UTMD Working Paper

The University of Tokyo
Market Design Center

UTMD-064

Endowments-swapping-proofness in Housing Markets with Exchange Constraints

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The first version: May 1, 2023
This version: June 7, 2024

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Abstract

This paper examines housing markets where constraints are placed on the number of agents involved in exchanges. We investigate the existence of exchange-constrained mechanisms that satisfy *endowments-swapping-proofness*, a condition ensuring that no pair of agents can benefit from swapping their endowments before the mechanism is applied. Our primary finding is that when preferences are strict, no exchange-constrained mechanism can simultaneously satisfy both *individual rationality* and *endowments-swapping-proofness*. To avoid this negative result, we next explore three well-known restricted domains of preferences: common ranking preferences, single-peaked preferences, and single-dipped preferences. Unfortunately, even when preferences are restricted to either common ranking preferences or single-peaked preferences, the two properties remain incompatible. However, a possibility arises if preferences are single-dipped: the well-known top trading cycles mechanism is the only exchange-constrained mechanism that satisfies *strategy-proofness* in addition to the two properties mentioned above. Notably, this characterization holds even without *strategy-proofness* when only pairwise exchanges are allowed.

Keywords: endowments-swapping-proofness; common ranking preferences; single-peaked preferences; single-dipped preferences; top trading cycles; housing markets.

JEL codes: C78; D47.

*This paper is a substantially revised version of our earlier paper entitled "Endowments-swapping-proof house allocation with feasibility constraints." We would like to thank Yuki Tamura for sending us a draft of her article (Tamura, 2023). We gratefully acknowledge financial support from the Joint Usage/Research Center at ISER, Osaka University, and the Grant-in-aid for Research Activity, Japan Society for the Promotion of Science (JP20K01555, JP22H00062, and JP24K04801).

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1 Introduction

1.1 Motivation and outline

We consider the object reallocation problem introduced by Shapley and Scarf (1974), known as the housing market. In the standard model of this problem, each agent initially owns one indivisible object (house) and has strict preferences over a set of objects. A "mechanism" reallocates the objects provided that each agent receives one and only one object. There are several real-life applications of this model: kidney exchange (Roth, Sönmez, and Ünver, 2004), on-campus housing (Abdulkadiroğlu and Sönmez, 1999), the reallocation of time slots (Moulin, 2003), and the reallocation of airport landing slots (Schummer and Vohra, 2013).

For the object reallocation problem with strict preferences, the top trading cycles mechanism (TTC) selects the unique core allocation via David Gale's TTC algorithm (Roth and Postlewaite, 1977). Moreover, TTC is the only mechanism that is *efficient* (a chosen assignment cannot be changed in a manner that no agent is worse off, and some agent is better off), *individually rational* (no agent is worse off after the reallocation), and *strategy-proof* (no agent ever benefits from misrepresenting his preferences). Following this characterization provided by Ma (1994), TTC has been widely characterized by various properties.[1]

In the context of object reallocation problems, we are interested in mechanisms that are immune to pairwise manipulation by the swapping of endowments. Fujinaka and Wakayama (2018) are the first to formulate this property as *endowments-swapping-proofness*, which states that no pair of agents obtain a strictly better outcome by swapping their endowments before entering the mechanism.[2] They then provide an alternative characterization of TTC in terms of this property: TTC is the only mechanism that is *individually rational*, *strategy-proof*, and *endowments-swapping-proof*.

[1]Examples of such properties include "Maskin monotonicity" (Takamiya, 2001), "anonymity" (Miyagawa, 2002), "no-envy" (Hashimoto and Saito, 2015), and a weak form of *efficiency* (Ekici, 2024). See also Morrill and Roth (2024) for the history of TTC, and its generalizations and extensions.

