UTMD Working Paper
The University of Tokyo
Market Design Center

UTMD-065

Beyond Quasilinearity: Exploring Nonlinear Scoring Rules in Procurement Auctions

Makoto Hanazono
Nagoya University

Jun Nakabayashi
Kyoto University

Ryuji Sano, Masanori Tsuruoka
Yokohama National University

July 29, 2024

Beyond Quasilinearity: Exploring Nonlinear Scoring Rules in Procurement Auctions*

Makoto Hanazono†, Jun Nakabayashi‡, Ryuji Sano§, Masanori Tsuruoka¶

July 29, 2024

Abstract

This study examines procurement auctions in which bidders submit price and quality, and bids are evaluated using a price-per-quality-ratio (PQR) scoring rule. We formulate a model of scoring auctions in which bidder cost is determined by a unidimensional type and a unidimensional quality and then characterize the equilibrium bidding behavior for the first-score and second-score auctions. In contrast to well-known quasilinear scoring rules in which price and quality are additively separable and the score is linear in price, the equivalence theorem does not hold between the auction formats in our setup. We show that the second-score auction yields a lower (better) expected score

*The authors thank Christian Hellwig for his insightful comments on an earlier version of this paper. We are also grateful to Nozomu Muto and seminar participants at the University of Tokyo for their helpful comments. We acknowledge financial support from the Japan Society for the Promotion of Science (KAKENHI 21K01401 and 23H00051). Editorial assistance was provided by Philip MacLellan. All remaining errors and omissions are our own.

†School of Economics, Nagoya University, Chikusa, Nagoya 464-8601, Japan. E-mail: hanazono@soec.nagoya-u.ac.jp

‡Department of Economics, Kyoto University, Yoshidahonmachi, Sakyo, Kyoto 606-8317, Japan. E-mail: nakabayashi.jun.8x@kyoto-u.ac.jp

§Department of Economics, Yokohama National University, Tokiwadai 79-4, Hodogaya, Yokohama 240-8501, Japan. E-mail: sano-ryuji-cx@ynu.ac.jp

¶Department of Economics, Yokohama National University, Tokiwadai 79-4, Hodogaya, Yokohama 240-8501, Japan. E-mail: matrok0603@gmail.com

than the first-score auction. We also provide a set of conditions under which expected quality and price are higher in the first-score auction than in the second-score auction. Finally, we show how these results can be extended to other non-quasilinear scoring rules.
Keywords: multidimensional bidding, scoring auctions, procurement, price-per-quality-ratio scoring rule
JEL codes: D44, H57, L13

1 Introduction

Due to the rapid accumulation of public debt in recent years, many countries face increased pressure to ensure value for money in their procurement practices. While low-price auctions have traditionally been used as a competitive and transparent allocation mechanism, there is a growing trend among procurement buyers to assess not only prices but also non-monetary attributes such as delivery time, design and quality. The *scoring auction* is one of the prevailing mechanisms that aims to achieve both price competition and value for money at the same time.

In a scoring auction, each bidder submits a multidimensional bid that consists of price and non-monetary attributes (henceforth, *quality*), and then a pre-announced scoring rule assigns a score to each bid to rank the bidders in the auction. The seminal paper by Che (1993) shows that scoring auctions under a symmetric independent private value setting can be reduced to the canonical model of unidimensional bid auctions as long as the scoring rule is quasilinear (QL);[1] that is, score is linear in price and additively separable from quality. Hence, the well-known revenue equivalence theorem applies to scoring auctions with QL scoring rules, and equivalence results with respect to price and quality hold between several auction formats. Since then, the simplicity of this framework has attracted a growing amount of theoretical and empirical attention.

[1] In this paper, we use the abbreviation QL only for *quasilinear scoring rules* but not for other situations such as the quasilinearity of the payoff function.

In real-world procurement auctions, however, a variety of scoring rules are used that are not quasilinear. A typical example is the "price-per-quality-ratio" (PQR) scoring rule in which a score is given by the price bid divided by the quality bid. Many state Departments of Transportation (DOTs) in the United States, including those in Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota, have adopted the PQR-equivalent "adjusted bid," and the Department of Health and Aging in Australia also employs a PQR awarding rule to achieve better returns on public investment (The Department of Health and Ageing, Australia, 2011). In addition, most public procurement contracts in Japan are allocated to the bidder with the highest price-per-quality bid ratio. However, despite the frequent use of PQR and other non-QL scoring rules in the real world, little is known about their properties.[2]

In this paper, we provide a theoretical analysis of PQR scoring auctions in order to acquire a deeper understanding of bidding behavior and outcomes of non-QL scoring auctions. We follow Che (1993) by focusing on settings with a unidimensional private signal and unidimensional quality level, and we characterize bidding behavior and compare auction outcomes for the following two auction formats: first-score (FS) and second-score (SS) auctions. In an FS auction, the winner delivers quality at the price specified in its bid, and in an SS auction, the winner is free to choose any price–quality pair as long as its score matches the score of the most-competitive rival. In our model, the winner of both auctions is the lowest-score bidder.

To date, the existing literature analyzes scoring auctions by transforming multidimensional-bid auctions into a unidimensional score-bid auction game (e.g., Che, 1993, Asker and Cantillon, 2008). In this reduced-form auction game, bidders select their profit-maximizing contract (i.e., a price–quality pair) for each feasible score, taking the scoring rule as given. The bidder's profit is reduced to an indirect payoff function

[2] While bid ranking is preserved in any monotonic transformation of the scoring rule, this transformation does not generally convert a non-QL scoring rule into a QL rule. If, for instance, we take a logarithm of the price-per-quality-ratio scoring rule, score is not linear in price anymore and so a necessary condition for quasilinearity is violated.

in score, and bidders play an auction game with respect to score bids. Che (1993) demonstrates that under a QL scoring rule, any scoring auction can be reduced to a score-bid auction in which the bidder's indirect payoff is quasilinear with respect to the score bid. This means that the well-understood results of auction theory such as revenue equivalence apply to QL scoring auctions. However, if the scoring function is not QL, the bidder's indirect payoff function is also generally not quasilinear, leading to a breakdown of equivalence and thus making it more challenging to characterize the properties of equilibrium price and quality and to draw implications.[3]

We first demonstrate that under a PQR scoring rule a symmetric Bayesian Nash equilibrium exists in both FS and SS auctions for a broad class of cost functions and we then show that FS auctions yield higher (worse) expected scores than SS auctions. Under the PQR scoring rule, bidders choose higher quality as the score increases, leading to the profit from winning bids being a convex function in score. This means that bidders are "risk-loving" in score, taking on greater risk for potentially larger winning profits in FS auctions. Consequently, the FS auction yields a higher expected score than the SS auction in equilibrium, which is similar to the findings of Maskin and Riley (1984) with non-risk-neutral bidders. This suggests that SS auctions are more favorable than FS auctions to buyers seeking to decrease price per quality.

To present more comprehensive arguments regarding the differences between FS and SS auctions, we analyze the equilibrium quality and price and then establish sets of conditions under which FS auctions produce a higher expected quality and price than in SS auctions. We observe that under these conditions both the quality and price chosen by the bidder increase with score, which leads to expected quality and price being higher in the FS auction than in the SS auction. These findings suggest that when the bidders' scores are sorted by price-per-quality, the contracted quality and prices are more likely to be skewed upward in the FS auction than in the SS auction.

[3]When analyzing FS auctions, for example, we need to clarify not only a scalar variable which is the equivalent of *type* in a standard auction model (*pseudotype* in Asker and Cantillon (2008) and *productive potential* in Che (1993)) but also the curvature of the indirect payoff function.

It is worth noting that, contrary to the apparent implications of the relative score ranking discussed above, an FS auction may actually be more beneficial for the buyer than an SS auction. We find certain specifications of the cost function in which the FS auction offers higher expected quality than the SS auction while the expected price is equivalent or even lower. This counterintuitive outcome can be explained as follows: First, given that the winner's type is fixed, the equilibrium score for the FS auction is deterministic while it remains random for the SS auction. Second, as quality increases with score and price is the product of quality and score, price is more sensitive than quality to a change in score and is likely to be convex. Combining these characteristics, the expected price ranking is more ambiguous than the expected quality ranking, leading to a possibility of higher prices on average in the SS auction compared to the FS auction. In summary, although the FS auction may appear disadvantageous initially, it can deliver favorable outcomes for buyers under certain conditions, which underscores the complexity of the PQR scoring auction and our motivation for studying it further.

Further, since price-only auctions are also widely used in practice, we compare PQR scoring to price-only auctions and find that PQR scoring auctions lead to higher quality and price than price-only auctions with minimum quality levels, which is consistent with studies of auctions using QL scoring rules. Moreover, by connecting traditional price theory to auction theory, we show that SS auctions using the PQR scoring rule yield a lower price-per-quality ratio than second-price price-only auctions at any quality level.

Lastly, we discuss how the equilibrium properties under the PQR scoring rule can be generalized to other non-QL scoring rules. We show that given a scoring rule, the expected score in an FS auction is higher (lower) than in an SS auction if the bidders' indirect payoff function is convex (concave) in score. Moreover, we examine key properties on scoring rules for convex/concave indirect payoffs in score. Although quality and price rankings are ambiguous under general scoring rules, we show that the property of the optimal quality function is crucial for our PQR scoring rule results. Thus, as for general scoring rules, it is an empirical question whether FS auctions lead to higher expected quality and price than SS auctions. In addition, we show that the equivalence theorem with respect to expected score, quality, and price is a feature unique to the QL scoring rule.

Related Literature This paper contributes to the theoretical literature on scoring auctions pioneered by Che (1993) which to date has focused on QL scoring auctions in which price and quality are additively separable and price enters linearly into the scoring rule. In Che (1993)'s original approach, scoring auctions are modelled in such a way that the price and quality bids reduce to a model of auctions in which bidders submit only scores as if it were a price-only auction. This research has been extended to cases of interdependent cost (Branco, 1997), multidimensional signals (Asker and Cantillon, 2008) and multidimensional quality (Nishimura, 2015). Furthermore, Asker and Cantillon (2008, 2010), Awaya, Fujiwara and Szabo (2022) and Sano (2023) compare the performance of QL scoring auctions and alternative mechanisms. While these previous studies focus on the properties of QL scoring auctions, we extend these studies by comparing the performance of FS and SS auctions under the non-QL PQR scoring rule.

