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Symmetric Equilibrium in Pre-Auction Investment

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Abstract

We characterize the symmetric equilibria of a pre-auction game of investment in an internet ad auction environment. For a wide class of auction formats, we show a symmetric equilibrium exists for the pre-auction investment game, is essentially unique, and is the same for all auction formats in the class; we give sufficient conditions for the symmetric equilibrium to be in pure strategies, and to achieve the efficient level of investment. Finally, we show comparative statics results with respect to a variety of parameters.

1 Introduction

We investigate symmetric equilibria of a pre-auction game of investment in an internet ad auction environment, with multiple goods (advertisement positions) for sale. Bidders have one-dimensional types, whose distribution is determined by pre-auction investments. We focus on auction formats in which, if bidders are symmetric going into the auction, the goods that are sold are allocated efficiently (although supply may be inefficiently restricted by a reserve price). If pre-auction investment is unobservable to competitors, a broad class of auction formats yield the same set of symmetric equilibria in investment. We establish existence and essential uniqueness of the symmetric equilibrium; relate the efficiency of investment to the reserve price; give sufficient conditions for the symmetric equilibrium to be in pure strategies; and give sufficient conditions for it to achieve first-best investment. Finally, we show comparative statics results with respect to a number of model parameters.

Many of our results follow from a useful fact we have not seen reported elsewhere: with independent private values, pre-auction investment in bidders' own valuations is a potential game, with the potential function closely related to total surplus. Aside from facilitating proofs, this fact is of independent interest, as play in a potential game has nice convergence properties under many standard learning dynamics.

2 Related Literature

Hausch and Li (1991) establish that in a standard single-item auction setting with symmetric bidders, the first- and second-price auctions yield the same level of pre-auction investment when investment is covert, which we extend to more general auction formats and environments. Stegeman (1996) gives an analogous

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result when investment takes the form of costly participation. Levin and Smith (1994) find a similar equivalence in auctions with costly entry, and find that the efficient reserve price induces the efficient amount of entry and maximizes both seller profit and social surplus, but that social surplus and seller profit could both be higher if the seller excluded some bidders altogether. (In their setting, bidders observe who has entered prior to bidding in the auction, so "investment" is observable rather than covert.) Arozamena and Cantillon (2004) contrast this equal-investment result with the case of observable investment beyond the yes-no entry decision, providing conditions under which the first-price auction leads to less (and less-than-efficient) value-enhancing (or cost-reducing) investment.[1] All these papers (like ours) assume a bidder's investment only affects their own valuation. In contrast, Matsushima and Noda (2023) study the case where each bidder's investment can affect all bidders' valuations in arbitrary ways, and find that unlike in our setting, the set of mechanisms that induce a particular outcome becomes small.

Rogerson (1992) observes that the Vickrey-Clarke-Groves mechanism provides efficient incentives for pre-auction investment. Given the computational burden of VCG in "large" auction settings, Hatfield, Kojima and Kominers (2025) and Akbarpour, Kominers, Li, and Milgrom (2023) study investment incentives in auctions which are approximations of VCG; depending on the details of the auction, these may or may not lead to approximately efficient investment. Gershkov, Moldovanu, Strack and Zhang (2021) study revenue-maximizing multi-unit auctions when bidders make investment decisions after learning the realization of their private information.

Another form of pre-auction investment is endogenous information acquisition (see, for example, Bergemann and Valimaki (2002) and Persico (2000)). Li and Tian (2008) establish existence and uniqueness in special cases of a symmetric model with information acquisition. Kim and Koh (2022) and Pernoud and Gleyze (2023) consider various flexible forms of information acquisition; Bobkova (2024) examines the interaction between auction formats and the incentives to acquire information about common versus private value components.

The internet ad auction framework we use was introduced by Edelman, Ostrovsky and Schwarz (2007) and Varian (2009), and discussed further by Athey and Ellison (2011), Levin and Milgrom (2010), and Arnosti, Beck and Milgrom (2016), among others. We note that the model, and therefore our results, apply to any setting with multiple goods of varying quality levels, unit demand by each bidder, and one-dimensional types; internet ad auctions are simply the leading example of such a setting. This model nests the standard single-item auction setting as a special case.

3 Model

Our environment is the one considered by Edelman, Ostrovsky and Schwarz (2007), Varian (2009), and others. There are N items (ad positions adjacent to search results) to be allocated, ordered most valuable to least valuable, with "clickthrough rates" (the fraction of consumers shown the ad who will click to visit the corresponding website) $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_N$. There is a set $\mathcal{K}$ of bidders, with $K = |\mathcal{K}| > N$. For $k \in \mathcal{K}$, bidder k's value s_k (per arriving customer) is drawn from a distribution F_k and independent of $\{s_j\}_{j \neq k}$. The value to bidder k of being assigned position n is $\alpha_n s_k$, minus the payment made. We allow for different possible auction formats, whose equilibrium allocation and payment may depend on the distributions $\{F_k\}$

[1] The key difference between these last two papers is that in Levin and Smith (1994), all bidders who choose to enter have the same private value distribution, so even "off the equilibrium path" of the entry stage, the auction is still symmetric. In the model of Arozamena and Cantillon (2004), the auction following a deviation in investment is asymmetric, so the two standard auction formats are not revenue-equivalent.

from which the various bidders' valuations are drawn.

We model pre-auction investment as follows. There is a set $\mathcal{F}$ of probability distributions with supports contained in an interval $[0, \bar{s}]$, and a cost function $c : \mathcal{F} \to \mathbb{R}^+$ assigning a cost to each, with $\min_{F \in \mathcal{F}} c(F) = 0$. Players simultaneously choose distributions $F_k \in \mathcal{F}$, incurring the corresponding costs $c(F_k)$. The auction then occurs, with each bidder's type s_k drawn (independently) from that bidder's chosen distribution F_k.

We assume investment is *covert*: bidders do not observe each others' choices of F_k, although they correctly infer them in equilibrium. (Thus, if a bidder plays a mixed strategy in equilibrium, at the time of the auction their opponents know their mixed strategy but not which distribution resulted.) We take as given equilibrium play in the auction itself, which may depend on the distribution the bidders' valuations are believed to come from; our focus is the symmetric equilibria of the game in which the bidders choose F_k.

We slightly abuse notation and associate each distribution $F \in \mathcal{F}$ with its CDF $F : [0, \bar{s}] \to [0, 1]$, and define a metric on $\mathcal{F}$ by defining the distance between two distributions as

$$d(F, F') \quad = \quad \inf \left\{ \epsilon : \left| \{ s : |F(s) - F'(s)| > \epsilon \} \right| \leq \epsilon \right\},$$

that is, F and F' are within ϵ of each other if their CDFs $F(s)$ and $F'(s)$ are within ϵ of each other except on a set of measure ϵ or less.[2] We impose standard restrictions to ensure best-responses exist:

Assumption 1. *$\mathcal{F}$ is compact and c is continuous.*

Note that $\mathcal{F}$ can be either finite or infinite; Assumption 1 holds trivially if $\mathcal{F}$ is finite.

We consider auctions with a reserve price $r \geq 0$, such that the minimum "payment per click" is r (or the minimum payment to win slot n is $\alpha_n r$). We consider a class of auction formats which includes all the standard ones, and which we define as follows. First, if bidder k plays a mixed strategy σ_k over $\mathcal{F}$, then from the point of view of the other bidders, his type is drawn from the compound lottery

$$F_{\sigma_k}(s) \quad \equiv \quad \int_{\mathcal{F}} F(s) d\sigma_k(F)$$

(or the analogous sum if σ_k has finite support), which lies in the convex hull $\text{conv}(\mathcal{F})$ of $\mathcal{F}$. For a particular auction format and a belief profile $\mathbf{F} = (F_{\sigma_1}, F_{\sigma_2}, \ldots, F_{\sigma_K}) \in (\text{conv}(\mathcal{F}))^K$, we will refer to an *equilibrium at* $\mathbf{F}$ as an equilibrium[3] of the auction when bidder types are commonly known to be drawn from the distributions $\mathbf{F}$. For a given $\mathbf{F}$ and a given equilibrium at $\mathbf{F}$, we say the equilibrium is *constrained efficient* if the resulting allocation maximizes ex post surplus among all allocations that only allocate prizes to bidders with valuations weakly above r – that is, if the number of prizes awarded is the lesser of N and the number of bidders with valuations of at least r, and among these bidders, the bidder with the n^{th} highest valuation is allocated the n^{th} prize. We define an auction format to be *constrained efficient when symmetric* if at every symmetric profile of distributions $\mathbf{F} = (F, F, \ldots, F)$, $F \in \text{conv}(\mathcal{F})$, the auction has a constrained efficient equilibrium. And we restrict our attention to auction formats which are constrained efficient when symmetric.

Note that we do *not* restrict ourselves to auctions for which an equilibrium exists at every $\mathbf{F}$, only for which a constrained-efficient equilibrium exists at every *symmetric* $\mathbf{F}$. This is analogous to focusing on equilibria for which revenue equivalence holds in the single-good case. Since our focus will be on symmetric

[2] This metric is reminiscent of the Ky Fan metric, whereby two random variables are within ϵ of each other if their realizations are within ϵ with probability at least $1 - \epsilon$. Our metric, however, is defined on CDFs rather than random variables. The exact metric used is not particularly important; what is needed is a metric by which $\mathcal{F}$ can be assumed to be compact and c continuous, and by which a bidder's expected payoffs of a VCG auction can be shown to be continuous in his, and his opponents', type distributions.

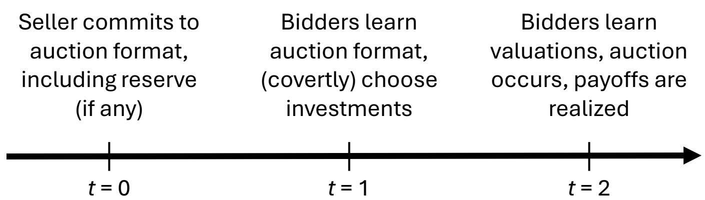
[3] A Bayes-Nash equilibrium if the auction format is static, or a Perfect Bayesian equilibrium if the auction format is dynamic.

equilibria of the investment stage, and since we consider the case where investment is covert, we need not worry about equilibrium play in the auction following asymmetric investment choices. As examples, the VCG mechanism, the equilibria of the Generalized Second Price Auction and Generalized English Auction studied by Edelman, Ostrovsky and Schwarz (2007), a first-price ad auction, and an all-pay ad auction are all constrained efficient when symmetric, though not all of them are constrained efficient when bidders are asymmetric.

Given a constrained-efficient-when-symmetric auction, we define a *symmetric investment equilibrium* to be a mixed strategy $\sigma \in \Delta\mathcal{F}$ such that there exists a constrained-efficient equilibrium of the auction β at $\mathbf{F} = (F_\sigma, F_\sigma, \ldots, F_\sigma)$, and such that when all bidders are expected to play σ and the equilibrium β will then be played in the auction, σ is a bidder's best-response at the investment stage. (Since investment is covert, the same equilibrium strategies β would be played during the auction following a unilateral deviation at the investment stage.) While it is not immediately obvious, it turns out that whether a strategy σ is a symmetric equilibrium of the investment stage does not depend on which constrained-efficient equilibrium will be subsequently played in the auction if there are more than one.[4]

Figure 1 summarizes the timing of our model.

Figure 1: Timing of our model. We focus on play at $t = 1$, and on auction formats for which given symmetric play at $t = 1$, the auction has a constrained-efficient equilibrium at $t = 2$.



How Our Model Relates to Others in the Literature

Within the class of models where bidders have independent private values and investment affects only one's own valuation (or information about one's own valuation), our model is extremely general, and nests several other models in the literature.

Much of the literature focuses on the model where $\mathcal{F}$ is a family of parameterized distributions $\mathcal{F} = \{F_\eta\}_{\eta \in [0,\bar{\eta}]}$, with $c(F_\eta) = C(\eta)$ increasing in η. Two special cases of this model are worth mentioning. First, when $F_{\eta'}$ first-order stochastically dominates F_η for $\eta' > \eta$, this is a classic model of value-enhancing investment. Second, when F_η second-order stochastically dominates $F_{\eta'}$ for $\eta' > \eta$, we can interpret this setting as a model of costly information acquisition: if each bidder receives a noisy signal x_k of their private value, the conditional expectation $E(s_k|x_k)$ serves as their private value in the auction, and a more precise signal x_k yields a mean-preserving spread in these interim expected values. Our general model requires no specific structure beyond Assumption 1, and nests both of these as special cases; the special case where $\{F_\eta\}$ are ranked by first-order stochastic dominance is used to provide intuition for many of our results below.

Some papers consider a different model, where each bidder k draws a type $\theta_k \in \Theta$ from a fixed distribution F and chooses an investment level $a_k \in A$ at cost $C(a_k)$, resulting in a private value $v(\theta_k, a_k)$. When (as in

[4]Think of the same auction format with two different constrained-efficient equilibria to be played following a particular symmetric investment strategy profile as two distinct constrained-efficient-when-symmetric auctions, and the proof of Proposition 1 given in the Appendix applies.

