UTMD Working Paper

The University of Tokyo
Market Design Center

UTMD-067

Reallocation-proofness in object reallocation problems with single-dipped preferences

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September 26, 2024

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Abstract

We examine the problem of reallocating indivisible objects among agents with single-dipped preferences with respect to a fixed order of objects. Our main axiom, *reallocation-proofness*, requires that no pair of agents can benefit by misrepresenting their preferences and swapping their assignments. In this setting, by invoking the recent work of Hu and Zhang (2024), we find that the top trading cycles rule (TTC) is characterized by *individual rationality*, *strategy-proofness*, and *reallocation-proofness*. Building on this, we extend the analysis in two directions. First, we explore the case where only pairwise exchanges are permitted, and show that under this constraint, the characterization of TTC holds even without *strategy-proofness*. Second, we consider a more general model where objects are arranged in a tree structure instead of in a line, and demonstrate that the characterization of TTC can be extended to this general setting as well.

Keywords: reallocation-proofness; single-dipped preferences; pair-efficiency; top trading cycles; housing markets.

JEL codes: C78; D47.

*We gratefully acknowledge the financial support from the Joint Usage/Research Center at ISER, Osaka University, and the Grant-in-Aid for Research Activity from the Japan Society for the Promotion of Science (JP20K01555, JP22H00062, and JP24K04801).

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1 Introduction

We examine the object reallocation problem á la Shapley and Scarf (1974), where each agent initially owns a heterogeneous, indivisible object and has preferences over the objects. A "rule" reallocates the objects such that each agent receives exactly one object, without any monetary transfers. For this problem, the top trading cycles rule (TTC), which selects the unique core allocation via David Gale's TTC algorithm (Roth and Postlewaite, 1977), has played a prominent role in the literature. The first characterization of TTC on the domain of strict preferences was provided by Ma (1994), based on *individual rationality* (no agent is worse off after the reallocation), *efficiency* (no chosen allocation can be improved such that no agent is worse off and some agent is better off), and *strategy-proofness* (no agent benefits from misrepresentation). Following Ma's study, various characterizations of TTC have been proposed.[1]

We focus on rules that satisfy *reallocation-proofness* (Moulin, 1995), a property requiring that no pair of agents can both strictly benefit from misrepresenting their preferences and swapping their assignments. Our previous study (Fujinaka and Wakayama, 2018) has already shown that when preferences are strict, TTC is the only rule that satisfies *individual rationality*, *reallocation-proofness*, and *strategy-proofness*. *Reallocation-proofness* can be weakened by excluding preference manipulations. Ekici (2024) introduces such a weaker version of *reallocation-proofness*, called *pair-efficiency*.[2] He then demonstrates that Fujinaka and Wakayama's characterization still holds when *reallocation-proofness* is weakened to *pair-efficiency*.

We confine our attention to the case where each agent has "single-dipped" preferences. Suppose the objects are ordered on a line. Each agent has single-dipped preferences with respect to the order; that is, he has a unique worst object, and his welfare strictly increases as his allocated object moves away from this object in either direction. For instance, consider the housing market problem, which is a classic example of the object reallocation problem. Suppose that all the houses are ordered from west to east by their locations. If each agent considers a house as being the worst owing to its location (having issues such as safety concerns,

[1]For additional characterizations of TTC on the domain of strict preferences, for example, see Takamiya (2001), Miyagawa (2002), Hashimoto and Saito (2015), Fujinaka and Wakayama (2018), and Chen and Zhao (2021). See also Morrill and Roth (2024) for the history of TTC and its generalizations and extensions.

[2]Ekici (2024) proposes *pair-efficiency* to weaken *efficiency* rather than *reallocation-proofness* and shows that this weakening does not affect Ma's (1994) characterization in the case of strict preferences.

the presence of a waste disposal facility, or city noise), and his welfare strictly increases as his allocated house moves away from this worst house, then his preferences are single-dipped.[3,4] Recently, Tamura (2023) (or Hu and Zhang (2024), respectively) finds that the characterization of TTC in terms of *efficiency* by Ma (1994) (or *pair-efficiency* by Ekici (2024), respectively) continues to hold even when agents' preferences are restricted to be single-dipped.[5] Considering that *pair-efficiency* is a weaker property than *reallocation-proofness*, Hu and Zhang's (2024) result implies that the characterization of TTC in terms of *reallocation-proofness* by Fujinaka and Wakayama (2018) continues to hold if preferences are restricted to be single-dipped (Corollary 2).[6]

This paper investigates the implications of *reallocation-proofness* in a framework that considers practical aspects of object reallocation problems. We first examine the case where exchange constraints are imposed. In many real-life applications, the number of agents involved in trading cycles is often limited because of legal or physical constraints.[7] Therefore, it is natural to impose constraints on the length of an exchange cycle. This paper focuses on the most stringent constraint: pairwise exchanges. It is revealed that under pairwise exchanges, TTC can be characterized by the combination of *individual rationality* and *reallocation-proofness* alone (Theorem 3). However, this characterization no longer holds when *reallocation-proofness* is weakened to *pair-efficiency* (Remark 3).

Next, we explore a more general model where objects are arranged on a "tree" (i.e., a connected graph with no cycles) instead of a line. For instance, in the hous-

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[3] In the UK, public housing exchange platforms, such as the website House Exchange (https://www.houseexchange.org.uk), allow tenants to swap homes. These platforms support exchanges involving three or more tenants as well as pairwise exchanges.

[4] Another example provided by Tamura (2023) is the scheduling of doctors for on-call emergency medical services during holiday seasons. If each doctor has the worst date (e.g., owing to family reasons), and his welfare strictly increases before and after this date, then his preferences are also single-dipped.

[5] Tamura (2023) also demonstrates that the characterization of TTC in terms of *endowments-swapping-proofess* (no pair of agents benefits from swapping their endowments before implementing the rule) still holds on the domain of single-dipped preferences. Fujinaka and Wakayama (2024) reveals that Tamura's result remains valid even when possible exchanges are limited.

[6] However, a gap remains in Hu and Zhang's (2024) proof. We fill this gap by offering a more rigorous proof of their theorem. For further details, see Remark 1.

[7] Vacation home exchange platforms, such as the website HomeExchange (https://www.homeexchange.com), allow only pairwise exchanges. As mentioned above, the UK's public housing exchange platforms impose no restrictions on the size of exchanges. However, as Balbuzanov (2015) points out, coordination difficulties (e.g., finding suitable moving dates) make longer exchange cycles infeasible. For the reallocation object problem with exchange constraints, see Nicolò and Rodríguez-Álvarez (2013, 2017), Balbuzanov (2015, 2020), Rodríguez-Álvarez (2023), and Fujinaka and Wakayama (2024).

ing market problem, suppose that all houses are located within a road network that has a tree structure. Similar to the case of preferences on a line, if each agent has the worst location and his welfare strictly increases as his allocated house moves away from this location toward an endpoint of the road network, then his preferences are single-dipped on the tree structure. Tamura (2023) reveals that the characterization of TTC in terms of *efficiency* for single-dipped preferences on a line also holds for single-dipped preferences on a tree. As in Tamura (2023), we can extend the characterization of TTC in terms of *pair-efficiency* (Ekici, 2024; Hu and Zhang, 2024) or *realocation-proofness* (Fujinaka and Wakayama, 2018) to this general model (Theorem 5 and Corollary 2).

The rest of the paper is organized as follows. Section 2 describes our model and axioms, and reviews existing results. Section 3 presents our characterization of TTC in the case of pairwise exchanges. Section 4 extends existing characterizations of TTC to the case where each agent has single-dipped preferences on a tree structure. Finally, Section 5 concludes with suggestions for future research. The proofs of our results are provided in Appendix A and Appendix B.

2 Preliminaries

2.1 Model

Let $N = \{1, 2, \ldots, n\}$ be the set of agents. Each agent $i \in N$ initially owns one indivisible object o_i and has a strict preference relation $\succ_i$ over the set of objects $O = \{o_1, o_2, \ldots, o_n\}$. Let $\mathscr{P}$ be the class of all strict preferences over O. For each $\succ_i \in \mathscr{P}$, let $\succsim_i$ represent the induced weak preference relation from $\succ_i$; that is, for each $\{o, o'\} \subseteq O$, $o \succsim_i o'$ if and only if either $o \succ_i o'$ or $o = o'$. Let $\mathscr{P}^N$ be the set of all strict preference profiles $\succ = (\succ_i)_{i \in N}$ where $\succ_i \in \mathscr{P}$ for each $i \in N$. Given a subset $\mathscr{D}$ of $\mathscr{P}$, we call $\mathscr{D}^N$ a **domain** of preferences. We often denote $N \setminus \{i\}$ by "$-i$," $N \setminus \{i, j\}$ by "$-i, j$," and $N \setminus S$ by "$-S$," respectively. With this notation, $(\succ'_i, \succ_{-i})$ represents the preference profile where agent i has $\succ'_i$ and each other agent j has $\succ_j$. We similarly define $(\succ'_i, \succ'_j, \succ_{-i,j})$ and $(\succ'_S, \succ_{-S})$. For each $i \in N$, each $\succ_i \in \mathscr{D}$, and each $O' \subseteq O$, let $b(\succ_i, O')$ be the **best object of agent i in O' according to $\succ_i$**; that is, $b(\succ_i, O') \in O'$ and for each $o \in O' \setminus \{b(\succ_i, O')\}$, $b(\succ_i, O') \succ_i o$.

This paper restricts our attention to single-dipped preferences. To define single-dipped preferences, we consider a linear order $\lhd$ on O. Without loss of generality, we fix a linear order $\lhd$ on O as follows:

$$o_1 \lhd o_2 \lhd \cdots \lhd o_n. \tag{1}$$

Given $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped** (with respect to $\lhd$) if there is an object, $d(\succ_i) \in O$, such that

(i) for each $o \in O \setminus \{d(\succ_i)\}$, $o \succ_i d(\succ_i)$;

(ii) for each $\{o, o'\} \subseteq O \setminus \{d(\succ_i)\}$, if either $o' \lhd o \lhd d(\succ_i)$ or $d(\succ_i) \lhd o \lhd o'$, then $o' \succ_i o$.

We denote the class of single-dipped preferences by $\mathscr{S}_\vee$. We call $\mathscr{S}_\vee^N$ the **single-dipped domain**.

An **allocation** is a bijection $x\colon N \to O$. We write x_i for $x(i)$. Here, x_i represents the object that agent i receives under x. We denote the set of allocations by X.

A **rule** on a domain $\mathscr{D}^N$ is a function $f\colon \mathscr{D}^N \to X$ that maps a preference profile $\succ \in \mathscr{D}^N$ to an allocation $f(\succ) \in X$. We denote the object allocated to agent i at $\succ$ under f by $f_i(\succ)$.

A rule that has played a central role in the literature is the top trading cycles rule. The **top trading cycles rule**, or simply TTC, is the rule $TTC\colon \mathscr{D}^N \to X$ that selects for each $\succ \in \mathscr{D}^N$, the allocation $TTC(\succ)$ obtained via the following TTC algorithm:

- **Round 1.** Each agent points to the agent who owns his best object, with the possibility of pointing to himself. Given that the number of agents is finite, at least one "cycle" is guaranteed. A **cycle** is a sequence of agents, $(i_1(= i_{C+1}), i_2, \ldots, i_C)$, where for each $c \in \{1, 2, \ldots, C\}$, agent i_c points to agent i_{c+1}. Each agent in a cycle is assigned the object along the cycle and then removed. If an agent remains, the algorithm proceeds to the next round; otherwise, it terminates.

- **Round $t \geq 2$.** Each remaining agent points to the agent who owns his best object among the remaining objects, with the possibility of pointing to himself. Given that the number of agents is finite, at least one cycle is guaranteed. Each agent in a cycle is assigned the object along the cycle and then removed. If an agent remains, the algorithm proceeds to the next round; otherwise, it terminates.

2.2 Axioms

The following three axioms are standard in the literature. The first is an efficiency property: no agent can be made better off without deteriorating someone else.

Efficiency: For each $\succ \in \mathscr{D}^N$, there is no $x \in X$ such that for each $i \in N$, $x_i \succsim_i f_i(\succ)$ and for some $j \in N$, $x_j \succ_j f_j(\succ)$.

The next axiom states that no agent should be worse off than he is at his endowment.

Individual rationality: For each $\succ \in \mathscr{D}^N$ and each $i \in N$, $f_i(\succ) \succsim_i o_i$.

The third axiom is the central incentive requirement in the literature: no agent should be able to gain by misrepresenting his preferences.

Strategy-proofness: For each $\succ \in \mathscr{D}^N$, each $i \in N$, and each $\succ'_i \in \mathscr{D}$, $f_i(\succ) \succsim_i f_i(\succ'_i, \succ_{-i})$.

We focus on the rules that are robust to pairwise manipulations through swapping their allocated objects. In our model, Moulin (1995) is the first to introduce the requirement that no pair of agents should benefit from misrepresenting their preferences and swapping their allocated objects.

Reallocation-proofness: There exist no $\succ \in \mathscr{D}^N$, $\{i, j\} \subseteq N$, and $(\succ'_i, \succ'_j) \in \mathscr{D} \times \mathscr{D}$ such that $f_j(\succ'_i, \succ'_j, \succ_{-i,j}) \succ_i f_i(\succ)$ and $f_i(\succ'_i, \succ'_j, \succ_{-i,j}) \succ_j f_j(\succ)$.

The following is a weaker version of *reallocation-proofness* introduced by Ekici (2024), which focuses on excluding pairwise collusions that swap allocated objects without manipulating preferences.[8]

Pair-efficiency: There exist no $\succ \in \mathscr{D}^N$ and $\{i, j\} \subseteq N$ such that $f_j(\succ) \succ_i f_i(\succ)$ and $f_i(\succ) \succ_j f_j(\succ)$.

[8]By definition, *pair-efficiency* is also weaker than *efficiency*. As mentioned in the introduction, Ekici (2024) introduces *pair-efficiency* to weaken *efficiency* rather than *reallocation-proofness*.

2.3 Existing characterizations of TTC on the single-dipped domain

It is established that when preferences are strict, TTC is characterized by the combination of *individual rationality*, *efficiency*, and *strategy-proofness* (Ma, 1994) or by the combination of *individual rationality*, *reallocation-proofness*, and *strategy-proofness* (Fujinaka and Wakayama, 2018). This characterization of TTC still holds even if *efficiency* or *reallocation-proofness* is weakened to *pair-efficiency* (Ekici, 2024).

Recently, it has been shown that the characterizations of TTC proposed by Ma (1994) and Ekici (2024) continue to hold even when the domain of preferences is restricted to the single-dipped domain (Tamura, 2023; Hu and Zhang, 2024).