In contrast to the research on scoring auctions using a QL scoring rule, there are few papers to date that study scoring auctions in which price does not enter linearly into the scoring rule, one of them being Wang and Liu (2014), who examine the relation between the number of bidders and equilibrium price and quality in a scoring auction with a non-QL scoring rule where price and quality are additively separable. Meanwhile, Dastidar (2014) analyzes scoring auctions with a general scoring rule and finds that the equilibrium bidding function of the FS auction is explicitly obtained if the bidder's cost function is additively separable in quality and their private information. We build upon these studies by considering a broader class of cost functions and by characterizing the equilibrium of PQR scoring auctions.

In another study, Hanazono, Hirose, Nakabayashi and Tsuruoka (2020), henceforth HHNT, discuss the equilibrium existence and the structural estimation of FS auctions incorporating general scoring rules and multidimensional private signals. This paper complements HHNT in two ways. First, the cost structure does not fall into that of HHNT despite multidimensionality: to ensure equilibrium existence, HHNT require that the cost function have a private fixed cost component. This paper covers the case of single-dimensional private signal on variable costs which is out of scope in HHNT. Second, the results of comparing expected price, quality, and score between different auction formats fails to obtain in HHNT because the monotonicity of equilibrium on a single-dimensional signal space is intrinsically different from that on a multidimensional signal space.

In addition to theoretical studies, empirical research on scoring auctions is growing as well (e.g., Lewis and Bajari, 2011; Koning and van de Meerendonk, 2014; Decarolis, Spagnolo and Pacini, 2016; Iimi, 2016; Andreyanov, 2018; Takahashi, 2018; Huang, 2019; Krasnokutskaya, Song and Tang, 2020; Ryan, 2020; Kong, Perrigne and Vuong, 2022 and Allen, Clark, Hickman and Richert, 2023). Building on the literature on scoring auctions, Bajari, Houghton and Tadelis (2014) and Bolotnyy and Vasserman (2023) develop structural auction models in which firms post unit price bids for each item required to complete a construction project. Among these studies, Takahashi (2018) examine scoring auctions with the PQR scoring rule and quantify the impact of uncertainty on reviewers' evaluations of quality. While theoretical, our paper compares performance between PQR scoring auctions and price-only auctions and so provides new empirically testable predictions that could have important policy implications.

The remainder of the paper is organized as follows. Section 2 describes the canonical model of scoring auctions in which they can be transformed into a unidimensional score-bid auction game. In Section 3, we focus on the PQR scoring rule and analyze symmetric equilibria in FS and SS auctions, comparing the expected winning score, quality, and price of the two auction formats as well as the relative performance of PQR scoring auctions and price-only auctions. Section 4 analyzes general scoring rules and characterizes the expected score rankings for FS and SS auctions, and the final section concludes the paper.

2 Model

Consider that a procurement buyer auctions off a procurement contract to $n \geq 2$ risk-neutral bidders who are all ex ante symmetric. Bidder i's private type is denoted by θ_i and is independently and identically drawn from a cumulative distribution F over $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ with a continuous density $f(\theta) > 0$ for every $\theta \in \Theta$. Let $q \in \mathbb{R}_+$ be a non-monetary attribute (quality) so that each bidder's production cost is given by $C(q, \theta_i)$. We assume that the cost function C is:

- thrice differentiable and strictly increasing in both q and θ (C_q, $C_\theta > 0$);
- strictly convex in quality ($C_{qq} > 0$);
- exhibits non-decreasing differences ($C_{q\theta} \geq 0$); and
- there exists a sufficiently large number $B > 0$, and for all θ, $C_q(q, \theta) \geq B$ for some $q > \underline{q}$.

Note that $C_\theta = \partial C / \partial \theta$ and that the other subscripts are defined in the same manner. The production cost is increasing in quality and type so that a bidder of a lower type is more efficient. The third assumption means that a bidder of a lower type has a smaller marginal cost,[4] and the last assumption guarantees that that an optimal quality exists under the scoring auctions we consider.

When bidder i wins the auction and signs a contract with a price p and a quality q, their payoff is given by

$$p - C(q, \theta_i),$$

and we suppose that every losing bidder's payoff is zero.

[4]The assumption $C_{q\theta} \geq 0$ is not necessary for most of the results. In particular, we can verify that a symmetric Bayesian Nash equilibrium exists for the FS auction under $C_{q\theta} \leq 0$.

In a scoring auction, each bidder submits a proposal (p, q), where $p \leq \bar{p}$ is a price bid and $q \geq \underline{q}$ is a quality bid, with reserve price and minimum quality denoted by $\bar{p} > 0$ and $\underline{q} > 0$. Each proposal is evaluated by a pre-announced scoring rule $S : [0, \bar{p}] \times [\underline{q}, \infty) \to \mathbb{R}$ which maps a multidimensional bid into a unidimensional score $s = S(p, q)$. The lowest-score bidder wins. We assume that the scoring rule is sufficiently smooth and satisfies $S_p > 0$ and $S_q < 0$.

We focus here on first-score (FS) and second-score (SS) auctions. In both types of auction, each bidder submits (p, q), and the bidder with the lowest score wins. In an FS auction, the winner's proposal is finalized as a contract whereas in an SS auction, the winner is required to match the highest rejected (i.e. the second lowest) score. To meet the score, the winner is free to choose any quality-price pair, so the finalized contract of an SS auction generally differs from both the winning bid (p, q) and the lowest losing bid.

Although our model allows general scoring rules, in Section 3 we focus on the *price-per-quality ratio* (PQR) scoring rule:

$$S(p, q) = \frac{p}{q}, \tag{1}$$

with $p \leq \bar{p}$ and $q \geq \underline{q}$.[5]

Remark 1 Score ranking is preserved in any monotone transformation of the scoring rule so that most properties of scoring auctions, equilibrium price and quality in particular, do not change in a monotone transformation of the scoring rule. However, in the following sections, we evaluate the expected scores of different auction formats which generally do change in monotone transformation. For example, the quality-per-price-ratio rule $S(p, q) = -q/p$ is a monotone (but not affine) transformation of the PQR scoring rule, so the equilibrium price and quality under such a scoring rule are the same as those presented in the next section, though the expected

[5]Quality q here is measured in terms of "quality score." One might consider a scoring rule $S(p, q) = p/V(q)$, where V is an increasing function. This is equivalent to the case in which a quality is defined as $\tilde{q} = V(q)$.

score ranking may differ. Note that the expected score ranking is preserved in any affine transformation.

Remark 2 We do not formulate the buyer's preferences. One interpretation of PQR and other non-QL scoring rules is that they are the buyer's (true) objective function. However, as Che (1993) shows, the buyer may be better off using a scoring rule that differs from the true objective function. Hence, scoring rules do not necessarily represent the buyer's preferences. Here we aim to characterize and compare equilibrium score, quality, and price under PQR and other non-QL rules without specifying the buyer's preferences.

2.1 Score-bid Auctions

The equilibrium of scoring auctions is derived in a manner similar to Che (1993). Given an arbitrary score s, every bidder will choose the optimal contract (p, q) that induces score s so that an auction with a multidimensional bid is reduced to a unidimensional auction in terms of the score bid.

Suppose that the winner of type θ needs to enforce a contract that fulfills score s. The winner determines a contract (p, q) that solves

$$\begin{aligned} &\max_{(p,q)} p - C(q, \theta) \\ &\text{s.t. } S(p, q) = s, \\ &\quad p \leq \bar{p},\ q \geq \underline{q}. \end{aligned} \tag{2}$$

Throughout the analysis, we assume that the reserve price and the minimum quality are not binding at (2). By substituting the score constraint into the objective function, payoff maximization is written as

$$\max_{q} P(s, q) - C(q, \theta), \tag{3}$$

where P is the inverse function of S with respect to p. When the objective function in (3) is strictly concave in q, the maximization problem has a unique solution, with the optimal quality denoted by

$$q^*(s,\theta) \in \arg\max_q P(s,q) - C(q,\theta) \tag{4}$$

and the indirect payoff function denoted by

$$u(s,\theta) \equiv P\left(s, q^*(s,\theta)\right) - C\left(q^*(s,\theta),\theta\right). \tag{5}$$

Note that as $S_p > 0$, we have $P_s > 0$. By the envelope theorem, the indirect payoff u is strictly increasing in s and strictly decreasing in θ. The equilibrium of the scoring auction is derived by solving standard auctions in terms of score bid s, where each bidder has the winning profit $u(s,\theta)$.

Let $z(\theta)$ be the unique solution of

$$u(z(\theta),\theta) = 0;$$

that is, z is the score bid such that the winner's indirect payoff is zero, which is the maximum willingness to accept for a bidder of type θ in the auction. We call $z(\theta)$ *the break-even score* for type θ. Because u is increasing in s and decreasing in θ, z is increasing in θ.

Lemma 1 *Suppose that $P(s,q) - C(q,\theta)$ is strictly concave in q. Then, $z(\theta)$ is well-defined and strictly increasing in θ.*

Proof See Appendix.

2.2 QL Scoring Rule

The seminal paper Che (1993) examines the *quasilinear (QL)* scoring rule $S(p,q) = p - q$ whereby optimal quality is given by the profit maximization problem

$$\max_q s + q - C(q,\theta).$$

When the optimal quality is determined by the first-order condition $1 - C_q(q^*,\theta) = 0$, q^* depends only on θ and is independent of s. The indirect payoff is reduced to a function

$$u(s,\theta) = s - k(\theta)$$

that is quasilinear in score, where

$$k(\theta) = -\max_{q} q - C(q, \theta)$$

is called the *productive potential* (Che, 1993) or *pseudotype* (Asker and Cantillon, 2008). Thus, in this framework, the QL scoring auction is reduced to a score-bid auction with a quasilinear payoff. Because k is increasing in θ, the bidder of the lowest type wins in both the FS and SS auctions and so the exercised quality is ex post equivalent between the two formats. The revenue equivalence theorem applies and thus, the expected scores of the FS and SS auctions are equivalent in equilibrium. As there is score equivalence and $p = s + q$, equivalence holds for the expected price too.