[2]*Endowments-swapping-proofness* applies only to two-agent coalitions. Postlewaite (1979) and Moulin (1995) have already formulated the version of *endowments-swapping-proofness* that involves all subsets of agents. However, the mechanism designer need not care about manipulations by large coalitions because such strategic cooperation is difficult for large coalitions. Therefore, this coalitional version of *endowments-swapping-proofness* may be too strong a requirement. Conversely, collusion by two agents is relatively easy, and hence, *endowments-swapping-proofness* is appealing if any pairs can form.

Many studies after Ma (1994) have attempted to identify the most desirable mechanisms in the standard object reallocation model. However, this standard model disregards some important aspects of reality, which prevent direct applications of its results to real-life problems. One such aspect is the constraint on the size of exchanges among agents. For example, in the context of kidney exchange, exchanges involving many donor-patient pairs may be infeasible due to the presence of logistic constraints (Roth, Sönmez, and Ünver, 2005; Nicolò and Rodríguez-Álvarez, 2012, 2017). Another example is the exchange of holiday houses, where legal restrictions may prevent exchanges of larger size than pairwise exchanges (Nicolò and Rodríguez-Álvarez, 2013). Based on these observations, this study aims to find *endowments-swapping-proof* mechanisms in an object reallocation model that incorporates exchange constraints.

We first establish a negative result on the domain of strict preferences: the presence of exchange constraints makes it impossible to construct a mechanism that satisfies *individual rationality* and *endowments-swapping-proofness* (Theorem 2).

The smaller the domain, the weaker is the requirement of *endowments-swapping-proofness*. Therefore, we ask whether the above negative result can be avoided on smaller domains. To analyze this issue, we first consider "common ranking" preferences (Nicolò and Rodríguez-Álvarez, 2017).[3] Unfortunately, the above negative result holds even if the domain is restricted to the class of common ranking preferences. In other words, no exchange-constrained mechanism on the domain of common ranking preferences satisfies *individual rationality* and *endowments-swapping-proofness* (Theorem 3).

We also consider two other well-known restricted domains, called "single-peaked" and "single-dipped" preferences.[4] The incompatibility of *individual rationality* and *endowments-swapping-proofness* persists even if the domain is restricted to be single-peaked (Theorem 4). However, a positive result emerges when each agent has single-dipped preferences: TTC is the only exchange-constrained mechanism on the domain of single-dipped preferences that is *individually rational*, *strategy-proof*, and *endowments-swapping-proof* (Theorem 6). More interestingly, when we focus on pairwise exchange, this characterization holds without *strategy-*

[3] We say that an agent has "common ranking" preferences if he ranks acceptable objects according to a common exogenous ranking of the objects.

[4] We say that an agent has "single-peaked" (resp. "single-dipped") preferences with respect to a fixed order of objects if he has a unique best (resp. worst) object, and his welfare is strictly decreasing (resp. increasing) away from this object on each side of this object according to the order.

proofness (Theorem 5). Note that TTC is well-defined on the domain of single-dipped preferences even when exchange constraints are imposed, for TTC on that domain involves only self-pointing cycles and pairwise trading cycles (Proposition 1 and Corollary 1). This nature of TTC leads to the characterization result of TTC on that domain with exchange constraints.

1.2 Related literature

This study is closely related to two branches of the literature. First, it contributes to the growing literature on the object reallocation problem with exchange constraints and its applications. Roth, Sönmez, and Ünver (2005) and Hatfield (2005) are the first to consider exchange constraints in the context of kidney exchange and propose *strategy-proof* exchange-constrained mechanisms on the domain of dichotomous preferences. Nicolò and Rodríguez-Álvarez (2012, 2017) and Balbuzanov (2020) also consider the object reallocation problem with exchange constraints, but assume acceptable objects are heterogeneous and preferences are strict. Nicolò and Rodríguez-Álvarez (2012) provide a negative result in that setting: no exchange-constrained mechanism satisfies *individual rationality*, *efficiency*, and *strategy-proofness*.[5] Balbuzanov (2020) derives another negative result, demonstrating the incompatibility between *efficiency* and a fairness property, called "anonymity." Our Theorem 2 can be considered as an *endowments-swapping-proofness* counterpart of these results.