Theorem 1 (Theorem 1 in Tamura (2023)). *A rule on $\mathscr{S}_{\vee}^N$ is individually rational, efficient, and strategy-proof if and only if it is TTC.*

Theorem 2 (Theorem 1 in Hu and Zhang (2024)). *A rule on $\mathscr{S}_{\vee}^N$ is individually rational, pair-efficient, and strategy-proof if and only if it is TTC.*

Remark 1. The proof provided by Hu and Zhang (2024) employs the fact that when preferences are single-dipped, each round of the TTC algorithm results in either self-pointing cycles or two-agent cycles (see Fact 1 below). While their proof is intuitive, the "only if" part is incomplete. To illustrate this, let $\succ \in \mathscr{S}_{\vee}^N$ and suppose that $S = \{\ell, h\}$ forms a cycle (ℓ, h) in Round r (≥ 2) of the TTC algorithm at $\succ$. Additionally, let $N^{r-1}(\succ)$ denote the set of agents that form cycles before Round r at $\succ$. Consider $\succ' \in \mathscr{S}_{\vee}^N$ such that for each $i \in N \setminus S$, $\succ'_i = \succ_i$ and for some $k \in S$, $\succ'_k \neq \succ_k$. Then, Hu and Zhang (2024) implicitly assume in their proof that for each $j \in N^{r-1}(\succ)$, $f_j(\succ') = TTC_j(\succ)$, where f is a rule that satisfies the three axioms. However, this assumption should be proved. We fill this gap in their argument and provide a complete proof of Theorem 2. The detailed proof can be found in Online Appendix C. $\lozenge$

As mentioned above, *pair-efficiency* is weaker than *reallocation-proofness*. Thus, a *reallocation-proofness* characterization of TTC can be obtained as a corollary of Theorem 2. In other words, Fujinaka and Wakayama's (2018) characterization of TTC still holds on the single-dipped domain.

Corollary 1. *A rule on $\mathscr{S}_{\vee}^N$ is individually rational, reallocation-proof, and strategy-proof if and only if it is TTC.*

Remark 2. Using the concept of "self-enforcing" introduced by Pápai (2000), we propose another weaker axiom that we call *self-enforcing reallocation-proofness*. This axiom applies the requirement of *reallocation-proofness* only to self-enforcing pairwise collusions, where two agents swap their allocated objects, but neither agent is worse off by misreporting his preferences if the other agent betrays the partner by reporting her true preferences. *Self-enforcing reallocation-proofness* is formally defined as follows: there exist no $\succ \in \mathscr{D}^N$, $\{i, j\} \subseteq N$, and $(\succ'_i, \succ'_j) \in \mathscr{D} \times \mathscr{D}$ such that

(i) $f_j(\succ'_i, \succ'_j, \succ_{-i,j}) \succ_i f_i(\succ)$ and $f_i(\succ'_i, \succ'_j, \succ_{-i,j}) \succ_j f_j(\succ)$;

(ii) for each $k \in \{i, j\}$, $f_k(\succ) = f_k(\succ'_k, \succ_{-k}) \neq f_k(\succ'_i, \succ'_j, \succ_{-i,j})$.

If we weaken *reallocation-proofness* to *self-enforcing reallocation-proofness*, then Corollary 1 no longer holds. The no-trade rule, which always assigns each agent his endowment, satisfies *individual rationality*, *strategy-proofness*, and *self-enforcing reallocation-proofness*.[9,10] $\diamondsuit$

3 Pairwise exchanges

We now focus on the case of pairwise exchanges, which is the most stringent exchange constraint. Given an allocation $x \in X$, we say that x is a **pairwise exchange** if for each $\{i, j\} \subseteq N$, $x_i = o_j$ implies $x_j = o_i$. Let X_2 be the set of all pairwise exchanges. Given $\mathscr{D} \subseteq \mathscr{P}$, we say that a rule f on $\mathscr{D}^N$ is a **pairwise exchange rule** if for each $\succ \in \mathscr{D}^N$, $f(\succ) \in X_2$. All the axioms defined in Section 2 are similarly defined in this setting. Thus, we omit their definitions.

Both Theorem 2 and Corollary 1 persist when only pairwise exchanges are allowed, because TTC is a pairwise exchange rule on the single-dipped domain (see, for example, Fujinaka and Wakayama (2024)). Significantly, Corollary 1 holds without *strategy-proofness*.

Theorem 3. *A pairwise exchange rule on $\mathscr{S}_\vee^N$ is individually rational and reallocation-proof if and only if it is TTC.*

[9] The no-trade rule will be formally defined later.

[10] While *pair-efficiency* is weaker than *reallocation-proofness*, there is no logical relationship between *pair-efficiency* and *self-enforcing reallocation-proofness*. The no-trade rule is *self-enforcing reallocation-proof* but not *pair-efficient*. We present a rule that is *pair-efficient* but not *self-enforcing reallocation-proof* in Online Appendix D.

Proof. The proof is presented in Appendix A. □

We verify that the two axioms in Theorem 3 are independent. If either of the two axioms in Theorem 3 is dropped, we find a non-TTC pairwise exchange rule that satisfies the remaining axiom. The no-trade rule defined below is a pairwise exchange rule that is *individually rational* but not *reallocation-proof*.

Example 1. The **no-trade rule** is the rule $NT\colon \mathscr{S}_\vee^N \to X$ such that for each $\succ \in \mathscr{S}_\vee^N$ and each $i \in N$, $NT_i(\succ) = o_i$. This rule is a pairwise exchange rule that is *individually rational* but not *reallocation-proof*. ■

The following pairwise exchange rule, $IR^{12\neg}$, is *reallocation-proof* but not *individually rational*.

Example 2. Suppose $n = 3$. Let $\succ^{12} \in \mathscr{S}_\vee^N$ be such that for each $i \in N$, $o_1 \succ_i^{12} o_2 \succ_i^{12} o_3$. Let $IR^{12\neg}$ be a pairwise exchange rule such that for each $\succ \in \mathscr{S}_\vee^N$,

$$IR^{12\neg}(\succ) = \begin{cases} (o_1, o_3, o_2) & \text{if } \succ = \succ^{12} \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that $TTC(\succ^{12}) = (o_1, o_2, o_3) \neq IR^{12\neg}(\succ^{12})$. This rule violates *individual rationality*, because $o_2 \succ_2^{12} o_3 = IR_2^{12\neg}(\succ^{12})$. For the proof of *reallocation-proofness* of this rule, see Online Appendix D. ■

Notably, Theorem 3 no longer holds when we consider exchanges involving more than two agents. We can construct a non-TTC rule that satisfies *individual rationality* and *reallocation-proofness*. The following example shows such a rule.

Example 3. Suppose $n = 4$ and exchanges involving at most three agents are only allowed. We denote by X_3 the set of allocations that satisfy this constraint. Let $\succ^\star \in \mathscr{S}_\vee^N$ be such that

$\succ_3^\star$	$\succ_{i\neq 3}^\star$
o_1	o_4
o_2	o_3
o_3	o_2
o_4	o_1

Let $f^{\overrightarrow{123}} \colon \mathscr{S}_\vee^N \to X_3$ be a rule such that for each $\succ \in \mathscr{S}_\vee^N$,

$$f^{\overrightarrow{123}}(\succ) = \begin{cases} (o_2, o_3, o_1, o_4) & \text{if } \succ = \succ^\star \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that $TTC(\succ^\star) = (o_3, o_2, o_1, o_4) \neq f^{\overrightarrow{123}}(\succ^\star)$. This rule is *individually rational*. For the proof of *reallocation-proofness* of this rule, see Online Appendix D.[11] ■

Remark 3. Theorem 2 does not hold without *strategy-proofness* when pairwise exchanges are only allowed. We can construct a non-TTC pairwise exchange rule that is *individually rational* and *pair-efficient*.[12] This also implies that Theorem 3 does not hold when *reallocation-proofness* is weakened to *pair-efficiency*. Furthermore, Theorem 3 does not hold when *reallocation-proofness* is weakened to *self-enforcing reallocation-proofness*. The no-trade rule is a non-TTC pairwise exchange rule that is *individually rational* and *self-enforcing reallocation-proof*. ♢

4 Tree structures

Thus far, we have considered single-dipped preferences defined on a "line." As mentioned in the introduction, this preference domain can be extended to a more general model where objects are arranged on a tree structure. Below, we show that three existing characterizations of TTC for single-dipped preferences on a line, mentioned in Section 2, also hold for single-dipped preferences on a tree.

4.1 Definitions and notation

We begin by introducing some graph theoretical concepts. An **(undirected) graph** is a pair $G = (O, E)$, where $E \subset \{\{o', o''\} \subset O \colon o' \neq o''\}$ is the set of **edges**. The **degree** of object $o \in O$ in $G = (O, E)$ is the number of edges that contain o;[13] that is,

$$\deg(o) = \left|\{\{o', o''\} \in E \colon o \in \{o', o''\}\}\right|.$$

[11] This rule violates *strategy-proofness*. To show this, let $\succ_1' \in \mathscr{S}_\vee$ be such that $o_4 \succ_1' o_3 \succ_1' o_1 \succ_1' o_2$. Then,

$$f_1^{\overrightarrow{123}}(\succ_1', \succ_{-1}^\star) = TTC_1(\succ_1', \succ_{-1}^\star) = o_3 \succ_1^\star o_2 = f_1^{\overrightarrow{123}}(\succ^\star),$$

and thus, $f^{\overrightarrow{123}}$ violates *strategy-proofness*.

[12] Example 4 below provides an example of such a rule.

[13] Given a set A, $|A|$ denotes the cardinality of A.

Given an object $o \in O$, we say that o is a **leaf in G** if $\deg(o) = 1$. We denote the set of leaves in G by $\mathbb{L}$.[14] Given $\{o', o''\} \subset O$ with $o' \neq o''$, a **path from o' to o'' in $G = (O, E)$** is a sequence $(o^1, o^2, \ldots, o^K)$ such that $o^1 = o'$, $o^K = o''$, $|\{o^1, o^2, \ldots, o^K\}| = K$, and for each $k \in \{1, 2, \ldots, K-1\}$, $\{o^k, o^{k+1}\} \in E$. A graph $G = (O, E)$ is a **tree** if

(i) it is connected (i.e., for each $\{o', o''\} \subset O$ with $o' \neq o''$, there is a path from o' to o'' in G); and

(ii) it has no cycle (i.e., there is no sequence $(o^1, o^2, \ldots, o^K)$ such that $K \geq 3$, $o^1 = o^K$, for each $k \in \{1, 2, \ldots, K-1\}$, $\{o^k, o^{k+1}\} \in E$, and for each $\{k', k''\} \subset \{1, 2, \ldots, K\}$ such that $k' \neq k''$ and $\{k', k''\} \neq \{1, K\}$, $o^{k'} \neq o^{k''}$).

If graph G is a tree, then for each $\{o', o''\} \subset O$ with $o' \neq o''$, a unique path exists from o' to o'' in G (see, for example, Theorem 2.1.4 in West (2001)). We denote the path from o' to o'' by $[o', o'']$. For each $\{o, o', o''\} \subset O$, we write $o \in [o', o'']$ if o lies on the path from o' to o''. That is, when $[o', o''] = (o^1 = o', o^2, \ldots, o^K = o'')$, there is $k \in \{1, 2, \ldots, K\}$ such that $o^k = o$

Given a tree $G = (O, E)$ and an agent $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped on the tree G** if there is an object, $d(\succ_i) \in O$, such that

(i) for each $o \in O \setminus \{d(\succ_i)\}$, $o \succ_i d(\succ_i)$;

(ii) for each $\{o, o'\} \subset O \setminus \{d(\succ_i)\}$ with $o \neq o'$, if $o \in [d(\succ_i), o']$, then $o' \succ_i o$.

Given a tree G, we denote the class of single-dipped preferences on the tree G by $\mathscr{T}_G \subset \mathscr{P}$.

Remark 4. Suppose that G is a tree. For each $i \in N$ and each $\succ_i \in \mathscr{T}_G$, the object $b(\succ_i, O)$ is a leaf in G. Moreover, the maximum size of any cycle formed under TTC on $\mathscr{T}_G^N$ is at most $|\mathbb{L}|$. See Fujinaka and Wakayama (2024) for the proofs of these results. $\Diamond$

4.2 Characterizations of TTC

Tamura (2023) recently demonstrated that Theorem 1 can be extended to the setting considered here.

[14] Formally, it should be $\mathbb{L}(G)$; however, unless otherwise specified, we omit G for simplicity.

Theorem 4 (Theorem 5 in Tamura (2023)). *Suppose that G is a tree. Then, a rule on $\mathscr{T}_G^N$ is individually rational, efficient, and strategy-proof if and only if it is TTC.*

The following result strengthens Theorem 4 by weakening *efficiency* to *pair-efficiency*. In other words, Hu and Zhang's result (Theorem 2) can be extended to the domain of single-dipped preferences on a tree.

Theorem 5. *Suppose that G is a tree. Then, a rule on $\mathscr{T}_G^N$ is individually rational, pair-efficient, and strategy-proof if and only if it is TTC.*

Proof. The proof of this theorem is presented in Appendix B. $\square$

Remark 5. By slightly modifying the proof of Theorem 5, we can apply the proof to the domain of strict preferences. Thus, we obtain an alternative proof of Ekici's characterization (Ekici, 2024). For more details, see Remark 7 in Appendix B.[15] $\Diamond$

Considering that *pair-efficiency* is weaker than *reallocation-proofness* or *efficiency*, Tamura's result (Theorem 4) and the *reallocation-proofness* characterization of TTC follow as corollaries of Theorem 5. Essentially, Corollary 1 can be extended to the domain of single-dipped preferences on a tree.

Corollary 2. *Suppose that G is a tree. Then, a rule on $\mathscr{T}_G^N$ is individually rational, strategy-proof, and reallocation-proof if and only if it is TTC.*

4.3 Independence of the axioms

Here, we verify the independence of the axioms in both Theorem 5 and Corollary 2. If any of the three axioms in Theorem 5 (or Corollary 2) is dropped, a non-TTC rule that satisfies the remaining two axioms exists. In doing so, we use the following notation in the three-agent case: given a graph G, let $\succ_0^{ij} \in \mathscr{T}_G$ be a preference relation such that

$$\begin{array}{c} \succ_0^{ij} \\ \hline o_i \\ o_j \\ o_k \end{array}$$

For each $j \in N$, let $\mathscr{T}_G^j \subset \mathscr{T}_G$ be the set of preference relations $\succ_0$ where $o_j \succ_0 o$ for each $o \in O \setminus \{o_j\}$.

[15] Ekici and Sethuraman (2024) also provide an alternative proof of Ekici's characterization.

We first verify the independence of the axioms in Theorem 5. The no-trade rule defined above is both *individually rational* and *strategy-proof*, but not *pair-efficient*. The following rule, $SP^{\neg}$, is both *individually rational* and *pair-efficient*, but not *strategy-proof*.

Example 4. Suppose $n = 3$. Let G be a tree with $\mathbb{L} = \{o_1, o_3\}$. Let $SP^{\neg}\colon \mathscr{T}_G^N \to X$ be a rule such that for each $\succ \in \mathscr{T}_G^N$,

$$SP^{\neg}(\succ) = \begin{cases} (o_2, o_1, o_3) & \text{if } \succ = (\succ_1^{32}, \succ_2^{12}, \succ_3^{13}) \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that $TTC(\succ_1^{32}, \succ_2^{12}, \succ_3^{13}) = (o_3, o_2, o_1) \neq SP^{\neg}(\succ_1^{32}, \succ_2^{12}, \succ_3^{13})$. This rule satisfies *individual rationality* and *pair-efficiency*. To show that $SP^{\neg}$ violates *strategy-proofness*, let $\succ = (\succ_1^{32}, \succ_2^{12}, \succ_3^{13})$ and $\succ_1' = \succ_1^{31}$. Then,

$$SP_1^{\neg}(\succ_1', \succ_{-1}) = TTC_1(\succ_1', \succ_{-1}) = o_3 \succ_1 o_2 = SP_1^{\neg}(\succ),$$

which implies that $SP^{\neg}$ violates *strategy-proofness*. ■

The following rule, $IR^{\neg}$, is both *strategy-proof* and *pair-efficient*, but not *individually rational*.