2.3 PQR Scoring Rule

Next we examine a scoring rule that is not QL. Consider the PQR scoring rule $S(p, q) = p/q$. The inverse function of S with respect to p is given by $P(s, q) = sq$, and the optimal quality is derived by the profit maximization problem

$$\max_{q \geq \underline{q}} sq - C(q, \theta). \tag{6}$$

It is clear that the objective function is strictly concave in q, and we assume that the optimal quality q^* always lies in the interior in equilibrium. This is satisfied if the optimal quality at $(z(\theta), \theta)$ is not binding.

Assumption 1 In the PQR scoring rule, for all θ, the optimal quality satisfies

$$q^* (z(\theta), \theta) > \underline{q}.$$

When the optimal quality q^* lies in the interior, it is determined by the first-order condition

$$s - C_q(q^*, \theta) = 0. \tag{7}$$

By the implicit function theorem, we have $q^*_s = 1/C_{qq} > 0$ and $q^*_\theta = -C_{q\theta}/C_{qq} \leq 0$, so the optimal quality is increasing in score s and non-increasing in type θ. It is immediately clear that the indirect payoff function is convex in score s.

Lemma 2 *Under the PQR scoring rule and Assumption 1, the indirect payoff function u is strictly convex in s.*

Proof By the envelope theorem, we have $u_s(s, \theta) = q^*(s, \theta) > 0$ and $u_{ss}(s, \theta) = q^*_s(s, \theta) > 0$. $\square$

Quality choice and the indirect payoff function under the PQR scoring rule are both closely related to standard producer theory whereby the maximization problem (6) is equivalent to the profit maximization problem of a firm in a competitive market when s is the price per unit of quality. The optimal quality supplied is thus determined by "price equals marginal cost" (7) and the supply function q^* is increasing in price s. Since the suppliers optimally adjust their quality supplied in response to price, the profit function u is convex in s. The break-even score $z(\theta)$ here corresponds to the break-even price for the firm.

From this interpretation, Assumption 1 requires that there exists a non-sunk fixed cost. As is well known, average cost is minimized and generally equals marginal cost at the break-even price. With the presence of fixed costs, average cost is U-shaped and minimized in the interior. When there are no fixed costs, the average cost is always smaller than the marginal cost. Hence, by ignoring the quality constraint $q \geq \underline{q}$, suppliers could always earn a positive profit by providing a small quality, and the quality supplied at the break-even point is zero. Thus, a non-sunk fixed cost is necessary to satisfy Assumption 1.

The PQR scoring rule is distinct from the QL scoring rule in two respects. First, the optimal quality under the PQR rule depends not only on bidder type but also on the required score s. Second, the indirect payoff function is not quasilinear, so the revenue equivalence theorem does not apply to the PQR rule.

3 Equilibrium Analysis of PQR Scoring Auctions

In this section, we examine the equilibrium properties of FS and SS auctions under the PQR scoring rule.

3.1 Equilibrium

We first characterize the equilibria of the SS and FS auctions, showing that in both auctions the bidder with the lowest type is selected as the winner.

In the SS auction, it is a weakly dominant strategy to bid $z(\theta)$ as in the standard second-price auction. The following proposition is shown in a standard manner and is similar to Maskin and Riley (1984), Saitoh and Serizawa (2008), and Sakai (2008), so the proof is omitted.

Proposition 1 *In the SS auction, it is a weakly dominant strategy for each bidder to submit* $s^{SS}(\theta) = z(\theta)$.

Under the PQR scoring rule, the score-bid auction game can be interpreted as competition among suppliers in terms of unit price per quality, and the supplier who submits the lowest price per quality ratio wins. From the perspective of standard producer theory, the break-even score is equal to the supplier's minimum average cost: $z(\theta) = \min_q C(q, \theta)/q$. In the SS auction, the unit price per quality for the winner is determined by the best rival offer, so suppliers submit their minimum average cost in equilibrium. The supplier with the lowest minimum average cost wins and supplies quality at the unit price equal to the second-lowest minimum average cost.

As for the FS auction, Maskin and Riley (1984) and Athey (2001) show that it has a symmetric, monotone Bayesian Nash equilibrium if the payoff function u is log-supermodular:

$$\frac{\partial^2 \log u(s, \theta)}{\partial s \partial \theta} > 0. \tag{8}$$

To meet this log-supermodularity condition, we additionally impose the technical conditions below.

Assumption 2 At least one of the following conditions holds.

1. C_θ / C_q is non-increasing in q, or

2. the cost function satisfies

$$1+q\left(\frac{C_{qqq}}{C_{qq}}-\frac{C_{qq\theta}}{C_{q\theta}}\right)>0. \tag{9}$$

A wide range of cost functions satisfy either of the above. The first case is equivalent to $C_qC_{q\theta}-C_\theta C_{qq}\leq 0$ and, roughly speaking, this condition is met when the marginal cost is more sensitive to a change in quality than to a change in type; that is, when C_{qq} is large and $C_{q\theta}$ is small. A special case is $C_{q\theta}=0$ in which bidder marginal cost is independent of θ whereby bidder variable costs for quality are identical but fixed costs are heterogeneous.[6] The second case is likely satisfied when the cost function is polynomial in q and type θ does not depend on the coefficient of the maximum degree of q. For example, this condition is met if $C(q,\theta)=q^2+\theta q+\kappa(\theta)$. Note that these two conditions are not disjoint. For example, a cost function $C(q,\theta)=c(q+\theta)$ in which c is a convex function satisfies both conditions because of $C_q=C_\theta$.

Under the log-supermodularity condition, the equilibrium bidding function is characterized by the first-order condition. Let $G(\theta)=1-(1-F(\theta))^{n-1}$ be the distribution of the lowest order statistic of $n-1$ independent draws from F. In addition, let $g=G'$ be its density.

Proposition 2 *If Assumptions 1 and 2 hold, there exists a symmetric Bayesian Nash equilibrium in the FS auction. Equilibrium score-bidding function s^{FS} is characterized by*

$$(s^{FS})'(\theta)=\frac{u\left(s^{FS}(\theta),\theta\right)}{u_s\left(s^{FS}(\theta),\theta\right)}\cdot\frac{g(\theta)}{1-G(\theta)} \tag{10}$$

with $s^{FS}(\bar{\theta})=z(\bar{\theta})$.

Proof See Appendix.

[6] Dastidar (2014) focuses on this type of cost function and examines the equilibrium of non-QL scoring auctions.

To see how each supplier determines its score bid in an FS auction, let us introduce a function

$$k(s,\theta) \equiv \frac{C(q^*(s,\theta),\theta)}{q^*(s,\theta)}, \tag{11}$$

which is the average cost that the supplier incurs when it wins at score s. Using this, we can also characterize the equilibrium bidding behavior in an FS auction which is analogous to the equilibrium bid in a first-price auction with risk-neutral bidders:

Corollary 1 *In an FS auction, the equilibrium score bid is the conditional expectation of the second-lowest bidder's average cost $k(s^{FS}(\cdot),\cdot)$:*

$$s^{FS}(\theta) = \int_{\theta}^{\bar{\theta}} \frac{g(\tau)}{1-G(\theta)} k\left(s^{FS}(\tau),\tau\right) d\tau. \tag{12}$$

Proof See Appendix.

Recall that a PQR scoring auction is a competition among suppliers in terms of unit price per quality. In an SS auction, the unit price of the winner is determined by the best rival offer, and suppliers submit their minimum average cost in equilibrium. In an FS auction, by contrast, unit price per quality is determined by the supplier's own offer, so bidding one's minimum average cost is not a best response for suppliers. Instead, suppliers submit a unit price higher than their minimum average cost, $s^{FS}(\theta) > z(\theta)$, and the equilibrium score bid is expressed by the expected average cost of the most-competitive rival bid (12).

When the bidder payoff function is quasilinear, the equilibrium strategy in the FS auction is expressed by an order statistic of a scalar variable that effectively summarizes the bidder's type and is referred to as *productive potential* by Che (1993) and *pseudotype* by Asker and Cantillon (2008). The break-even score is interpreted as this scalar variable for general scoring rules, and it characterizes the equilibrium in the SS auction. However, because of the nonlinearity of the payoff function, the break-even score is not sufficient to characterize the equilibrium strategy in the FS auction. Note that by linearizing the indirect payoff in s at $(\hat{s}, \theta)$, we have

$$\begin{aligned} u^l(s; \hat{s}, \theta) &\equiv u_s(\hat{s}, \theta)(s - \hat{s}) + u(\hat{s}, \theta) \\ &= u_s(\hat{s}, \theta) \left(s - \hat{s} + \frac{\hat{s}q^*(\hat{s}, \theta) - C(q^*(\hat{s}, \theta), \theta)}{q^*(\hat{s}, \theta)} \right) \\ &= u_s(\hat{s}, \theta)(s - k(\hat{s}, \theta)) \end{aligned}$$

by $u_s(\hat{s}, \theta) = q^*(\hat{s}, \theta)$. Thus, $k(\hat{s}, \theta)$ is the break-even score of the bidder that has the linearized utility at $(\hat{s}, \theta)$. The first order condition for the optimal score bid under the linearized utility u^l coincides with (10) at $\hat{s} = s^{FS}(\theta)$; thus, $k(s, \theta)$ is a key variable to characterize the equilibrium strategy. If $u(s, \theta)$ is already linear in s (as it is with QL scoring rules), $k(s, \theta)$ is independent of s and indeed coincides with the original break-even score; $k(s, \theta) = z(\theta)$. However, under the PQR scoring rule, $k(s, \theta)$ is the average cost for the optimal quality at score s, so it varies with s and differs from the break-even score $z(\theta)$. Thus, the equilibrium bidding strategy is not explicitly expressed by an order statistic of a variable, but expression (12) is an implicit form.

3.2 Comparison of FS and SS Auctions

We now compare the equilibrium performance of FS and SS auctions under the PQR scoring rule. In contrast to a QL scoring rule, equivalence between the two formats does not hold, so we evaluate the two formats with respect to expected score, quality and price.

3.2.1 Score Ranking

Buyers for whom the price per quality ratio is the true objective function prefer an auction format that yields a lower (expected) score. The expected score rankings of the FS and SS auctions depends on the curvature of the bidder's indirect payoff, u. Maskin and Riley (1984) show that if u is *concave* in payment, the expected revenue from the first-price auction is higher than that of the second-price auction. Here, by Lemma 2, u is *convex* in score in a PQR scoring auction, so we have a similar but reverse expected score ranking, which is shown in an analogous manner to Maskin and Riley (1984). The following theorem implies that buyers prefer the SS to the FS auction when they wish to reduce the price per quality ratio.

