and students exist because regions are constructed from a base of regions. Consider a preference profile such that:

$$\succ_i\colon s, \quad \succ_s\colon i, i',$$
$$\succ_{i'}\colon s,$$

and the capacity of school s is one, while all other schools and students prefer $\emptyset$ the most.

By feasibility, Pareto efficiency and the fact that $i, s \in r$ and $i', s \in r'$, we have $\varphi_i(\succ, R) = s$ and $\varphi_{i'}(\succ, R') = s$. However, since the capacity of s is one, the assumption that φ is individually rational implies that we must have either $\varphi_i(\succ, R'') \neq s$, which implies $\varphi_i(\succ, R'') = \emptyset$, or $\varphi_{i'}(\succ, R'') \neq s$, which implies $\varphi_{i'}(\succ, R'') = \emptyset$. This implies that either i is worse off under R'' compared to under R, or i' is worse off under R'' compared to under R'. Since $r \subseteq r''$ and $r' \subseteq r''$, this implies that φ is not monotone.

"If" direction:

Suppose that $\mathcal{Q}$ is weakly hierarchical. We construct a monotone mechanism φ that is Pareto efficient and individually rational.

For this purpose, let $\mathcal{R} = \bigcup_{R \in \mathcal{Q}} R$ and define a directed graph with the set of nodes being $\mathcal{R}$ and the set of edges being:

$$E = \{rr' | r, r' \in \mathcal{R}, r \subsetneq r' \text{ and } \nexists r'' \in \mathcal{R} \text{ s.t. } r \subsetneq r'' \subsetneq r'\}.$$

For every $r \in \mathcal{R}$, let $c(r)$ be the maximum length of a path in the graph that leads to r. Formally, $c : \mathcal{R} \to \{0\} \cup \mathbb{N}$ is a unique function that satisfies the following: (i) $c(r) = 0$ if there is no $\tilde{r} \in \mathcal{R}$ with $\tilde{r}r \in E$, and (ii) for any $r \in \mathcal{R}$ such that there is at least one $\tilde{r} \in \mathcal{R}$ with $\tilde{r}r \in E$,

$$c(r) = 1 + \max_{\tilde{r} \in \mathcal{R} \text{ s.t. } \tilde{r}r \in E} c(\tilde{r}).$$

Say that a matching μ is feasible for $r \in \mathcal{R}$ if $\mu_i \in r \cup \{\emptyset\}$ for every student $i \in r \cap I$ and $\mu_s \subseteq r$ for every school $s \in r \cap S$.

We define φ inductively as follows. Fix $\succ$.

Step 0: Consider r such that $c(r) = 0$. Take an arbitrary matching, denoted μ^r, that is feasible for r, Pareto efficient for r and individually rational.[7] (Such a matching exists because the set of all feasible and individually rational matchings is nonempty and finite.[8]) For every $a \in r$, we let $\varphi_a(\succ, R) = \mu^r_a$ for every $R \in \mathcal{Q}$ such that $r \in R$.

For any $n \geq 1$ such that there is $r \in \mathcal{R}$ such that $c(r) = n$, we define Step n as follows.

Step n: Consider r such that $c(r) = n$. Let $S(r) = \{\tilde{r} \in \mathcal{R} | \tilde{r}r \in E\}$. Since $\mathcal{Q}$ is weakly hierarchical, any two $\tilde{r}, \hat{r} \in S(r)$ are disjoint.

Consider a matching that is feasible for r, denoted by $\mu^{r,0}$, such that, for each $\tilde{r} \in S(r)$ and each $a \in \tilde{r}$, we set $\mu^{r,0}_a = \varphi_a(\succ, R)$ for some $R \in \mathcal{Q}$ satisfying $\tilde{r} \in R$ (the choice of R does not matter because $\varphi_a(\succ, R) = \varphi_a(\succ, R')$ for any $R, R' \in \mathcal{Q}$ satisfying $\tilde{r} \in R$ and $\tilde{r} \in R'$ from Steps $0, \dots, n-1$). Note that this $\mu^{r,0}_a$ is well defined due to Steps $0, \dots, n-1$ and the fact that any $\tilde{r}, \hat{r} \in S(r)$ are disjoint. Then, take an arbitrary matching, denoted μ^r, that is feasible for r and Pareto efficient for r and satisfies $\mu^r_a \succeq_a \mu^{r,0}_a$ for all $a \in I \cup S$ such that there exists $\tilde{r}$ with $a \in \tilde{r} \in S(r)$.[9] For every $a \in r$, we let $\varphi_a(\succ, R) = \mu^r_a$ for every $R \in \mathcal{Q}$ such that $r \in R$.