Example 5. Suppose $n = 3$. Let G be a tree with $\mathbb{L} = \{o_1, o_3\}$. Let $IR^{\neg}\colon \mathscr{T}_G^N \to X$ be a rule such that for each $\succ \in \mathscr{T}_G^N$,

$$IR^{\neg}(\succ) = \begin{cases} (o_3, o_2, o_1) & \text{if } \succ \in \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1 \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that for each $\succ \in \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$, $TTC(\succ) = (o_1, o_2, o_3) \neq IR^{\neg}(\succ)$. This rule violates *individual rationality* as there is $\succ \in \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$ such that $o_1 \succ_1 o_3 = IR_1^{\neg}(\succ)$ by $\succ_1 \in \mathscr{T}_G^1$. For the proof of *strategy-proofness* of this rule, see Online Appendix D. Below, we show that $IR^{\neg}(\succ)$ satisfies *pair-efficiency*. Let $\succ \in \mathscr{T}_G^N$. If $\succ \notin \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$, then by $IR^{\neg}(\succ) = TTC(\succ)$ and *pair-efficiency* of TTC, no pair has an incentive to collude. If $\succ \in \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$, then $IR_3^{\neg}(\succ) = o_1 = b(\succ_3, O)$. Thus, agent 3 has no incentive to collude with another agent. Additionally, by $IR_2^{\neg}(\succ) = o_2 \succ_2 o_3 = IR_1^{\neg}(\succ)$, agent 2 has no incentive to collude with agent 1. Hence, no pair has an incentive to collude. ■

Next, we verify the independence of the axioms in Corollary 2. The no-trade rule is both *individually rational* and *strategy-proof*, but not *reallocation-proof*. The following rule, $SP^{*\neg}$, is both *individually rational* and *reallocation-proof*, but not *strategy-proof*.

Example 6. Suppose $n = 3$. Let G be a tree with $\mathbb{L} = \{o_1, o_3\}$. Let $SP^{*\neg} \colon \mathscr{T}_G^N \to X$ be a rule such that for each $\succ \in \mathscr{T}_G^N$,

$$SP^{*\neg}(\succ) = \begin{cases} (o_2, o_3, o_1) & \text{if } \succ = (\succ_1^{32}, \succ_2^{32}, \succ_3^{12}) \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that $TTC(\succ_1^{32}, \succ_2^{32}, \succ_3^{12}) = (o_3, o_2, o_1) \neq SP^{*\neg}(\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$. This rule satisfies *individual rationality*. For the proof of *reallocation-proofness* of this rule, see Online Appendix D. To show that $SP^{*\neg}$ violates *strategy-proofness*, let $\succ = (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$ and $\succ_1' = \succ_1^{31}$. Then,

$$SP_1^{*\neg}(\succ_1', \succ_{-1}) = TTC_1(\succ_1', \succ_{-1}) = o_3 \succ_1 o_2 = SP_1^{*\neg}(\succ),$$

which implies that $SP^{*\neg}$ violates *strategy-proofness*. ■

The following rule, $IR^{*\neg}$, is both *strategy-proof* and *reallocation-proof*, but not *individually rational*.

Example 7. Suppose $n = 3$. Let G be a tree with $\mathbb{L} = \{o_1, o_3\}$. Let $\mathscr{T}_G^{-12} = \mathscr{T}_G \setminus \{\succ_0^{12}\}$. Let $IR^{*\neg} \colon \mathscr{T}_G^N \to X$ be a rule such that for each $\succ \in \mathscr{T}_G^N$,

$$IR^{*\neg}(\succ) = \begin{cases} (o_2, o_3, o_1) & \text{if } \succ \in \mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1 \\ TTC(\succ) & \text{otherwise.} \end{cases}$$

Note that for each $\succ \in \mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1$,

$$IR^{*\neg}(\succ) \neq TTC(\succ) \in \{(o_1, o_2, o_3), (o_1, o_3, o_2), (o_3, o_2, o_1)\}.$$

For the proof of *strategy-proofness* and *reallocation-proofness* of this rule, see Online Appendix D. Given that there is $\succ \in \mathscr{T}_G^1 \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1$ such that $o_1 \succ_1 o_2 = IR_1^{*\neg}(\succ)$, this rule violates *individual rationality*. ■

Remark 6. Both $SP^{\neg}$ (Example 4) and $IR^{\neg}$ (Example 5) violate *reallocation-proofness* (see Online Appendix D for the proof). This demonstrates that *pair-efficiency* is

Table 1: Satisfaction of axioms of rules. The notation "+" (resp. "−") in a cell indicates that the axiom is satisfied (resp. violated) by the corresponding rule.

	Rules					
Axioms	TTC	NT (Example 1)	$SP^{\neg}$ (Example 4)	$IR^{\neg}$ (Example 5)	$SP^{*\neg}$ (Example 6)	$IR^{*\neg}$ (Example 7)
individual rationality	+	+	+	−	+	−
strategy-proofness	+	+	−	+	−	+
efficiency	+	−	+	+	+	+
pair-efficiency	+	−	+	+	+	+
reallocation-proofness	+	−	−	−	+	+

considerably weaker than *reallocation-proofness* even when combined with either *individual rationality* or *strategy-proofness*. ◊

For each axiom, Table 1 shows which of the rules satisfy the axiom.[16]

5 Concluding comments

We conclude with two comments on possible directions for future research.

1. **Single-peaked preferences.** It would be interesting to identify the set of *reallocation-proof* rules when preferences are single-peaked. We say that an agent has single-peaked preferences (with respect to a fixed order of objects) if he has a unique best object, and his welfare strictly decreases as one moves away from this object in either direction according to the given order. In contrast to the case of single-dipped preferences, Ma's (1994) characterization of TTC does not hold for the case of single-peaked preferences: There are many non-TTC rules that satisfy *individual rationality*, *efficiency*, and *strategy-proofness* (Bade, 2019; Tamura, 2022; Tamura and Hosseini, 2022; Liu, 2022; Huang and Tian, 2023). We conjecture that several non-TTC rules satisfy *individual rationality*, *reallocation-proofness*, and *strategy-proofness*. Thus, future research should identify various rules that satisfy these three axioms on the single-peaked preferences domain.

2. **Social endowments.** It would be interesting to examine *reallocation-proof* rules for indivisible object allocation problems without private endowments.

[16] In this subsection, we mainly consider the case where $n = 3$ and G is a tree with $\mathbb{L} = \{o_1, o_3\}$. Note that this tree structure G is equivalent to a line structure where the objects are ordered according to (1), and additionally, $\mathscr{T}_G = \mathscr{S}_\vee$. Thus, this figure also demonstrates the independence of axioms in Theorem 2 and Corollary 1.

In this context, Pápai (2000) investigates a strong version of *reallocation-proofness*, which we call *strong reallocation-proofness*. In this version, one agent in a deviating pair may be indifferent after reallocating objects ex-post. Pápai's result reveals that no rule satisfies *efficiency*, *strategy-proofness*, *strong reallocation-proofness*, and an auxiliary axiom. Mandal and Roy (2021) further confirm that Pápai's negative result still holds even when the domain is restricted to the domain of (minimally rich) single-peaked preferences. It remains an open question whether the negative results can be avoided by weakening *strong reallocation-proofness* to *reallocation-proofness*.

A Appendix: Proof of Theorem 3

Before proving Theorem 3, we introduce some additional notation. Let $\succ \in \mathscr{S}_{\vee}^{N}$ and $t \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of natural numbers. We denote the set of groups of agents that form cycles in Round t of the TTC algorithm at $\succ$ by

$$\mathbb{S}_t(\succ) \subseteq 2^N \setminus \{\emptyset\}.$$

We denote the set of agents who are assigned objects in Round t of the TTC algorithm at $\succ$ by

$$N_t(\succ) = \bigcup_{S \in \mathbb{S}_t(\succ)} \{S\}.$$

We denote the set of objects that are assigned to agents in Round t of the TTC algorithm at $\succ$ by

$$O_t(\succ) = \{o \in O \colon \exists\, i \in N_t(\succ), o = o_i\}.$$

Define $N^t(\succ)$ and $O^t(\succ)$ as follows:

$$N^t(\succ) = \bigcup_{z=1}^{t} N_z(\succ) \quad \text{and} \quad O^t(\succ) = \bigcup_{z=1}^{t} O_z(\succ).$$

For convenience, let $N^0(\succ) = O^0(\succ) = \emptyset$. With a slight abuse of notation, each $S \in \mathbb{S}_t(\succ)$ also represents a cycle; that is, "$S = \{i_1(= i_{K+1}), i_2, \dots, i_K\} \in \mathbb{S}_t(\succ)$" denotes that for each $k \in \{1, 2, \dots, K\}$, $i_k \in N \setminus N^{t-1}(\succ)$, $o_{i_k} \in O \setminus O^{t-1}(\succ)$, and $b\left(\succ_{i_k}, O \setminus O^{t-1}(\succ)\right) = o_{i_{k+1}}$. We denote by $\ell(\succ, t)$ (resp. $h(\succ, t)$) the lowest (resp. highest) index among the set of remaining agents in Round t of the TTC algorithm at $\succ$. Note that for each $\succ \in \mathscr{S}_{\vee}^{N}$, $\ell(\succ, 1) = 1$ and $h(\succ, 1) = n$.

In proving our theorems, we frequently invoke the following fact: in the case of single-dipped preferences, the TTC algorithm generates either self-pointing cycles or a two-agent cycle in each round.

Fact 1 (Proposition 1 in Fujinaka and Wakayama (2024)). *For each $\succ \in \mathscr{S}_{\vee}^{N}$ and each $t \in \mathbb{N}$,*

$$\mathbb{S}_t(\succ) \in \Big\{\{\{\ell(\succ,t), h(\succ,t)\}\}, \{\{\ell(\succ,t)\}, \{h(\succ,t)\}\}, \{\{\ell(\succ,t)\}\}, \{\{h(\succ,t)\}\}\Big\}.$$

We also present two key lemmas. The first one states that joint preference manipulation by a group of agents that forms a cycle in Round r of the TTC algorithm does not affect the outcome of agents forming cycles before Round r at the original preference profile.

Lemma 1. *Let $\succ \in \mathscr{S}_{\vee}^{N}$, $r \in \mathbb{N}$, $S \in \mathbb{S}_r(\succ)$, and $\tilde{\succ}_S \in \mathscr{S}_{\vee}^{S}$. Then, for each $i \in N^{r-1}(\succ)$, $TTC_i(\tilde{\succ}_S, \succ_{-S}) = TTC_i(\succ)$.*

Proof. Let $t \in \{1, 2, \ldots, r-1\}$ and $M \in \mathbb{S}_t(\succ)$. Suppose that

$$\forall\, i \in N^{t-1}(\succ),\ TTC_i(\tilde{\succ}_S, \succ_{-S}) = TTC_i(\succ). \tag{2}$$

Note that by $M \in \mathbb{S}_t(\succ)$ and $t \leq r-1$, $M \subseteq N \setminus S$. For simplicity, we write $\ell(t)$ (resp. $h(t)$) for $\ell(\succ, t)$ (resp. $h(\succ, t)$). By Fact 1, $M \in \big\{\{\ell(t), h(t)\}, \{\ell(t)\}, \{h(t)\}\big\}$. There are two cases.

- **Case 1: $M \in \{\{\ell(t)\}, \{h(t)\}\}$.** Without loss of generality, we assume $M = \{\ell(t)\}$. Then, $TTC_{\ell(t)}(\succ) = o_{\ell(t)}$. Let $o \in O$ be such that $o \succ_{\ell(t)} o_{\ell(t)}$. Then, $o \in O^{t-1}(\succ)$. Thus, there is $i \in N^{t-1}(\succ)$ such that $TTC_i(\succ) = o$. By (2), $TTC_i(\tilde{\succ}_S, \succ_{-S}) = TTC_i(\succ) = o$ and $TTC_{\ell(t)}(\tilde{\succ}_S, \succ_{-S}) \neq o$. Hence, by *individual rationality*, $TTC_{\ell(t)}(\tilde{\succ}_S, \succ_{-S}) = o_{\ell(t)} = TTC_{\ell(t)}(\succ)$.
- **Case 2: $M = \{\ell(t), h(t)\}$.** Then, $TTC_{\ell(t)}(\succ) = o_{h(t)}$ and $TTC_{h(t)}(\succ) = o_{\ell(t)}$. Let $\{\underline{t}, \bar{t}\} \subset \mathbb{N}$ be such that

$$\ell(t) \in N_{\underline{t}}(\tilde{\succ}_S, \succ_{-S}) \quad \text{and} \quad h(t) \in N_{\bar{t}}(\tilde{\succ}_S, \succ_{-S}).$$

Let $o \in O$ be such that $o \succ_{\ell(t)} o_{h(t)}$. By $TTC_{\ell(t)}(\succ) = o_{h(t)}$ and $\ell(t) \in M \in \mathbb{S}_t(\succ)$, $o \in O^{t-1}(\succ)$. Thus, there is $i \in N^{t-1}(\succ)$ with $TTC_i(\succ) = o$. By (2), $TTC_i(\tilde{\succ}_S, \succ_{-S}) = TTC_i(\succ) = o$ and $TTC_{\ell(t)}(\tilde{\succ}_S, \succ_{-S}) \neq o$. Hence, $o_{h(t)} \tilde{\succsim}_{\ell(t)}$

$TTC_{\ell(t)}(\widetilde{\succ}_S, \succ_{-S})$, which implies

$$\bar{t} \leq \underline{t}. \tag{3}$$

Similarly, $o_{\ell(t)} \succsim_{h(t)} TTC_{h(t)}(\widetilde{\succ}_S, \succ_{-S})$ and

$$\underline{t} \leq \bar{t}. \tag{4}$$

By (3) and (4), $\underline{t} = \bar{t}$, which implies $M \subset N \setminus N^{\underline{t}-1}(\widetilde{\succ}_S, \succ_{-S})$. Furthermore, by $o_{h(t)} \succsim_{\ell(t)} TTC_{\ell(t)}(\widetilde{\succ}_S, \succ_{-S})$, $TTC_{\ell(t)}(\widetilde{\succ}_S, \succ_{-S}) = o_{h(t)} = TTC_{\ell(t)}(\succ)$.[17] Similarly, $TTC_{h(t)}(\widetilde{\succ}_S, \succ_{-S}) = o_{\ell(t)} = TTC_{h(t)}(\succ)$. □

The second lemma states that joint preference manipulation by a group of agents forming a cycle in Round r of the TTC algorithm neither affects the cycles formed before Round r at the original preference profile nor delays the formation of these cycles at the new preference profile.