Hausch and Li 1991) a_k is chosen before a bidder learns his type, our model nests this possibility by defining F_{a_k} as the distribution of $v(\theta_k, a_k)$, and $\mathcal{F}$ as $\{F_{a_k}\}_{a_k \in A}$. When a bidder learns his type θ_k before choosing his investment level (as in Gershkov et al. 2021), the mapping to our model is less obvious, but we can still nest such a model by considering a bidder making an ex ante choice of a *contingent* investment plan $\alpha_k : \Theta \to A$, defining a planned investment level $\alpha_k(\theta_k)$ for each possible realization of θ_k. For each such mapping, we can define F_{α_k} as the resulting distribution of private values $v(\theta_k, \alpha_k(\theta_k))$, with the corresponding (expected) investment cost $c(F_{\alpha_k}) = E_{\theta_k} C(\alpha_k(\theta_k))$. In Appendix A.1, we show what this looks like for a particular example and solve for the symmetric investment equilibrium.

4 Equilibrium Characterization

4.1 Symmetric Investment Equilibria are the Same Across Auction Formats

Our first result is that all constrained-efficient-when-symmetric auction formats have the same set of symmetric investment equilibria. Hausch and Li (1991) consider a standard single-item setting with independent private values and symmetric, one-dimensional pre-auction investment opportunities. They show that, subject to a second-order condition, there is a level of pre-auction investment that is a symmetric pure-strategy equilibrium for all "standard auctions." Here, we generalize this result somewhat: multiple items are allowed, the *entire set* of symmetric investment equilibria (pure or mixed) is the same for all auctions in our class, and we do not require a parametrization of $\mathcal{F}$ or a second-order condition. (We explore existence, and whether the equilibrium is in pure strategies, separately). Still, we see the main insight of identical pre-auction investment across auction formats as being Hausch and Li's, and a similar idea appears in Stegeman (1996); we give the result here to justify characterizing symmetric investment equilibria for just one auction format.

Proposition 1. *If investment is covert and two auctions are both constrained efficient when symmetric, they have the same set of symmetric investment equilibria.*

The proof is in the Appendix, but the intuition is as follows. Given incentive compatibility of equilibrium bids in an auction, up to an additive constant, a bidder's ex interim expected surplus at a given realized type (in expectation over his opponents' types) depends only on the equilibrium allocation rule. This means that at any profile of anticipated investment strategies $(\sigma_1, \ldots, \sigma_K)$, a bidder's expected gain from deviating to a different investment strategy depends only on the equilibrium allocation rule that will result at $(F_{\sigma_1}, \ldots, F_{\sigma_K})$. The incentive to deviate away from a potential symmetric investment equilibrium, then, must be the same for all auction formats, since they all implement the same on-path allocation rule at any symmetric investment profile.

Proposition 1 depends critically on three assumptions: that bidders' types are independent, that investment is covert, and that a bidder's investment decision affects only his own type (or his information about his own type), not his opponents' types or his knowledge of them. When any of these three assumptions is violated, Proposition 1 fails, and different auction formats will lead to different investment levels. For example, Persico (2000) studies information acquisition – think of our model, but interpreting a bidder's type as the interim expected value of his type, given his signal, with a more precise signal corresponding to more investment. In a setting with affiliated values, he finds that first price auctions typically lead to greater investment incentives than second price auctions. This is because with affiliated values, more precise information about a bidder's own type also conveys more information about his opponents' types, and this information is more valuable in a first-price than in a second-price auction. Arozamena and Cantillon (2004) study cost-reducing investments prior to a procurement auction, and assume that bidders' pre-auction investments are observable to their opponents. This time, the first price auction gives less of an incentive for investment, because a bidder who is known to be "strong" in the auction is penalized by having his opponent bid more aggressively, a response that is absent in a second price auction. Matsushima and Noda (2023) study more abstract mechanism design, and focus on the case where a bidder's set of possible invesment choices is particularly rich, in the sense that each bidder can unilaterally affect the distribution of *all* bidders' valuations in any direction. In that case, the set of mechanisms that induce a particular outcome becomes small, or in our sense, different auctions lead to different outcomes.

Under these assumptions – independent private types, covert investment, and a bidder's investment only affecting his own type distribution – the extension of Proposition 1 from standard single-item auctions to the ad auction setting with multiple slots is straightforward, and does not seem to involve any new trade-offs that were absent in the single-good case. We have not explored whether this result extends to multiple-good settings where a single bidder may want multiple prizes, or where bidder types are multi-dimensional, though in those settings, we expect the set of auctions which are constrained-efficient-when-symmetric to be much smaller, and the result therefore less powerful.

In light of Proposition 1, to characterize the symmetric investment equilibria of all constrained-efficient-when-symmetric auctions, it suffices to characterize the symmetric investment equilibria of any one of them. For convenience, we therefore focus on characterizing the symmetric investment equilibria of the Vickrey-Clarke-Groves auction with a reserve price.

4.2 In VCG, Pre-Auction Investment is a Potential Game

In a VCG auction with a possibly-binding reserve price $r \geq 0$, the equilibrium allocation rule does not depend on bidders' beliefs about the distributions chosen by their opponents: since it's an equilibrium behavior for each bidder to report their type truthfully, the n^{th} prize goes to the bidder with the n^{th} highest valuation, provided it exceeds the reserve. For ease of exposition, we focus on the case where the distributions in $\mathcal{F}$ have no point masses, and the probability of a tie is therefore zero.[5] For $s \geq r$, then, if each opponent j's valuation is drawn from F_{σ_j},

$$\Pr(k \;\; wins \;\; prize \;\; n | s_k = s) \quad = \quad \sum_{A \subset \mathcal{K} \setminus \{k\} : |A| = n-1} \left(\prod_{j \in A} (1 - F_{\sigma_j}(s)) \prod_{j \in \mathcal{K} - \{k\} - A} F_{\sigma_j}(s) \right)$$

where the sum is taken over the sets A that contain exactly $n-1$ of bidder k's opponents. (This probability also depends on the opponent strategies σ_{-k}, which we suppress to keep the notation manageable. The summand is the probability that the $n-1$ opponents in the set A have valuations above s, while the other $K-n$ opponents have valuations below s; by summing over all subsets of k's opponents of size $n-1$, we get the probability that exactly $n-1$ opponents have valuations above s, and therefore the probability that bidder k has the n^{th} highest valuation and will win prize n.)

[5]This assumption is made just for expositional simplicity. Proposition 1 has already been proved without this restriction; here, we're specifically considering the VCG mechanism, not a generic auction format, and under truthtelling in VCG, a bidder is indifferent between "winning" and "losing" in the case of a tie, and the expression for their expected surplus at each realized type is therefore still valid if their opponents' value distributions have point masses.

For a given profile of mixed strategies $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_K)$, define

$$F_{\boldsymbol{\sigma}}^{(1)}(s) = \prod_{k=1}^{K} F_{\sigma_k}(s)$$

as the CDF of the highest valuation among the K bidders, and more generally,

$$F_{\boldsymbol{\sigma}}^{(n)}(s) = \sum_{i=1}^{n} \sum_{A \subset \mathcal{K}: |A|=i-1} \prod_{k \in A} (1 - F_{\sigma_k}(s)) \prod_{k \in \mathcal{K}-A} F_{\sigma_k}(s)$$

as the CDF of the n^{th} highest valuation. Extend the cost function c to mixed strategies in the natural way, defining the cost of a mixed strategy as the expected cost of the pure strategy selected,

$$c(\sigma_k) = \int_{\mathcal{F}} c(F) d\sigma_k(F)$$

for $\sigma_k \in \Delta\mathcal{F}$; and define a function $P : (\Delta\mathcal{F})^K \to \mathbb{R}$ over the space of mixed strategy profiles by

$$P(\sigma_1, \ldots, \sigma_K) = -\sum_{n=1}^{N} \alpha_n \int_r^{\overline{s}} F_{\sigma}^{(n)}(s) ds - \sum_{k=1}^{K} c(\sigma_k)$$

Our next result is that $P(\cdot)$ is a potential function for the investment game preceding a VCG auction.

Let $U_k(F, \sigma_{-k})$ denote bidder k's ex ante expected surplus from playing the strategy $F \in \mathcal{F}$ prior to a VCG auction, when his opponents are playing the possibly mixed strategies σ_{-k}. Note that this is well-defined because in a VCG auction, equilibrium bidding does not depend on bidders' beliefs, and payoffs are therefore only a function of the actual investment strategies played.[6]

Proposition 2. *In VCG with reserve price r, for every $k \in \mathcal{K}$, every F_k and $F_k' \in \mathcal{F}$, and every profile σ_{-k} of bidder k's opponents' strategies,*

$$U_k(F_k', \sigma_{-k}) - U_k(F_k, \sigma_{-k}) = P(F_k', \sigma_{-k}) - P(F_k, \sigma_{-k}) \tag{1}$$

Thus, the pre-auction investment game is a potential game, with potential function P.

Proof. We will show that $U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k})$ does not depend on F_k, so that $U_k(F_k', \sigma_{-k}) - P(F_k', \sigma_{-k}) = U_k(F_k, \sigma_{-k}) - P(F_k, \sigma_{-k})$, which is equivalent to (1). Using the expression for U_k we de-

[6]Throughout the paper and the appendix, we abuse notation somewhat by using the same expression U_k for bidder k's ex ante expected payoff (as a function of his own and his opponents' investment choices) regardless of whether it is defined over pure strategies or mixed strategies, and whether it depends additionally on bidders' beliefs about each others' strategies or not (as in the case of VCG, since bidding is in dominant strategies).

rive in the proof of Proposition 1 in the appendix,

$$
\begin{aligned}
U_k(F_k,\sigma_{-k}) - P(F_k,\sigma_{-k}) &= \int_r^{\overline{s}} (1-F_k(s)) \sum_{n=1}^N \alpha_n \Pr(k \ \ wins \ \ prize \ \ n|s_k = s) ds - c(F_k) \\
&\quad + \sum_{n=1}^N \alpha_n \int_r^{\overline{s}} F_\sigma^{(n)}(s) ds + c(F_k) + \sum_{k' \neq k} c(\sigma_{k'}) \\
&= \int_r^{\overline{s}} \sum_{n=1}^N \alpha_n \Pr(k \ \ wins \ \ prize \ \ n|s_k = s) ds + \sum_{k' \neq k} c(\sigma_{k'}) \\
&\quad + \sum_{n=1}^N \alpha_n \int_r^{\overline{s}} \left[F_\sigma^{(n)}(s) - F_k(s) \Pr(k \ \ wins \ \ prize \ \ n|s_k = s) \right] ds
\end{aligned}
$$

The first line of the final expression in this chain of equalities does not depend on F_k, so it suffices to show that the expression in square brackets doesn't either.

Define $q_i(s)$ as the probability, given σ_{-k}, that exactly i of bidder k's opponents have valuations above s. Then we can decompose $F_\sigma^{(n)}(s)$, the probability the n^{th}-highest valuation is below s, into two cases, based on whether $s_k > s$ or $s_k \leq s$. Summing the probabilities of those two cases,

$$
\begin{aligned}
F_\sigma^{(n)}(s) &= \Pr(s_k > s \text{ and at most } n-2 \text{ other bidders have values above } s) \\
&\quad + \Pr(s_k \leq s \text{ and at most } n-1 \text{ other bidders have values above } s) \\
&= (1-F_k(s)) \sum_{i=0}^{n-2} q_i(s) + F_k(s) \sum_{i=0}^{n-1} q_i(s)
\end{aligned}
$$

Since k's probability of winning prize n when $s_k = s$ is $q_{n-1}(s)$, the term in the square brackets above, $F_\sigma^{(n)}(s) - F_k(s) \Pr(k \ \ wins \ \ prize \ \ n|s_k = s)$, simplifies to

$$
(1-F_k(s)) \sum_{i=0}^{n-2} q_i(s) + F_k(s) \sum_{i=0}^{n-1} q_i(s) - F_k(s) q_{n-1}(s) = \sum_{i=0}^{n-2} q_i(s)
$$

Since this does not depend on F_k, $U_k(F_k,\sigma_{-k}) - P(F_k,\sigma_{-k})$ does not depend on F_k, proving the result. □

As noted in the introduction, investment being a potential game is of independent interest due to the convergence properties of learning in potential games. Swenson, Murray and Kar (2018), for example, show that for a broad class of potential games, best-response dynamics converge at an exponential rate to a pure-strategy Nash equilibrium; see section 13.6 of Sandholm (2015) for more.[7]

In special cases of congestion games (Rosenthal 1973), the potential function has a straightforward interpretation as total welfare. In our setting, as we will explore in Section 5.1, the potential function is equal to total welfare distorted by a term proportional to $(r - v_0)$, the difference between the reserve price and the seller's cost. Thus, when the seller sets a reserve price that may make the auction inefficient ex post, this will also tend to distort pre-auction investment away from the welfare-maximizing level, in a way that

[7]Potential games also yield a strong characterization of informationally robust play when rationally-inattentive players have unrestricted opportunities to learn about each others' information, see Denti (2023).

the potential function will make fairly transparent.

4.3 Existence and Essential Uniqueness of Symmetric Investment Equilibrium

Knowing that the investment game preceding VCG is a potential game helps us characterize symmetric investment equilibrium. We begin with an existence result, to ensure later results aren't vacuous. Proofs of all lemmas are in the appendix.