The above procedure pins down $\varphi_a(\succ, R)$ for all $a \in I \cup S$ and $R \in \mathcal{Q}$. Note that it follows from the construction that $\varphi(\succ, R)$ is a feasible matching and it is Pareto

[7]Say that a matching is Pareto efficient for r if it is feasible for r and there exists no other matching μ' that is feasible for r such that $\mu'_a \succeq_a \mu_a$ for all $a \in r$.

[8]One way to find such a matching is to implement the DA mechanism for the students and schools in r.

[9]Again, there is such a matching due to finiteness.

efficient. It is individually rational because at each n and any $r \in R$ such that $c(r) = n$, the matching $\mu^{r,0}$ is individually rational. Finally, φ is monotone because for any $r, r' \in \mathcal{R}$ such that $r \subsetneq r'$ and $i \in r \cap I$, the construction implies that there is a sequence $(r^1, \dots, r^K)$ for some K such that (i) $r^k \in \mathcal{R}$ for every $k = 1, \dots, K$, (ii) $rr^1, r^1r^2, \dots, r^{K-1}r^K, r^Kr' \in E$, and (iii) $\mu_i^{r'} \succeq_i \mu_i^{r^K} \succeq \cdots \succeq_i \mu^{r^1} \succeq \mu_i^r$.

This completes the proof. □

This result shows that there is a limit to monotonicity even when the requirement for strategy-proofness is lifted. Moreover, the result completely characterizes such a limit, providing a guidance to the policymaker about when one can guarantee an existence of a monotone mechanism that satisfies other desirable properties.

4. When Is DA Monotone?

The preceding section showed senses in which monotonicity is hard to guarantee because of the competitive effect of integration. Intuitively, the effect of competition is present when schools in a given region prefer the students in other regions than the students in their own region. We investigate the validity of this intuition by considering a number of standard mechanisms in school choice. We begin by defining basic concepts for this investigation.

Definition 4. Let $\succ_S$ be a profile of school preferences. A mechanism φ is **monotone at** $\succ_S$ if, for all $R, R' \in \mathcal{Q}$, $r \in R$, $r' \in R'$ such that $r \subseteq r'$, $i \in r \cap I$, and $\succ'$ such that $\succ'_S = \succ_S$, we have $\varphi_i(\succ', R') \succeq_i \varphi_i(\succ', R)$.

Definition 5. A school preference relation $\succ_s$ **favors locals** if there exist no $R, R' \in \mathcal{Q}, r \in R, r' \in R'$ with $s \in r \subseteq r'$, $i \in r \cap I$ with $i \succ_s \emptyset$, $I' \subseteq I$ with $I' \subseteq r'$, $I' \not\subseteq r$, and $|I'| = q_s$, such that $i' \succ_s i$ for all $i' \in I'$.

Intuitively, a school s fails to favor locals if a local student i is ranked lower than some non-local students in a manner that "matters" for matching. Specifically, we require that there be a set of competing students $I' \subseteq I$ such that (i) there are enough students in I' to fill the capacity of the school ($|I'| = q_s$), (ii) all students in I' are ranked higher by s than i ($i' \succ_s i$ for all $i' \in I'$), and (iii) some students in I' can compete for a seat with i only after the expansion of the region ($I' \subseteq r'$ and $I' \not\subseteq r$).

Proposition 1. *Fix an environment $(I, S, \mathcal{Q})$. The DA mechanism is monotone at $\succ_S$ if and only if $\succ_s$ favors locals for all $s \in S$.*

Proof. "If" direction:

Suppose that $\succ_s$ favors locals for each $s \in S$, and let $R, R' \in \mathcal{Q}$, $r \in R$ and $r' \in R'$ be such that $r \subseteq r'$. First, consider the DA mechanism between all schools in r and all students in r'.[10] More specifically, consider a version of Gale and Shapley (1962)'s algorithm that outputs the outcome of the DA mechanism in which applications by students in $r' \setminus r$ are made only after all students in r either are tentatively matched or have been rejected by all schools that they find acceptable.[11] Note that, because $\succ_s$ favors locals for each $s \in r \cap S$, no student in r is rejected after students in $r' \setminus r$ begin to make applications.[12] Therefore, at the end of this algorithm, each student in r is matched to a school that she is matched with at the DA mechanism between all schools in r and all students in r. Now, because of the well-known comparative statics result that adding schools make students weakly better off under the DA mechanism (Crawford, 1991), the DA mechanism between all schools in r' and all students in r' places each student in r to a school that she weakly prefers, showing the monotonicity of the DA mechanism.