Lemma 2. *Let* $\succ \in \mathscr{S}_\vee^N$, $r \in \mathbb{N}$, $S \in \mathbb{S}_r(\succ)$, *and* $\widetilde{\succ}_S \in \mathscr{S}_\vee^S$. *Then, for each* $t \in \{1, 2, \ldots, r-1\}$ *and each* $M \in \mathbb{S}_t(\succ)$, *there is* $t_M \in \{1, 2, \ldots, t\}$ *such that* $M \in \mathbb{S}_{t_M}(\widetilde{\succ}_S, \succ_{-S})$.

Proof. For convenience, let $\widetilde{\succ} = (\widetilde{\succ}_S, \succ_{-S})$. Suppose on the contrary that there are $t' \in \{1, 2, \ldots, r-1\}$ and $M' \in \mathbb{S}_{t'}(\succ)$ such that

$$\forall\, t'' \in \{1, 2, \ldots, t'\},\ M' \notin \mathbb{S}_{t''}(\widetilde{\succ}). \tag{5}$$

Note that for each $t \in \{1, 2, \ldots, r-1\}$ and each $M \in \mathbb{S}_t(\succ)$, Lemma 1 leads to the following fact:

$$\forall\, i \in M,\ TTC_i(\widetilde{\succ}) = TTC_i(\succ). \tag{6}$$

By (6), there is $t_{M'} \in \mathbb{N}$ with $M' \in \mathbb{S}_{t_{M'}}(\widetilde{\succ})$. Then, by (5), this implies $t_{M'} > t'$. We denote the set of rounds having such a property by

$$T = \{t \in \{1, 2, \ldots, r-1\} : \exists\, M \in \mathbb{S}_t(\succ),\ \exists\, t_M \in \{t+1, t+2, \ldots\},\ M \in \mathbb{S}_{t_M}(\widetilde{\succ})\}.$$

Let $t^* = \min T$. Then, we can choose $t^* \in \{1, 2, \ldots, r-1\}$ that satisfies the following two conditions:

[17] If $TTC_{\ell(t)}(\widetilde{\succ}_S, \succ_{-S}) \neq o_{h(t)}$, $o_{h(t)} \succ_{\ell(t)} TTC_{\ell(t)}(\widetilde{\succ}_S, \succ_{-S})$, which implies that agent $\ell(t)$ with $\succ_{\ell(t)}$ does not receive $b\left(\succ_{\ell(t)}, O \setminus O^{\underline{t}-1}(\widetilde{\succ}_S, \succ_{-S})\right)$ in Round $\underline{t} = \bar{t}$ of the TTC algorithm at $(\widetilde{\succ}_S, \succ_{-S})$, a contradiction.

T1. There is $M^* \in \mathbb{S}_{t^*}(\succ)$ such that for some $t_{M^*} \in \{t^*+1, t^*+2, \dots\}$, $M^* \in \mathbb{S}_{t_{M^*}}(\widetilde{\succ})$.

T2. For each $t \in \{1, 2, \dots, t^*-1\}$ and each $M \in \mathbb{S}_t(\succ)$, there is $t_M \in \{1, 2, \dots, t\}$ such that $M \in \mathbb{S}_{t_M}(\widetilde{\succ})$.

Note that by $M^* \in \mathbb{S}_{t^*}(\succ)$ and $t^* \leq r-1$, for each $i \in M^*$, i's preference relation at $\succ$ and $\widetilde{\succ}$ is $\succ_i$. In Round $t_{M^*}-1$ of the TTC algorithm at $\widetilde{\succ}$, M^* does not form a cycle. Thus, there is $i^* \in M^*$ who points to agent j such that

$$o_j \succ_{i^*} TTC_{i^*}(\widetilde{\succ}) \overset{\text{(by (6))}}{=} TTC_{i^*}(\succ).$$

By $i^* \in M^* \in \mathbb{S}_{t^*}(\succ)$, there is $\widehat{t} \in \mathbb{N}$ such that $j \in \widehat{M} \in \mathbb{S}_{\widehat{t}}(\succ)$ and

$$\widehat{t} < t^*. \tag{7}$$

Given that $j \in \widehat{M} \in \mathbb{S}_{\widehat{t}}(\succ)$ and $\widehat{t} < t^* \leq r-1$, it follows that $\widehat{M} \subset N^{r-1}(\succ)$. This together with Lemma 1 implies that there is $t_{\widehat{M}} \in \{1, 2, \dots, \widehat{t}\}$ with $\widehat{M} \in \mathbb{S}_{t_{\widehat{M}}}(\widetilde{\succ})$. Additionally, recall that agent j is involved in Round $t_{M^*}-1$ of the TTC algorithm at $\widetilde{\succ}$. Then,

$$t_{M^*} - 1 \leq t_{\widehat{M}}. \tag{8}$$

Hence,

$$\widehat{t} \overset{\text{(by (7))}}{<} t^* \overset{\text{(by T1)}}{\leq} t_{M^*} - 1 \overset{\text{(by (8))}}{\leq} t_{\widehat{M}} \overset{\text{(by T2)}}{\leq} \widehat{t},$$

which is a contradiction. □

We now prove Theorem 3.

Proof of Theorem 3. It suffices to prove the "only if" part, as the "if" part follows from Corollary 1. We now show that for each $t \in \mathbb{N}$, each $\succ \in \mathscr{S}_{\vee}^N$, and each $i \in N_t(\succ)$, it holds that $f_i(\succ) = TTC_i(\succ)$. We prove this by induction on t.

Base step. $t = 1$. Let $\succ \in \mathscr{S}_{\vee}^N$ and $S \in \mathbb{S}_1(\succ)$. By Fact 1, $S \in \{\{1, n\}, \{1\}, \{n\}\}$. There are two cases.

- **Case 1: $S \in \{\{1\}, \{n\}\}$.** Without loss of generality, we assume $S = \{1\}$. Then, $b(\succ_1, O) = o_1$. Hence, by *individual rationality*, $f_1(\succ) = o_1 = TTC_1(\succ)$.

• **Case 2: $S = \{1, n\}$.** Then, $b(\succ_1, O) = o_n$ and $b(\succ_n, O) = o_1$. Suppose on the contrary that

$$(f_1(\succ), f_n(\succ)) \neq (TTC_1(\succ), TTC_n(\succ)) = (o_n, o_1).$$

Without loss of generality, we assume $f_1(\succ) \neq o_n$. Because f is a pairwise exchange rule, $f_n(\succ) \neq o_1$. Let $\succ_S^{\leftrightarrow} = (\succ_1^{\leftrightarrow}, \succ_n^{\leftrightarrow}) = (\succ_n, \succ_1)$. Then, $b(\succ_1^{\leftrightarrow}, O) = o_1$ and $b(\succ_n^{\leftrightarrow}, O) = o_n$. Note that $(\succ_S^{\leftrightarrow}, \succ_{-S}) \in \mathscr{S}_\vee^N$. By *individual rationality*,

$$f_1(\succ_S^{\leftrightarrow}, \succ_{-S}) = o_1 \quad \text{and} \quad f_n(\succ_S^{\leftrightarrow}, \succ_{-S}) = o_n,$$

which imply that

$$f_n(\succ_S^{\leftrightarrow}, \succ_{-S}) = o_n \succ_1 f_1(\succ) \quad \text{and} \quad f_1(\succ_S^{\leftrightarrow}, \succ_{-S}) = o_1 \succ_n f_n(\succ),$$

in violation of *reallocation-proofness*.

Induction hypothesis. For each $t \in \{1, 2, \ldots, r-1\}$, each $\succ \in \mathscr{S}_\vee^N$, and each $i \in N_t(\succ)$, $f_i(\succ) = TTC_i(\succ)$.

Induction step. Let $t = r$. By the induction hypothesis, for each $\succ' \in \mathscr{S}_\vee^N$,

$$O^{r-1}(\succ') = \left\{ o \in O \colon \exists\, i \in N^{r-1}(\succ'),\ o = f_i(\succ') \right\}. \tag{9}$$

Let $\succ \in \mathscr{S}_\vee^N$. For each $t \in \mathbb{N}$, we simply write $\ell(t)$ (resp. $h(t)$) for $\ell(\succ, t)$ (resp. $h(\succ, t)$). Let $S \in \mathbb{S}_r(\succ)$. By Fact 1, $S \in \{\{\ell(r), h(r)\}, \{\ell(r)\}, \{h(r)\}\}$. There are two cases.

• **Case 1: $S \in \{\{\ell(r)\}, \{h(r)\}\}$.** Without loss of generality, we assume $S = \{\ell(r)\}$. Then, $TTC_{\ell(r)}(\succ) = o_{\ell(r)}$ and

$$b\left(\succ_{\ell(r)}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)}. \tag{10}$$

It follows from (9) that

$$f_{\ell(r)}(\succ) \in O \setminus O^{r-1}(\succ). \tag{11}$$

By (10) and (11), *individual rationality* implies $f_{\ell(r)}(\succ) = o_{\ell(r)} = TTC_{\ell(r)}(\succ)$.

• **Case 2: $S = \{\ell(r), h(r)\}$.** Then, $TTC_{\ell(r)}(\succ) = o_{h(r)}$ and $TTC_{h(r)}(\succ) = o_{\ell(r)}$, and

$$b\left(\succ_{\ell(r)}, O \setminus O^{r-1}(\succ)\right) = o_{h(r)} \quad \text{and} \quad b\left(\succ_{h(r)}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)}. \tag{12}$$

It follows from (9) that

$$\{f_{\ell(r)}(\succ), f_{h(r)}(\succ)\} \subseteq O \setminus O^{r-1}(\succ). \tag{13}$$

Suppose on the contrary that

$$(f_{\ell(r)}(\succ), f_{h(r)}(\succ)) \neq (TTC_{\ell(r)}(\succ), TTC_{h(r)}(\succ)) = (o_{h(r)}, o_{\ell(r)}).$$

Without loss of generality, we assume $f_{\ell(r)}(\succ) \neq o_{h(r)}$. Because f is a pairwise exchange rule, $f_{h(r)}(\succ) \neq o_{\ell(r)}$. Hence, by (13),

$$\begin{aligned} f_{\ell(r)}(\succ) &\in O \Big\backslash \left(O^{r-1}(\succ) \cup \{o_{h(r)}\}\right); \\ f_{h(r)}(\succ) &\in O \Big\backslash \left(O^{r-1}(\succ) \cup \{o_{\ell(r)}\}\right). \end{aligned} \tag{14}$$

Now we proceed in four steps.

Step 1: Defining a preference profile $\succ_S^{\leftrightarrow}$. Let

$$\succ_S^{\leftrightarrow} = (\succ_{\ell(r)}^{\leftrightarrow}, \succ_{h(r)}^{\leftrightarrow}) = (\succ_{h(r)}, \succ_{\ell(r)}).$$

Then, by (12),

$$b\left(\succ_{\ell(r)}^{\leftrightarrow}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)} \quad \text{and} \quad b\left(\succ_{h(r)}^{\leftrightarrow}, O \setminus O^{r-1}(\succ)\right) = o_{h(r)}. \tag{15}$$

Let $\succ^{\leftrightarrow} = (\succ_S^{\leftrightarrow}, \succ_{-S})$. Note that $\succ^{\leftrightarrow} \in \mathscr{S}_{\vee}^{N}$.

Step 2: For each $i \in N^{r-1}(\succ)$, $f_i(\succ^{\leftrightarrow}) = TTC_i(\succ)$. Let $i \in N^{r-1}(\succ)$. Then, there is $t \in \{1, 2, \ldots, r-1\}$ with $i \in M \in \mathbb{S}_t(\succ)$. By Lemma 2, there is $t_M \in \{1, 2, \ldots, t\}$ with $M \in \mathbb{S}_{t_M}(\succ^{\leftrightarrow})$. Thus, it follows that $i \in N^{r-1}(\succ^{\leftrightarrow})$, which together with the induction hypothesis implies

$$f_i(\succ^{\leftrightarrow}) = TTC_i(\succ^{\leftrightarrow}). \tag{16}$$

Hence,

$$f_i(\succ^{\leftrightarrow}) \overset{\text{(by (16))}}{=} TTC_i(\succ^{\leftrightarrow}) \overset{\text{(by Lemma 1)}}{=} TTC_i(\succ).$$

Step 3: $\{f_{\ell(r)}(\succ^{\leftrightarrow}), f_{h(r)}(\succ^{\leftrightarrow})\} \subseteq O \setminus O^{r-1}(\succ)$. Let $o \in O^{r-1}(\succ)$. Then, there is $i \in N^{r-1}(\succ)$ such that $TTC_i(\succ) = o$. By Step 2, $f_i(\succ^{\leftrightarrow}) = o$, which implies that

$$f_{\ell(r)}(\succ^{\leftrightarrow}) \neq o \quad \text{and} \quad f_{h(r)}(\succ^{\leftrightarrow}) \neq o.$$

Therefore, $\{f_{\ell(r)}(\succ^{\leftrightarrow}), f_{h(r)}(\succ^{\leftrightarrow})\} \subseteq O \setminus O^{r-1}(\succ)$.

Step 4: Concluding. By (15) and Step 3, *individual rationality* implies

$$f_{\ell(r)}(\succ^{\leftrightarrow}) = o_{\ell(r)} \quad \text{and} \quad f_{h(r)}(\succ^{\leftrightarrow}) = o_{h(r)}. \tag{17}$$

Thus, it follows from (12), (14), and (17) that

$$f_{h(r)}(\succ^{\leftrightarrow}) = o_{h(r)} \succ_{\ell(r)} f_{\ell(r)}(\succ) \quad \text{and} \quad f_{\ell(r)}(\succ^{\leftrightarrow}) = o_{\ell(r)} \succ_{h(r)} f_{h(r)}(\succ),$$

in violation of *reallocation-proofness*.

From Cases 1 and 2, for each $i \in N_r(\succ)$, $f_i(\succ) = TTC_i(\succ)$. $\square$

B Appendix: Proof of Theorem 5

We begin by introducing two lemmas that are useful for establishing Theorem 5. Lemma 3 states that every object removed in the first round of the TTC algorithm is a leaf.

Lemma 3. *For each $\succ \in \mathscr{T}_G^N$, $O_1(\succ) \subseteq \mathbb{L}$.*

Proof. As stated in Remark 4, for each $i \in N$ and each $\succ_i \in \mathscr{T}_G$, $b(\succ_i, O)$ must be a leaf in G. Hence, $N_1(\succ) \subseteq \{i \in N \colon o_i \in \mathbb{L}\}$ and $O_1(\succ) \subseteq \mathbb{L}$. $\square$

Lemma 4 states that for every agent that forms a cycle in the first round of the TTC algorithm, he receives his TTC assignment under any rule satisfying the three axioms.

Lemma 4. *Let f be a rule that is individually rational, pair-efficient, and strategy-proof. Then, for each $\succ \in \mathscr{T}_G^N$, each $S \in \mathbb{S}_1(\succ)$, and each $i \in S$, $f_i(\succ) = TTC_i(\succ)$.*

Proof. We now show that for each $k \in \{1, 2, \ldots, |\mathbb{L}|\}$, each $\succ \in \mathscr{T}_G^N$, each $S \in \mathbb{S}_1(\succ)$ with $|S| = k$, and each $i \in S$, $f_i(\succ) = TTC_i(\succ)$. For each $\succ \in \mathscr{T}_G^N$ and $S \in \mathbb{S}_1(\succ)$, if $S = \{i\}$ (i.e., $|S| = 1$), then *individual rationality* implies $f_i(\succ) = o_i =$

$TTC_i(\succ)$ because of $b(\succ_i, O) = o_i$. Below, we focus on the case where $|S| \geq 2$. We prove this case by induction on $|S|$.