Theorem 1 *Suppose that Assumptions 1 and 2 hold. The expected score of the FS auction is higher than that of the SS auction. Moreover, for every winner's type* θ*, we have*

$$s^{FS}(\theta) \geq E[s^{SS}(\tau) \mid \tau > \theta], \tag{13}$$

where τ *is the lowest order statistic of* $n-1$ *independent draws from* F*.*

Proof This is shown in a manner parallel to Theorem 4 of Maskin and Riley (1984). Although Maskin and Riley (1984) consider a concave payoff function, it is not necessary to assume concavity to ensure the existence of a symmetric equilibrium. □

Theorem 1 is also proved by using Corollary 1. Note that $k(s, \theta)$ is the average cost for the optimal quality at score s. Given that the winner's type is θ, (12) yields

$$\begin{aligned} s^{FS}(\theta) = E\left[k\left(s^{FS}(\tau), \tau\right) \mid \tau > \theta\right] &= E\left[\left.\frac{C\left(q^*(s^{FS}(\tau), \tau), \tau\right)}{q^*(s^{FS}(\tau), \tau)}\right| \tau > \theta\right] \\ &\geq E\left[\left.\min_q \frac{C(q, \tau)}{q}\right| \tau > \theta\right] \\ &= E\left[z(\tau) \mid \tau > \theta\right], \end{aligned}$$

where τ is the lowest order statistic of $n-1$ independent draws from F. Because bidders submit a higher score than the break-even score in the FS auction, the associated average cost is not minimized. Thus, the equilibrium score in the FS auction is higher than the expected break-even score of the most competitive rival.

A convex payoff function is associated with risk-loving bidders; when bidders are risk-loving, they take more risk on winning; that is, they want to increase the winning profit even if they lose more often. Hence, risk-loving bidders submit a higher score (which provides a larger profit) than risk-neutral bidders.[7]

[7] Another model that results in a convex payoff function is Board (2007), which analyzes auctions

3.2.2 Quality Ranking

Because optimal quality depends on score s, and score equivalence does not hold for the PQR scoring rule, the equilibrium quality obtained by the two auction formats also differs. Note that the optimal quality function q^* is increasing in score. Therefore, as the FS auction yields a higher expected score, it is thus likely to provide a higher quality than the SS auction.

The expected quality is ranked under additional conditions. Note that in an FS auction, the winner's quality is deterministic at the bidding stage because the winner's quality bid is enforced. In contrast, in an SS auction, the winner's quality is stochastic because the optimal quality depends on the second-lowest score which is uncertain for the winner. Hence, to obtain the expected quality ranking, we need a condition on the curvature of the optimal quality function q^*. The following theorem states that the FS auction provides a higher expected quality than the SS auction when the optimal quality q^* is weakly concave in score.

Theorem 2 *Suppose that Assumptions 1 and 2 hold and that $C_{qqq} \geq 0$. Then, the expected quality in the FS auction is higher than that in the SS auction.*

Proof See Appendix.

The condition $C_{qqq} \geq 0$ means that marginal cost is weakly convex, which implies that the optimal quality function q^* is weakly concave in s. The optimal quality is determined by (7) whereby the unit price per quality equals the marginal cost. When marginal cost is convex, it rapidly increases as q increases. Hence, the optimal quality does not increase very much when the score or unit price per quality is increased, meaning that it is weakly concave.

3.2.3 Price Ranking

Given that an FS auction yields a higher expected score and quality when marginal cost is convex, it is natural to conjecture that expected price would also be higher

of risky assets in which bidders limited liability.

for an FS auction. However, the expected price ranking is more ambiguous than the quality ranking. Under the PQR scoring rule, the equilibrium price is given by

$$\pi(s, \theta) \equiv sq^*(s, \theta).$$

Analogous to the quality ranking, we have an expected price ranking if the optimal price π is weakly concave in score. However, because q^* is increasing in s, π is more sensitive to a change in s and is likely to be convex. Thus, the concavity of π is more stringent than the concavity of q^*.

We provide two sufficient conditions for ranking the expected prices of FS and SS auctions. The first one is when the optimal price is weakly concave in score.

Theorem 3 *Suppose that Assumptions 1 and 2 hold and*

$$C_q C_{qqq} \geq 2(C_{qq})^2. \tag{14}$$

Then, the expected price in the FS auction is higher than that in the SS auction.

Proof See Appendix.

The price function π is weakly concave under (14) above. An example of such a cost function is

$$C(q, \theta) = \log \frac{a}{a - \theta - q},$$

where $a > \bar{\theta}$ is constant. This cost function satisfies all the basic assumptions and Assumption 2. The optimal quality and price are given by

$$q^*(s, \theta) = a - \theta - \frac{1}{s}$$

and

$$\pi(s, \theta) = (a - \theta)s - 1,$$

respectively.

We provide another condition under which the expected price can be ranked even when the equilibrium price π is convex, assuming that the bidders' type represents their fixed costs, or $C_{q\theta} = 0$. In this case, the optimal quality q^* is independent of type; that is, $q^*(s,\theta) = q^*(s)$, so the optimal price is also independent of type and $\pi(s) = sq^*(s)$. Because the quality function q^* is increasing in s, the optimal price $\pi(s) = sq^*(s)$ is also increasing so there is a one-to-one correspondence between score and optimal price. Thus we transform the indirect payoff function $u(s,\theta)$ in terms of s into one in terms of price p; with $\hat{u}(p,\theta) \equiv u(\pi^{-1}(p),\theta)$. The payoff function $\hat{u}(p,\theta)$ is the winner's payoff when they sign a contract under which they optimally choose the price as p. As the bidder of the lowest score bid also makes the lowest price bid, the score-bid auction is transformed into a unidimensional price-bid auction. The equilibrium price of the two auction formats can be ranked when the bidder payoff $\hat{u}$ is convex (or concave) for the associated price-bid auction.

Theorem 4 *Consider the PQR scoring rule. Suppose that Assumption 1 holds and* $C_{q\theta} = 0$*. The expected price in the FS auction is at least as high as that in the SS auction if* qC_{qq}/C_q *is nondecreasing in* q*, or equivalently,*

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0 \tag{15}$$

holds for all $q \geq \underline{q}$*. The expected price in the SS auction is at least as high as that in the FS auction if* qC_{qq}/C_q *is nonincreasing in* q*, or equivalently,*

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \leq 0 \tag{16}$$

holds for all $q \geq \underline{q}$*.*

Proof See Appendix.

Given $C_{q\theta} = 0$, condition (15) is weaker than the concavity of π, (14). Indeed, when (14) holds, we have

$$\begin{aligned} C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 &= C_q C_{qq} + q\left(C_q C_{qqq} - 2(C_{qq})^2 + (C_{qq})^2\right) \\ &\geq (C_q + qC_{qq})C_{qq} \\ &> 0. \end{aligned}$$

Note that (15) for the price ranking is relatively stronger than that for the quality ranking because the price function π is more likely to be convex than the quality function. Thus, although the FS auction yields a higher expected score than the SS auction, the convex price function could lead to a higher expected price in the SS auction than the FS auction. In sum, while the expected quality is higher for the FS auction than the SS auction, the expected price of the FS auction may be equal to or even lower than that of the SS auction.

To see this, consider a specific cost function $C(q,\theta) = q^a + bq + \theta$ with $a \geq 2$ and $b \in \mathbb{R}$.[8] Since $C_{qqq} \geq 0$, expected quality is higher in the FS auction than in the SS auction. Also, since

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 = a(a-1)^2 bq^{a-2},$$

the expected price is higher in the FS auction than in the SS auction if $b > 0$ and, conversely, is lower in the FS auction if $b < 0$. When $b = 0$, the optimal quality and price are explicitly given by

$$q^*(s) = a^{-\frac{1}{a-1}} s^{\frac{1}{a-1}}$$

and

$$\pi(s) = sq^*(s) = a^{-\frac{1}{a-1}} s^{\frac{a}{a-1}},$$

respectively. The indirect payoff function is

$$u(s,\theta) = \left(1 - a^{-1}\right) a^{-\frac{1}{a-1}} s^{\frac{a}{a-1}} - \theta$$

which can be transformed into

$$\hat{u}(p,\theta) = \frac{a-1}{a}\left(p - \frac{a\theta}{a-1}\right),$$

where $p = \pi(s)$. That is, the score-bid auction is transformed into a price-bid auction with a quasilinear payoff function and a pseudotype $a\theta/(a-1)$. Thus, we can apply the revenue equivalence theorem, and so the equilibrium price is the same in the FS and SS auctions.

Corollary 2 *Consider the PQR scoring rule, and suppose that Assumption 1 and* $C_{q\theta} = 0$ *hold. If* $C_{qqq} \geq 0$ *and* qC_{qq}/C_q *is nonincreasing in* $q \geq \underline{q}$*, then the expected*

[8] We focus on the region where the cost is increasing in q when $b < 0$.

quality is higher in the FS auction than in the SS auction, and the expected price in the FS auction is at most as high as in the SS auction. Thus, the FS auction achieves a higher expected quality with a weakly lower expected price.

At first glance, Corollary 2 seems inconsistent with Theorem 1 which shows that the SS auction yields a lower expected score than the FS auction. However, even though the expected price per quality ratio is higher, the FS auction can lead to a higher expected quality and lower expected price than the SS auction. Thus, if the buyer's true objective is to achieve a higher expected quality at a lower expense rather than to minimize the price per quality ratio, the FS auction can be more beneficial for the buyer than the SS auction.

3.3 Comparison of PQR Scoring and Price-only Auctions

We next compare the performance of PQR scoring auctions and price-only auctions. The following proposition comparing the performance of PQR scoring auctions and price-only auctions with minimum quality levels is that PQR scoring auctions lead to higher winning prices but higher quality levels.

Proposition 3 *Under Assumption 1, SS auctions using the PQR scoring rule lead to higher winning prices and quality levels than price-only auctions with minimum quality levels. In addition, under Assumptions 1, 2, and the conditions presented in Theorems 3 or 4, FS auctions using the PQR scoring rule lead to higher winning prices and quality levels than price-only auctions with minimum quality levels.*

Proof See Appendix.