"Only if" direction:

We begin by letting φ denote the DA mechanism. Suppose that there exists $s \in S$ such that $\succ_s$ does not favor locals. Then, there exist $R, R' \in \mathcal{Q}, r \in R, r' \in R'$ with $s \in r \subseteq r'$, $i \in r \cap I$ with $i \succ_s \emptyset$, $I' \subseteq r' \cap I$ with $|I'| = q_s$ and $I' \not\subseteq r$ such that $i' \succ_s i$ for all $i' \in I'$. Take such s, R, R', r, r', i, and I'.

Consider student preferences such that

(1) $s \succ_{i''} \emptyset \succ_{i''} s'$ for every $s' \in S \setminus s$ and $i'' \in \{i\} \cup I'$,
(2) $\emptyset \succ_{i''} s'$ for every $s' \in S$ and $i'' \in I \setminus (\{i\} \cup I')$.

First, consider R and region r. Because $|I' \cap r| \leq q_s - 1$ by $|I'| = q_s$ and $I' \not\subseteq r$, stability implies $\varphi_i(\succ, R) = s$. Next, consider R' and r'. Because $|I' \cap r'| = q_s$ and $i' \succ_s i$ for all $i' \in I'$, stability implies $\varphi_i(\succ, R') = \emptyset$. Therefore, we have shown $\varphi_i(\succ, R) = s \succ_i \emptyset = \varphi_i(\succ, R')$, so monotonicity is violated. □

[10] Strictly speaking, we defined the DA mechanism only for each region structure R. However, it is straightforward to extend the definition to the one that operates between any set of students and any set of schools.

[11] We note that the outcome of Gale and Shapley (1962)'s algorithm does not depend on the order of applications (McVitie and Wilson, 1970).

[12] Otherwise, the condition in favoring locals is violated by setting I' as follows. Consider the first step after students in $r' \setminus r$ begin to make applications at which a student in r gets rejected, and let s be the school that made that rejection. Let I' be the set of all students that are tentatively accepted at s at that step.

In the context of the DA mechanism, this result verifies the intuition that the negative effect of competition is caused precisely by schools that do not favor local students. Intuitively, when the school preferences favor locals, no student who is matched with her local school would be "kicked out" when students from other districts can make applications to the school. Conversely, if school preferences do not favor locals, then there must be an instance where some students are kicked out when a region expands, violating monotonicity.

The conclusion of no kicking out when schools favor locals holds because we consider the DA mechanism, and indeed, other mechanisms may fail to have monotonicity even when schools favor locals.[13] We discuss this point in Examples 1, 2 and 3.

For TTC, the same proof as the "only if" direction of the Proposition shows that it is not monotone if $\succ_S$ does not favor locals. It is not necessarily monotone, however, even if $\succ_S$ favors locals.

Example 1 (Non-monotonicity of TTC under favoring locals)**.** Let $I = \{i_1, i_2, i'\}, S = \{s, s'\}$, $\mathcal{Q} = \{R, R'\}$, $R = \{r, r'\}$ where $r = \{i_1, i_2, s\}, r' = \{i', s'\}$, $R' = \{r \cup r'\}$.[14] Let

$$\succ_{i_1}: s', \qquad \succ_s: i_1, i_2, i',$$
$$\succ_{i_2}: s, \qquad \succ_{s'}: i', i_1, i_2,$$
$$\succ_{i'}: s, s'$$

where each school's capacity is 1. Note that $\succ_S$ favors locals. Under this preference profile, the TTC mechanism returns

$$\begin{pmatrix} s & s' & \emptyset \\ i_2 & i' & i_1, \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s & s' & \emptyset \\ i' & i_1 & i_2 \end{pmatrix},$$

under region structures R and R', respectively. Hence, in particular, student i_2 is assigned to s under R and she is unmatched under R', while $s \succ_{i_2} \emptyset$. Thus, monotonicity is violated.

To get the intuition, first note that monotonicity of a mechanism may fail when merging regions result in a situation where a student who could secure a seat at a local school before

[13]One might conjecture that the conclusion of the proposition holds for every stable mechanism. We note that this is not the case. For example, if we replace the DA mechanism with a mechanism that produces the same outcome as the DA mechanism under some region structure while producing the outcome of the school-proposing deferred acceptance algorithm under another region structure, then the "if" direction of the proposition does not hold for some environments.