Base step. Let $\succ \in \mathscr{T}_G^N$ and $S \in \mathbb{S}_1(\succ)$ with $|S| = 2$. Suppose $S = \{i, j\}$. Then, $TTC_i(\succ) = o_j$ and $TTC_j(\succ) = o_i$, and $b(\succ_i, O) = o_j$ and $b(\succ_j, O) = o_i$. Suppose on the contrary that

$$(f_i(\succ), f_j(\succ)) \neq (TTC_i(\succ), TTC_j(\succ)) = (o_j, o_i).$$

Without loss of generality, we assume $f_i(\succ) \neq o_j$. Because $\{o_i, o_j\} \subseteq \mathbb{L}$ by Lemma 3, we can pick $(\succ_i^{\uparrow}, \succ_j^{\uparrow}) \in \mathscr{T}_G \times \mathscr{T}_G$ such that

$$\begin{array}{cc}
\succ_i^{\uparrow} & \succ_j^{\uparrow} \\
\hline
o_j & o_i \\
o_i & o_j \\
\vdots & \vdots
\end{array}$$

By *strategy-proofness* and *individual rationality*,

$$f_i(\succ_i^{\uparrow}, \succ_{-i}) = o_i \quad \text{and} \quad f_j(\succ_i^{\uparrow}, \succ_{-i}) \neq o_i.$$

Moreover, by *strategy-proofness* and *individual rationality*,

$$f_i(\succ_S^{\uparrow}, \succ_{-S}) = o_i \quad \text{and} \quad f_j(\succ_S^{\uparrow}, \succ_{-S}) = o_j.$$

Hence,

$$\begin{aligned}
f_j(\succ_S^{\uparrow}, \succ_{-S}) = o_j \succ_i^{\uparrow} o_i = f_i(\succ_S^{\uparrow}, \succ_{-S});\\
f_i(\succ_S^{\uparrow}, \succ_{-S}) = o_i \succ_j^{\uparrow} o_j = f_j(\succ_S^{\uparrow}, \succ_{-S}),
\end{aligned}$$

in violation of *pair-efficiency*.

Induction hypothesis. Let $K \in \{3, 4, \ldots, |\mathbb{L}|\}$. For each $\succ \in \mathscr{T}_G^N$, each $S \in \mathbb{S}_1(\succ)$ with $|S| \leq K - 1$, and each $i \in S$, $f_i(\succ) = TTC_i(\succ)$.

Induction step of induction on $|S|$. Let $K \in \{3, 4, \ldots, |\mathbb{L}|\}$. Let $\succ \in \mathscr{T}_G^N$ and $S \in \mathbb{S}_1(\succ)$ with $|S| = K$. Without loss of generality, we assume $S = \{1, 2, \ldots, K\}$. Then, for each $k \in \{1, 2, \ldots, K-1\}$, $TTC_k(\succ) = b(\succ_k, O) = o_{k+1}$ and $TTC_K(\succ) = b(\succ_K, O) = o_1$. Suppose on the contrary that there is $i \in S$ such that $f_i(\succ) \neq TTC_i(\succ)$. Without loss of generality, we assume $i = 1$; that is,

$$f_1(\succ) \neq TTC_1(\succ) = o_2. \tag{18}$$

Because $\bigcup_{i \in S}\{o_i\} \subseteq \mathbb{L}$ by Lemma 3, we can pick any $(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_1^{3}) \in \mathscr{T}_G \times \mathscr{T}_G \times \mathscr{T}_G$ such that

$\succ_1^{\uparrow}$	$\succ_2^{\uparrow}$	$\succ_1^{3}$
o_2	o_3	o_3
o_3	o_2	$\vdots$
o_1	$\vdots$	
$\vdots$		

Note that because $|S| = K \geq 3$, o_1, o_2, and o_3 are distinct objects. We proceed in four steps.

Step 1: $f_1(\succ_1^{3}, \succ_{-1}) = o_3$. Note that $S \setminus \{2\} \in \mathbb{S}_1(\succ_1^{3}, \succ_{-1})$ and $|S \setminus \{2\}| = K - 1$. By the induction hypothesis,

$$\forall\, i \in S \setminus \{2\},\ f_i(\succ_1^{3}, \succ_{-1}) = TTC_i(\succ_1^{3}, \succ_{-1}).$$

Hence, $f_1(\succ_1^{3}, \succ_{-1}) = o_3$.

Step 2: $f_1(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_3$ and $f_2(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_2$. Note that $S \setminus \{2\} \in \mathbb{S}_1(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2})$ and $|S \setminus \{2\}| = K - 1$. By the induction hypothesis, for each $i \in S \setminus \{2\}$, $f_i(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = TTC_i(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2})$. This together with *individual rationality* implies that $f_1(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_3$ and $f_2(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_2$.

Step 3: $f_1(\succ_1^{\uparrow}, \succ_{-1}) = o_3$ and $f_2(\succ_1^{\uparrow}, \succ_{-1}) \neq o_3$. By (18) and Step 1, *strategy-proofness* implies $f_1(\succ_1^{\uparrow}, \succ_{-1}) = o_3$, and thus, $f_2(\succ_1^{\uparrow}, \succ_{-1}) \neq o_3$.

Step 4: Concluding. By $f_2(\succ_1^{\uparrow}, \succ_{-1}) \neq o_3$ (Step 3), *strategy-proofness* and *individual rationality* together imply $f_2(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_2$. Thus, $f_1(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}) \neq o_2$. Then, by $f_1(\succ_1^{3}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_3$ (Step 2) and *strategy-proofness*,

$$f_1(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_3.$$

Hence,

$$f_2(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_2 \succ_1^{\uparrow} o_3 = f_1(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2});$$

$$f_1(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}) = o_3 \succ_2^{\uparrow} o_2 = f_2(\succ_1^{\uparrow}, \succ_2^{\uparrow}, \succ_{-1,2}),$$

in violation of *pair-efficiency*. □

Now we prove Theorem 5.

Proof of Theorem 5. Because *pair-efficiency* is weaker than *efficiency*, the "if" part follows from Tamura (2023). Therefore, we show the "only if" part. Let f be a rule satisfying the three axioms. We prove this by induction on the number of agents n.

BASE STEP. Let $n = 2$. Each $\succ \in \mathscr{T}_G^N$ falls into one of the following three categories:

$$\text{(i)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline o_j & o_i \\ o_i & o_j \end{array} \qquad \text{(ii)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline o_i & o_i \\ o_j & o_j \end{array} \qquad \text{(iii)} \quad \begin{array}{cc} \succ_i & \succ_j \\ \hline o_i & o_j \\ o_j & o_i \end{array}$$

By Lemma 4, in case (i), $f_i(\succ) = o_j$ and $f_j(\succ) = o_i$; and in cases (ii) and (iii), $f_i(\succ) = o_i$, which implies $f_j(\succ) = o_j$. Hence, $f(\succ) = TTC(\succ)$.

INDUCTION HYPOTHESIS. The theorem holds for each $n \in \{3, 4, \ldots, m-1\}$.

INDUCTION STEP. Let $n = m$ and f be a rule on $\mathscr{T}_G^N$ satisfying the three axioms. Pick any $\succ^* \in \mathscr{T}_G^N$. Note that by Lemma 3, $O_1(\succ^*) \subseteq \mathbb{L}$. Let $\overline{N} = N \setminus N_1(\succ^*)$ and $\overline{O} = O \setminus O_1(\succ^*)$. If $N_1(\succ^*) = N$ and $O_1(\succ^*) = O$, then by Lemma 4, $f(\succ^*) = TTC(\succ^*)$. Thus, we only consider the case where $N_1(\succ^*) \neq N$ and $O_1(\succ^*) \neq O$ (that is, $\overline{N} \neq \emptyset$ and $\overline{O} \neq \emptyset$). We denote by $X_{\overline{O}}$ the set of allocations when the sets of agents and objects are $\overline{N}$ and $\overline{O}$, respectively; specifically,

$$X_{\overline{O}} = \left\{ y \in \overline{O}^{\overline{N}} : y \text{ is bijective} \right\},$$

where $\overline{O}^{\overline{N}}$ is the set of all mappings from $\overline{N}$ to $\overline{O}$.

Given $S \in \mathbb{S}_1(\succ^*)$, it follows that for each $\succ_{\overline{N}} \in \mathscr{T}_G^{\overline{N}}$, $S \in \mathbb{S}_1(\succ^*_{N_1(\succ^*)}, \succ_{\overline{N}})$. This together with Lemma 4 implies that for each $\succ_{\overline{N}} \in \mathscr{T}_G^{\overline{N}}$ and each $i \in N_1(\succ^*)$,

$$f_i(\succ^*_{N_1(\succ^*)}, \succ_{\overline{N}}) = TTC_i(\succ^*_{N_1(\succ^*)}, \succ_{\overline{N}}) = f_i(\succ^*). \tag{19}$$

By (19), we focus on the "reduced" economy consisting of the set of agents $\overline{N}$ and the set of objects $\overline{O}$. Then, we construct a graph $\overline{G} = (\overline{O}, \overline{E})$, where

$\overline{E} = \{\{o,o'\} \in E\colon \{o,o'\} \subset \overline{O}\}$. Because $\overline{O} = O \setminus O_1(\succ^*)$ and $O_1(\succ^*) \subseteq \mathbb{L}$, by Lemma 2.1.3 in West (2001), $\overline{G}$ is a tree. For each $\{o',o''\} \subset \overline{O}$ with $o' \neq o''$, we denote by $\overline{[o',o'']}$ the unique path from o' to o'' in the tree $\overline{G}$. Let $\mathscr{T}_{\overline{G}}$ be the class of single-dipped preference relations on the tree $\overline{G}$. For each $i \in \overline{N}$, let $\succ_i^*|_{\overline{O}}$ be a preference relation over $\overline{O}$ such that for each pair $\{o,o'\} \subseteq \overline{O}$,

$$o \succ_i^*|_{\overline{O}}\, o' \iff o \succ_i^* o'. \tag{20}$$

However, just because $\succ_i^*$ is single-dipped on the tree G does not guarantee that $\succ_i^*|_{\overline{O}}$ will be single-dipped on the tree $\overline{G}$. The following claim states that $\succ_i^*|_{\overline{O}}$ is in fact single-dipped on the tree $\overline{G}$.

Claim 1. *For each $i \in \overline{N}$, $\succ_i^*|_{\overline{O}} \in \mathscr{T}_{\overline{G}}$.*

Proof of Claim 1. Let $\overline{d}$ be i's worst object in $\overline{O}$ according to $\succ_i^*|_{\overline{O}}$. Then, $\overline{d} \in \overline{O}$ and for each $o \in \overline{O} \setminus \{\overline{d}\}$, $o \succ_i^*|_{\overline{O}}\, \overline{d}$. Next, let $\{o',o''\} \subset \overline{O} \setminus \{\overline{d}\}$ be such that $o' \neq o''$ and $o' \in \overline{[\overline{d},o'']} = (o^1 = \overline{d}, o^2, \ldots, o^K = o'')$. For each $k \in \{1,2,\ldots,K-1\}$, by $\{o^k, o^{k+1}\} \in \overline{E}$, $\{o^k, o^{k+1}\} \in E$. Hence, $[\overline{d},o''] = \overline{[\overline{d},o'']}$. There are two cases.

- **Case 1: $\boldsymbol{d(\succ_i^*) \in \overline{O}}$.** It is evident that $\overline{d} = d(\succ_i^*)$. By $\succ_i^* \in \mathscr{T}_G$ and $o' \in [\overline{d} = d(\succ_i), o'']$, $o'' \succ_i^* o'$. Thus, by $\{o',o''\} \subset \overline{O}$ and (20), $o'' \succ_i^*|_{\overline{O}}\, o'$.

- **Case 2: $\boldsymbol{d(\succ_i^*) \notin \overline{O}}$.** Then, $d(\succ_i^*) \in O_1(\succ^*) \subset \mathbb{L}$. Thus, $\deg(d(\succ_i^*)) = 1$. Let $o^* \in O$ be the unique object such that $\{d(\succ_i^*), o^*\} \in E$. Then, $o^* \in \overline{O}$.[18] We now show that $o^* = \overline{d}$; that is, for each $o \in \overline{O} \setminus \{o^*\}$, $o \succ_i^*|_{\overline{O}}\, o^*$. Let $o \in \overline{O} \setminus \{o^*\}$. Note that by $d(\succ_i^*) \notin \overline{O}$, $o \neq d(\succ_i^*)$. By $o \in O$ and $o \neq d(\succ^*)$, we find $[d(\succ_i^*), o] = (\hat{o}^1 = d(\succ_i^*), \hat{o}^2, \ldots, \hat{o}^{\hat{K}} = o)$. Because o^* is the unique object such that $\{d(\succ_i^*), o^*\} \in E$, $\hat{o}^2 = o^*$. Thus, $o^* \in [d(\succ_i^*), o]$. Given that $\succ_i^* \in \mathscr{T}_G$, $o \succ_i^* o^*$. Therefore, by $\{o^*, o\} \subset \overline{O}$ and (20), $o \succ_i^*|_{\overline{O}}\, o^*$. Moreover, since $[\overline{d} = o^*, o'']$ and $\{d(\succ_i^*), o^*\} \in E$, we find the path from $d(\succ_i^*)$ to o'', $[d(\succ_i^*), o''] = (d(\succ_i^*), \overline{d} = o^*, \ldots, o'')$. By $o' \in \overline{[\overline{d},o'']} = [\overline{d},o'']$, $o' \in [d(\succ_i^*), o'']$. Since $\succ_i^* \in \mathscr{T}_G$, $o'' \succ_i^* o'$. Hence, by $\{o',o''\} \subset \overline{O}$ and (20), $o'' \succ_i^*|_{\overline{O}}\, o'$. □

For each $i \in \overline{N}$ and each $\succ_i^{\overline{G}} \in \mathscr{T}_{\overline{G}}$, we associate it with the preference relation over $\succ_i \in \mathscr{P}$ that satisfies the following two conditions:

PG1. If $\succ_i^{\overline{G}} = \succ_i^*|_{\overline{O}}$, then $\succ_i = \succ_i^*$.

[18] If $o^* \in O_1(\succ^*)$, then $o^* \in \mathbb{L}$ and $\deg(o^*) = 1$. This implies that $O = \{d(\succ_i^*), o^*\}$, $E = \{\{d(\succ_i^*), o^*\}\}$, and $O_1(\succ^*) = O$, a contradiction.

PG2. If $\succ_i^{\overline{G}} \neq \succ_i^*|_{\overline{O}}$, then $\succ_i$ satisfies the following:

(i) for each $o \in O_1(\succ^*)$ and $o' \in \overline{O}$,

$$o \succ_i o';$$

(ii) for each pair $\{o, o'\} \subseteq \overline{O}$,

$$o \succ_i o' \iff o \succ_i^{\overline{G}} o';$$

(iii) for each pair $\{o, o'\} \subseteq O_1(\succ^*)$,

$$o \succ_i o' \iff o \succ_i^* o'.$$

Note that because of Claim 1, we observe that PG1 makes sense. The following claim states that under PG1 and PG2, the preference relation $\succ_i \in \mathscr{P}$ associated with each $\succ_i^{\overline{G}} \in \mathscr{T}_{\overline{G}}$ is single-dipped on G.