Lemma 1. *In the covert investment game preceding VCG with reserve price r, a symmetric investment equilibrium exists.*

The proof is by applying an appropriate fixed point theorem to the correspondence mapping a mixed strategy $\sigma \in \Delta\mathcal{F}$ to a bidder's best response set when the other $K-1$ bidders play σ.

Next, we establish essential uniqueness of the symmetric investment equilibrium, which takes several steps. We begin by showing that bidders' investment choices are strategic substitutes:

Lemma 2. *Pick two bidders j and k, and fix a profile $\sigma_{-j,k}$ of the other bidders' strategies.*

- *The private gain to bidder k of switching from σ_k to σ'_k is weakly greater when $\sigma_j = \sigma_k$ than when $\sigma_j = \sigma'_k$.*
- *The difference in the gain is strict if $F_{\sigma'_k}$ and F_{σ_k} differ on a range that "matters," i.e., if there exists $s \geq r$ at which $F_{\sigma_k}(s) \neq F_{\sigma'_k}(s)$ and $|\{k' \notin \{j,k\} : F_{\sigma_{k'}}(s) = 0\}| < N$.*

The proof is mostly algebra: we explicitly calculate the difference in bidder k's payoff from switching from σ_k to σ'_k, and how this difference changes when a rival also switches between the same two investment strategies. Intuitively, a bidder benefits more from a higher valuation when he is more likely to win, or to win a "bigger" prize, which is when his rival is more likely to have a smaller valuation, making two bidders' choices between the same two options strategic substitutes.

Next, define

$$\tilde{P}(\sigma) \quad = \quad P(\sigma, \sigma, \ldots, \sigma)$$

as the potential function evaluated at a symmetric strategy profile.

Lemma 3. *If a symmetric investment equilibrium exists where all players play $\hat{\sigma}$, then $\hat{\sigma} \in \arg\max_{\sigma \in \Delta\mathcal{F}} \tilde{P}(\sigma)$.*

Intuition for the proof is that if $\hat{\sigma}$ is not a symmetric maximizer of P, then switching all bidders' strategies from $\hat{\sigma}$ to some other strategy σ' strictly increases P. Since the change in a bidder's own payoff matches the change in P, and since bidders' investments are strategic substitutes, this implies that switching a single bidder from $\hat{\sigma}$ to σ' must strictly increase that bidder's payoff, so $\hat{\sigma}$ is not an equilibrium.

Finally, we can use the results above to establish essential uniqueness of the symmetric investment equilibrium:

Lemma 4. *If σ' and σ'' are both symmetric investment equilibria, then $F_{\sigma'}(s) = F_{\sigma''}(s)$ for all $s \in (r, \overline{s})$.*

Lemma 4 implies that all symmetric investment equilibria give the same distribution of valuations above r; moreover, to give the same value of $\tilde{P}$, they must also have the same cost. Thus, the symmetric investment equilibrium is effectively unique. Note that this uniqueness follows from the existence of a potential function, its symmetry, and the fact that bidders' investments are strategic substitutes. Combining Lemma 4 with the existence result (Lemma 1) and Proposition 1 gives the following:

Theorem 1. *Suppose Assumption 1 holds and investment is covert. Consider any auction format which is constrained efficient when symmetric.*

1. *A symmetric investment equilibrium exists (possibly in mixed strategies).*
2. *The symmetric investment equilibrium is essentially unique, in that all symmetric investment equilibria have the same expected cost per bidder and the same distribution of bidder valuations above r.*
3. *The symmetric investment equilibrium is the same for all constrained-efficient-when-symmetric auction formats with the same reserve price.*

An immediate implication of Theorem 1 is that assuming a symmetric investment equilibrium will be played, any constrained-efficient-when-symmetric auction format leads to the same distribution of bidder types and the same type-dependent allocation, although they need not necessarily all lead to the same expected payments.

5 Efficiency and Comparative Statics

5.1 Efficiency and Reserve Price

Next, we consider total surplus, which is the expected surplus of the bidders plus the expected profit of the seller. Let $v_0 \geq 0$ denote the seller's cost per click, so that $\alpha_n v_0$ is the cost to the seller of assigning slot n, or the value the seller earns from not assigning it.

Total surplus at a strategy profile $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_K)$ is

$$W(\boldsymbol{\sigma}) \quad = \quad \sum_{n=1}^{N} \alpha_n \int_r^{\overline{s}} (s - v_0) dF_{\boldsymbol{\sigma}}^{(n)}(s) - \sum_{k=1}^{K} c(\sigma_k),$$

or, integrating by parts,

$$W(\boldsymbol{\sigma}) \quad = \quad (\overline{s} - v_0) \sum_{n=1}^{N} \alpha_n - \sum_{n=1}^{N} \alpha_n \int_r^{\overline{s}} F_{\boldsymbol{\sigma}}^{(n)}(s) ds - \sum_{k=1}^{K} c(\sigma_k) - (r - v_0) \sum_{n-1}^{N} \alpha_n F_{\boldsymbol{\sigma}}^{(n)}(r)$$

The use of the potential function to characterize equilibrium investment, and the relationship between the potential function and total surplus, give a straightforward way to see that a reserve price equal to the seller's cost is not only ex post efficient but also ex ante. Since all constrained-efficient-when-symmetric equilibrium yield the same investment, we focus on the VCG mechanism.

Proposition 3. *If the symmetric investment equilibrium will be played at each r, then a reserve price $r = v_0$ maximizes ex ante surplus.*

Proof. Suppose not, i.e., suppose there is some r_1 such that $r = r_1$ leads to greater total surplus than $r = v_0$. Letting F_1 and F_0 denote the symmetric investment equilibria at $r = r_1$ and $r = v_0$, respectively, this means

$$-\sum_{n=1}^{N} \alpha_n \int_{r_1}^{\overline{s}} F_1^{(n)}(s) ds - Kc(F_1) - (r_1 - v_0) \sum_{n-1}^{N} \alpha_n F_1^{(n)}(r_1) \quad > \quad -\sum_{n=1}^{N} \alpha_n \int_{v_0}^{\overline{s}} F_0^{(n)}(s) ds - Kc(F_0)$$

Suppose first that $r_1 < v_0$. Rewriting the integral on the left as $\int_{r_1}^{\overline{s}} = \int_{r_1}^{v_0} + \int_{v_0}^{\overline{s}}$, and combining the integral over (r_1, v_0) with the $(r_1 - v_0)$ term, this becomes

$$-\sum_{n=1}^{N} \alpha_n \int_{r_1}^{v_0} \left[F_1^{(n)}(s) - F_1^{(n)}(r_1)\right] ds - \sum_{n=1}^{N} \alpha_n \int_{v_0}^{\overline{s}} F_1^{(n)}(s) ds - Kc(F_1) \quad > \quad -\sum_{n=1}^{N} \alpha_n \int_{v_0}^{\overline{s}} F_0^{(n)}(s) ds - Kc(F_0)$$

Since each $F_1^{(n)}$ is increasing, the integrand in square brackets is nonnegative, so this requires

$$-\sum_{n=1}^{N} \alpha_n \int_{v_0}^{\overline{s}} F_1^{(n)}(s) ds - Kc(F_1) \quad > \quad -\sum_{n=1}^{N} \alpha_n \int_{v_0}^{\overline{s}} F_0^{(n)}(s) ds - Kc(F_0)$$

or $\tilde{P}(F_1) > \tilde{P}(F_0)$ at $r = v_0$; by Lemma 3, this contradicts F_0 being the symmetric investment equilibrium at $r = v_0$. For $r > v_0$, the same proof holds, just with $\int_{r_1}^{\overline{s}} = -\int_{v_0}^{r_1} + \int_{v_0}^{\overline{s}}$ and some flipped signs. □

At $r = v_0$, the $(r - v_0)$ term in the expression for total welfare vanishes, and up to a constant, the potential function and total surplus function match. This (and Lemma 3) implies that not only is $r = v_0$ ex post efficient, it also leads to the symmetric investment profile giving the highest expected surplus ex ante. At $r \neq v_0$, this need not be true: the reserve price may cause a surplus loss due to an ex post inefficient allocation, and also distort investment away from the efficient level.

Continuing to focus on the VCG mechanism with reserve price, we define *first-best* (in the ex ante sense) as any combination of reserve price and investment choices that maximizes ex ante expected total surplus. For a given reserve price r, we define *efficient investment* as any investment profile (symmetric or asymmetric) that maximizes expected surplus given r, and *constrained-efficient investment* as any *symmetric* investment profile that maximizes expected surplus among symmetric profiles.

Theorem 2. *Fix a constrained-efficient-when-symmetric auction with reserve price r. Let $\tilde{\sigma}$ be the symmetric investment equilibrium; and let σ^* be any constrained-efficient investment at r.*

- *If $r = v_0$ then $F_{\tilde{\sigma}}$ and F_{σ^*} coincide on $[r, \overline{s}]$.*
- *If $r > v_0$ then $F_{\tilde{\sigma}}(r) \geq F_{\sigma^*}(r)$; if $r < v_0$ then $F_{\tilde{\sigma}}(r) \leq F_{\sigma^*}(r)$.*

Proof. From the calculations above,

$$W(\boldsymbol{\sigma}) \quad = \quad P(\boldsymbol{\sigma}) + (\overline{s} - v_0) \sum_{n=1}^{N} \alpha_n - (r - v_0) \sum_{n=1}^{N} \alpha_n F_\sigma^{(n)}(r),$$

If $r = v_0$, then the last term on the right vanishes and up to an additive constant, $W = P$; since F_{σ^*} is a symmetric maximizer of W, it's therefore also a symmetric maximizer of P, as is $F_{\tilde{\sigma}}$ (Lemma 3); Lemma 4 establishes that all symmetric maximizers of P coincide above r.

For the second part, suppose $r > v_0$ and $F_{\tilde{\sigma}}(r) < F_{\sigma^*}(r)$. This implies $F_{\tilde{\sigma}}^{(n)}(r) < F_{\sigma^*}^{(n)}(r)$ for each n. Since $\tilde{\sigma}$ is the symmetric maximizer of P, $P(\tilde{\sigma}, \tilde{\sigma}, \ldots, \tilde{\sigma}) \geq P(\sigma^*, \sigma^*, \ldots, \sigma^*)$. Together, these would imply $W(\tilde{\sigma}, \tilde{\sigma}, \ldots, \tilde{\sigma}) > W(\sigma^*, \sigma^*, \ldots, \sigma^*)$, contradicting σ^* being the symmetric maximizer of W. An analogous contradiction occurs if $r < v_0$ and $F_{\tilde{\sigma}}(r) > F_{\sigma^*}(r)$. □

To interpret Theorem 2, consider a special case where $\mathcal{F}$ is a family $\{F_\eta\}$ of distributions parameterized by a one-dimensional parameter $\eta \in [0, \overline{\eta}]$, and where $\eta' > \eta$ corresponds to greater investment $c(F_{\eta'}) > c(F_\eta)$

and stochastically higher valuations $F_{\eta'} >_{FOSD} F_\eta$. (This same special case will be useful in interpreting many of our results, and we refer to it as the case where $\mathcal{F}$ is a *first-order stochastic dominance-ranked family of distributions*, or more succinctly, a *FOSD-ranked family*.) When $\mathcal{F}$ is a FOSD-ranked family and the symmetric investment equilibrium is in pure strategies, Theorem 2 says that when $r > v_0$, the equilibrium level of investment $\tilde{\eta}$ will be weakly less than the surplus-maximizing level η^*, and when $r < v_0$, it will be weakly higher.[8] When F_η varies continuously and smoothly with η, these inequalities will typically be strict. Thus, Theorem 2 says that when $r \neq v_0$, in addition to distorting the ex post allocation, the reserve price may distort equilibrium investment away from the efficient symmetric level given r.

While Proposition 3 makes explicit that a reserve price of $r = v_0$ maximizes total surplus, less is clear about the reserve that maximizes the *seller's* profit. We offer two well-known examples in the Appendix, one where the seller's optimal reserve is equal to v_0 and one where it is strictly above v_0. Combining this with Theorem 2 establishes that the seller-optimal reserve price may or may not induce efficient bidder investment. We have not found an example where $r < v_0$ is optimal for the seller, but we also do not have a proof that it can't be. A reserve below v_0 yields lower total surplus than a reserve of v_0, so it can only benefit the seller if it lowers bidder surplus by more; while it seems unlikely for bidders to be hurt by a lower reserve price, since they compete with each other in choosing their pre-auction investments, we have not yet been able to rule it out.[9]

5.2 Efficiency of Investment with Efficient Reserve Price

When $r = v_0$, the symmetric investment equilibrium is always the symmetric investment profile that maximizes total surplus. This still may or may not, however, be the global maximizer of total surplus, because total surplus may be maximized at an asymmetric investment profile. Three examples below will help build intuition for when this will happen.[10] For all three examples, we consider a simple environment with $K = 2$ bidders and $N = 1$ prize with $\alpha_1 = 1$. Let $\mathcal{F} = \{F_\eta\}_{\eta \in [0,1/2]}$, and let each F_η be a discrete distribution putting weight on just two points, $s_i = 0$ and $s_i = 1$, with the likelihood of $s_i = 1$ increasing in η. Normalize $c(F_\eta) = \eta$, and assume $r = v_0 = 0$. In all three examples, the symmetric investment equilibrium will give each bidder a probability $\frac{1}{2}$ of a valuation $s_i = 1$; but the equilibria will look different, based on how the distributions F_η vary with η.