The properties in Proposition 3 are similar to that of the QL scoring rule reported in Asker and Cantillon (2008) in that when choosing auction formats, a buyer faces a trade-off between price and quality. Empirical analysis of highway construction procurement in California (Lewis and Bajari, 2011) suggests that scoring auctions using a QL scoring rule leads to shorter delivery times but higher winning prices relative to price-only auctions (i.e. higher quality but at a higher price). Because PQR scoring rule auctions as well as QL scoring auctions are used worldwide, Proposition 3 has practical implications.

When the buyer's objective is to decrease the price per quality ratio, it would also be worth comparing price per quality as well. The following result shows that an SS auction using a PQR scoring rule yields a lower price per quality ratio than a second-price auction, even if the buyer specifies any quality level in the price-only auction.

Proposition 4 *Under Assumption 1, the equilibrium price per quality ratio is lower in the SS auction with a PQR scoring rule than in the second-price auction with any quality requirement $\hat{q}$.*

Proof See Appendix.

The intuition for Proposition 4 is as follows. In the price-only auction, bidders provide only the quality level required. In the second-price auction, bidders submit their true cost for providing $\hat{q}$, with an associated average cost of $C(\hat{q}, \theta)/\hat{q}$. In the SS PQR auction, however, bidders compete in terms of average cost, submitting their minimum average cost. Hence, the SS auction with the PQR scoring rule achieves a lower price per quality ratio than the second-price auction.

Proposition 4 is related to Awaya, Fujiwara and Szabo (2022), who compare a QL scoring auction with a price-only auction in which the quality is optimally specified by the buyer. Awaya, Fujiwara and Szabo (2022) show that the QL scoring auction even yields a higher expected quality and price than the "optimal price-only auction." Similarly, Proposition 4 implies that even if the buyer optimally specifies the quality level in a price-only auction, the PQR scoring auction achieves a lower price per quality ratio. This result may support the use of the PQR scoring rule when the buyer aims to reduce the price per quality ratio, and may also suggest why PQR scoring auctions are prevalent worldwide.

4 General Scoring Rules

In this section, we consider general scoring rules and examine how the properties of the PQR scoring rule can be generalized to other non-QL scoring rules. We begin with scoring rule S which is increasing in p and decreasing in q. The inverse function in terms of p is denoted by $P(s,q)$, which is the price function given score s and quality q, with $P_s > 0$ and $P_q > 0$. The bidder indirect payoff function is given by

$$u(s,\theta) \equiv \max_q P(s,q) - C(q,\theta).$$

We assume that the payoff function $P - C$ is strictly concave in q and that the payoff maximization problem always has a (unique) interior solution. That is, the optimal quality q^* is determined by the first-order condition

$$P_q(s,q^*) - C_q(q^*,\theta) = 0. \tag{17}$$

We further assume that the indirect payoff function u is well behaved and satisfies the log-supermodularity condition $\partial^2 \log u / \partial s \partial \theta > 0$. The equilibrium of the SS and FS auctions is characterized in the same manner with the PQR scoring rule.

Proposition 5 *Suppose that $P(s,q) - C(q,\theta)$ is strictly concave in q and that the optimal quality q^* is determined by the first-order condition (17). In the SS auction, it is a weakly dominant strategy for each bidder to submit $s^{SS}(\theta) = z(\theta)$. In the FS auction, the symmetric equilibrium score-bidding function s^{FS} is characterized by (10) with $s^{FS}(\bar{\theta}) = z(\bar{\theta})$ if u is log-supermodular.*

Proof The proof is the same as Propositions 1 and 2. See Appendix B for sufficient conditions for the log-supermodularity of u. □

4.1 Expected Score

The argument of Section 3.2.1 can be directly applied to general scoring rules. Namely, the expected score is higher (lower) in the FS than in the SS auction if

$u(s, \theta)$ is convex (concave) in s. The following proposition is shown in the same manner as Theorem 1.

Proposition 6 *Suppose that the FS auction has a symmetric Bayesian Nash equilibrium. Then, the expected score in the FS auction is weakly higher (lower) than in the SS auction if $u(s, \theta)$ is convex (concave) in s for all θ.*

Note that bidders with a convex (concave) utility function place score bids less (more) aggressively in the FS auction than in the SS auction. This is analogous to the comparison of the bidding behaviors between first- and second-price auctions with non-risk-neutral bidders.

Two factors affect the curvature of the bidder's indirect payoff function. Note that, by differentiation, we have

$$u_{ss}(s, \theta) = P_{ss}(s, q^*(s, \theta)) + P_{sq}(s, q^*(s, \theta))q^*_s(s, \theta). \tag{18}$$

The first term on the right-hand side of (18) captures the direct effect on u_{ss} of a change in the marginal payments with respect to s given q, while the second term in (18) captures the indirect effect of the change in the marginal payments with respect to s through the change in q.

Regarding the direct effect, the curvature of the scoring rule directly affects the bidder's induced utility function. Since $P_{ss}(s, q) = -S_{pp}/(S_p)^3$, as the scoring function becomes more concave (convex) in p, $u(s, \theta)$ becomes more (less) convex in s, ceteris paribus. Note that this direct effect is independent of the properties of the cost function.

On the other hand, the indirect effect, $P_{sq}(s, q^*(s, \theta))q^*_s(s, \theta)$, is always nonnegative. Indeed, by the first-order condition (17) for optimal quality and the implicit function theorem, we have

$$q^*_s = -\frac{P_{sq}}{P_{qq} - C_{qq}}. \tag{19}$$

Hence, $P_{sq}q^*_s$ is always nonnegative because of the strict concavity of the payoff function in q. Intuitively, with a scoring rule in which the associated P_s falls (rises) as q rises, the bidder will optimally choose a smaller (larger) q as s becomes larger. Moreover, as the indirect effect increases, $u(s, \theta)$ becomes more convex in s, ceteris paribus. Thus, given that the indirect effect is always nonnegative, $u(s, \theta)$ is convex if $S_{pp} \leq 0$.

4.2 Expected Quality and Price

An interesting feature of the PQR scoring rule is that the optimal quality q^* is increasing in score s. This suggests that under a PQR scoring auction, the lower-type bidders compete on price at the expense of quality. Note that the lower-type bidder submits a lower-score bid in equilibrium, so these bidders propose a lower quality with a much lower price as they become more efficient. This property may not be desirable for the procurer unless the scoring function represents their true preferences over a price-quality choice.

Note that by (19), the signs of q_s^* and P_{sq} coincide. Also, we have

$$P_{sq} = -\frac{S_{pp}P_q + S_{pq}}{(S_p)^2} = \frac{S_{pp}S_q - S_pS_{pq}}{(S_p)^3}. \tag{20}$$

The sign of S_{pq} is crucial for the slope of the optimal quality in s. In particular, if the scoring rule is linear in p (i.e., $S_{pp} = 0$), then the sign of q_s^* is determined by $-S_{pq}$. In the PQR scoring rule, $S_{pq} < 0$ and the optimal quality is increasing in score s.[9]

A scoring rule with $S_{pq} < 0$ implies that S_p, the marginal score with respect to price, increases as quality decreases. That is, when quality is already relatively low, a lower price lowers the score even more. In other words, the lower the quality, the more price competition is encouraged. Thus, even though lower-type bidders choose higher quality, scoring rules such as PQR are prone to price competition at the expense of quality.

[9]Note that the optimal quality is not affected by any monotone transformation of scoring rule S. Hence, we can focus on scoring rules with $S_{pp} = 0$ because every reasonable scoring rule can be transformed into this.

Additionally, the quality ranking between FS and SS auctions depends on the curvature of the quality function q^*. When the indirect payoff u is convex, the FS auction yields a higher expected score than the SS auction. Similar to the discussion in the previous section, the expected quality is higher in the FS than in the SS auction if q^* is increasing and weakly concave in s but is higher in the SS auction if q^* is decreasing and weakly convex in s. However, because the condition for the concavity or convexity of q^* is complicated, it is difficult to obtain a clear comparison of quality between FS and SS auctions.

Moreover, the price ranking between FS and SS auctions is more ambiguous than quality. Let $\pi(s,\theta) \equiv P(s, q^*(s,\theta))$ be the price associated with score s and the optimal quality q^*. Then,

$$\pi_s = P_s + q_s^* P_q$$

and

$$\pi_{ss} = P_{ss} + q_{ss}^* P_q - \frac{(P_{sq})^2}{P_{qq} - C_{qq}} \left(2 - \frac{P_{qq}}{P_{qq} - C_{qq}}\right)$$

by (19). The last term of π_{ss} is positive if $P_{qq} \geq 0$. Hence, the optimal price π is likely to be convex and so the expected price ranking becomes ambiguous when u is convex and q^* is increasing in s. This is analogous to Theorem 3 for the PQR scoring rule.

Further, for a scoring rule in which the associated optimal quality q^* is decreasing in s, the price function π may no longer be monotone, which makes the price ranking more ambiguous. Thus, with respect to general scoring rules, whether the expected price (quality) in the FS auction is lower relative to that in the SS auction is an empirical question.

4.3 Characterization of QL Scoring Rules

As Che (1993) shows, QL scoring auctions can be reduced to score-bid auction games with quasilinear indirect payoff functions so that the equivalence theorem holds with respect to price and quality. Our inspections thus far also support the converse: if a scoring rule induces the equivalence theorem with respect to price and quality, it is a QL scoring auction.

Note that if we consider the ex post score rankings over (p, q), we can suppose that $S_{pp} = 0$ without any loss in generality because score ordering over price-quality choices is preserved in any monotone transformation.[10] For any reasonable class of scoring rules S, we are thus able to construct another scoring rule $\hat{S}$ which is affine in p, $\hat{S}_{pp} = 0$, by taking an appropriate monotone transformation on S.

Suppose $S_{pp} = 0$. Then, a scoring rule is QL if and only if $S_{pq} = 0$. Because the slope of the optimal quality q^* is determined by (19) and (20), q^* is independent of s if and only if the scoring rule is QL. Also, as the second derivative of u in s is $u_{ss} = P_{sq} q^*_s$, the indirect payoff u is quasilinear only for QL scoring rules. Therefore, we have confirmed that the QL scoring rule is a unique rule that induces a quasilinear indirect payoff and under which the optimal quality is independent of s.