[14] Strictly speaking, our model assumes that each region is constructed from a base of regions and hence contains at least two students, a condition violated by r'. This is just for expositional simplicity, and it is straightforward to modify the present example such that the condition is satisfied.

a merger is displaced by some non-local student after the merger. Such displacement does not occur under the DA mechanism if school preference favor locals. However, the TTC mechanism is not stable, so such displacement may happen even if school preferences favor locals. In fact, other unstable mechanisms such as the Boston mechanism (Abdulkadiroğlu and Sönmez, 2003) and the serial dictatorship also fail to be monotone even if school preferences favor locals for the same reason.[15] Overall, the intuition that the negative effect of competition is only caused by schools not favoring their local students is valid under the DA mechanism but not under other standard (but unstable) mechanisms.

5. Conclusion

The present paper investigated the scope for welfare gain in integrating fragmented matching markets, identifying possible barriers against integration. Our analysis revealed difficulties with integrating markets in a "monotone" manner in the sense of leaving no students worse off. Specifically, we found that whether a monotone mechanisms exists depends on other properties of mechanisms to be required, the class of possible region structures, and school preferences. Further investigations are in order so as to better understand the implications of integrations as well as desirable matching mechanisms in the face of barriers to integration.

References

Abdulkadiroğlu, Atila. 2005. "College admissions with affirmative action." *International Journal of Game Theory*, 33(4): 535–549.

Abdulkadiroğlu, Atila, and Tayfun Sönmez. 2003. "School Choice: A Mechanism Design Approach." *American Economic Review*, 93: 729–747.

Abraham, David J., Robert W. Irving, and David F. Manlove. 2007. "Two algorithms for the student-project allocation problem." *Journal of Discrete Algorithms*, 5: 73–90.

Aygün, Orhan, and Bertan Turhan. 2020. "Dynamic reserves in matching markets." *Journal of Economic Theory*, 188: 105069.

Biro, Peter, Tamas Fleiner, Robert W. Irving, and David F. Manlove. 2010. "The College Admissions Problem with Lower and Common Quotas." *Theoretical Computer Science*, 411(34-36): 3136–3153.

Crawford, Vincent P. 1991. "Comparative statics in matching markets." *Journal of Economic Theory*, 54(2): 389–400.

[15] Specific examples are presented in the Appendix.

Dur, Umut Mert, and M Utku Ünver. 2019. "Two-sided matching via balanced exchange." *Journal of Political Economy*, 127(3): 1156–1177.

Ergin, Haluk, and Tayfun Sönmez. 2006. "Games of School Choice under the Boston Mechanism." *Journal of Public Economics*, 90: 215–237.

Gale, David, and Lloyd S. Shapley. 1962. "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69: 9–15.

Goto, Masahiro, Atsushi Iwasaki, Yujiro Kawasaki, Yosuke Yasuda, and Makoto Yokoo. 2014. "Improving Fairness and Efficiency in Matching with Distributional Constraints: An Alternative Solution for the Japanese Medical Residency Match." https://mpra.ub.uni-muenchen.de/53409/1/MPRA_paper_53409.pdf, last accessed on October 13, 2023.

Hafalir, Isa E, Fuhito Kojima, and M Bumin Yenmez. 2022. "Interdistrict school choice: A theory of student assignment." *Journal of Economic Theory*, 201: 105441.

Hafalir, Isa E, M Bumin Yenmez, and Muhammed A Yildirim. 2013. "Effective affirmative action in school choice." *Theoretical Economics*, 8(2): 325–363.

Kamada, Yuichiro, and Fuhito Kojima. 2015. "Efficient Matching under Distributional Constraints: Theory and Applications." *American Economic Review*, 105(1): 67–99.

Kamada, Yuichiro, and Fuhito Kojima. 2017. "Stability concepts in matching with distributional constraints." *Journal of Economic Theory*, 168: 107–142.

Kamada, Yuichiro, and Fuhito Kojima. 2018. "Stability and Strategy-Proofness for Matching with Constraints: A Necessary and Sufficient Condition." *Theoretical Economics*, 13: 761–793.

Kamada, Yuichiro, and Fuhito Kojima. 2023*a*. "Ekkyo Matching: How to Integrate Fragmented Matching Markets for Welfare Improvement." Working Paper.

Kamada, Yuichiro, and Fuhito Kojima. 2023*b*. "Fair Matching under Constraints: Theory and Applications." Forthcoming, *The Review of Economic Studies*.