Example 1. *Suppose that for each $\eta \in [0, \frac{1}{2}]$, the distribution F_η puts probability $4\eta^2$ on $s_i = 1$ and probability $1 - 4\eta^2$ on $s_i = 0$.*

In that case, any interior investment level $\eta \in (0, \frac{1}{2})$ is dominated by a mixed strategy using the two extreme strategies F_0 and $F_{1/2}$.[11] *The symmetric investment equilibrium involves each bidder mixing between*

[8] For $\mathcal{F}$ a FOSD-ranked family, Theorem 2 part 2 appears to still allow the possibility that $r > v_0$ and $\tilde{\eta} > \eta^*$ as long as $F_{\tilde{\eta}}(r) = F_{\eta^*}(r)$. However, this is not in fact possible: if $F_{\tilde{\eta}}(r) = F_{\eta^*}(r)$, then from the first equation in the proof of Theorem 2, $W(\tilde{\eta}) - P(\tilde{\eta}) = W(\eta^*) - P(\eta^*)$; since $\tilde{\eta}$ uniquely maximizes P (Lemma 4) and η^* maximizes W, this rules out the two distributions being distinct anywhere above r; and if they coincide everywhere above r, the one with lower cost would uniquely maximize both P and W.

In more general environments, the interpretation of Theorem 2 part 2 is not as clear. For example, even when $\mathcal{F} = \{F_\eta\}$ a one-dimensional family of distributions, if $\eta' > \eta$ corresponds to acquiring more precise information about an unknown private value, we will typically have $F_{\eta'}$ being a mean-preserving spread around F_η; in that case, $F_{\tilde{\eta}}(r) \geq F_{\eta^*}(r)$ can correspond to either "underinvestment" ($\tilde{\eta} < \eta^*$) or "overinvestment" ($\tilde{\eta} > \eta^*$), depending on whether the reserve price is in the low or the high range of the value distribution.

[9] Quint (2017) contains examples of settings where a seller benefits from a reserve price below his own valuation, and notes how a different result in Vincent (1995) relies on similar logic, but these are the result of bidders having common values rather than investment being endogenous.

[10] The three examples use discrete rather than continuous value distributions. While we have focused on the continuous case with no point masses for expositional ease, footnote 5 above explains why the discrete case does not pose a problem.

[11] An interior strategy $\eta \in (0, \frac{1}{2})$ gives probability $4\eta^2$ of a high valuation at cost η. A mixed strategy putting probability 2η

F_0 and $F_{1/2}$ with equal probabilities, resulting in equal probabilities of $s_i = 1$ and $s_i = 0$.[12] *This equilibrium is inefficient, however. Total surplus is the valuation of the winner minus total investment costs, so expected surplus is the probability at least one bidder has $s_i = 1$ minus costs, or $\frac{3}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4}$. But total surplus is maximized at the asymmetric profiles $(\eta_1, \eta_2) = (0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$, which give expected surplus $1 - \frac{1}{2} - 0 = \frac{1}{2}$.*

Note that in a sense, investment in this example has increasing returns to scale – the likelihood of a high valuation is convex in the investment made, so it is sensible to either invest the maximal amount ($\eta = \frac{1}{2}$) or nothing ($\eta = 0$), rather than some intermediate amount; this leads to mixing in equilibrium. Mixing in equilibrium turns out to *always* lead to inefficiency, as we formalize next.

To establish a relationship between efficiency and whether the symmetric investment equilibrium involves mixing, there is one subtlety to discuss first. Since our general model puts few restrictions on $\mathcal{F}$, there is a possibility that two different distributions $F, F' \in \mathcal{F}$ might be identical above the reserve price, $F(s) = F'(s)$ for $s \geq r$, and have the same cost $c(F) = c(F')$, differening only in the distribution of valuations below the reserve. In that case, those two pure strategies, and any mixture of the two, would be payoff-equivalent for bidders, and the distinction between pure and mixed strategy equilibrium would be lost. To be able to give stronger results without qualifying them, we assume away this possibility:

Assumption 2. *For any two elements $F, F' \in \mathcal{F}$, either $c(F') \neq c(F)$ or there is some $s > r$ such that $F'(s) \neq F(s)$.*

Theorem 3. *Under Assumptions 1 and 2, if the symmetric investment equilibrium is in mixed strategies, it does not achieve efficient investment.*

Proof. For simplicity, we focus on the case where $r = v_0$, so that P and W coincide up to a constant. Consider first a mixed strategy over two pure strategies, $\sigma = pF' + (1-p)F''$. Since it's an equilibrium, player 1 is indifferent among F', F'', and the mixture σ, meaning $P(\sigma, \sigma) = P(F', \sigma)$. But with player 1 mixing between F' and F'', player 2 was indifferent between F' and F''. If player 1 plays F', then player 2 strictly prefers F'' to F' (Lemma 2). So starting from any strictly mixed equilibrium, P strictly increases if we switch player 1 to one of the two pure strategies and player 2 to the other.

For mixtures over more than two strategies, simply think of the strategy as a mixture over two distinct mixed strategies, and the same argument holds. The proof for $r \neq v_0$ is in the appendix. □

The inverse of Theorem 3 does not hold: a symmetric investment equilibrium in pure strategies still need not be efficient, as the next example demonstrates.

Example 2. *Suppose that for each $\eta \in [0, \frac{1}{2}]$, the distribution F_η gives a bidder a probability $\frac{1}{2}\left(\sqrt{16\eta + 1} - 1\right)$ of $s_i = 1$, and the remaining probability $\frac{1}{2}\left(3 - \sqrt{16\eta + 1}\right)$ of $s_i = 0$.*

This time, the symmetric investment equilibrium is in pure strategies, and is $\eta = \frac{3}{16}$, which gives each bidder a probability $\frac{1}{2}\left(\sqrt{3 + 1} - 1\right) = \frac{1}{2}$ of a high valuation.[13] *Although it uses pure strategies, the symmetric investment equilibrium is again inefficient. Expected surplus at the symmetric investment equilibrium is $\frac{3}{4} - \frac{3}{16} - \frac{3}{16} = \frac{3}{8}$, while the asymmetric profiles $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ again give expected surplus $1 - \frac{1}{2} - 0 = \frac{1}{2}$.*

on $F_{1/2}$ and probability $1 - 2\eta$ on F_0 gives probability $2\eta > 4\eta^2$ of a high valuation, at the same cost η.

[12] A bidder's expected payoff in the auction is the probability that he gets a valuation $s_i = 1$ while his opponent gets a valuation $s_{-i} = 0$; so when one bidder mixes 50/50, the other bidder's expected payoff from a strategy η is $\frac{1}{2}\Pr(s_i = 1|\eta) - \eta = 2\eta^2 - \eta$; since this is maximized at $\eta \in \{0, 1/2\}$, mixing 50/50 between these two strategies is indeed a best-response.

[13] Given this strategy from the opponent, a bidder's pre-auction problem is to maximize $\frac{1}{2}\left(\frac{1}{2}\left(\sqrt{16\eta + 1} - 1\right)\right) - \eta$, which is strictly concave and has first-order condition $\frac{16}{8\sqrt{16\eta+1}} - 1 = 0$, giving $\sqrt{16\eta + 1} = 2$ and therefore $\eta = \frac{3}{16}$.

This time, the probability of a high valuation is concave in the amount invested, or investment has decreasing returns; this makes interior levels of investment optimal, leading to a pure-strategy equilibrium. However, the equilibrium is still inefficient, as an asymmetric investment profile gives higher surplus. This is not always the case, however, as the final example illustrates.

Example 3. *Suppose that for each $\eta \in [0, \frac{1}{2}]$, the distribution F_η gives a bidder a probability $\frac{8\eta}{1+8\eta}$ of $s_i = 1$, and the remaining probability $\frac{1}{1+8\eta}$ of $s_i = 0$.*

The symmetric investment equilibrium is now $\eta_1 = \eta_2 = \frac{1}{8}$, again giving each bidder a probability $\frac{1}{2}$ of a high valuation.[14] *This time, the symmetric investment equilibrium achieves efficient investment. While it's not immediately obvious, expected total surplus is strictly concave in (η_1, η_2),*[15] *and globally maximized at the symmetric investment equilibrium $\eta_1 = \eta_2 = \frac{1}{8}$.*

Informally, what determines whether total surplus is maximized at a symmetric investment profile is how strongly the returns to investment decrease with more investment. This determines whether it is more efficient for a subset of bidders to invest a lot and others to invest little, or more efficient for all bidders to invest an intermediate amount.

Returning to the general model, we formalize this intuition by giving sufficient conditions for the symmetric investment equilibrium to be in pure strategies, and for it to achieve the efficient level of investment. The gap between the sufficient conditions in Theorem 4 and Theorem 5 sheds light on when the symmetric investment equilibrium will be in pure strategies but inefficient, as in Example 2 above.

Theorem 4. *Suppose that for any two distributions $F, F' \in \mathcal{F}$, there exists $F'' \in \mathcal{F}$ such that:*

1. $c(F'') \leq \frac{1}{2}c(F) + \frac{1}{2}c(F')$
2. $F''(s) \leq \frac{1}{2}F(s) + \frac{1}{2}F'(s)$ *for all s, with strict inequality somewhere on $(r, \overline{s})$*

Then the symmetric investment equilibrium must be in pure strategies.

The proof is in the appendix: intuitively, if σ were a mixed-strategy equilibrium, we could generate a profitable deviation by shifting weight away from two distinct pure strategies F and F' and onto another strategy F'' that dominates them, in the sense of the two conditions in the theorem.

The condition in Theorem 4 is difficult to interpret when $\mathcal{F}$ is abstract and multi-dimensional, but better understood intuitively for the case where $\mathcal{F}$ is a FOSD-ranked family and costs are normalized such that $c(F_\eta) = \eta$. In that case, $c(F_{\eta''}) \leq \frac{1}{2}c(F_\eta) + \frac{1}{2}c(F_{\eta'})$ requires $\eta'' \leq \frac{1}{2}\eta + \frac{1}{2}\eta'$, so the two requirements will be satisfied if and only if

$$F_{\frac{1}{2}\eta + \frac{1}{2}\eta'}(s) \quad \leq \quad \frac{1}{2}F_\eta(s) + \frac{1}{2}F_{\eta'}(s)$$

Thus, the condition is essentially equivalent to $F_\eta(s)$ being convex in η (with strict convexity for some s).

On the other hand, in the FOSD-ranked family case with $c(F_\eta) = \eta$, if $F_\eta(s)$ is strictly concave in η for every s, then like in Example 1, every interior strategy is dominated by a mixture over the two extreme strategies 0 and $\bar{\eta}$,[16] and the symmetric investment equilibrium will be in mixed strategies unless it is a corner solution $\eta = 0$ or $\eta = \bar{\eta}$.

[14] Given this opponent strategy, bidder i now maximizes $\frac{1}{2}\left(1 - \frac{1}{1+8\eta}\right) - \eta$, which has FOC $\frac{1}{2}\frac{8}{(1+8\eta)^2} - 1 = 0$, giving $\eta = \frac{1}{8}$.

[15] Expected surplus is $1 - \frac{1}{1+8\eta_1}\frac{1}{1+8\eta_2} - \eta_1 - \eta_2$; its Hessian matrix $-64\begin{bmatrix} 2(1+8\eta_1)^{-3}(1+8\eta_2)^{-1} & (1+8\eta_1)^{-2}(1+8\eta_2)^{-2} \\ (1+8\eta_1)^{-2}(1+8\eta_2)^{-2} & 2(1+8\eta_1)^{-1}(1+8\eta_2)^{-3} \end{bmatrix}$ is negative definite, so expected surplus is strictly concave and and maximized where the FOC holds, at $\eta_1 = \eta_2 = \frac{1}{8}$.

[16] Specifically, for any interior strategy $\eta' \in (0, \bar{\eta})$, the mixed strategy putting weight $\frac{\eta'}{\bar{\eta}}$ on $\eta = \bar{\eta}$ and weight $\frac{\bar{\eta} - \eta'}{\bar{\eta}}$ on $\eta = 0$ will have the same expected cost, but lead to a distribution of types that strictly first-order stochastically dominates $F_{\eta'}$.

Returning to Example 2, for $s < 1$, $F_\eta(s) = \Pr(s_i = 0|\eta) = \frac{1}{2}\left(3 - \sqrt{16\eta + 1}\right)$, which is indeed strictly convex in η. As Example 2 illustrates, this convexity is sufficient to ensure the symmetric investment equilibrium is in pure strategies, but not sufficient to ensure efficiency. A similar but stronger condition guarantees efficiency:

Theorem 5. *Suppose that for any two distributions $F, F' \in \mathcal{F}$, there exists $F'' \in \mathcal{F}$ such that:*

1. $c(F'') \leq \frac{1}{2}c(F) + \frac{1}{2}c(F')$
2. $F''(s) \leq \sqrt{F(s)F'(s)}$ *for all s, with strict inequality somewhere on $(r, \bar{s})$*

Then when $r = v_0$, the symmetric investment equilibrium achieves first-best.