Claim 2. *For each $\succ_i^{\overline{G}} \in \mathscr{T}_{\overline{G}}$, let $\succ_i \in \mathscr{P}$ be a preference relation associated with $\succ_i^{\overline{G}}$ under PG1 and PG2. Then, $\succ_i \in \mathscr{T}_G$.*

Proof of Claim 2. Let $\succ_i^{\overline{G}} \in \mathscr{T}_{\overline{G}}$ and $\succ_i \in \mathscr{P}$ be a preference relation associated with $\succ_i^{\overline{G}}$ under PG1 and PG2. If $\succ_i^{\overline{G}} = \succ_i^*|_{\overline{O}}$, it follows from PG1 that $\succ_i = \succ_i^* \in \mathscr{T}_G$. Thus, we consider the case $\succ_i^{\overline{G}} \neq \succ_i^*|_{\overline{O}}$. Let $\overline{d} \in \overline{O}$ be such that for each $o \in \overline{O} \setminus \{\overline{d}\}$, $o \succ_i^{\overline{G}} \overline{d}$. By PG2, $d(\succ_i) = \overline{d}$. Let $\{o, o'\} \subset O \setminus \{\overline{d}\}$ be such that $o \neq o'$ and $o \in [\overline{d}, o'] = (o^1 = \overline{d}, o^2, \dots, o^K = o')$. Note that for each $k \in \{2, 3, \dots, K-1\}$, by $\{\{o^{k-1}, o^k\}, \{o^k, o^{k+1}\}\} \subset E$, $\deg(o^k) \geq 2$. This together with $O_1(\succ^*) \subset \mathbb{L}$ and Lemma 3 implies that for each $k \in \{2, 3, \dots, K-1\}$, $o^k \in \overline{O}$. By $o \notin \{\overline{d}, o'\}$, there is $k \notin \{1, K\}$ such that $o^k = o$ and thus, $o \in \overline{O}$. There are two cases.

• **Case 1: $o' \in O_1(\succ^*)$.** By $o \in \overline{O}$ and PG2(i), $o' \succ_i o$.

• **Case 2: $o' \in \overline{O}$.** By $\{o^1 = \overline{d}, o^2, \dots, o^K = o'\} \subset \overline{O}$, $\overline{[\overline{d}, o']} = (o^1, o^2, \dots, o^K) = [\overline{d}, o']$. Because $o = o^k \in \overline{[\overline{d}, o']}$ and $\succ_i^{\overline{G}} \in \mathscr{T}_{\overline{G}}$, $o' \succ_i^{\overline{G}} o$. Hence, by $\{o, o'\} \subset \overline{O}$ and PG2(ii), $o' \succ_i o$.

From Cases 1 and 2, $\succ_i \in \mathscr{T}_G$. □

We now define a rule $g \colon \mathscr{T}_{\overline{G}}^{\overline{N}} \to X_{\overline{O}}$ as follows:

$$\forall \succ^{\overline{G}} \in \mathscr{T}_{\overline{G}}^{\overline{N}},\ \forall i \in \overline{N},\ g_i(\succ^{\overline{G}}) = f_i(\succ^*_{N_1(\succ^*)}, \succ_{\overline{N}}), \tag{21}$$

where for each $i \in \overline{N}$, $\succ_i$ is a preference relation associated with $\succ_i^{\overline{G}}$ under PG1 and PG2. Note that by (19), Claim 2, and the definition of f_i, g is well-defined. Because f satisfies *individual rationality*, *strategy-proofness*, and *pair-efficiency*, g also satisfies these three axioms. According to the induction hypothesis,

$$\forall \succ^{\overline{G}} \in \mathscr{T}_{\overline{G}}^{\overline{N}},\ \forall i \in \overline{N},\ g_i(\succ^{\overline{G}}) = TTC_i(\succ^{\overline{G}}). \tag{22}$$

Let $\succ^*|_{\overline{O}} = (\succ_i^*|_{\overline{O}})_{i \in \overline{N}}$. Note that by Claim 1, $\succ^*|_{\overline{O}} \in \mathscr{T}_{\overline{G}}^{\overline{N}}$, and by (20) and PG1, for each $i \in \overline{N}$, $\succ_i^*$ is associated with $\succ_i^*|_{\overline{O}}$. Hence, it follows from (21) and (22) that

$$\forall i \in \overline{N},\ TTC_i(\succ^*|_{\overline{O}}) = g_i(\succ^*|_{\overline{O}}) = f_i(\succ^*_{N_1(\succ^*)}, \succ^*_{\overline{N}}) = f_i(\succ^*). \tag{23}$$

It is evident that

$$\forall i \in \overline{N},\ TTC_i(\succ^*|_{\overline{O}}) = TTC_i(\succ^*).$$

Hence, by (19) and (23), we have $f(\succ^*) = TTC(\succ^*)$. $\square$

Remark 7. The structure of our proof of Theorem 5 is similar to that of Fujinaka and Wakayama (2018), who provide a characterization of TTC on the domain of strict preferences. We can apply the proof of Theorem 5 to the domain of strict preferences by abbreviating some steps in the proof of Theorem 5. As mentioned in Remark 5, this allows us to provide an alternative proof for the characterization of TTC proposed by Ekici (2024). Specifically, when handling the unrestricted domain of strict preferences, we can omit proving both Claim 1 and Claim 2, which state that $\succ_i^*|_{\overline{O}}$ and a preference relation $\succ_i \in \mathscr{P}$ associated with $\succ_i^{\overline{G}}$ under PG1 and PG2 is single-dipped on G. $\lozenge$

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Online Appendix to "Reallocation-proofness in object reallocation problems with single-dipped preferences" by Fujinaka and Wakayama (September 26, 2024)

C Appendix: Proof of Theorem 2

Because *pair-efficiency* is weaker than *efficiency*, the "if" part follows from Tamura (2023). Thus, we show the "only if" part. We now demonstrate that for each $t \in \mathbb{N}$, each $\succ \in \mathscr{S}_{\vee}^{N}$, and each $i \in N_t(\succ)$, $f_i(\succ) = TTC_i(\succ)$. We prove this by induction on t.

Base step. Let $t = 1$. Let $\succ \in \mathscr{S}_{\vee}^{N}$ and $S \in \mathbb{S}_1(\succ)$. By Fact 1, $S \in \{\{1, n\}, \{1\}, \{n\}\}$. There are two cases.

• **Case 1: $S \in \{\{1\}, \{n\}\}$.** Without loss of generality, we assume $S = \{1\}$. Then, $b(\succ_1, O) = o_1$. Hence, by *individual rationality*, $f_1(\succ) = o_1 = TTC_1(\succ)$.

• **Case 2: $S = \{1, n\}$.** Then, $b(\succ_1, O) = o_n$ and $b(\succ_n, O) = o_1$. Suppose on the contrary that

$$(f_1(\succ), f_n(\succ)) \neq (TTC_1(\succ), TTC_n(\succ)) = (o_n, o_1).$$

Without loss of generality, we assume $f_1(\succ) \neq o_n$. Let $(\succ_1^{\uparrow}, \succ_n^{\uparrow}) \in \mathscr{S}_{\vee} \times \mathscr{S}_{\vee}$ be such that

$\succ_1^{\uparrow}$	$\succ_n^{\uparrow}$
o_n	o_1
o_1	o_n
$\vdots$	$\vdots$

By *strategy-proofness* and *individual rationality*,

$$f_1(\succ_1^{\uparrow}, \succ_{-1}) = o_1 \quad \text{and} \quad f_n(\succ_1^{\uparrow}, \succ_{-1}) \neq o_1.$$

Additionally, by *strategy-proofness* and *individual rationality*,

$$f_1(\succ_S^{\uparrow}, \succ_{-S}) = o_1 \quad \text{and} \quad f_n(\succ_S^{\uparrow}, \succ_{-S}) = o_n.$$

Hence,

$$f_n(\succ_S^{\uparrow}, \succ_{-S}) = o_n \succ_1^{\uparrow} o_1 = f_1(\succ_S^{\uparrow}, \succ_{-S});$$
$$f_1(\succ_S^{\uparrow}, \succ_{-S}) = o_1 \succ_n^{\uparrow} o_n = f_n(\succ_S^{\uparrow}, \succ_{-S}),$$

in violation of *pair-efficiency*.

Induction hypothesis. For each $t \in \{1, 2, \ldots, r-1\}$, each $\succ \in \mathscr{S}_{\vee}^{N}$, and each $i \in N_t(\succ)$, $f_i(\succ) = TTC_i(\succ)$.

Induction step. Let $t = r$. By the induction hypothesis, for each $\succ' \in \mathscr{S}_{\vee}^{N}$,

$$O^{r-1}(\succ') = \left\{o \in O \colon \exists\, i \in N^{r-1}(\succ'),\ o = f_i(\succ')\right\}. \tag{24}$$

Let $\succ \in \mathscr{S}_{\vee}^{N}$. For each $t \in \mathbb{N}$, we simply write $\ell(t)$ (resp. $h(t)$) for $\ell(\succ, t)$ (resp. $h(\succ, t)$). Let $S \in \mathbb{S}_r(\succ)$. By Fact 1, $S \in \big\{\{\ell(r), h(r)\}, \{\ell(r)\}, \{h(r)\}\big\}$. There are two cases.

• **Case 1: $S \in \{\{\ell(r)\}, \{h(r)\}\}$.** Without loss of generality, we assume $S = \{\ell(r)\}$. Then, $TTC_{\ell(r)}(\succ) = o_{\ell(r)}$ and

$$b\left(\succ_{\ell(r)}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)}. \tag{25}$$

It follows from (24) that

$$f_{\ell(r)}(\succ) \in O \setminus O^{r-1}(\succ). \tag{26}$$

By (25) and (26), *individual rationality* implies $f_{\ell(r)}(\succ) = o_{\ell(r)} = TTC_{\ell(r)}(\succ)$.

• **Case 2: $S = \{\ell(r), h(r)\}$.** Then, $TTC_{\ell(r)}(\succ) = o_{h(r)}$ and $TTC_{h(r)}(\succ) = o_{\ell(r)}$, and

$$b\left(\succ_{\ell(r)}, O \setminus O^{r-1}(\succ)\right) = o_{h(r)} \quad \text{and} \quad b\left(\succ_{h(r)}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)}. \tag{27}$$

It follows from (24) that

$$\{f_{\ell(r)}(\succ), f_{h(r)}(\succ)\} \subseteq O \setminus O^{r-1}(\succ). \tag{28}$$

Suppose on the contrary that

$$(f_{\ell(r)}(\succ), f_{h(r)}(\succ)) \neq (TTC_{\ell(r)}(\succ), TTC_{h(r)}(\succ)) = (o_{h(r)}, o_{\ell(r)}).$$

Without loss of generality, we assume

$$f_{\ell(r)}(\succ) \neq o_{h(r)}. \tag{29}$$

By Fact 1, for each $t \in \mathbb{N}$, $N_t(\succ) \subset \{\ell(t), h(t)\}$. This implies

$$O \setminus O^{r-1}(\succ) = \{o_i \in O\colon \ell(r) \leq i \leq h(r)\}. \tag{30}$$

Now we proceed in four steps.

Step 1: Defining a preference profile $\succ_S^{\uparrow}$. Let $N_L = \{i \in N^{r-1}(\succ)\colon 1 \leq i < \ell(r)\}$ and $N_R = \{i \in N^{r-1}(\succ)\colon h(r) < i \leq n\}$. Note that $N_L \cup N_R = N^{r-1}(\succ)$. Let $\succ_S^{\uparrow} \in \mathscr{S}_{\vee} \times \mathscr{S}_{\vee}$ be such that:

(i) for each $i \in N_L$, each $j \in N_R$, and each $k \in N \setminus \big(N^{r-1}(\succ) \cup \{\ell(r), h(r)\}\big)$,

$$o_i \succ_{\ell(r)}^{\uparrow} o_j \succ_{\ell(r)}^{\uparrow} o_{h(r)} \succ_{\ell(r)}^{\uparrow} o_{\ell(r)} \succ_{\ell(r)}^{\uparrow} o_k,$$
$$o_i \succ_{h(r)}^{\uparrow} o_j \succ_{h(r)}^{\uparrow} o_{\ell(r)} \succ_{h(r)}^{\uparrow} o_{h(r)} \succ_{\ell(r)}^{\uparrow} o_k;$$

(ii) for each $\{i, i'\} \subseteq N_L$, if $i < i'$, then $o_i \succ_{\ell(r)}^{\uparrow} o_{i'}$ and $o_i \succ_{h(r)}^{\uparrow} o_{i'}$;

(iii) for each $\{j, j'\} \subseteq N_R$, if $j < j'$, then $o_{j'} \succ_{\ell(r)}^{\uparrow} o_j$ and $o_{j'} \succ_{h(r)}^{\uparrow} o_j$.

Figure 1 illustrates profile $\succ_S^{\uparrow}$. Furthermore, this profile can be represented as follows:

$\succ_{\ell(r)}^{\uparrow}$	$\succ_{h(r)}^{\uparrow}$
o_1	o_1
$\vdots$	$\vdots$
$o_{\ell(r)-1}$	$o_{\ell(r)-1}$
o_n	o_n
$\vdots$	$\vdots$
$o_{h(r)+1}$	$o_{h(r)+1}$
$o_{h(r)}$	$o_{\ell(r)}$
$o_{\ell(r)}$	$o_{h(r)}$
$\vdots$	$\vdots$

By (30) and the conditions (i)–(iii) of $\succ_S^{\uparrow}$, we observe that $\succ_S^{\uparrow}$ satisfies the following:

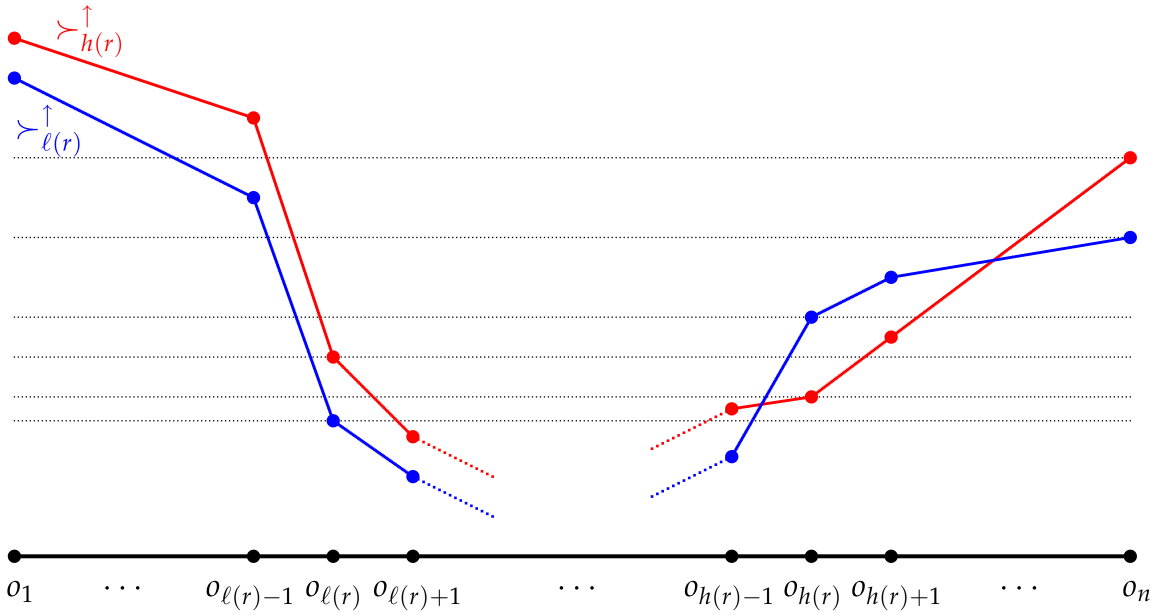

Figure 1: An illustration of $\succ^{\uparrow}_S$ in the proof of Theorem 2.