Proposition 7 *Suppose $S_{pp} = 0$. Then, the following statements are equivalent.*

1. *The scoring rule is QL; $S_{pq} = 0$.*

2. *The optimal quality q^* is independent of score s.*

3. *The indirect payoff u is quasilinear in s.*

5 Concluding Remarks

This study has examined scoring auctions using PQR and more general non-QL scoring rules. For the PQR scoring rule, we have characterized the equilibrium bidding strategies in FS and SS auctions and have found that the expected score is higher in FS auctions and that under a set of conditions expected quality and price are also higher. We also provided an example in which the expected quality in an FS auction is higher than in an SS auction while the expected price is equivalent or lower. These results suggest that if the price per quality ratio is the procurement

[10] Note, however, that we do lose generality regarding the expected score.

buyer's true objective function, an SS auction is better for that buyer than an FS auction. However, the results also imply that the FS auction may perform better than the SS auction with respect to expected quality and price.

We have also showed that a PQR scoring auction provides higher quality at a higher price than a price-only auction with a minimum quality level. In addition, an SS auction using a PQR scoring rule generates a lower price per quality ratio than a second-price auction with any minimum quality level.

We have further examined other non-QL scoring rules. The expected score ranking is characterized by the curvature of the indirect payoff function. When the scoring function is weakly concave in price, the payoff function is reduced to convex, so an SS auction yields a lower expected score than an FS auction. However, expected quality and price rankings are ambiguous because the curvature of the optimal quality function is complicated. Hence, for auctions with general scoring rules, it is an empirical question whether SS auctions generate a lower expected quality (price) than FS auctions.

There are several potential extensions for further research. One important extension would be a theoretical consideration of a scoring auction with an interdependent scoring rule. In this study, we have restricted our attention to scoring rules in which each bidder's score depends only on its own price and quality. However, in practice, the buyer sometimes uses an interdependent scoring rule in which the score depends not only on the bidder's own price and quality bid but also on some or all competitors' price and quality bids. Another would be to incorporate the uncertainty of buyer's quality bid evaluation. Our model, following Che (1993), assumes that bidders do not face uncertainty in how their quality bids are evaluated by the buyer but, in practice, the bids are evaluated by reviewers and hence the scores of quality bids include noise (Takahashi, 2018). These theoretical analyses are left to future research.

References

Allen, Jason, Robert Clark, Brent Hickman, and Eric Richert, "Resolving failed banks: Uncertainty, multiple bidding & auction design," *Review of Economic Studies*, 2023, p. rdad062.

Andreyanov, Pasha, "Mechanism choice in scoring auctions," Technical Report 2018.

Asker, John and Estelle Cantillon, "Properties of Scoring Auctions," *RAND Journal of Economics*, 2008, *39* (1), 69–85.

_ **and** _ , "Procurement when price and quality matter," *The Rand journal of economics*, 2010, *41* (1), 1–34.

Athey, Susan, "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, July 2001, *69* (4), 861–889.

Awaya, Yu, Naoki Fujiwara, and Marton Szabo, "Quality and Price in Scoring Auctions," Technical Report May 2022.

Bajari, Patrick, Stephanie Houghton, and Steven Tadelis, "Bidding for Incomplete Contracts: An Empirical Analysis of Adaptation Costs," *American Economic Review*, April 2014, *104* (4), 1288–1319.

Board, Simon, "Bidding into the red: A model of post-auction bankruptcy," *The Journal of Finance*, 2007, *62* (6), 2695–2723.

Bolotnyy, Valentin and Shoshana Vasserman, "Scaling auctions as insurance: A case study in infrastructure procurement," *Econometrica*, 2023, *91* (4), 1205–1259.

Branco, Fernando, "The Design of Multidimensional Auctions," *RAND Journal of Economics*, 1997, *28* (1), 63–81.

Che, Yeon-Koo, "Design Competition through Multidimensional Auctions," *RAND Journal of Economics*, Winter 1993, *24* (4), 668–680.

Dastidar, Krishnendu Ghosh, "Scoring auctions with non-quasilinear scoring rules," ISER Discussion Paper 0902, Institute of Social and Economic Research, Osaka University June 2014.

Decarolis, Francesco, Giancarlo Spagnolo, and Riccardo Pacini, "Past performance and procurement outcomes," Technical Report, National Bureau of Economic Research 2016.

Hanazono, Makoto, Yosuke Hirose, Jun Nakabayashi, and Masanori Tsuruoka, "Theory, Identification and Estimation of Scoring Auctions," Technical Report August 2020.

Huang, Yangguang, "An empirical study of scoring auctions and quality manipulation corruption," *European Economic Review*, 2019, *120*, 103322.

Iimi, Atsushi, "Multidimensional auctions for public energy efficiency projects: Evidence from Japanese ESCO market," *Review of Industrial Organization*, 2016, *49*, 491–514.

Kong, Yunmi, Isabelle Perrigne, and Quang Vuong, "Multidimensional auctions of contracts: An empirical analysis," *American Economic Review*, 2022, *112* (5), 1703–1736.

Koning, Pierre and Arthur van de Meerendonk, "The impact of scoring weights on price and quality outcomes: An application to the procurement of Welfare-to-Work contracts," *European Economic Review*, 2014, *71* (C), 1–14.

Krasnokutskaya, Elena, Kyungchul Song, and Xun Tang, "The role of quality in internet service markets," *Journal of Political Economy*, 2020, *128* (1), 75–117.

Lewis, Gregory and Patrick Bajari, "Procurement Contracting With Time Incentives: Theory and Evidence," *The Quarterly Journal of Economics*, 2011, *126* (3), 1173–1211.

Maskin, Eric and John Riley, "Optimal Auctions with Risk Averse Buyers," *Econometrica*, 1984, *52* (6), pp. 1473–1518.

Nishimura, Takeshi, "Optimal design of scoring auctions with multidimensional quality," *Review of Economic Design*, 2015, *19* (2), 117–143.

Ryan, Nicholas, "Contract enforcement and productive efficiency: Evidence from the bidding and renegotiation of power contracts in India," *Econometrica*, 2020, *88* (2), 383–424.

Saitoh, Hiroki and Shigehiro Serizawa, "Vickrey allocation rule with income effect," *Economic Theory*, 2008, *35*, 391–401.

Sakai, Toyotaka, "Second price auctions on general preference domains: two characterizations," *Economic Theory*, 2008, *37*, 347–356.

Sano, Ryuji, "Post-auction investment by financially constrained bidders," *Journal of Economic Theory*, 2023, *213*, 105742.

Takahashi, Hidenori, "Strategic design under uncertain evaluations: structural analysis of design-build auctions," *The RAND Journal of Economics*, 2018, *49* (3), 594–618.

The Department of Health and Ageing, Australia, "Tender Evaluation Plan," 2011. `http://www.health.gov.au/internet/main/publishing.nsf/Content/205B1A69101B75C3CA257909000720F1/$File/FOI%20264_1011%20doc%2013.pdf`.

Wang, Mingxi and Shulin Liu, "Equilibrium bids in practical multi-attribute auctions," *Economics Letters*, 2014, *123* (3), 352–355.

A Proofs

A.1 Proof of Lemma 1

Consider the following minimization problem:

$$\min_{q \geq \underline{q}} S(C(q,\theta),q).$$

Given an arbitrary q_0, set $s_0 = S(C(q_0,\theta),q_0)$. We can restrict the constraint set to $\{q \geq \underline{q} | S(C(q,\theta),q) \leq s_0\}$ without affecting the solution. We show that the restricted set is compact: Suppose not. Since the set is closed, it must be unbounded. Then we can take an arbitrarily large q_1 such that $S(C(q_1,\theta),q_1) \leq s_0$, which implies that $P(s_0,q_1) \geq C(q_1,\theta)$. Thus

$$\int_{q_0}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi = P(s_0,q_1) - C(q_1,\theta) - \{\overbrace{P(s_0,q_0) - C(q_0,\theta)}^{0}\} \geq 0.$$

Because $P(s,q) - C(q,\theta)$ is strictly concave in q, $P_q(s_0,q) < C_q(q,\theta)$ for all $q > q_2$, where $P_q(s_0,q_2) = C_q(q_2,\theta)$. Therefore

$$\int_{q_0}^{q_2} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi + \int_{q_2}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi \geq 0. \tag{21}$$

The second term of the left-hand side is negative and has a sufficiently large absolute value as $q_1 \to \infty$, which is a contradiction to inequality (21). By the Weierstrass Theorem, a solution to the score minimization exists, and the value is the break-even score.

To show that $z(\cdot)$ is strictly increasing, let $q^z(\theta)$ denote a solution to the above score-minimization problem. Then $z(\theta) = S(C(q^z(\theta),\theta),q^z(\theta))$. Note that $P(z(\theta),q) \leq C(q,\theta)$ for all q (with equality at $q = q^z(\theta)$). Consider $\tilde{\theta} > \theta$. Since $C(q,\theta) < C(q,\tilde{\theta})$, we must have $P(z(\theta),q) < C(q,\tilde{\theta})$ for all q, implying that there is no intersection between $P(z(\theta),\cdot)$ and $C(\cdot,\tilde{\theta})$. Since $P_s(s,q) > 0$ and $P(z(\tilde{\theta}),q^z(\tilde{\theta})) = C(q^z(\tilde{\theta}),\tilde{\theta})$, $z(\tilde{\theta}) > z(\theta)$. $\square$

A.2 Proof of Proposition 2

Note that $q_s^* = 1/C_{qq}$ and $q_\theta^* = -C_{q\theta}/C_{qq}$. By differentiation, we have

$$\frac{\partial \log u(s,\theta)}{\partial s} = \frac{q^*(s,\theta)}{u(s,\theta)}$$

and

$$\begin{aligned}\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} &= \frac{1}{u(s,\theta)^2}\left(q_\theta^*(s,\theta)u(s,\theta) + q^*(s,\theta)C_\theta(q^*(s,\theta),\theta)\right)\\ &= \frac{1}{u(s,\theta)^2}\left(-q_s^*(s,\theta)C_{q\theta}(q^*,\theta)u(s,\theta) + q^*(s,\theta)C_\theta(q^*,\theta)\right).\end{aligned}$$

It is immediately clear that log-supermodularity holds if $C_{q\theta} \leq 0$. In what follows, we assume $C_{q\theta} > 0$ and provide two sufficient conditions under which the log-supermodularity condition holds.