The proof is a lot of algebra, but intuitively, the aim is to show that no asymmetric investment profile can maximize total surplus, by showing that for any asymmetric profile, there's another profile (replacing one bidder's choice of F and another bidder's choice of F' with both bidders choosing F'') that gives strictly higher total surplus. This implies that total surplus must be maximized at a symmetric profile; since when $r = v_0$, the symmetric investment equilibrium is the highest-surplus symmetric profile, it achieves efficient investment. Since $r = v_0$ also yields the ex post efficient outcome in the auction, this implies the symmetric investment equilibrium with $r = v_0$ achieves first-best.

Once again, this condition is easiest to interpret in case of a FOSD-ranked family with normalized costs, where this requirement now becomes

$$F_{\frac{1}{2}\eta + \frac{1}{2}\eta'}(s) \quad \leq \quad \sqrt{F_\eta(s)F_{\eta'}(s)}$$

or, taking logs,

$$\log F_{\frac{1}{2}\eta + \frac{1}{2}\eta'}(s) \quad \leq \quad \frac{1}{2}\log F_\eta(s) + \frac{1}{2}\log F_{\eta'}(s)$$

Thus, a sufficient condition is now that $F_\eta(s)$ is *log*-convex in η. This was the difference between Examples 2 and 3: $F_\eta(s)$ is convex but not log-convex in η in Example 2, while it is log-convex in Example 3.

Note that this interpretation of Theorem 5 for the FOSD-ranked family case is the exact analogue to Proposition 3 in Piccione and Tan (1996). The condition there (translated to our notation) is that $\frac{\partial F_\eta(s)/\partial\eta}{1 - F_\eta(s)}$ is strictly decreasing in η, while log-convexity means that $\frac{\partial F_\eta(s)/\partial\eta}{F_\eta(s)}$ is increasing in η. The difference is that Piccione and Tan analyze a procurement setting, where F_η is the distribution of sellers' costs and the low-cost seller wins, so the roles of "high" and "low" vaulations are reversed. Aside from that, Theorem 5 is a generalization of their Proposition 3 to a broader set of environments.

5.3 Comparative Statics of Equilibrium Investment

Finally, we examine how the symmetric investment equilibrium responds to changes in the environment. We establish a number of comparative statics results for the general model where $\mathcal{F}$ is unrestricted; once again, most of them are best understood intuitively in the special case where $\mathcal{F}$ is a FOSD-ranked family.

First, we consider how equilibrium investment responds to reserve price.

Proposition 4. *Let σ_1 and σ_2 be the symmetric investment equilibria at reserve prices $r = r_1$ and $r = r_2$, respectively. If $r_2 > r_1$ and $F_{\sigma_2}(s) \neq F_{\sigma_1}(s)$ for some $s > r_2$, then $F_{\sigma_2} \not\succ_{FOSD} F_{\sigma_1}$.*

This result may appear weak when $\mathcal{F}$ is unrestricted – if $\mathcal{F}$ is multidimensional, there's no reason to expect two of its elements to be ranked in either direction via first-order stochastic dominance. However, when $\mathcal{F}$ is a FOSD-ranked family of distributions and the symmetric investment equilibrium is in pure strategies, Proposition 4 is the result that equilibrium investment is weakly decreasing in the reserve price r.

Proof of Proposition 4. We first establish an inequality and an expression that will be used to prove all the results in this section; we then proceed to the proof of this particular Proposition.

Let $U_k(F, F', \gamma)$ denote a bidder k's expected surplus from playing F when his opponents are playing F', given parameter value γ. (For the moment, γ represents any relevant parameter; for each proof, we will specialize γ to a particular parameter.) Let σ_1 and σ_2 be equilibrium strategies under the symmetric investment equilibria at $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively, and assume $F_{\sigma_2}(s)$ and $F_{\sigma_1}(s)$ do not coincide everywhere above r. At $\gamma = \gamma_2$, σ_1 must not be a profitable deviation when others are playing σ_2, so

$$U_k(F_{\sigma_2}, F_{\sigma_2}, \gamma_2) - U_k(F_{\sigma_1}, F_{\sigma_2}, \gamma_2) \quad \geq \quad 0$$

Likewise, since σ_1 is a symmetric investment equilibrium at $\gamma = \gamma_1$,

$$U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma_1) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma_1) \quad \leq \quad 0$$

The strategic substitutes result (Lemma 2) tells us that holding fixed the parameter value, switching from σ_1 to σ_2 is worth strictly more when one's opponents are playing σ_1 than σ_2, or

$$U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma_2) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma_2) \quad > \quad U_k(F_{\sigma_2}, F_{\sigma_2}, \gamma_2) - U_k(F_{\sigma_1}, F_{\sigma_2}, \gamma_2)$$

Since (from above) $U_k(F_{\sigma_2}, F_{\sigma_2}, \gamma_2) - U_k(F_{\sigma_1}, F_{\sigma_2}, \gamma_2) \geq 0 \geq U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma_1) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma_1)$, this therefore implies

$$U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma_2) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma_2) \quad > \quad U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma_1) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma_1) \tag{2}$$

As we derive in the appendix, fixing opponent strategies and a constrained-efficient equilibrium at a symmetric profile, we can write bidder k's private gain from switching from σ_1 to σ_2 as

$$U_k(F_{\sigma_2}, F_{\sigma_1}, \gamma) - U_k(F_{\sigma_1}, F_{\sigma_1}, \gamma) \quad = \quad \int_r^{\overline{s}} \sum_{n=1}^{N} \alpha_n (F_{\sigma_1}(s) - F_{\sigma_2}(s)) y_n(s) ds - c(F_{\sigma_2}) + c(F_{\sigma_1}) \tag{3}$$

where $y_n(s)$ is the probability that exactly $n-1$ opposing bidders have types above s when they're all playing σ_1.

Combining Equations 2 and 3, we've now shown that when σ_1 and σ_2 are symmetric investment equilibrium strategies at parameter values γ_1 and γ_2, the right-hand side of Equation 3 must strictly increase when γ changes from γ_1 to γ_2. Returning to Proposition 4, γ now refers to the reserve price r, with (by assumption) $r_2 > r_1$. If $F_{\sigma_2} >_{FOSD} F_{\sigma_1}$, then $F_{\sigma_1}(s) - F_{\sigma_2}(s) \geq 0$, making the integrand everywhere weakly positive; since the region of integration is $[r, \overline{s}]$, increasing r from r_1 to r_2 would therefore cause the right-hand side of Equation 3 to weakly fall, contradicting Equation 2 and therefore proving the result by contradiction. □

We can derive similar comparative statics for how the symmetric investment equilibrium changes with the number of bidders K, the cost function $c(\cdot)$, and the clickthrough rates $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ that determine bidders' valuations. The remaining proofs are in the appendix, but they all follow similar logic to the proof of Proposition 4. First, consider an increase in the number of bidders:

Proposition 5. *Let σ_1 and σ_2 be the symmetric investment equilibria when there are $K = K_1$ and $K = K_2$ bidders, respectively. If $K_2 > K_1$ and $F_{\sigma_2}(s) \neq F_{\sigma_1}(s)$ for some $s > r$, then $F_{\sigma_2} \not\succ_{FOSD} F_{\sigma_1}$.*

Next, consider a uniform scaling-up of the cost function. Formally, we parametrize the cost function $c(\cdot)$ as $c(F) = \kappa\bar{c}(F)$, with $\bar{c} : \mathcal{F} \to \mathbb{R}^+$ and $\kappa \in \mathbb{R}^+$ a constant.

Proposition 6. *Let σ_1 and σ_2 be the symmetric investment equilibria when $\kappa = \kappa_1$ and $\kappa = \kappa_2$, respectively. If $\kappa_2 > \kappa_1$, then $\bar{c}(F_{\sigma_2}) \leq \bar{c}(F_{\sigma_1})$, and if $F_{\sigma_2}(s) \neq F_{\sigma_1}(s)$ for some $s > r$ then $F_{\sigma_2} \not\succ_{FOSD} F_{\sigma_1}$.*

Finally, we consider decreases in the clickthrough rates of some or all of the positions (or decreases in the value of some of the auction prizes). For two vectors $\alpha^1 = (\alpha_1^1, \alpha_2^1, \ldots, \alpha_N^1)$ and $\alpha^2 = (\alpha_1^2, \alpha_2^2, \ldots, \alpha_N^2)$ of clickthrough rates, we'll say $\alpha^2 < \alpha^1$ if $\alpha^2 \neq \alpha^1$ and $\alpha_n^2 \leq \alpha_n^1$ for every n.

Proposition 7. *Let σ_1 and σ_2 be the symmetric investment equilibria when $\alpha = \alpha^1$ and $\alpha = \alpha^2$, respectively. If $\alpha^2 < \alpha^1$ and $F_{\sigma_2}(s) \neq F_{\sigma_1}(s)$ for some $s > r$, then $F_{\sigma_2} \not\succ_{FOSD} F_{\sigma_1}$.*

Again, all of these comparative statics are most intuitively understood through the special case where $\mathcal{F}$ is a FOSD-ranked family. In that case, if the symmetric investment equilibrium is in pure strategies, then these results tell us that equilibrium investment must weakly decrease as the number of bidders increases, investment costs increase, or the clickthrough rates (or the value of any of the prizes) decreases.

6 Conclusion

We've established that for any auction format implementing the constrained-efficient allocation when bidders make identical investment choices, a symmetric equilibrium in the covert pre-auction investment game exists, is essentially unique, and is the same across all such auction formats. Further, when this equilibrium is in mixed strategies, the equilibrium investment profile is inefficient; but when the investment technology has decreasing returns in a particular sense, this equilibrium is in pure strategies and achieves first-best investment.

One interpretation of these results is that when pre-auction investment is covert, revenue equivalence extends to auctions with endogenous valuations. Of course, this depends heavily on both a symmetric environment and a focus on symmetric equilibrium. When different auction formats yield different equilibrium allocations when valuations are asymmetric, they will have different asymmetric equilibria for the pre-auction investment game; in cases where the efficient investment profile is asymmetric, the choice of auction format may therefore be important.

Appendix

A.1 Example – Ex Ante Private Types

Here, we give an example to illustrate how our model nests a model with deterministic investment and ex ante private types. There are K bidders and one prize, with $\alpha_1 = 1$. A bidder's type consists of two components, $s_k = \theta_k + a_k$. The θ_k are *i.i.d.* draws from the uniform distribution on $[0, 1]$; after observing θ_k but before the auction, bidder k chooses $a_k \in \mathbb{R}^+$ at cost $C(a_k) = \gamma a_k^2$. Then a second-price auction is held, with $r = 0$.

We can think of a bidder's strategy before knowing his type as a measurable mapping $a : [0, 1] \rightarrow \mathbb{R}^+$ indicating the a_k he will choose for each realization of θ_k. Mapping this to our model, any strategy $a(\cdot)$ induces a distribution of types $\theta_k + a(\theta_k)$ with CDF

$$F_a(s) \quad = \quad \|\{\theta_k \in [0,1] : \theta_k + a(\theta_k) \leq s\}\|$$

and a corresponding cost

$$c(F_a) \quad = \quad \int_0^1 C(a(\theta_k))d\theta_k \quad = \quad \int_0^1 \gamma(a(\theta_k))^2 d\theta_k$$

Let $\mathcal{A}$ denote the set of mappings $a : [0, 1] \rightarrow \mathbb{R}^+$ with the property that $t + a(t)$ is nondecreasing.[17] The set $\mathcal{F} = \{F_a\}_{a \in \mathcal{A}}$ is compact, and c is continuous, so our results hold – we know a symmetric investment equilibrium must exist, be essentially unique, and be the same across standard auction formats.

We can build further intuition by solving explicitly for this symmetric investment equilibrium. Rather than choosing a_k at cost γa_k^2, think of a bidder with ex ante type θ_k choosing a final type $s_k \geq \theta_k$ at cost $\gamma(s_k - \theta_k)^2$. Knowing θ_k, the bidder will solve

$$\max_{s_k \geq \theta_k} \left\{ U(s_k) - \gamma(s_k - \theta_k)^2 \right\}$$

where $U(s_k)$ is the expected surplus in the auction (which depends of course on the other bidders' strategies). From the usual envelope theorem argument, at any symmetric investment strategy of one's rivals,

$$U(s_k) \quad = \quad \int_0^{s_k} F^{K-1}(t)dt$$

where F is the distribution of each rival's type s_j given their investment strategy. So the bidder will be solving

$$\max_{s_k \geq \theta_k} \left\{ \int_0^{s_k} F^{K-1}(t)dt - \gamma(s_k - \theta_k)^2 \right\}$$

This is differentiable, with first-order condition

$$F^{K-1}(s_k) - 2\gamma(s_k - \theta_k) \quad = \quad 0$$

Guessing that $s_k(\theta_k)$ will be strictly increasing in equilibrium, $F(s_k)$ (the probability in a symmetric investment equilibrium that $s_j < s_k$) is the probability that $\theta_j < \theta_k$, which (since $\theta_j \sim U[0, 1]$) is θ_k; the first-order

[17] Mappings without this property are at least weakly dominated, as a bidder can get the same distribution of valuations at weakly lower cost by choosing a mapping $a \in \mathcal{A}$.

condition therefore becomes $\theta_k^{K-1} = 2\gamma a(\theta_k)$, giving $a(\theta_k) = \frac{1}{2\gamma}\theta_k^{K-1}$ as the symmetric investment equilibrium.