PL. $b\left(\succ^{\uparrow}_{\ell(r)}, O \setminus O^{r-1}(\succ)\right) = o_{h(r)}$ and $b\left(\succ^{\uparrow}_{\ell(r)}, O \Big\backslash \left(O^{r-1}(\succ) \cup \{o_{h(r)}\}\right)\right) = o_{\ell(r)}$.

PH. $b\left(\succ^{\uparrow}_{h(r)}, O \setminus O^{r-1}(\succ)\right) = o_{\ell(r)}$ and $b\left(\succ^{\uparrow}_{h(r)}, O \Big\backslash \left(O^{r-1}(\succ) \cup \{o_{\ell(r)}\}\right)\right) = o_{h(r)}$.

Let $\tilde{\succ} \in \{(\succ^{\uparrow}_S, \succ_{-S}), (\succ^{\uparrow}_{\ell(r)}, \succ_{-\ell(r)})\}$. Note that $\tilde{\succ} \in \mathscr{S}^N_{\vee}$.

Step 2: For each $i \in N^{r-1}(\succ)$, $f_i(\tilde{\succ}) = TTC_i(\succ)$. Let $i \in N^{r-1}(\succ)$. Then, there is $t \in \{1, 2, \ldots, r-1\}$ with $i \in M \in \mathbb{S}_t(\succ)$. By Lemma 2, there is $t_M \in \{1, 2, \ldots, t\}$ with $M \in \mathbb{S}_{t_M}(\tilde{\succ})$. Thus, it follows that $i \in N^{r-1}(\tilde{\succ})$, which together with the induction hypothesis implies

$$f_i(\tilde{\succ}) = TTC_i(\tilde{\succ}). \tag{31}$$

Hence,

$$f_i(\tilde{\succ}) \overset{\text{(by (31))}}{=} TTC_i(\tilde{\succ}) \overset{\text{(by Lemma 1)}}{=} TTC_i(\succ).$$

Step 3: $\{f_{\ell(r)}(\tilde{\succ}), f_{h(r)}(\tilde{\succ})\} \subseteq O \setminus O^{r-1}(\succ)$. Let $o \in O^{r-1}(\succ)$. Then, there is $i \in N^{r-1}(\succ)$ such that $TTC_i(\succ) = o$. By Step 2, $f_i(\tilde{\succ}) = o$, which implies that $f_{\ell(r)}(\tilde{\succ}) \neq o$ and $f_{h(r)}(\tilde{\succ}) \neq o$. Therefore, $\{f_{\ell(r)}(\tilde{\succ}), f_{h(r)}(\tilde{\succ})\} \subseteq O \setminus O^{r-1}(\succ)$.

Step 4: Concluding. By (27)–(29), *strategy-proofness* implies $f_{\ell(r)}(\succ^{\uparrow}_{\ell(r)}, \succ_{-\ell(r)}) \neq o_{h(r)}$; otherwise, $f_{\ell(r)}(\succ^{\uparrow}_{\ell(r)}, \succ_{-\ell(r)}) = o_{h(r)} \succ_{\ell(r)} f_{\ell(r)}(\succ)$, in violation of *strategy-*

proofness. Then, by PL and Step 3, *individual rationality* implies that

$$f_{\ell(r)}(\succ^{\uparrow}_{\ell(r)}, \succ_{-\ell(r)}) = o_{\ell(r)};$$
$$f_{h(r)}(\succ^{\uparrow}_{\ell(r)}, \succ_{-\ell(r)}) \in O \Big\backslash \left(O^{r-1}(\succ) \cup \{o_{\ell(r)}\}\right).$$

Furthermore, by (27), PL, PH, and Step 3, *strategy-proofness* and *individual rationality* together imply that

$$f_{\ell(r)}(\succ^{\uparrow}_{S}, \succ_{-S}) = o_{\ell(r)} \quad \text{and} \quad f_{h(r)}(\succ^{\uparrow}_{S}, \succ_{-S}) = o_{h(r)}.$$

Hence, by PL and PH,

$$f_{h(r)}(\succ^{\uparrow}_{S}, \succ_{-S}) = o_{h(r)} \succ^{\uparrow}_{\ell(r)} o_{\ell(r)} = f_{\ell(r)}(\succ^{\uparrow}_{S}, \succ_{-S});$$
$$f_{\ell(r)}(\succ^{\uparrow}_{S}, \succ_{-S}) = o_{\ell(r)} \succ^{\uparrow}_{h(r)} o_{h(r)} = f_{h(r)}(\succ^{\uparrow}_{S}, \succ_{-S}),$$

in violation of *pair-efficiency*.

From Cases 1 and 2, for each $i \in N_r(\succ)$, $f_i(\succ) = TTC_i(\succ)$. □

D Appendix: Omitted proofs in the main text

D.1 Remark 2

Here, we provide a rule that is *pair-efficient* but not *self-enforcing reallocation-proof*. Suppose $n = 4$. Let $\succ^{ijk}_0 \in \mathscr{S}_\vee$ be a preference relation such that

$$\begin{array}{c} \succ^{ijk}_0 \\ \hline o_i \\ o_j \\ o_k \\ o_m \end{array}$$

and let $f^* \colon \mathscr{S}^N_\vee \to X$ be a rule such that for each $\succ \in \mathscr{S}^N_\vee$,

$$f^*(\succ) = \begin{cases} (o_3, o_1, o_2, o_4) & \text{if } \succ \in \mathscr{S}^*_\vee \\ (o_4, o_2, o_1, o_3) & \text{if } \succ = (\succ^{432}_1, \succ^{123}_2, \succ^{123}_3, \succ^{432}_4) \\ TTC(\succ) & \text{otherwise,} \end{cases}$$

where

$$\mathscr{S}^*_{\vee} = \Big\{ (\succ_1^{123}, \succ_2^{412}, \succ_3^{123}, \succ_4^{432}), (\succ_1^{432}, \succ_2^{412}, \succ_3^{123}, \succ_4^{432}), (\succ_1^{123}, \succ_2^{123}, \succ_3^{123}, \succ_4^{432}) \Big\}.$$

It is easy to see that this rule satisfies *pair-efficiency*. To observe that f^* violates *self-enforcing reallocation-proofness*, let $\succ' = (\succ_1^{123}, \succ_2^{412}, \succ_3^{123}, \succ_4^{432}) \in \mathscr{S}^*_{\vee}$. Note that $\{(\succ_1^{432}, \succ'_{-1}), (\succ_2^{123}, \succ'_{-2})\} \subset \mathscr{S}^*_{\vee}$ and $(\succ_1^{432}, \succ_2^{123}, \succ'_{-1,2}) \notin \mathscr{S}^*_{\vee}$. It then follows that

$$\begin{aligned}
&f_1^*(\succ') = f_1^*(\succ_1^{432}, \succ'_{-1}) = o_3 \neq o_4 = f_1^*(\succ_1^{432}, \succ_2^{123}, \succ'_{-1,2});\\
&f_2^*(\succ') = f_2^*(\succ_2^{123}, \succ'_{-2}) = o_1 \neq o_2 = f_2^*(\succ_1^{432}, \succ_2^{123}, \succ'_{-1,2});\\
&f_2^*(\succ_1^{432}, \succ_2^{123}, \succ'_{-1,2}) = o_2 \succ'_1 o_3 = f_1^*(\succ');\\
&f_1^*(\succ_1^{432}, \succ_2^{123}, \succ'_{-1,2}) = o_4 \succ'_2 o_1 = f_2^*(\succ').
\end{aligned}$$

This implies that f^* violates *self-enforcing reallocation-proofness*. □

D.2 Example 2

Here, we show that $IR^{12\neg}$ satisfies *reallocation-proofness*. Let $\succ \in \mathscr{S}^N_{\vee}$, $\{i,j\} \subset N$, and $\succ' = (\succ'_i, \succ'_j, \succ_k) \in \mathscr{S}^N_{\vee}$.

• **Case 1:** $\boldsymbol{\succ = \succ^{12}}$**.** Then, $IR^{12\neg}(\succ) = (o_1, o_3, o_2)$. Because $IR_1^{12\neg}(\succ) = b(\succ_1, O)$, agent 1 has no incentive to collude with another agent at $\succ$. Hence, we consider the case where $\{i,j\} = \{2,3\}$. Then, $\succ'_1 = \succ_1 = \succ_1^{12}$. Thus, if $\succ' \neq \succ^{12}$, by *individual rationality*, $IR_1^{12\neg}(\succ') = TTC_1(\succ') = o_1$; if $\succ' = \succ^{12}$, $IR_1^{12\neg}(\succ') = o_1$. That is, in both cases, $IR_1^{12\neg}(\succ') = o_1$. Because $IR_2^{12\neg}(\succ') \in \{o_2, o_3\}$,

$$IR_3^{12\neg}(\succ) = o_2 \succsim_3 (= \succsim_3^{12})\ IR_2^{12\neg}(\succ').$$

This implies that agent 3 has no incentive to collude with agent 2 at $\succ$.

• **Case 2:** $\boldsymbol{\succ \neq \succ^{12}}$ **and** $\boldsymbol{\succ' \neq \succ^{12}}$**.** Then, $IR^{12\neg}(\succ) = TTC(\succ)$ and $IR^{12\neg}(\succ') = TTC(\succ')$. Hence, by *reallocation-proofness* of TTC, no pair of agents has an incentive to collude at $\succ$.

• **Case 3:** $\boldsymbol{\succ \neq \succ^{12}}$ **and** $\boldsymbol{\succ' = \succ^{12}}$**.** Then, $IR^{12\neg}(\succ) = TTC(\succ)$ and $IR^{12\neg}(\succ') = (o_1, o_3, o_2)$. We further distinguish three subcases.

◦ *Subcase 3-1*: $\{i,j\} = \{1,2\}$. Then, $\succ_3 = \succ'_3 = \succ_3^{12}$. In Round 1 of the TTC algorithm at $\succ$, agent 1 points to either himself or agent 3 and agent 3 points to agent 1. This implies $IR_1^{12\neg}(\succ) = TTC_1(\succ) = b(\succ_1, O)$. Hence, agent 1 has no incentive to collude with agent 2 at $\succ$.

∘ *Subcase 3-2*: $\{i, j\} = \{1, 3\}$. In Round 1 of the TTC algorithm at $\succ$, agent 1 points to either himself or agent 3 and agent 3 points to either himself or agent 1. This implies either $IR_1^{12\neg}(\succ) = TTC_1(\succ) = b(\succ_1, O)$ or $IR_3^{12\neg}(\succ) = TTC_3(\succ) = b(\succ_3, O)$. Hence, this pair has no incentive to collude at $\succ$.

∘ *Subcase 3-3*: $\{i, j\} = \{2, 3\}$. By *individual rationality* of TTC,

$$IR_2^{12\neg}(\succ) = TTC_2(\succ) \succsim_2 o_2 = IR_3^{12\neg}(\succ').$$

Hence, agent 2 has no incentive to collude with agent 3 at $\succ$. □

D.3 Example 3

Here, we show that $f^{\overrightarrow{123}}$ satisfies *reallocation-proofness*. Let $\succ \in \mathscr{S}_\vee^N$, $\{i, j\} \subset N$, and $\succ' = (\succ'_i, \succ'_j, \succ_{-i,j}) \in \mathscr{S}_\vee^N$. There are three cases.

• **Case 1: $\succ = \succ^\star$.** Then, $f^{\overrightarrow{123}}(\succ) = (o_2, o_3, o_1, o_4)$. Because $f_3^{\overrightarrow{123}}(\succ) = b(\succ_3, O)$ and $f_4^{\overrightarrow{123}}(\succ) = b(\succ_4, O)$, it suffices to consider the case $\{i, j\} = \{1, 2\}$. If $\succ' = \succ^\star$, then $f_2^{\overrightarrow{123}}(\succ) = o_3 \succ_2 o_2 = f_1^{\overrightarrow{123}}(\succ')$, which implies that agent 2 has no incentive to collude with agent 1. If $\succ' \neq \succ^\star$, then $f^{\overrightarrow{123}}(\succ') = TTC(\succ')$. Note that $f_4^{\overrightarrow{123}}(\succ') = o_4$. This implies $f_1^{\overrightarrow{123}}(\succ') \neq o_4$. Hence, $f_2^{\overrightarrow{123}}(\succ) = o_3 \succsim_2 f_1^{\overrightarrow{123}}(\succ')$, which implies that agent 2 has no incentive to collude with agent 1 at $\succ$.

• **Case 2: $\succ \neq \succ^\star$ and $\succ' \neq \succ^\star$.** Then, $f^{\overrightarrow{123}}(\succ) = TTC(\succ)$ and $f^{\overrightarrow{123}}(\succ') = TTC(\succ')$. Hence, by *reallocation-proofness* of TTC, no pair of agents has an incentive to collude at $\succ$.

• **Case 3: $\succ \neq \succ^\star$ and $\succ' = \succ^\star$.** Then, $f^{\overrightarrow{123}}(\succ) = TTC(\succ)$ and $f^{\overrightarrow{123}}(\succ') = (o_2, o_3, o_1, o_4)$. We further distinguish two cases.

∘ *Subcase 3-1*: $\{i, j\} \cap \{4\} = \emptyset$. Then, $\{i, j\} \subset \{1, 2, 3\}$. Without loss of generality, assume $\{i, j\} = \{1, 2\}$. Then, by *individual rationality* of TTC,

$$f_2^{\overrightarrow{123}}(\succ) = TTC_2(\succ) \succsim_2 o_2 = f_1^{\overrightarrow{123}}(\succ').$$

Thus, agent 2 has no incentive to collude with agent 1 at $\succ$.

∘ *Subcase 3-2*: $\{i, j\} \cap \{4\} \neq \emptyset$. If $\{i, j\} = \{1, 4\}$, then there is $k \in \{1, 4\}$ such that agent k receives $b(\succ_k, O)$ in Round 1 of the TTC algorithm at $\succ$. This implies that agent k has no incentive to collude with another agent. Next, we consider the case where $\{i,j\} \in \{\{2,4\},\{3,4\}\}$. By $1 \notin \{i,j\}$, $\succ_1 = \succ'_1 = \succ^\star_1$. In Round 1 of the TTC algorithm at $\succ$, agent 1 points to agent 4 and agent 4 points to either himself or agent 1. This implies $f_4^{\overrightarrow{123}}(\succ) = TTC_4(\succ) = b(\succ_4, O)$. Hence, agent 4 has no incentive to collude with any other agent at $\succ$. $\square$

D.4 Example 5

Here, we show that $IR^\neg$ satisfies *strategy-proofness*. Let $\succ \in \mathscr{T}_G^N$, $i \in N$, $\succ'_i \in \mathscr{T}_G$, and $\succ' = (\succ'_i, \succ_{-i})$. There are four cases.