Condition 1. Suppose that C_θ/C_q is non-increasing in q. That is, we have

$$C_{q\theta}C_q - C_\theta C_{qq} \leq 0 \Leftrightarrow -\frac{C_{q\theta}}{C_{qq}} \geq -\frac{C_\theta}{C_q}$$

for all q and θ. By evaluating this at $q = q^*(s,\theta)$, we have

$$q_\theta^*(s,\theta) > -\frac{C_\theta(q^*,\theta)}{C_q(q^*,\theta)}. \tag{22}$$

Because $u(s,\theta) \geq 0$ for $s \geq z(\theta)$, we have

$$\begin{aligned}\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} &= \frac{1}{u(s)^2}\left(q_\theta^*(s)u(s) + q^*(s)C_\theta(q^*(s))\right)\\ &\geq \frac{1}{u(s)^2}\left(-\frac{C_\theta(q^*(s))}{C_q(q^*(s))}u(s) + q^*(s)C_\theta(q^*(s))\right)\\ &= \frac{C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}\left(q^*(s)C_q(q^*(s)) - (sq^*(s) - C(q^*(s)))\right) \qquad (23)\\ &> \frac{q^*(s)C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}(C_q(q^*(s)) - s)\\ &= 0.\end{aligned}$$

Note that we omit the parameter θ from the presentation. The second line is derived from (22). The third line comes from the definition of the indirect payoff $u(s,\theta)$. The strict inequality is due to $C(q^*,\theta) > 0$ under Assumption 1. Finally, the last line comes from the first-order condition for the optimal quality $s - C_q(q^*, \theta) = 0$. Thus, log-supermodularity condition holds.

Condition 2. Fix an arbitrary θ and define a function V of score s by[11]

$$V(s) \equiv -q_s^*(s) C_{q\theta}\left(q^*(s)\right) u(s) + q^*(s) C_\theta\left(q^*(s)\right).$$

What we want to show is that $V(s) > 0$ for all $s \geq z(\theta)$. Note that $V(z(\theta)) = q^* C_\theta > 0$ by $u(z(\theta)) = 0$. Hence, it suffices to show that $V(s) = 0 \Rightarrow V'(s) > 0$ for every $s > z(\theta)$.

By differentiation, we have

$$\begin{aligned} V'(s) &= -q_{ss}^* C_{q\theta} u - (q_s^*)^2 C_{qq\theta} u - q_s^* C_{q\theta} q^* + q_s^* C_\theta + q^* q_s^* C_{q\theta} \\ &= q_s^* C_\theta - q_{ss}^* C_{q\theta} u - (q_s^*)^2 C_{qq\theta} u. \end{aligned} \tag{24}$$

Suppose $V(s) = 0 \Leftrightarrow u = q^* C_\theta / q_s^* C_{q\theta}$. By substituting this into (24), we have

$$V'(s)\big|_{V(s)=0} = q_s^* C_\theta - \frac{q_{ss}^* q^* C_\theta}{q_s^*} - \frac{q_s^* C_{qq\theta} q^* C_\theta}{C_{q\theta}}. \tag{25}$$

Note that $q_s^* = 1/C_{qq}$ and $q_{ss}^* = -C_{qqq}/(C_{qq})^3$. By substituting them into (25), we have

$$\begin{aligned} V'(s)\big|_{V(s)=0} &= \frac{C_\theta}{C_{qq}} + \frac{C_{qqq} q^* C_\theta}{(C_{qq})^2} - \frac{q^* C_{qq\theta} C_\theta}{C_{qq} C_{q\theta}} \\ &= \frac{C_\theta}{C_{qq}} \left(1 + q^* \left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}}\right)\right). \end{aligned} \tag{26}$$

Because C_θ, $C_{qq} > 0$, we conclude that log-supermodularity holds if

$$1 + q\left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}}\right) > 0$$

for all q.

If the log-supermodularity condition (8) holds, there exists a monotone pure-strategy Bayesian Nash equilibrium in FPA (Athey, 2001). The equilibrium strategy is symmetric and characterized by the first-order condition as shown by Maskin and

[11] We omit the fixed parameter θ from the presentation.

Riley (1984, Theorem 2).[12] Suppose that the equilibrium is symmetric and let s^{FS} be the symmetric equilibrium strategy. Suppose that every bidder other than i follows s^{FS}. The interim expected payoff when bidder i makes an equilibrium bid of type τ is

$$(1 - G(\tau))\, u\left(s^{FS}(\tau), \theta\right).$$

The first-order condition for the payoff maximization is

$$-g(\tau)u\left(s^{FS}(\tau), \theta\right) + (s^{FS})'(\tau)\,(1 - G(\tau))\, u_s\left(s^{FS}(\tau), \theta\right) = 0.$$

Because the first-order condition should hold with $\tau = \theta$, we have

$$-g(\theta)u\left(s^{FS}(\theta), \theta\right) + (s^{FS})'(\theta)((1 - G(\theta))\, u_s\left(s^{FS}(\theta), \theta\right) = 0, \tag{27}$$

which is (10). The terminal condition for the differential equation is $u(s^{FS}(\bar{\theta}), \bar{\theta}) = 0$. Thus, $s^{FS}(\bar{\theta}) = z(\bar{\theta})$. Under the log-supermodularity condition (8), the monotonicity of a strategy and the first-order condition are sufficient for the best response. Hence, the strategy s^{FS} characterized by (10) is the symmetric equilibrium. □

A.3 Proof of Corollary 1

Note that

$$\frac{u(s, \theta)}{u_s(s, \theta)} = \frac{sq^*(s, \theta) - C(q^*(s, \theta), \theta)}{q^*(s, \theta)} = s - k(s, \theta).$$

The first-order condition (27) of the bidder's problem in an FS auction yields

$$-\left(s^{FS}(\theta) - k(s^{FS}(\theta), \theta)\right) g(\theta) + (s^{FS})'(\theta)(1 - G(\theta)) = 0.$$

Solving the differential equation gives (12). □

[12] Although Maskin and Riley (1984) assume that U is concave, this is not used nor is it necessary to obtain the FSA equilibrium. For instance, Board (2007, Lemma 3) is an example of a convex payoff function.

A.4 Proof of Theorem 2

By the first-order condition for the optimal quality $s = C_q(q^*, \theta)$, we have

$$q^*_{ss} = -\frac{C_{qqq}}{(C_{qq})^3}.$$

Hence, the optimal quality function q^* is weakly concave if $C_{qqq} \geq 0$. Let $\theta_{(1)}$ and $\theta_{(2)}$ be the lowest and second lowest order statistics of bidder types. When q^* is weakly concave in s, we have

$$\begin{aligned} E\left[q^*\left(s^{SS}(\theta_{(2)}), \theta_{(1)}\right)\right] &= E_{\theta_{(1)}}\left[E_{\theta_{(2)}}\left[q^*\left(s^{SS}(\theta_{(2)}), \theta_{(1)}\right) \mid \theta_{(2)} > \theta_{(1)}\right]\right] \\ &\leq E_{\theta_{(1)}}\left[q^*\left(E_{\theta_{(2)}}\left[s^{SS}(\theta_{(2)}) \mid \theta_{(2)} > \theta_{(1)}\right], \theta_{(1)}\right)\right] \\ &\leq E\left[q^*\left(s^{FS}(\theta_{(1)}), \theta_{(1)}\right)\right]. \end{aligned}$$

Note that E_X means that we take an expectation regarding X. The first inequality is Jensen's inequality. The second inequality comes from Theorem 1. □

A.5 Proof of Theorem 3

Let $\pi(s, \theta) = sq^*(s, \theta)$ be the optimal price given score s and type θ. Then, by differentiation, we have

$$\pi_{ss}(s, \theta) = sq^*_{ss} + 2q^*_s.$$

Bu substituting $q^*_s = 1/C_{qq}$, $q^*_{ss} = -C_{qqq}/(C_{qq})^3$, and the first-order condition $s = C_q$, we have

$$\pi_{ss} = \frac{2(C_{qq})^2 - C_q C_{qqq}}{(C_{qq})^3}.$$

Thus, the optimal price is weakly concave if (14) holds. When π is weakly concave in s, we have the expected price ranking in the same manner with the quality ranking Theorem 2. □

A.6 Proof of Theorem 4

Suppose that $C_{q\theta} = 0$. Then, it is clear that the optimal quality q^* is independent of θ and is denoted by $q^*(s)$. Let π be the optimal price function $\pi(s) = sq^*(s)$. Because

q^* is increasing in s, π is also increasing in s. Thus, each price bid corresponds to a score bid in the one-to-one sense. That is, for every score s, we have a unique associated price $p = \pi(s)$. We define a payoff function in terms of the price bid $\hat{u}$ as

$$\hat{u}(p,\theta) \equiv u(\pi^{-1}(p),\theta).$$

Abusing notation, the cost function is denoted by $C = C(q) + \theta$, where $C(q)$ is variable cost and θ is the fixed cost.[13] Then, we have

$$\hat{u}(p,\theta) = p - C\left(\frac{p}{\pi^{-1}(p)}\right) - \theta.$$

By differentiation, we have

$$\hat{u}_p = 1 - \left(\frac{p}{\pi^{-1}(p)}\right)' C'\left(\frac{p}{\pi^{-1}(p)}\right)$$

and

$$\hat{u}_{pp} = -\left(\frac{p}{\pi^{-1}(p)}\right)'' C'\left(\frac{p}{\pi^{-1}(p)}\right) - \left\{\left(\frac{p}{\pi^{-1}(p)}\right)'\right\}^2 C''\left(\frac{p}{\pi^{-1}(p)}\right).$$

By differentiation, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{\pi^{-1}(p) - p(\pi^{-1})'(p)}{(\pi^{-1}(p))^2} = \frac{\pi'(\pi^{-1}(p))\pi^{-1}(p) - p}{\pi'(\pi^{-1}(p))(\pi^{-1}(p))^2}$$

and

$$\begin{aligned}\left(\frac{p}{\pi^{-1}(p)}\right)'' &= \frac{1}{(\pi^{-1})^3}\left[-p(\pi^{-1})''\pi^{-1} - 2(\pi^{-1})'\pi^{-1} + 2p((\pi^{-1})')^2\right] \\ &= \frac{1}{(\pi^{-1})^3}\left[\frac{p\pi^{-1}\pi''}{(\pi')^3} - \frac{2\pi^{-1}}{\pi'} + \frac{2p}{(\pi')^2}\right] \\ &= \frac{1}{(\pi')^3(\pi^{-1})^3}\left[p\pi^{-1}\pi'' + 2p\pi' - 2\pi^{-1}(\pi')^2\right].\end{aligned}$$