A.2 Examples of $r = v_0$ and $r > v_0$ seller-optimal

For "classic" examples of seller-optimal reserve prices being either at or above the seller's cost, we port over to our notation the classic entry models of Levin and Smith (1994) and Samuelson (1985), respectively. For both, we assume a single prize ($N = 1$ with $\alpha_1 = 1$) and two bidders. In either case, a bidder's valuation s_i is drawn from the uniform distribution on $[0, 1]$, and a bidder must incur a cost c to participate in the auction. The difference is that in the model of Levin and Smith, bidders must decide whether to enter before learning their type, while in Samuelson, bidders learn their types before making their entry decision.

To map this to our model, think of $\mathcal{F} = \{F_\eta\}_{\eta\in[0,1]}$, where η corresponds to a bidder's probability of entry. (In the Samuelson model, this means that bidders with types above $1 - \eta$ will enter, and those with types below $1 - \eta$ will not.) For the Levin and Smith model,

$$F_\eta(s) \quad = \quad (1 - \eta) + \eta s$$

and for the Samuelson model,

$$F_\eta(s) \quad = \quad \max\{s, 1 - \eta\}$$

and for either model, $c(F_\eta) = c\eta$ with a constant $c > 0$.

In the former model, a reserve price of r induces an equilibrium "investment" (entry) level of $\eta^* = \frac{1.5}{1-r} - \frac{3c}{(1-r)^3}$ when this is between 0 and 1.[18] Whenever $\eta^* \in (0, 1)$, it is straightforward to see that $r = v_0$ maximizes seller profit, since $r = v_0$ is efficient and expected bidder surplus is zero. As an example, at $c = 0.2$, reserve prices between 0 and 0.367 induce interior η^*, and therefore $r = v_0$ is seller-optimal for v_0 in this range.

In the latter model, the equilibrium entry level is determined by the indifference of the marginal entrant, which requires $(1 - \eta^*)(1 - \eta^* - r) = c$, or $\eta^* = 1 - \frac{1}{2}\left(r + \sqrt{r^2 + 4c}\right)$. We can verify numerically that this time, at say $c = 0.2$ and $v_0 = 0$, seller profit is strictly increasing in r up to the profit-maximizing level of about 0.415.

A.3 Proof of Proposition 1

We begin by establishing that a bidder's incentives in the pre-auction investment game are determined by the auction's equilibrium allocation rule following the anticipated investment choices:

Lemma 5. *Fix a (not necessarily symmetric) profile of beliefs* $\mathbf{F} = (F_{\sigma_1}, F_{\sigma_2}, \ldots, F_{\sigma_K}) \in (\text{conv}(\mathcal{F}))^K$*. If two auctions will yield the same equilibrium allocation rule at* $\mathbf{F}$*, then the private benefit to bidder* k *of covertly switching from* F_k *to* F'_k *is the same for the two auctions.*

Proof. Let $u_k(t_k, \mathbf{F})$ denote bidder k's equilibrium interim expected payoff when it's common knowledge that bidder valuations are drawn from $\mathbf{F}$ and bidder k's realized valuation is t_k. By usual envelope theorem

[18] Entry gives a bidder expected payoff $\int_r^1 (s-r)ds$ when the opposing bidder does not enter, and expected payoff $\int_r^1 s(1-s)ds$ when he does; solving $(1-\eta^*)\int_r^1 (s-r)ds + \eta^* \int_r^1 s(1-s)ds = c$ for gives the level of entry η^* for one bidder that makes the other bidder indifferent between entering and not entering.

arguments,

$$u_k(t_k, \mathbf{F}) \quad = \quad u_k(0, \mathbf{F}) + \int_0^{t_k} \sum_{n=1}^{N} \alpha_n \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds$$

where $\Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F})$ is based on the equilibrium allocation rule at $\mathbf{F}$. If investment is covert, then whatever distribution bidder k chooses, once his valuation t_k is realized, this is his expected payoff: his opponents bid as if his value was drawn from F_{σ_k}, so he faces the same optimization problem as if he had drawn t_k from F_{σ_k}, so his optimal bid and expected payoff are the same. If he actually chooses F'_k, his ex ante expected payoff is

$$U_k(F'_k, \mathbf{F}) \quad = \quad \int_0^{\bar{s}} u_k(t_k, \mathbf{F}) dF'_k(t_k) - c(F'_k)$$

whether or not $F'_k = F_{\sigma_k}$. Similar to Myerson (1981), we can plug the expression above for $u_k(t_k, \mathbf{F})$ into this last expression, switch the order of integration, and evaluate the (new) integral to get

$$U_k(F'_k, \mathbf{F}) \quad = \quad u_k(0, \mathbf{F}) + \int_0^{\bar{s}} (1 - F'_k(s)) \sum_{n=1}^{N} \alpha_n \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds - c(F'_k)$$

and therefore

$$U_k(F'_k, \mathbf{F}) - U_k(F_k, \mathbf{F}) \quad = \quad \int_0^{\bar{s}} \sum_{n=1}^{N} \alpha_n (F_k(s) - F'_k(s)) \Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F}) ds - c(F'_k) + c(F_k)$$

If two auctions implement the same equilibrium allocation at $\mathbf{F}$, then $\Pr(k \;\; wins \;\; prize \;\; n | s_k = s, \mathbf{F})$ is the same for both, so they have the same value for $U_k(F'_k, \mathbf{F}) - U_k(F_k, \mathbf{F})$, giving the result. □

From here, we can prove Proposition 1, which is the claim that if investment is covert and two auctions are both constrained efficient when symmetric, they have the same set of symmetric investment equilibria. Given a mixed strategy $\hat{\sigma}$ over $\mathcal{F}$, let $F_{\hat{\sigma}} \in \text{conv}(\mathcal{F})$ be the corresponding compound distribution, and $\hat{\mathbf{F}} = (F_{\hat{\sigma}}, F_{\hat{\sigma}}, \ldots, F_{\hat{\sigma}})$. All bidders playing $\hat{\sigma}$ is an equilibrium if and only if

$$U_k(F', \hat{\mathbf{F}}) - U_k(F, \hat{\mathbf{F}}) \quad \leq \quad 0 \qquad \text{for every } F \in \text{supp}(\hat{\sigma}) \text{ and } F' \in \mathcal{F} \tag{4}$$

Two auctions which are constrained efficient when symmetric implement the same allocation at $\hat{\mathbf{F}}$; by Lemma 5, $U_k(F', \hat{\mathbf{F}}) - U_k(F, \hat{\mathbf{F}})$ is the same for the two auctions, and (4) either holds for both or for neither, giving the result. □

A.4 Proof of Lemma 1 (existence of symmetric investment equilibrium)

The result is a direct application of Glicksberg's (1952) or Fan's (1952) extensions of Kakutani's fixed point theorem to infinite-dimensional strategy spaces. Glicksberg (1952) proves existence of a Nash equilibrium in a two-player game with compact Hausdorff pure strategy spaces A_1 and A_2 by applying his fixed point theorem to the correspondence from $\Delta A_1 \times \Delta A_2$ to itself that takes a mixed strategy profile (σ_1, σ_2) to the set $(BR_1(\sigma_2), BR_2(\sigma_1))$, whose fixed point is a mixed strategy equilibrium. To prove a *symmetric* investment equilibrium exists for a game where the players all have pure strategy space $\mathcal{F}$, we make the same argument, but for the correspondence from $\Delta\mathcal{F}$ to $\Delta\mathcal{F}$ taking a mixed strategy σ to the set of one player's best responses when the other $K-1$ players are playing σ.[19] The only requirements for this to work are that the pure strategy space $\mathcal{F}$ be compact (assumed) and Hausdorff (trivial since we placed a metric on it), and that the payoff function is continuous as a function of the pure strategies.[20]

A.5 Proof of Lemma 2 (strategic substitutes)

The private gain from switching from σ_k to σ_k' is

$$U_k(\sigma_k', \cdot) - U_k(\sigma_k, \cdot) = \int_0^{\bar{s}} \sum_{n=1}^{N} \alpha_n (F_{\sigma_k}(s) - F_{\sigma_k'}(s)) \Pr(k \;\; wins \;\; prize \;\; n | s_k = s) ds - c(\sigma_k') + c(\sigma_k)$$

Letting $z_i(s)$ be the probability (given $\sigma_{-j,k}$) that exactly i bidders other than j and k have valuations above s, then for $s \geq r$,

$$\Pr(k \;\; wins \;\; prize \;\; n | s_k = s) = F_{\sigma_j}(s) z_{n-1}(s) + (1 - F_{\sigma_j}(s)) z_{n-2}(s)$$

for $n > 1$, and $F_{\sigma_j}(s) z_0(s)$ for $n = 1$. Defining $z_{-1} = 0$, we can then write

$$\begin{aligned} U_k(\sigma_k', \cdot) - U_k(\sigma_k, \cdot) &= \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) \sum_{n=1}^{N} \alpha_n \left[F_{\sigma_j}(s) z_{n-1}(s) + (1 - F_{\sigma_j}(s)) z_{n-2}(s)\right] ds - c(\sigma_k') + c(\sigma_k) \\ &= \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) \sum_{n=1}^{N} \alpha_n F_{\sigma_j}(s) \left[z_{n-1}(s) - z_{n-2}(s)\right] ds \\ &\quad + \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) \sum_{n=1}^{N} \alpha_n z_{n-2}(s) ds - c(\sigma_k') + c(\sigma_k) \end{aligned}$$

Define Z as the entire second line, and note that it does not depend on F_{σ_j}, so

$$\begin{aligned} U_k(\sigma_k', \cdot) - U_k(\sigma_k, \cdot) &= Z + \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) \sum_{n=1}^{N} \alpha_n F_{\sigma_j}(s) \left[z_{n-1}(s) - z_{n-2}(s)\right] ds \\ &= Z + \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) F_{\sigma_j}(s) \left[\sum_{n=1}^{N} \alpha_n z_{n-1}(s) - \sum_{n=0}^{N-1} \alpha_{n+1} z_{n-1}(s)\right] ds \\ &= Z + \int_r^{\bar{s}} \left(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\right) F_{\sigma_j}(s) \left[\alpha_N z_{N-1}(s) + \sum_{n=1}^{N} (\alpha_n - \alpha_{n+1}) z_{n-1}(s)\right] ds \end{aligned}$$

[19]For good intuition on the underlying mechanics of the proof, see also lecture 6 of Asu Ozdaglar's 2010 lecture notes for the MIT course "Game Theory with Engineering Applications," available at https://ocw.mit.edu/courses/6-254-game-theory-with-engineering-applications-spring-2010/pages/lecture-notes/

[20]To see the latter, note that for the VCG mechanism,

$$U_k(F_k, F_{-k}) = \sum_{n=1}^{N} \alpha_n \int_0^{\bar{s}} (1 - F_k(s)) \Pr(n-1 \text{ of bidder } k\text{'s opponents have valuations above } s) ds - c(F_k)$$

This is continuous in F_k, since c is continuous, and a change from F_k to F_k' with $\|F_k - F_k'\| \leq \epsilon$ changes $1 - F_k(s)$ by less than ϵ except on a set of measure ϵ, and changes it by less than 1 everywhere, so the overall change in the n^{th} integral is no more than $2\epsilon\alpha_n$. Likewise, it's continuous in F_{-k} because an ϵ change in one of the other bidders' value distributions likewise changes each of the probabilities by less than ϵ except on a set of s having measure less than ϵ.

Letting $U_k(\sigma_k, \sigma_j)$ denote bidder k's expected payoff as a function of his own and bidder j's strategies (holding fixed the other $K-2$ bidders' strategies),

$$
\begin{aligned}
& (U_k(\sigma_k', \sigma_k) - U_k(\sigma_k, \sigma_k)) - (U_k(\sigma_k', \sigma_k') - U_k(\sigma_k, \sigma_k')) \\
& \quad = \int_r^{\bar{s}} \Big(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\Big)\Big(F_{\sigma_k}(s) - F_{\sigma_k'}(s)\Big)\left[\alpha_N z_{N-1}(s) + \sum_{n=1}^{N}(\alpha_n - \alpha_{n+1}) z_{n-1}(s)\right] ds
\end{aligned}
$$

Since α_n is decreasing in n, the term in square brackets is positive, and therefore the integrand is always positive, ensuring that $U_k(\sigma_k', \sigma_k) - U_k(\sigma_k, \sigma_k) \geq U_k(\sigma_k', \sigma_k') - U_k(\sigma_k, \sigma_k')$. Further, the integrand is strictly positive wherever both $F_{\sigma_k}(s) - F_{\sigma_k'}(s) \neq 0$ and $\sum_{n-1}^{N} z_{n-1}(s) \neq 0$, making $U_k(\sigma_k', \sigma_k) - U_k(\sigma_k, \sigma_k)$ *strictly* greater than $U_k(\sigma_k', \sigma_k') - U_k(\sigma_k, \sigma_k')$ as long as F_{σ_k} and $F_{\sigma_k'}$ differ over any range where $\sum_{n=1}^{N} z_n(s) > 0$, i.e., where there is positive probability that fewer than N rivals have valuations above s.