Second, this paper is connected with the recent literature on the object reallocation problem with restricted domains. Axiomatic characterizing mechanisms in the object reallocation problems with the following three restricted domains, which are considered in this paper, has been a growing research agenda.

• **Common ranking preferences:** Assuming that each agent has common ranking preferences, Nicolò and Rodríguez-Álvarez (2017) propose a pairwise exchange mechanism that is *individually rational*, *efficient*, and *strategy-proof*, while ending up with a negative result under more general exchange constraints.[6] Our Theorem 3 shows that a negative result holds even for pairwise exchanges when *efficiency* and *strategy-proofness* are replaced by *endowments-swapping-proofness*.

[5] Nicolò and Rodríguez-Álvarez (2013) show that one cannot escape this negative result by weakening *strategy-proofness* to "ordinal Bayesian incentive compatibility."

[6] Rodríguez-Álvarez (2023) specifies the extent to which the domain of common ranking preferences can be enlarged to permit the existence of pairwise exchange mechanisms that satisfy the three properties.

• **Single-peaked preferences:** When preferences are single-peaked but exchange constraints are not allowed, there are many non-TTC mechanisms that satisfy certain desirable properties as well as TTC does (Bade, 2019; Liu, 2022; Tamura, 2022; Tamura and Hosseini, 2022; Huang and Tian, 2023). However, our Theorem 4 tells us that once exchange constraints are imposed, there is no nice mechanism in terms of *endowments-swapping-proofness*.

• **Single-dipped preferences:** Tamura (2023) shows that the characterizations of TTC proposed by Ma (1994) and Fujinaka and Wakayama (2018) persist even if preferences are restricted to being single-dipped. However, she does not consider constraints on the size of exchanges. Our Theorem 5 and Theorem 6 show that Tamura's characterization of TTC holds even when exchange constraints are imposed; furthermore, *strategy-proofness* can be dropped from the list of axioms in Tamura's characterization of TTC when we focus on pairwise exchanges.

1.3 Organization

The rest of the paper is organized as follows. Section 2 describes the model and introduces our properties for mechanisms. Section 3 states our impossibility result on the domain of strict preferences in the presence of exchange constraints. Section 4 considers three types of restricted domains of preferences and presents the results on these domains. Section 5 concludes with some suggestions for future research. Appendix A contains the proofs that are omitted from the main text.

2 Preliminaries

2.1 Model

Let $N = \{1, 2, \ldots, n\}$ be a finite set of agents and $H = \{h_1, h_2, \ldots, h_n\}$ be a finite set of objects. In this paper, we assume $n \geq 3$. An **assignment** is a bijection $x \colon N \to H$. For simplicity of notation, we write x_i for $x(i)$. As usual, x_i represents the object agent i receives at x. Let X be the set of assignments. An **endowment** is denoted by $\omega = (\omega_i)_{i \in N} \in X$, where ω_i represents the object owned by agent i.

Given an assignment and an endowment $(x, \omega) \in X \times X$, we call a sequence $(i_1 (= i_{S+1}), i_2, \ldots, i_S)$ of agents a **trading cycle** at (x, ω) if one of the following holds:

(a) $S = 1$ and $x_{i_1} = \omega_{i_1}$, or

(b) $S \geq 2$, for each $\{s, s'\} \subseteq \{1, 2, \ldots, S\}$ with $s \neq s'$, $i_s \neq i_{s'}$, and for each $s \in \{1, 2, \ldots, S\}$, $x_{i_s} = \omega_{i_{s+1}}$

Given an endowment $\omega \in X$ and an integer $\ell \in N$, we say that an assignment $x \in X$ is **ℓ-feasible with respect to ω** if for each trading cycle $(i_1, i_2, \ldots, i_S)$ at (x, ω), $|\{i_1, i_2, \ldots, i_S\}| \leq \ell$.[7] We denote the set of ℓ-feasible assignments with respect to ω by $X_\ell(\omega)$. Note that for each $\omega \in X$, $X_1(\omega) = \{\omega\}$ and $X_n(\omega) = X$.