Note that by definition, we have $p = \pi(s) = sq^*(s)$, $\pi^{-1}(p) = s$, $\pi'(s) = q^* + sq^*_s$, and $\pi''(s) = sq^*_{ss} + 2q^*_s$. By substituting them into the above, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{(q^* + sq^*_s)s - sq^*}{(q^* + sq^*_s)s^2} = \frac{q^*_s}{q^* + sq^*_s}$$

[13] Because the cost function is increasing in θ (by assumption), θ can be defined by fixed cost without any loss of generality.

and

$$\left(\frac{p}{\pi^{-1}(p)}\right)'' = \frac{1}{(q^* + sq_s^*)^3 s^3}\left[sq^* \cdot s(sq_{ss}^* + 2q_s^*) + 2sq^*(q^* + sq_s^*) - 2s(q^* + sq_s^*)^2\right]$$
$$= \frac{q^* q_{ss}^* - 2(q_s^*)^2}{(q^* + sq_s^*)^3}.$$

By the first-order condition for the optimal quality $s = C_q$, we have

$$\hat{u}_p = 1 - \frac{q_s^* C_q}{q^* + sq_s^*} = \frac{q^* C_{qq}}{q^* C_{qq} + C_q} > 0.$$

Also, we have

$$\hat{u}_{pp} = \frac{2(q_s^*)^2 - q^* q_{ss}^*}{(q^* + sq_s^*)^3} C_q(q^*) - \frac{(q_s^*)^2}{(q^* + sq_s^*)^2} C_{qq}(q^*)$$
$$= \frac{1}{(q^* + sq_s^*)^3}\left[2(q_s^*)^2 C_q - q^* q_{ss}^* C_q - (q_s^*)^2 C_q q(q^* + sq_s^*)\right]$$
$$= \frac{1}{(q^* + sq_s^*)^3}\left[\frac{2C_q}{(C_{qq})^2} + \frac{qC_q C_{qqq}}{(C_{qq})^3} - \frac{qC_{qq} + C_q}{(C_{qq})^2}\right]$$
$$= \frac{C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2}{(q^* + sq_s^*)^3 (C_{qq})^3}.$$

The third line comes from $q_s^* = 1/C_{qq}$ and $q_{ss}^* = -C_{qqq}/(C_{qq})^3$. Hence, $\hat{u}$ is convex in p if $C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0$. Then, the expected price in the FS auction is higher than in the SS auction, which is analogous to Theorem 1 and Maskin and Riley (1984). □

A.7 Proof of Proposition 3

In price-only auctions, it is clear that bidders choose the minimum quality $\underline{q}$ in equilibrium. Thus, it is immediate by Assumption 1 that the equilibrium quality in FS and SS auctions is $q^*(s, \theta) > \underline{q}$, where $s \geq z(\theta)$ is the winning score. The price associated with the break-even score $z(\theta)$ is $\pi(z(\theta), \theta) = C(q^*(z(\theta), \theta), \theta)$. Note that the equilibrium bid in the second-price auction is given by $p^{PO}(\theta) \equiv C(\underline{q}, \theta)$. Let bidders i and j be the lowest and the second lowest bidders, respectively. By

Proposition 1, the winner i's price in the SS auction is

$$\begin{aligned}\pi(z(\theta_j),\theta_i) = z(\theta_j)q^*(z(\theta_j),\theta_i) &\geq z(\theta_j)q^*(z(\theta_j),\theta_j) \\ &= \pi(z(\theta_j),\theta_j) \\ &> p^{PO}(\theta_j).\end{aligned}$$

The first inequality comes from $q^*_\theta \leq 0$ and $\theta_j \geq \theta_i$. The last line comes from Assumption 1: $q^*(z(\theta),\theta) > \underline{q}$ and $C(q^*(z(\theta),\theta),\theta) > C(\underline{q},\theta)$. Thus, the equilibrium price in the SS auction is higher than in the second-price auction for every type profile. Because the revenue equivalence theorem holds for price-only auctions, the expected price in the FS auction is higher than in the first-price auction if the conditions presented in Theorems 3 or 4 hold. □

A.8 Proof of Proposition 4

Consider a price-only auction in which bidders must enforce a quality at least as high as $\hat{q}$. Because bidders have no incentive to provide a higher quality than $\hat{q}$, they submit $p_i = C(\hat{q},\theta_i)$ in a second-price auction. Let j be the second lowest type bidder. The equilibrium price per quality in the second-price auction is $\frac{C(\hat{q},\theta_j)}{\hat{q}}$.

In the SS auction with a PQR scoring rule, bidders submit their break-even score $z(\theta_i)$. By the definition of the PQR rule, the equilibrium price per quality is

$$\frac{\pi(z(\theta_j),\theta_i)}{q^*(z(\theta_j),\theta_i)} = z(\theta_j) = \min_q \frac{C(q,\theta_j)}{q} \leq \frac{C(\hat{q},\theta_j)}{\hat{q}}.$$

The second equality comes from the fact that the average cost is minimized at the break-even price by interpreting score s as a unit price per quality. □

B Sufficient Conditions for the Equilibrium Existence of the FS Auction

In this appendix, we explore a set of conditions on primitives that guarantees the log-supermodularity condition for general scoring rules. We can restrict the domain to $\{(s,\theta)|u(s,\theta)>0\}$, since otherwise, the score bid is clearly suboptimal for a type θ bidder. Suppose that $u_{s\theta}$ exists. We suppose that the payoff function $P(s,q)-C(q,\theta)$ is strictly concave in q and that the optimal quality q^* always lies in the interior $q^*(s,\theta)>\underline{q}$.

Proposition 8 *The log-supermodularity condition holds if the optimal quality (and price) are not binding for all* (s,θ) *and*

1. $P_{sq}C_{q\theta}\leq 0$, *or*

2. $P_{sq}>0$, $C_{q\theta}\geq 0$, P/P_s *weakly increasing in* q*,and* $C_{q\theta}/(C_{qq}-P_{qq})<C_\theta/C_q$.

Proof The log-supermodular condition holds if and only if

$$\frac{u(s,\theta)}{u_s(s,\theta)}u_{s\theta}(s,\theta)-u_\theta(s,\theta)>0. \tag{28}$$

Note that by the envelope theorem, we have $u_s(s,\theta)=P_s(s,q^*)$, $u_\theta(s,\theta)=-C_\theta(q^*,\theta)$, and $u_{s\theta}(s,\theta)=P_{sq}(s,q^*)q^*_\theta(s,\theta)$. Thus, (28) holds if

$$\frac{u(s,\theta)}{u_s(s,\theta)}P_{sq}(s,q^*(s,\theta))q^*_\theta(s,\theta)+C_\theta(q^*(s,\theta),\theta)>0. \tag{29}$$

Because $q^*_\theta(s,\theta)=-C_{q\theta}/[C_{qq}(q^*(s,\theta),\theta)-P_{qq}(s,q^*(s,\theta))]$, we have condition 1 by the concavity of $P(s,q)-C(q,\theta)$ in q.

In what follows, we provide the proof for condition 2. We assume that $C_{q\theta}(q,\theta)\geq 0$ and that $P(s,q)/P_s(s,q)$ is weakly increasing in q. Let us further assume that

$$C_q(\cdot)\left[-\frac{C_{q\theta}(\cdot)}{C_{qq}(\cdot)-P_{qq}(\cdot)}\right]+C_\theta(\cdot)>0$$

holds for all q and θ. Then we evaluate this inequality at $q=q^*(s,\theta)$. Recall that the square-bracket term equals $q^*_\theta(s,\theta)$ if $q=q^*(s,\theta)$. Hence, we obtain

$$C_q(q^*(s,\theta),\theta)q^*_\theta(s,\theta)+C_\theta(q^*(s,\theta),\theta)>0. \tag{30}$$

Next, we show that if $P(s,q)/P_s(s,q)$ is weakly increasing in q for all s and q and $P_{sq}(\cdot)\geq 0$, then $[u(s,\theta)/u_s(s,\theta)]P_{sq}(\cdot)\leq C_q$. First, the condition that P/P_s is weakly increasing in q implies that

$$\frac{d}{dq}\frac{P(s,q)}{P_s(s,q)}=\frac{1}{(P_s(s,q))^2}\left[P_q(s,q)P_s(s,q)-P(s,q)P_{sq}(s,q)\right]\geq 0$$

for all s and q. Given the fact that $P_s > 0$, this inequality is equivalent to

$$\frac{P(s,q)}{P_s(s,q)} P_{sq}(s,q) \leq P_q(s,q) \quad \text{for all } s \text{ and } q.$$

Then we consider this (weak) inequality, replacing $P(s,q)$ with $P(s,q) - C(q,\theta)$ on the left-hand side. Given that $P_{sq} \geq 0$ and that $C(q,\theta)$ is nonnegative, the inequality implies that

$$\frac{[P(s,q) - C(q,\theta)]P_{sq}(s,q)}{P_s(s,q)} \leq P_q(s,q) \tag{31}$$

for all s and q.

By substituting $q = q^*(s,\theta)$ into (31), we have

$$\begin{aligned} \frac{u(s,\theta)}{u_s(s,\theta)} P_{sq}(s, q^*(s,\theta)) &= \frac{P(s,q^*) - C(q^*,\theta)}{P_s(s,q^*)} P_{sq}(s,q^*) \\ &\leq P_q(s,q^*) \\ &= C_q(q^*,\theta). \end{aligned} \tag{32}$$

The last equality comes from the first-order condition for the optimal quality q^*. Expressions (30) and (32) imply

$$\begin{aligned} &\frac{u(s,\theta)}{u_s(s,\theta)} P_{sq}(s, q^*(s,\theta)) q^*_\theta(s, q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta) \\ &\geq C_q(q^*(s,\theta),\theta) q^*_\theta(s, q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta) \\ &> 0 \end{aligned}$$

by $P_{sq} > 0$ and $q^*_\theta \leq 0$. Thus, log-supermodularity holds. □