A.6 Proof of Lemma 3 (symmetric investment equilibrium maximizes $\tilde{P}(\cdot)$)

The claim is that the symmetric investment equilibrium maximizes $\tilde{P}(\cdot)$, so suppose not for contradiction: suppose $\hat{\sigma}$ is a symmetric investment equilibrium, but there is some σ' with $\tilde{P}(\sigma') > \tilde{P}(\hat{\sigma})$. Then

$$
\begin{aligned}
0 \quad < \quad & P(\sigma', \sigma', \sigma', \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \hat{\sigma}) \\
\\
= \quad & P(\sigma', \sigma', \sigma', \ldots, \sigma') - P(\hat{\sigma}, \sigma', \sigma', \ldots, \sigma') \\
& + P(\hat{\sigma}, \sigma', \sigma', \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \sigma', \ldots, \sigma') \\
& + P(\hat{\sigma}, \hat{\sigma}, \sigma', \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \sigma') \\
& + \ldots \\
& + P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \hat{\sigma})
\end{aligned}
$$

Since P is symmetric in its arguments, we can rewrite this as

$$
\begin{aligned}
0 \quad < \quad & P(\sigma', \sigma', \sigma', \ldots, \sigma') - P(\hat{\sigma}, \sigma', \sigma', \ldots, \sigma') \\
& + P(\sigma', \hat{\sigma}, \sigma', \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \sigma', \ldots, \sigma') \\
& + P(\sigma', \hat{\sigma}, \hat{\sigma}, \ldots, \sigma') - P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \sigma') \\
& + \ldots \\
& + P(\sigma', \hat{\sigma}, \hat{\sigma}, \ldots, \hat{\sigma}) - P(\hat{\sigma}, \hat{\sigma}, \hat{\sigma}, \ldots, \hat{\sigma})
\end{aligned} \tag{5}
$$

where each line is the change in P from bidder 1 switching from $\hat{\sigma}$ to σ', with some of his opponents playing σ' and the rest $\hat{\sigma}$. Since P is a potential function (Proposition 2), $P(\sigma', \sigma_{-1}) - P(\hat{\sigma}, \sigma_{-1}) = U_1(\sigma', \sigma_{-1}) - U_1(\hat{\sigma}, \sigma_{-1})$; so each line on the right-hand side of Equation 5 is bidder 1's change in payoff from switching from $\hat{\sigma}$ to σ'. Lemma 2 then tells us that the last line on the right-hand side is weakly higher than any of the others, as a bidder's gain from switching from $\hat{\sigma}$ to σ' is higher when more of his opponents are playing $\hat{\sigma}$. But since the differences on the right sum to a strictly positive number, that means that the biggest one is strictly positive, or $U_1(\sigma', \hat{\sigma}, \ldots, \hat{\sigma}) - U_1(\hat{\sigma}, \hat{\sigma}, \ldots, \hat{\sigma}) > 0$, making σ' a profitable deviation away from a symmetric profile of $\hat{\sigma}$, and therefore contradicting $\hat{\sigma}$ being a symmetric investment equilibrium.

A.7 Proof of Lemma 4 (uniqueness)

By Lemma 3, σ' and σ'' must both maximize $\tilde{P}$, so $P(\sigma'', \sigma'', \dots, \sigma'') - P(\sigma', \sigma', \dots, \sigma') = 0$. Like in the proof of Lemma 3. we decompose that difference into a sum of smaller differences, and then use the symmetry of P and the fact that P is a potential function to get

$$\begin{aligned} 0 &= P(\sigma'', \sigma'', \sigma'', \dots, \sigma'') - (\sigma', \sigma', \sigma', \dots, \sigma') \\ \\ &= U_1(\sigma'', \sigma'', \sigma'', \dots, \sigma'') - U_1(\sigma', \sigma'', \sigma'', \dots, \sigma'') \\ &\quad + U_1(\sigma'', \sigma', \sigma'', \dots, \sigma'') - U_1(\sigma', \sigma', \sigma'', \dots, \sigma'') \\ &\quad + U_1(\sigma'', \sigma', \sigma', \dots, \sigma'') - U_1(\sigma', \sigma', \sigma', \dots, \sigma'') \\ &\quad + \dots \\ &\quad + U_1(\sigma'', \sigma', \sigma', \dots, \sigma') - U_1(\sigma', \sigma', \sigma', \dots, \sigma') \end{aligned}$$

Lemma 2 tells us that if $F_{\sigma'}(s) \neq F_{\sigma''}(s)$ for some $s \in (r, \bar{s})$, then the last line is strictly greater than any of the others (the value to a bidder of switching from σ' to σ'' is higher when his opponents play σ' than σ''); since the lines all sum to zero, the last line would have to be strictly positive. This would imply that σ'' is a profitable deviation from the symmetric investment equilibrium σ', creating a contradiction; so $F_{\sigma'}(s)$ and $F_{\sigma''}(s)$ can't be different anywhere on $(r, \bar{s})$, proving the result.

A.8 Proof of Theorem 3 when $r \neq v_0$

In the text, we proved Theorem 3 for $r = v_0$. When $r \neq v_0$, that $P(\cdot)$ and $W(\cdot)$ differ by a term that depends on $\{F_\sigma^{(n)}(s)\}_{n=1,2,\dots,N}$, so "not maximizing P" is different from "not maximizing W". Define

$$A(\sigma) \quad = \quad -(r - v_0) \sum_{n=1}^{N} \alpha_n F_\sigma^{(n)}(r)$$

and recall that $W(\sigma) = P(\sigma) + A(\sigma)$ plus a constant.

As in the text, consider the case where $\sigma = pF' + (1-p)F''$ is a symmetric investment equilibrium. We will show that for some small (β, γ),

$$\sigma' \quad = \quad ((p+\beta)F' + (1-p-\beta)F'', (p-\gamma)F' + (1-p+\gamma)F'', pF' + (1-p)F'', \cdots, pF' + (1-p)F'')$$

gives strictly higher total surplus than σ. We will do this by finding (β, γ) such that $P(\sigma') > P(\sigma)$ and $A(\sigma') = A(\sigma)$.

By the same logic as in the text, $P(\sigma') > P(\sigma)$ as long as β and γ are both strictly positive. This is because when we start at σ, moving bidder 1 incrementally toward F' does not change P at all (since at equilibrium, bidder 1 must be indifferent among F', F'', and any mix of the two); but once bidder 1 is more likely to be playing F', bidder 2 strictly prefers F'' to F', and so increasing γ incrementally now increases P. Thus, to prove the result, we need only show that we can find $(\beta, \gamma) \gg 0$ such that $A(\sigma') \geq A(\sigma)$.

Now, since r is fixed, it must be that either $F'(r) > F''(r)$, $F'(r) = F''(r)$, or $F'(r) < F''(r)$. If $F'(r) = F''(r)$, then $A(\cdot)$ does not depend on β or γ and $A(\sigma') = A(\sigma)$, so we're done. Without loss of generality, then, focus on the csae where $F'(r) > F''(r)$. This means $A(\sigma')$ is decreasing in β and increasing in γ. So $A(\sigma') < A(\sigma)$ when $\beta > 0 = \gamma$, and $A(\sigma') > A(\sigma)$ when $\gamma > 0 = \beta$. But then by continuity, we can find β, γ both strictly positive such that $A(\sigma') = A(\sigma)$. This completes the proof.

A.9 Proof of Theorem 4 (pure strategy equilibrium)

The result is that the symmetric investment equilibrium is in pure strategies, so suppose not – suppose there exists a mixed strategy equilibrium σ^*.

Think of σ^* as a measure on $\mathcal{F}$, with $\sigma^*(\mathcal{F}) = 1$. The support of σ^* is defined as the set of points $F \in \mathcal{F}$ such that $\sigma(O) > 0$ for all open neighborhoods O of F. Note that if the support of σ^* does not contain two distinct points, then σ^* is a pure strategy.[21]

So pick two distinct distributions F_1 and F_2 in the support of σ^* such that $\sigma^*(O) > 0$ for any open neighborhood around either F_1 or F_2. Find the distribution F_3 that "dominates" F_1 and F_2 in the sense given in the Theorem: $c(F_3) \leq \frac{1}{2}c(F_1) + \frac{1}{2}c(F_2)$ and $F_3(s) \leq \frac{1}{2}F_1(s) + \frac{1}{2}F_2(s)$, with strict inequality somewhere on $(r, \bar{s})$. Letting $U_k(\sigma, \sigma^*)$ continue to denote the expected surplus (net of pre-auction investment cost) of playing strategy σ (either pure or mixed) against opponents all playing σ^*, define

$$
\begin{aligned}
\delta \quad &\equiv \quad U_k(F_3, \sigma^*) - U_k\left(\tfrac{1}{2}F_1 + \tfrac{1}{2}F_2, \sigma^*\right) \\
&= \quad \int_0^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left(\frac{1}{2}F_1(s) + \frac{1}{2}F_2(s) - F_3(s)\right) \Pr(k \text{ wins prize } n \mid s_k = s,\, \sigma^*) ds - c(F_3) + \frac{1}{2}c(F_1) + \frac{1}{2}c(F_2) \\
&\geq \quad \int_0^{\bar{s}} \sum_{n=1}^{N} \alpha_n \left(\frac{1}{2}F_1(s) + \frac{1}{2}F_2(s) - F_3(s)\right) \Pr(k \text{ wins prize } n \mid s_k = s,\, \sigma^*) ds
\end{aligned}
$$

(using expressions from the proof of Proposition 1). Now, since CDFs are right-continuous, saying $\frac{1}{2}F_1(s) + \frac{1}{2}F_2(s) - F_3(s) > 0$ somewhere on $(r, \bar{s})$ implies it's strictly positive on an interval with positive measure; and since σ^* involves playing strategies near F_1 and F_2, anywhere $\frac{1}{2}F_1(s) + \frac{1}{2}F_2(s) > 0$ (which must hold wherever $\frac{1}{2}F_1(s) + \frac{1}{2}F_2(s) - F_3(s) > 0$), a bidder with valuation s whose opponents are playing σ^* has a positive probability of winning each of the prizes. So the integrand is strictly positive on an interval with positive measure (and non-negative everywhere), so $\delta > 0$.

Next, choose $\epsilon^* > 0$ small enough so that $|U_k(F_1', \sigma^*) - U_k(F_1, \sigma^*)| \leq \frac{1}{2}\delta$ for every $F_1' \in B_{\epsilon^*}(F_1)$, $|U_k(F_2', \sigma^*) - U_k(F_2, \sigma^*)| \leq \frac{1}{2}\delta$ for every $F_2' \in B_{\epsilon^*}(F_2)$, and $B_{\epsilon^*}(F_1) \cap B_{\epsilon^*}(F_2) = \emptyset$. (We know this is possible because $U_k(\cdot, \sigma^*)$ is continuous in its first argument.) Let $m_1 = \sigma^*(B_{\epsilon^*}(F_1))$ and $m_2 = \sigma^*(B_{\epsilon^*}(F_2))$; by our choice of F_1 and F_2, both are strictly positive; choose $m > 0$ smaller than both of them.

Now, starting with σ^*, we'll construct a new mixed strategy σ' by shifting probability from $B_{\epsilon^*}(F_1)$ onto F_1 itself, and shifting probability from $B_{\epsilon^*}(F_2)$ onto F_2 itself. Formally, define a new measure σ' on $\mathcal{F}$ by assigning for any set $U \subseteq \mathcal{F}$,

$$
\sigma'(U) \quad = \quad \sigma^*(U) - \frac{m}{m_1}\sigma^*(U \cap B_{\epsilon^*}(F_1)) + m\mathbf{1}(F_1 \in U) - \frac{m}{m_2}\sigma^*(U \cap B_{\epsilon^*}(F_2)) + m\mathbf{1}(F_2 \in U)
$$

Our goal here was to construct a new mixed strategy that is "close to" σ^* but puts positive probability mass on the two strategies F_1 and F_2, by shifting probability onto those two strategies from nearby ones. Since

[21] If F^* is the only element in the support of σ^*, then for each $F \neq F^*$, pick O_F an open neighborhood of F with $\sigma^*(O_F) = 0$. For any n, the set $\{B_{1/n}(F^*)\} \cup \bigcup_{F \neq F^*} O_F$ is an open cover of $\mathcal{F}$; since $\mathcal{F}$ is compact, a finite subcover exists, call it $\mathcal{O}$. Then

$$
1 \quad = \quad \sigma^*(\mathcal{F}) \quad \leq \quad \sigma^*\left(\cup_{O \in \mathcal{O}} O\right) \quad \leq \quad \sum_{O \in \mathcal{O}} \sigma^*(O) \quad \leq \quad \sigma^*(B_{1/n}(F^*))
$$

since $\sigma^*(O) = 0$ for all the other sets in $\mathcal{O}$. So $\sigma^*(B_{1/n}(F^*)) = 1$ for every $n = 1, 2, \ldots$, and therefore σ^* puts weight 1 on F^*. If σ^* has no points in its support, a similar argument implies $\sigma^*(\mathcal{F}) = 0$, a contradiction.

we are shifting a total probability of m from strategies within ϵ^* of F_1 to F_1, and total probability of m from strategies within ϵ^* of F_2 to F_2, by our construction of ϵ^*,

$$|U_k(\sigma', \sigma^*) - U_k(\sigma^*, \sigma^*)| \quad \leq \quad m\frac{\delta}{2} + m\frac{\delta}{2} \quad = \quad m\delta$$

Next, we define an additional mixed strategy σ'' by shifting equal probability weight away from F_1 and F_2 and onto F_3. Formally, define σ'' as a measure on $\mathcal{F}$ by

$$\sigma''(U) \quad = \quad \sigma'(U) - m\mathbf{1}(F_1 \in U) - m\mathbf{1}(F_2 \in U) + 2m\mathbf{1}(F_3 \in U)$$

That is, having constructed σ' such that $\sigma'(F_1) \geq m$ and $\sigma'(F_2) \geq m$, we now construct σ'' from σ' by replacing probability mass m on each of F_1 and F_2 with probability mass $2m$ on F_3. By the definition of δ,

$$U_k(\sigma'', \sigma^*) - U_k(\sigma', \sigma^*) \quad = \quad 2m\delta$$

and therefore

$$U_k(\sigma'', \sigma^*) - U_k(\sigma^*, \sigma^*) \quad = \quad U_k(\sigma'', \sigma^*) - U_k(\sigma', \sigma^*) + U_k(\sigma', \sigma^*) - U_k(\sigma^*, \sigma^*) \quad \geq \quad 2m\delta - m\delta \quad > \quad 0$$

Thus, σ^* can't be a symmetric investment equilibrium if it has two strategies F_1 and F_2 in its support, so the symmetric investment equilibrium must be in pure strategies.