Kojima, Fuhito, Akihisa Tamura, and Makoto Yokoo. 2018. "Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis." *Journal of Economic Theory*, 176: 803–833.

McVitie, D. G., and L.B. Wilson. 1970. "Stable Marriage Assignments for Unequal Sets." *BIT*, 10: 295–309.

Pathak, Parag A, Tayfun Sönmez, M Utku Ünver, and M Bumin Yenmez. 2021. "Fair allocation of vaccines, ventilators and antiviral treatments: leaving no ethical value behind in health care rationing." Forthcoming, *Management Science*.

Ricardo, David. 1821. *On the Principles of Political Economy and Taxation.* John Murray.

Robinson-Cortes, Alejandro. 2019. "Who gets placed where and why? An empirical framework for foster care placement." Working Paper.

Roth, Alvin E. 1984. "The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory." *Journal of Political Economy*, 92: 991–1016.

Roth, Alvin E. 1985. "The College Admission Problem is not Equivalent to the Marriage Problem." *Journal of Economic Theory*, 36: 277–288.

Roth, Alvin E., and Elliott Peranson. 1999. "The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design." *American Economic Review*, 89: 748–780.

Shapley, Lloyd, and Herbert Scarf. 1974. "On cores and indivisibility." *Journal of mathematical economics*, 1(1): 23–37.

Slaugh, Vincent W, Mustafa Akan, Onur Kesten, and M Utku Ünver. 2016. "The Pennsylvania Adoption Exchange improves its matching process." *Interfaces*, 46(2): 133–153.

Stolper, Wolfgang F, and Paul A Samuelson. 1941. "Protection and real wages." *The Review of Economic Studies*, 9(1): 58–73.

Westkamp, Alexander. 2013. "An Analysis of the German University Admissions System." *Economic Theory*, 53(3): 561–589.

Appendix A. Boston Mechanism and Serial Dictatorship

We provide examples in which the Boston mechanism and the serial dictatorship fail to be monotone although school preferences favor locals. The intuition for the lack of monotonicity is essentially the same as for the TTC mechanism (Example 1).

Example 2 (Non-monotonicity of Boston under favoring locals)**.** Let $I = \{i_1, i_2, i'\}, S = \{s_1, s_2, s'\}$, $\mathcal{Q} = \{R, R'\}$, $R = \{r, r'\}$ where $r = \{i_1, i_2, s_1, s_2\}, r' = \{i', s'\}$, $R' = \{r \cup r'\}$.[16] Let

$$\begin{aligned} &\succ_{i_1}: s_1, s_2 && \succ_{s_1}: i_1, i_2, i', \\ &\succ_{i_2}: s_1, s_2 && \succ_{s_2}: i_1, i_2, i', \\ &\succ_{i'}: s_2, s' && \succ_{s'}: i', i_1, i_2, \end{aligned}$$

where each school's capacity is 1. Note that $\succ_S$ favors locals. Under this preference profile, the Boston mechanism returns

$$\begin{pmatrix} s_1 & s_2 & s' \\ i_1 & i_2 & i' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s_1 & s_2 & s' & \emptyset \\ i_1 & i' & \emptyset & i_2 \end{pmatrix},$$

under region structures R and R', respectively. Hence, in particular, student i_2 is assigned to s_2 under R and she is unmatched under R', while $s_2 \succ_{i_2} \emptyset$. Thus, monotonicity is violated.

Example 3 (Non-monotonicity of serial dictatorship under favoring locals)**.** Let $I = \{i, i'\}, S = \{s, s'\}$, $\mathcal{Q} = \{R, R'\}$, $R = \{r, r'\}$ where $r = \{i, s\}, r' = \{i', s'\}$, $R' = \{r \cup r'\}$.[17] Let

$$\begin{aligned} &\succ_{i}: s && \succ_{s}: i, i', \\ &\succ_{i'}: s, s' && \succ_{s'}: i', i, \end{aligned}$$

where each school's capacity is 1. Note that $\succ_S$ favors locals. Consider serial dictatorship such that the serial order is i', i. Under this preference profile, the serial dictatorship returns

$$\begin{pmatrix} s & s' & \emptyset \\ i & i' & \emptyset \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} s & s' & \emptyset \\ i' & \emptyset & i \end{pmatrix},$$

under region structures R and R', respectively. Hence, in particular, student i is assigned to s under R and she is unmatched at R', while $s \succ_i \emptyset$. Thus, monotonicity is violated.

[16] The same remark as in footnote 14 applies to this example as well.

[17] Again, the same remark as in footnote 14 applies to this example as well.