• **Case 1:** $\{\succ, \succ'\} \cap (\mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1) = \emptyset$**.** Then, $IR^\neg(\succ) = TTC(\succ)$ and $IR^\neg(\succ') = TTC(\succ')$. Hence, *strategy-proofness* of TTC implies that agent i has no incentive to misrepresent his preference relation.

• **Case 2:** $\{\succ, \succ'\} \subset \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$**.** Then, $IR^\neg(\succ) = IR^\neg(\succ') = (o_3, o_2, o_1)$. Hence, agent i has no incentive to misrepresent his preference relation.

• **Case 3:** $\{\succ, \succ'\} \cap (\mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1) = \{\succ'\}$**.** Then, $IR^\neg(\succ') = (o_3, o_2, o_1)$ and $IR^\neg(\succ) = TTC(\succ)$. We further distinguish three subcases.

∘ *Subcase 3-1*: $i = 1$. Then, $\succ_1 \in \mathscr{T}_G^3$, $\succ'_1 \in \mathscr{T}_G^1$, and $(\succ_2, \succ_3) \in \{\succ_2^{12}\} \times \mathscr{T}_G^1$. Hence, $IR_1^\neg(\succ) = TTC_1(\succ) = o_3 = IR_1^\neg(\succ')$, which implies that agent 1 has no incentive to misrepresent his preference relation.

∘ *Subcase 3-2*: $i = 2$. By *individual rationality* of TTC, $IR_2^\neg(\succ) = TTC_2(\succ) \succsim_2 o_2 = IR_2^\neg(\succ')$, which implies that agent 2 has no incentive to misrepresent his preference relation.

∘ *Subcase 3-3*: $i = 3$. Then, $\succ_3 \in \mathscr{T}_G^3$, $\succ'_3 \in \mathscr{T}_G^1$, and $(\succ_1, \succ_2) \in \mathscr{T}_G^1 \times \{\succ_2^{12}\}$. Hence, $IR_3^\neg(\succ) = TTC_3(\succ) = o_3 \succ_3 o_1 = IR_3^\neg(\succ')$, which implies that agent 3 has no incentive to misrepresent his preferences relation.

• **Case 4:** $\{\succ, \succ'\} \cap (\mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1) = \{\succ\}$**.** Then, $IR^\neg(\succ) = (o_3, o_2, o_1)$ and $IR^\neg(\succ') = TTC(\succ')$. We further distinguish three subcases.

∘ *Subcase 4-1*: $i = 1$. Then, $\succ_1 \in \mathscr{T}_G^1$, $\succ'_1 \in \mathscr{T}_G^3$, and $(\succ_2, \succ_3) \in \{\succ_2^{12}\} \times \mathscr{T}_G^1$. Hence, $IR_1^\neg(\succ) = o_3 = TTC_1(\succ') = IR_1^\neg(\succ')$, which implies that agent 1 has no incentive to misrepresent his preference relation.

∘ *Subcase 4-2*: $i = 2$. Then, $\succ_2 = \succ_2^{12}$. Because $\succ_1 \in \mathscr{T}_G^1$ and $IR^\neg(\succ') = TTC(\succ')$, $IR_2^\neg(\succ') \neq o_1$. Hence, $IR_2^\neg(\succ) = o_2 \succsim_2 IR_2^\neg(\succ')$, which implies that agent 2 has no incentive to misrepresent his preference relation.

∘ *Subcase 4-3*: $i = 3$. Then, $\succ_3 \in \mathcal{T}_G^1$, $\succ'_3 \in \mathcal{T}_G^3$, and $(\succ_1, \succ_2) \in \mathcal{T}_G^1 \times \{\succ_2^{12}\}$. Hence, $IR_3^{\neg}(\succ) = o_1 \succ_3 o_3 = TTC_3(\succ') = IR_3^{\neg}(\succ')$, which implies that agent 3 has no incentive to misrepresent his preference relation. □

D.5 Example 6

Here, we show that $SP^{*\neg}$ satisfies *reallocation-proofness*. Let $\succ \in \mathcal{T}_G^N$, $\{i,j\} \subset N$, and $\succ' = (\succ'_i, \succ'_j, \succ_k) \in \mathcal{T}_G^N$. There are three cases.

• **Case 1: $\succ = (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$.** Then, $SP^{*\neg}(\succ) = (o_2, o_3, o_1)$. Note that $SP_2^{*\neg}(\succ) = o_3 = b(\succ_2 = \succ_2^{32}, O)$ and $SP_3^{*\neg}(\succ) = o_1 = b(\succ_3 = \succ_3^{12}, O)$. Hence, no pair of agents has an incentive to collude at $\succ$.

• **Case 2: $\succ \neq (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$ and $\succ' \neq (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$.** Then, $SP^{*\neg}(\succ) = TTC(\succ)$ and $SP^{*\neg}(\succ') = TTC(\succ')$. Hence, by *reallocation-proofness* of TTC, no pair of agents has an incentive to collude at $\succ$.

• **Case 3: $\succ \neq (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$ and $\succ' = (\succ_1^{32}, \succ_2^{32}, \succ_3^{12})$.** Then, $SP^{*\neg}(\succ) = TTC(\succ)$ and $SP^{*\neg}(\succ') = (o_2, o_3, o_1)$. Without loss of generality, we assume $\{i,j\} = \{1,2\}$. By *individual rationality* of TTC, $SP_2^{*\neg}(\succ) = TTC_2(\succ) \succsim_2 o_2 = SP_1^{*\neg}(\succ')$. Hence, agent 2 has no incentive to collude with agent 1 at $\succ$. □

D.6 Example 7

Here, we show that $IR^{*\neg}$ satisfies both *strategy-proofness* and *reallocation-proofness*.

D.6.1 Strategy-proofness

Let $\succ \in \mathcal{T}_G^N$, $i \in N$, $\succ'_i \in \mathcal{T}_G$, and $\succ' = (\succ'_i, \succ_{-i})$. There are four cases.

• **Case 1: $\{\succ, \succ'\} \cap \left(\mathcal{T}_G \times \mathcal{T}_G^{-12} \times \mathcal{T}_G^1\right) = \emptyset$.** Then, $IR^{*\neg}(\succ) = TTC(\succ)$ and $IR^{*\neg}(\succ') = TTC(\succ')$. Hence, *strategy-proofness* of TTC implies that agent i has no incentive to misrepresent his preference relation.

• **Case 2: $\{\succ, \succ'\} \subset \left(\mathcal{T}_G \times \mathcal{T}_G^{-12} \times \mathcal{T}_G^1\right)$.** Then, $IR^{*\neg}(\succ) = IR^{*\neg}(\succ')$. Hence, agent i has no incentive to misrepresent his preference relation.

• **Case 3: $\{\succ, \succ'\} \cap \left(\mathcal{T}_G \times \mathcal{T}_G^{-12} \times \mathcal{T}_G^1\right) = \{\succ'\}$.** Then, $i \in \{2,3\}$. We observe that $IR^{*\neg}(\succ') = (o_2, o_3, o_1)$ and $IR^{*\neg}(\succ) = TTC(\succ)$. We further distinguish two subcases.

∘ *Subcase 3-1*: $i = 2$. Then, $\succ_2 = \succ_2^{12}$ and $\succ'_2 \in \mathscr{T}_G^{-12}$. Because object o_3 is agent 2's worst one according to $\succ_2 = \succ_2^{12}$, $IR_2^{*\neg}(\succ) \succsim_2 o_3 = IR_2^{*\neg}(\succ')$, which implies that agent 2 has no incentive to misrepresent his preference.

∘ *Subcase 3-2*: $i = 3$. Then, $\succ_3 \in \mathscr{T}_G^3$ and $\succ'_3 \in \mathscr{T}_G^1$. Hence, $IR_3^{*\neg}(\succ) = o_3 \succ_3 o_1 = IR_3^{*\neg}(\succ')$, which implies that agent 3 has no incentive to misrepresent his preference.

• **Case 4:** $\boldsymbol{\{\succ, \succ'\} \cap \left(\mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1\right) = \{\succ\}}$**.** Then, $i \in \{2, 3\}$. Moreover, we observe that $IR^{*\neg}(\succ) = (o_2, o_3, o_1)$ and $IR^{*\neg}(\succ') = TTC(\succ')$. We further distinguish two subcases.

∘ *Subcase 4-1*: $i = 2$. Then, $\succ_2 \in \mathscr{T}_G^{-12}$ and $\succ'_2 = \succ_2^{12}$. We first consider the case $\succ_2 \in \mathscr{T}_G^3$. Then, $IR_2^{*\neg}(\succ) = o_3 = b(\succ_2, O)$. Thus, he has no incentive to misrepresent his preference. We next consider the case $\succ_2 = \succ_2^{13}$. In Round 1 of the TTC algorithm at $\succ'$, agent 1 points to either himself or agent 3 and both agent 2 and 3 point to agent 1. In either case, $IR_2^{*\neg}(\succ') \neq o_1$. Because $IR_2^{*\neg}(\succ') \in \{o_2, o_3\}$, $IR_2^{*\neg}(\succ) = o_3 \succsim_2 (= \succsim_2^{13})\ IR_2^{*\neg}(\succ')$, which implies that agent 2 has no incentive to misrepresent his preference.

∘ *Subcase 4-2*: $i = 3$. Then, $\succ_3 \in \mathscr{T}_G^1$ and $\succ'_3 \in \mathscr{T}_G^3$. This implies $IR_3^{*\neg}(\succ) = o_1 = b(\succ_3, O)$. Hence, he has no incentive to misrepresent his preference. □

D.6.2 Reallocation-proofness

Let $\succ \in \mathscr{T}_G^N$, $\{i, j\} \subset N$, and $\succ' = (\succ'_i, \succ'_j, \succ_k) \in \mathscr{T}_G^N$. There are four cases.

• **Case 1:** $\boldsymbol{\{\succ, \succ'\} \cap \left(\mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1\right) = \emptyset}$**.** Then, $IR^{*\neg}(\succ) = TTC(\succ)$ and $IR^{*\neg}(\succ') = TTC(\succ')$. Hence, by *reallocation-proofness* of TTC, no pair of agents has an incentive to collude.

• **Case 2:** $\boldsymbol{\{\succ, \succ'\} \cap \left(\mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1\right) = \{\succ\}}$**.** Then, $IR^{*\neg}(\succ) = (o_2, o_3, o_1)$ and $IR^{*\neg}(\succ') = TTC(\succ')$. By $\succ_3 \in \mathscr{T}_G^1$, $IR_3^{*\neg}(\succ) = o_1 = b(\succ_3, O)$, which implies that he has no incentive to collude with another agent. Moreover, if $\succ_2 \in \mathscr{T}_G^3$, then $IR_2^{*\neg}(\succ) = o_3 = b(\succ_2, O)$, which implies that he has no incentive to collude with another agent. Thus, we consider the case $\{i, j\} = \{1, 2\}$ and $\succ_2 = \succ_2^{13}$. By $\succ \in \mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1$ and $(\succ'_1, \succ'_2, \succ_3) \notin \mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1$, $\succ'_2 = \succ_2^{12}$. In Round 1 of the TTC algorithm at $\succ'$, agent 1 points to either himself or agent 3 and both agents 2 and 3 point to agent 1. Hence, $TTC_2(\succ') \neq o_1$. By *inidividual rationality* of TTC and $\succ'_2 = \succ_2^{12}$, $IR_2^{*\neg}(\succ') = TTC_2(\succ') = o_2$. Hence, $IR_1^{*\neg}(\succ) =$

$o_2 = IR_2^{*\neg}(\succ')$, which implies that agent 1 has no incentive to collude with agent 2.

• **Case 3: $\{\succ, \succ'\} \cap \left(\mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1\right) = \{\succ'\}$.** Then, $IR^{*\neg}(\succ) = TTC(\succ)$ and $IR^{*\neg}(\succ') = (o_2, o_3, o_1)$. Without loss of generality, we assume $\{i, j\} = \{1, 2\}$. By *individual rationality* of TTC, $IR_2^{*\neg}(\succ) = TTC_2(\succ) \succsim_2 o_2 = IR_1^{*\neg}(\succ')$. Hence, agent 2 has no incentive to collude with agent 1 at $\succ$.

• **Case 4: $\{\succ, \succ'\} \subset \mathscr{T}_G \times \mathscr{T}_G^{-12} \times \mathscr{T}_G^1$.** Then, $IR^{*\neg}(\succ) = IR^{*\neg}(\succ') = (o_2, o_3, o_1)$. By $\succ_3 \in \mathscr{T}_G^1$, $IR_3^{*\neg}(\succ) = o_1 = b(\succ_3, O)$, which implies that he has no incentive to collude with another agent. Additionally, if $\succ_2 \in \mathscr{T}_G^3$, because $IR_2^{*\neg}(\succ) = o_3 = b(\succ_2, O)$, which implies that he has no incentive to collude with another agent. We now consider the case $\{i, j\} = \{1, 2\}$ and $\succ_2 = \succ_2^{13}$. Then, $IR_2^{*\neg}(\succ) = o_3 \succ_2 (= \succ_2^{13})\ o_2 = IR_1^{*\neg}(\succ')$. Hence, agent 2 has no incentive to collude with agent 1. □

D.7 Remark 6

We first show that $SP^\neg$ violates *reallocation-proofness*. Let $\succ = (\succ_1^{32}, \succ_2^{12}, \succ_3^{13})$ and $\succ' = (\succ_1^{12}, \succ_2 = \succ_2^{12}, \succ_3^{32})$. Note that $SP^\neg(\succ) = (o_2, o_1, o_3)$ and $SP^\neg(\succ') = TTC(\succ') = (o_1, o_2, o_3)$. Hence,

$$SP_3^\neg(\succ') = TTC_3(\succ') = o_3 \succ_1 (= \succ_1^{32})\ o_2 = SP_1^\neg(\succ);$$
$$SP_1^\neg(\succ') = TTC_1(\succ') = o_1 \succ_3 (= \succ_3^{13})\ o_3 = SP_3^\neg(\succ),$$

which implies that $SP^\neg$ violates *reallocation-proofness*.

Next, we show $IR^\neg$ violates *reallocation-proofness*. Let $\succ = (\succ_1^{12}, \succ_2^{12}, \succ_3^{13}) \in \mathscr{T}_G^1 \times \{\succ_2^{12}\} \times \mathscr{T}_G^1$ and $\succ' = (\succ_1^{12}, \succ_2^{13}, \succ_3^{13})$. Note that $IR^\neg(\succ) = (o_3, o_2, o_1)$ and $IR^\neg(\succ') = TTC(\succ') = (o_1, o_2, o_3)$. Hence,

$$IR_2^\neg(\succ') = o_2 \succ_1 (= \succ_1^{12})\ o_3 = IR_1^\neg(\succ);$$
$$IR_1^\neg(\succ') = o_1 \succ_2 (= \succ_2^{12})\ o_2 = IR_2^\neg(\succ),$$

which implies that $IR^\neg$ violates *reallocation-proofness*. □