We assume that each agent $i \in N$ has a strict preference relation $\succ_i$ over H. Let $\mathscr{P}$ be the set of all strict preferences over H. For each $\succ_0 \in \mathscr{P}$, $\succsim_0$ represents the induced weak preference relation from $\succ_0$; that is, for each $\{h, h'\} \subset H$, $h \succsim_0 h'$ if and only if either $h \succ_0 h'$ or $h = h'$. Let $\mathscr{P}^N$ be the set of all strict preference profiles $\succ = (\succ_i)_{i \in N}$ such that for each $i \in N$, $\succ_i \in \mathscr{P}$. We often denote $N \setminus \{i\}$ by "$-i$." With this notation, $(\succ'_i, \succ_{-i})$ denotes the preference profile where agent i has $\succ'_i$ and each other agent j has $\succ_j$. For each $i \in N$ and each $(\succ_i, \omega_i) \in \mathscr{P} \times H$, let $A(\succ_i, \omega_i) = \{h \in H \colon h \succ_i \omega_i\}$ be the set of **acceptable** objects for i at $(\succ_i, \omega_i)$.

An **economy** is a pair of a preference profile and an endowment $e = (\succ, \omega) \in \mathscr{P}^N \times X$. Let $\mathscr{E} \subseteq \mathscr{P}^N \times X$ be a set of admissible economies, called a **domain**. We denote the **strict domain** by $\mathscr{E}^{\text{st}} = \mathscr{P}^N \times X$.

Given a domain $\mathscr{E} \subseteq \mathscr{E}^{\text{st}}$, a **mechanism** on $\mathscr{E}$ is a function $f \colon \mathscr{E} \to X$ that maps each economy $e = (\succ, \omega) \in \mathscr{E}$ to an assignment $f(e) \in X$. Given an integer $\ell \in N$, we say that a mechanism f on $\mathscr{E}$ is **ℓ-feasible** if for each $e = (\succ, \omega) \in \mathscr{E}$, $f(e) \in X_\ell(\omega)$. In particular, we say that a mechanism f on $\mathscr{E}$ is a **pairwise exchange mechanism** if it is 2-feasible.

2.2 Properties

In this subsection, we list our properties for mechanisms. Our main property is as follows: no pair of agents can strictly benefit from swapping their endowments before they enter the mechanism. To define this property, we require additional notation. Given an economy $e = (\succ, \omega) \in \mathscr{E}$ and a pair $\{i, j\} \subset N$, let $e^{i,j} = (\succ, \omega^{i,j}) \in \mathscr{P}^N \times X$ be such that $\omega^{i,j}_i = \omega_j$, $\omega^{i,j}_j = \omega_i$, and for each $k \in N \setminus \{i, j\}$, $\omega^{i,j}_k = \omega_k$.

***Endowments-swapping-proofness*:** There are no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset N$ such that (i) $e^{i,j} \in \mathscr{E}$, and (ii) $f_i(e^{i,j}) \succ_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

[7] Given a set Z, $|Z|$ denotes the cardinality of Z.

Remark 1. Fujinaka and Wakayama (2018) do not include Condition (i), $e^{i,j} \in \mathscr{E}$, in their definition of *endowments-swapping-proofness* because they only consider the strict domain that includes any "swapping economy" where a pair of agents swaps their endowments. Unlike Fujinaka and Wakayama (2018), this paper considers both the strict domain and its restricted domains. There is no guarantee that such restricted domains necessarily include any swapping economy.[8] This makes it necessary for us to require Condition (i) in the definition of *endowments-swapping-proofness*. $\Diamond$

We also consider the following allocative property, which states that no one is made worse off by participating in a mechanism.