A.10 Proof of Theorem 5 (first best)

We begin with the following lemma:

Lemma 6. *Fix the reserve price $r = v_0$. For any three distributions $F_1, F_2, F' \in \mathcal{F}$, if (i) $c(F') \leq \frac{1}{2}c(F_1) + \frac{1}{2}c(F_2)$, and (ii) $F'(s) \leq \sqrt{F_1(s)F_2(s)}$ everywhere, with strict inequality somewhere on $(v_0, \bar{s})$, the profile $(F', F', F_3, \ldots, F_K)$ gives strictly higher total surplus than $(F_1, F_2, F_3, \ldots, F_K)$.*

Lemma 6 therefore establishes that under the conditions of the theorem, no asymmetric investment profile can be the global maximizer of total surplus with $r = v_0$; since total surplus is continuous on a compact space, a maximizer exists, so it must be symmetric.

Proof of Lemma 6. Recall that with $r = v_0$, total surplus coincides (up to a constant) with the potential function, which we can write as

$$W \quad = \quad P \quad = \quad -\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n F^{(n)}(s) ds - \sum_{k=1}^{K} c(F_k)$$

where $F^{(n)}(s)$ is the CDF of the n^{th} highest valuation, given the investment profile chosen. We'll rewrite this as

$$W \quad = \quad -\int_r^{\bar{s}} I(s) ds - \sum_{k=1}^{K} c(F_k)$$

where

$$I(s) \quad = \quad \sum_{n=1}^{N} \alpha_n F^{(n)}(s)$$

Note that we can write $F^{(n)}(s)$ as

$$F^{(n)}(s) = \sum_{\|A\|<n} \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s)$$

where the A are subsets of bidders (since this is the combined probability of all the ways that fewer than n bidders have valuations above s, and therefore that the n^{th} highest is below s), and therefore

$$I(s) = \sum_{n=1}^{N} \alpha_n \sum_{\|A\|<n} \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s)$$

Changing the order of sums gives

$$I(s) = \sum_{\|A\|<N} \left(\sum_{n=\|A\|+1}^{N} \alpha_n \right) \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s)$$

Next, we'll break up this sum over different subsets of bidders, based on whether the set A includes bidders 1 and 2, one of them, or neither of them:

$$\begin{aligned} I(s) = & \sum_{\|A\|<N;1,2\in A} \left(\sum_{n=\|A\|+1}^{N} \alpha_n \right) \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s) \\ & + \sum_{\|A\|<N;1\in A,2\notin A} \left(\sum_{n=\|A\|+1}^{N} \alpha_n \right) \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s) \\ & + \sum_{\|A\|<N;2\in A,1\notin A} \left(\sum_{n=\|A\|+1}^{N} \alpha_n \right) \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s) \\ & + \sum_{\|A\|<N;1,2\notin A} \left(\sum_{n=\|A\|+1}^{N} \alpha_n \right) \prod_{k\in A}(1-F_k(s)) \prod_{k\notin A} F_k(s) \end{aligned}$$

Next, to each set of bidders A, we associate the subset A' which is all the bidders other than 1 and 2 in A; so a set A of size n that contains bidders 1 and 2, corresponds to a set A' of size $n-2$ that does not. For each set A' of bidders excluding 1 and 2, we also associate the function $\zeta_{A'} = \prod_{k\in A'}(1-F_k(s)) \prod_{k\notin A',k\neq 1,2} F_k(s)$ so that we can shorten our expressions. We rewrite $I(s)$ as

$$\begin{aligned} I(s) = & \sum_{\|A'\|<N-2} \left(\sum_{n=\|A'\|+3}^{N} \alpha_n \right) (1-F_1(s))(1-F_2(s))\zeta_{A'} + \sum_{\|A'\|<N-1} \left(\sum_{n=\|A'\|+2}^{N} \alpha_n \right) (1-F_1(s))F_2(s)\zeta_{A'} \\ & + \sum_{\|A'\|<N-1} \left(\sum_{n=\|A'\|+2}^{N} \alpha_n \right) F_1(s)(1-F_2(s))\zeta_{A'} + \sum_{\|A'\|<N} \left(\sum_{n=\|A'\|+1}^{N} \alpha_n \right) F_1(s)F_2(s)\zeta_{A'} \end{aligned}$$

Next, we regroup across sums by sets A', rewriting $I(s)$ as

$$\begin{aligned}
I(s) &= \sum_{\|A'\|=N-1} \alpha_N F_1 F_2 \zeta_{A'} \\
&\quad + \sum_{\|A'\|=N-2} \left(\alpha_N(1-F_1)F_2 + \alpha_N F_1(1-F_2) + (\alpha_{N-1}+\alpha_N)F_1F_2\right)\zeta_{A'} \\
&\quad + \sum_{\|A'\|<N-2} \left[\left(\sum_{n=\|A'\|+3}^{N} \alpha_n\right)(1-F_1)(1-F_2) + \left(\sum_{n=\|A'\|+2}^{N} \alpha_n\right)((1-F_1)F_2 + F_1(1-F_2)) \right. \\
&\quad \left. + \left(\sum_{n=\|A'\|+1}^{N} \alpha_n\right) F_1F_2\right]\zeta_{A'} \\
&= \sum_{\|A'\|=N-1} \alpha_N F_1 F_2 \zeta_{A'} \\
&\quad + \sum_{\|A'\|=N-2} \alpha_{N-1} F_1F_2\zeta_{A'} + \sum_{\|A'\|=N-2} \alpha_N\left((1-F_1)F_2 + F_1(1-F_2) + F_1F_2\right)\zeta_{A'} \\
&\quad + \sum_{\|A'\|<N-2} \alpha_{\|A'\|+1} F_1F_2\zeta_{A'} + \sum_{\|A'\|<N-2} \alpha_{\|A'\|+2}\left[(1-F_1)F_2 + F_1(1-F_2) + F_1F_2\right]\zeta_{A'} \\
&\quad + \sum_{\|A'\|<N-2} \left(\sum_{n=\|A'\|+3}^{N} \alpha_n\right)\left[(1-F_1)(1-F_2) + (1-F_1)F_2 + F_1(1-F_2) + F_1F_2\right]\zeta_{A'} \\
&= \sum_{\|A'\|=N-1} \alpha_N F_1F_2\zeta_{A'} + \sum_{\|A'\|=N-2} \alpha_{N-1}F_1F_2\zeta_{A'} + \sum_{\|A'\|<N-2} \alpha_{\|A'\|+1}F_1F_2\zeta_{A'} \\
&\quad + \sum_{\|A'\|=N-2} \alpha_N\left(F_1+F_2-F_1F_2\right)\zeta_{A'} + \sum_{\|A'\|<N-2} \alpha_{\|A'\|+2}\left[F_1+F_2-F_1F_2\right]\zeta_{A'} \\
&\quad + \sum_{\|A'\|<N-2} \left(\sum_{n=\|A'\|+3}^{N} \alpha_n\right)[1]\,\zeta_{A'}
\end{aligned}$$

Now, what we want is for $I(s)$ to weakly fall when F_1 and F_2 are both replaced by $F' \leq \sqrt{F_1F_2}$. This last string of simplifications make it clear that this will happen if $F'F' \leq F_1F_2$ and $2F' - (F')^2 \leq F_1 + F_2 - F_1F_2$. Now, the first is by assumption. As for the second, note that the geometric mean of two numbers is always weakly lower than the arithmetic mean, so if we let $F' = \sqrt{F_1F_2} - \epsilon$ for some $\epsilon \geq 0$, then $F' \leq \frac{1}{2}F_1 + \frac{1}{2}F_2 - \epsilon$, and therefore

$$\begin{aligned}
2F' - (F')^2 &\leq 2\left(\tfrac{1}{2}F_1 + \tfrac{1}{2}F_2 - \epsilon\right) - \left(\sqrt{F_1F_2} - \epsilon\right)^2 \\
&= F_1 + F_2 - 2\epsilon - F_1F_2 + 2\epsilon\sqrt{F_1F_2} - \epsilon^2 \\
&= F_1 + F_2 - F_1F_2 - 2\epsilon(1 - \sqrt{F_1F_2}) - \epsilon^2 \\
&\leq F_1 + F_2 - F_1F_2
\end{aligned}$$

Thus, under the conditions of the Theorem, replacing (F_1, F_2) with (F', F') weakly increases $I(s)$.

Further, we've assumed $F'(s) < \sqrt{F_1(s)F_2(s)}$ on some range; so $I(s)$ strictly decreases on some range when (F_1, F_2) are replaced by (F', F').

By assumption, this switch also weakly decreases aggregate investment costs, since $2c(F') \leq c(F_1)+c(F_2)$. Thus, it strictly increases total surplus. This means that any asymmetric investment profile can't be optimal; since the global maximizer exists, it must be symmetric. □

To complete the proof of Theorem 5, we note that if W has a symmetric global maximizer when $r = v_0$, that global maximizer is also the symmetric maximizer of P, and therefore the symmetric investment equilibrium. Starting at any reserve price r' and investment profile $\mathbf{F}'$, note that $W(r', \mathbf{F}') \leq W(v_0, \mathbf{F}') \leq W(v_0, \mathbf{F}_{SIE})$, where $\mathbf{F}_{SIE}$ is the symmetric investment equilibrium at $r = v_0$ – the first inequality because $r = v_0$ is ex post efficient given any investment profile, the second inequality because we just established $\mathbf{F}_{SIE}$ maximizes surplus given $r = v_0$ – so the symmetric investment equilibrium at $r = v_0$ achieves first-best.

A.11 Proofs of Propositions 4, 5, 6, and 7

Proposition 4 was proved in the body. The key step was demonstrating that the payoff difference

$$\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n (F_{\sigma_1}(s) - F_{\sigma_2}(s)) y_n(s) ds - c(F_{\sigma_2}) + c(F_{\sigma_1})$$

must strictly increase as the parameter value γ goes from γ_1 to γ_2. That same fact will be essential in proving the other comparative statics.

Proof of Proposition 5. Let γ refer to the number of bidders K. Defining $\alpha_{N+1} = 0$, we can rewrite the expression above as

$$\int_r^{\bar{s}} (F_{\sigma_1}(s) - F_{\sigma_2}(s)) \sum_{n=1}^{N} (\alpha_n - \alpha_{n+1}) y_n^+(s) ds - c(F_{\sigma_2}) + c(F_{\sigma_1})$$

where $y_n^+(s) = \sum_{i=1}^{n} y_n(s)$ is the probability that $n-1$ *or fewer* opposing bidders have types above s when they're all playing σ_1. If $F_{\sigma_2} >_{FOSD} F_{\sigma_1}$, then the integrand is weakly positive everywhere; since $y_n^+(s)$ is obviously decreasing in K, the integral would therefore be weakly decreasing in K, and could not increase as K rose from K_1 to K_2. □

Proof of Proposition 6. Now let the parameter γ represent the cost parameter κ. Then the expression above is

$$\int_r^{\bar{s}} \sum_{n=1}^{N} \alpha_n (F_{\sigma_1}(s) - F_{\sigma_2}(s)) y_n(s) ds + \kappa[\bar{c}(F_{\sigma_1}) - \bar{c}(F_{\sigma_2})]$$

By inspection, this can only increase in κ if $\bar{c}(F_{\sigma_2}) \leq \bar{c}(F_{\sigma_1})$. Moreover, if $F_{\sigma_2}(s) \neq F_{\sigma_1}(s)$ for some $s > r$ and $F_{\sigma_2} >_{FOSD} F_{\sigma_1}$ then, together with $\bar{c}(F_{\sigma_2}) \leq \bar{c}(F_{\sigma_1})$, it follows that σ_1 cannot be chosen in symmetric investment equilibrium at parameter κ_1, a contradiction. □

Proof of Proposition 7. Let the parameter γ represent α. If $F_{\sigma_2} >_{FOSD} F_{\sigma_1}$, then $F_{\sigma_2}(s) \leq F_{\sigma_1}(s)$ everywhere, so the integrand is everywhere weakly positive and the integral therefore weakly decreases when α changes from α_1 to α_2, giving the same contradiction. □

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