***Individual rationality*:** For each $e = (\succ, \omega) \in \mathscr{E}$ and each $i \in N$, $f_i(e) \succsim_i \omega_i$.

The following property is the standard incentive requirement, which requires that no agent should ever be made better off than by telling the truth.

***Strategy-proofness*:** For each $e = (\succ, \omega) \in \mathscr{E}$, each $i \in N$, and each $e' = ((\succ'_i, \succ_{-i}), \omega) \in \mathscr{E}$, $f_i(e) \succsim_i f_i(e')$.

3 Impossibility result on the strict domain

A prominent mechanism on the strict domain is the so-called top trading cycles mechanism. The **top trading cycles mechanism**, or TTC for short, is the mechanism $TTC\colon \mathscr{E}^{\text{st}} \to X$ that selects for each $e \in \mathscr{E}^{\text{st}}$, the assignment $TTC(e)$ obtained via the following algorithm known as the TTC algorithm:

- **Round 1.** Each agent points to the agent who owns his best object. Here each agent is allowed to point to himself. Then, there is at least one trading cycle as the number of agents is finite. Each agent involved in a cycle is assigned an object along the cycle and then removed. If an agent remains, the algorithm continues to the next round; otherwise, it terminates.

- **Round $t \geq 2$.** Each remaining agent points to the agent who owns his best object among those remaining. Here each agent is allowed to point

[8] We illustrate this point with a simple example. Consider the common ranking domain, $\mathscr{E}^{\text{cm}}$ (this domain will be formally defined in Section 4.1). Let $n = 3$ and $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$ be such that $h_2 \succ_1 h_1 \succ_1 h_3$ and $\omega = (h_1, h_2, h_3)$. We now consider $e^{1,3}$. Then, $\succ_1$ is not common ranking preferences at $e^{1,3}$, and hence, $e^{1,3} \notin \mathscr{E}^{\text{cm}}$.

to himself. Then, at least one trading cycle exists. Each agent involved in a cycle is assigned an object along the cycle and then removed. If an agent remains, the algorithm continues to the next round; otherwise, it terminates.

An *endowments-swapping-proofness* characterization of TTC on the strict domain in the absence of exchange constraints has been already presented in Fujinaka and Wakayama (2018).

Theorem 1 (Theorem 4 in Fujinaka and Wakayama (2018)). *A mechanism on $\mathscr{E}^{\mathrm{st}}$ satisfies individual rationality, strategy-proofness, and endowments-swapping-proofness if and only if it is TTC.*

The question that naturally arises from Theorem 1 is whether we can find an *endowments-swapping-proof* mechanism that satisfies certain desirable properties when we impose exchange constraints on the size of trading cycles. Unfortunately, the next result indicates that *individual rationality* and *endowments-swapping-proofness* are incompatible once exchange constraints are imposed; that is, Theorem 1 breaks down as soon as the exchange size is limited.

Theorem 2. *Let $\ell \in \{1, 2, \ldots, n-1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^{\mathrm{st}}$ satisfies individual rationality and endowments-swapping-proofness.*

Proof. Suppose on the contrary that there is an ℓ-feasible mechanism f on $\mathscr{E}^{\mathrm{st}}$ that satisfies the two properties. Let $N_\ell = \{1, 2, \ldots, \ell+1\}$.[9] We derive a contradiction. The proof is in three steps.

Step 1: Constructing economies. Let $\succ \in \mathscr{P}^N$ be such that:

(i) for each $i \in N_\ell$, each $j \in \{i, i+1, \ldots, \ell+1\}$, each $j' \in \{1, 2, \ldots, i-1\}$, and each $j'' \in N \setminus N_\ell$, $h_j \succ_i h_{j'} \succ_i h_{j''}$;

(ii) for each $i \in N_\ell$ and each $\{j, j'\} \subset N$, if $j < j'$ and either $\{j, j'\} \subseteq \{i, i+1, \ldots, \ell+1\}$ or $\{j, j'\} \subseteq \{1, 2, \ldots, i-1\}$, then $h_j \succ_i h_{j'}$;

(iii) for each $i \in N \setminus N_\ell$ and each $\{j, j'\} \subset N$, if $j < j'$, then $h_j \succ_i h_{j'}$

The preference profile $\succ$ can be represented as follows:

[9] Since $\ell \leq n-1$, agent $\ell+1$ exists.

$\succ_1$	$\succ_2$	$\succ_3$	$\cdots$	$\succ_\ell$	$\succ_{\ell+1}$	$\succ_{j\geq \ell+2}$
h_1	h_2	h_3	$\cdots$	h_ℓ	$h_{\ell+1}$	h_1
h_2	h_3	h_4	$\cdots$	$h_{\ell+1}$	h_1	h_2
h_3	h_4	h_5	$\cdots$	h_1	h_2	h_3
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$h_{\ell-1}$	h_ℓ	$h_{\ell+1}$	$\cdots$	$h_{\ell-3}$	$h_{\ell-2}$	$h_{\ell-1}$
h_ℓ	$h_{\ell+1}$	h_1	$\cdots$	$h_{\ell-2}$	$h_{\ell-1}$	h_ℓ
$h_{\ell+1}$	h_1	h_2	$\cdots$	$h_{\ell-1}$	h_ℓ	$h_{\ell+1}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$h_{\ell+2}$
						$\vdots$
						h_n

For each $k \in N_\ell$, let $\omega^k \in X$ be such that for each $i \in N$,

$$\omega_i^k = \begin{cases} h_{i+1} & \text{if } i \leq k-1 \\ h_1 & \text{if } i = k \\ h_i & \text{if } i \geq k+1. \end{cases} \tag{1}$$

That is, $\omega^1 = (h_1, h_2, \ldots, h_n)$, $\omega^2 = (h_2, h_1, h_3, \ldots, h_n)$, $\omega^3 = (h_2, h_3, h_1, h_4, \ldots, h_n)$, $\omega^4 = (h_2, h_3, h_4, h_1, h_5, \ldots, h_n)$, and so forth. Note that for each $k \in N_\ell \setminus \{1\}$, $\omega^{k-1} = (\omega^k)^{k-1,k}$. For each $k \in N_\ell$, let $e^k = (\succ, \omega^k)$.

Step 2: For each $k \in N_\ell$ and each $i \in N_\ell$, $f_i(e^k) = h_i$. We use the induction to prove this step.

BASE STEP. Let $K = 1$ and $i \in N_\ell$. By the definition of $\succ_i$ (Step 1), $A(\succ_i, \omega_i^1 = h_i) = \emptyset$. Thus, *individual rationality* implies $f_i(e^1) = h_i$.

INDUCTION HYPOTHESIS. Let $K \in N_\ell \setminus \{1\}$. For each $k \in \{1, 2, \ldots, K-1\}$ and each $i \in N_\ell$, $f_i(e^k) = h_i$.

INDUCTION STEP. Let $K \in N_\ell \setminus \{1\}$.

► **Substep 2-1: For each $i \in \{K+1, K+2, \ldots, \ell+1\}$, $f_i(e^K) = h_i$.** This substep can be proved in the same way as the base step.

► **Substep 2-2: $f_1(e^K) = h_1$.** Suppose on the contrary that $f_1(e^K) \neq h_1$. Then, by the definition of $\succ_1$ (Step 1), $A(\succ_1, \omega_1^K = h_2) = \{h_1\}$. Thus, $f_1(e^K) \neq h_1$ and *individual rationality* together imply $f_1(e^K) = h_2$. Also, by the definition of $\succ_2$ (Step 1), $A(\succ_2, \omega_2^K = h_3) = \{h_2\}$. Thus, $f_2(e^K) \neq h_2$ and *individual rationality* together imply $f_2(e^K) = h_3$. By repeating this argument, for each $i \in \{1, 2, \ldots, K-1\}$, $f_i(e^K) = h_{i+1}$. In particular, $f_{K-1}(e^K) = h_K$. Note that by $K \geq 2$ and the definition of $\succ_K$ (Step 1), $A(\succ_K, \omega_K^K = h_1) = \{h_K, h_{K+1}, \ldots, h_{\ell+1}\}$. By $f_{K-1}(e^K) = h_K$ and Substep 2-1, for each $i \in \{K, K+1, \ldots, \ell+1\}$, $f_K(e^K) \neq h_i$. Thus, *individual rationality* implies that $f_K(e^K) = \omega_K^K = h_1$. Since $\omega^{K-1} = (\omega^K)^{K-1,K}$ and $(f_{K-1}(e^{K-1}), f_K(e^{K-1})) = (h_{K-1}, h_K)$ by the induction hypothesis,

$$f_{K-1}(e^{K-1}) = h_{K-1} \succ_{K-1} h_K = f_{K-1}(e^K)$$
$$f_K(e^{K-1}) = h_K \succ_K h_1 = f_K(e^K),$$

in violation of *endowments-swapping-proofness*.

▶ **Substep 2-3: For each $i \in \{2, 3, \ldots, K\}$, $f_i(e^K) = h_i$.** By the definition of $\succ_K$ (Step 1), $A(\succ_K, \omega_K^K = h_1) = \{h_K, h_{K+1}, \ldots, h_{\ell+1}\}$. It follows from Substeps 2-1 and 2-2 that for each $i \in \{1, K+1, \ldots, \ell+1\}$, $f_K(e^K) \neq h_i$. Thus, *individual rationality* implies $f_K(e^K) = h_K$. Additionally, by the definition of $\succ_{K-1}$ (Step 1), $A(\succ_{K-1}, \omega_{K-1}^K = h_K) = \{h_{K-1}\}$. Thus, $f_{K-1}(e^K) \neq h_K$ and *individual rationality* together implies $f_{K-1}(e^K) = h_{K-1}$. Continuing in the similar way, we see that for each $k \in \{2, 3, \ldots, K-2\}$, $f_k(e^K) = h_k$.

From Substeps 2-1, 2-2, and 2-3, we have that for each $i \in N_\ell$, $f_i(e^K) = h_i$.

Step 3: Concluding. Step 2 implies that for each $i \in N_\ell$, $f_i(e^{\ell+1}) = h_i$; that is, $f_1(e^{\ell+1}) = h_1 = \omega_{\ell+1}^{\ell+1}$ and for each $i \in N_\ell \setminus \{1\}$, $f_i(e^{\ell+1}) = h_i = \omega_{i-1}^{\ell+1}$. Since $(1, 2, \ldots, \ell+1)$ is a trading cycle at $(f(e^{\ell+1}), \omega^{\ell+1})$, $f(e^{\ell+1}) \notin X_\ell(\omega^{\ell+1})$, which is a contradiction. □

Now, we verify that the two properties in Theorem 2 are independent; that is, if any of the two properties in Theorem 2 is relaxed, there is a mechanism that satisfies the remaining property. The no-trade mechanism, which always assigns each agent his endowment, is an ℓ-feasible ($\ell \leq n-1$) mechanism that is *individually rational* but not *endowments-swapping-proof*. The following example illustrates a pairwise exchange mechanism that is *endowments-swapping-proof* but not *individually rational* in the three-agent case.

Example 1. Suppose $n = 3$. Let $\mathscr{P}_{23} \subset \mathscr{P}^N$ be the set of preference profiles represented as

$\succ_i$	$\succ_j$	$\succ_k$
h_1	h_2	h_3
h_2	$\vdots$	$\vdots$
h_3		

and $\mathscr{P}_{32} \subset \mathscr{P}^N$ the set of preference profiles represented as

$\succ_i$	$\succ_j$	$\succ_k$
h_1	h_2	h_3
h_3	$\vdots$	$\vdots$
h_2		

Note that $\mathscr{P}_{13} \cup \mathscr{P}_{23}$ is the set of preference profiles where the three agents have the different best objects. Consider the following mechanism f^{nir} defined by, for each $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$,

$$f^{\text{nir}}(e) = \begin{cases} \omega & \text{if } [\succ \in \mathscr{P}_{23} \text{ and } \omega_k = h_1] \text{ or } [\succ \in \mathscr{P}_{32} \text{ and } \omega_j = h_1] \\ \omega^{i,k} & \text{if } \succ \in \mathscr{P}_{23} \text{ and } \omega_i = h_1 \\ \omega^{j,k} & \text{if } [\succ \in \mathscr{P}_{23} \text{ and } \omega_j = h_1] \text{ or } [\succ \in \mathscr{P}_{32} \text{ and } \omega_k = h_1] \\ \omega^{i,j} & \text{if } \succ \in \mathscr{P}_{32} \text{ and } \omega_i = h_1 \\ TTC(e) & \text{otherwise.} \end{cases}$$

This mechanism is indeed a pairwise exchange mechanism.[10] One can easily verify that f^{nir} violates *individual rationality*. For the proof of *endowments-swapping-proofness* of this mechanism, see Online Appendix B. ■

4 Restricted domains

We have shown that when the size of trading cycles is limited, *individual rationality* and *endowments-swapping-proofness* are incompatible on the strict domain. However, these two properties might be compatible if one restricts the domain of strict preferences to a special class of preferences, considering *endowments-swapping-proofness* is weaker on smaller domains. Therefore, we examine whether

[10]To see this, let $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$. If $\succ \in \mathscr{P}_{23} \cup \mathscr{P}_{32}$, by $f^{\text{nir}}(e) \in \{\omega, \omega^{i,j}, \omega^{i,k}, \omega^{j,k}\}$, at most two agents exchange their endowments, that is, $f^{\text{nir}}(e) \in X_2(\omega)$. If $\succ \in \mathscr{P}^N \setminus (\mathscr{P}_{23} \cup \mathscr{P}_{32})$, then at least two agents' best objects are the same, and hence, the size of a cycle formed in Round 1 of the TTC algorithm is at most two. This implies that the size of a trading cycle at $(TTC(e), \omega)$ is at most two, that is, $f^{\text{nir}}(e)(= TTC(e)) \in X_2(\omega)$.

the two properties are compatible on a restricted domain. We consider three well-known restricted domains in the literature on the object reallocation problem: common ranking preferences (Nicolò and Rodríguez-Álvarez, 2017; Rodríguez-Álvarez, 2023), single-peaked preferences (Bade, 2019; Liu, 2022; Tamura, 2022; Tamura and Hosseini, 2022; Huang and Tian, 2023), and single-dipped preferences (Tamura, 2023).

4.1 Common ranking preferences

In this subsection, we first define the common ranking preferences. An agent who has common ranking preferences orders acceptable objects according to an exogenous ranking of objects that is common to all agents. Here, we consider the common ranking wherein objects are naturally ordered; that is, for each $\{j,k\} \subset N$ with $j < k$, object h_j is ranked higher than object h_k. Given $i \in N$ and $\omega_i \in H$, we say that agent i's preference relation $\succ_i \in \mathscr{P}$ is a **common ranking preference with respect to $\boldsymbol{\omega_i}$** if for each $\{h_j, h_k\} \subseteq A(\succ_i, \omega_i)$,

$$h_j \succ_i h_k \iff j < k.$$

Let $\mathscr{P}_{\omega_i} \subset \mathscr{P}$ be the set of common ranking preferences with respect to ω_i. Given $\omega \in X$, let $\mathscr{P}_\omega = \prod_{i=1}^n \mathscr{P}_{\omega_i}$. We denote the **common ranking domain** by $\mathscr{E}^{\text{cm}} = \bigcup_{\omega \in X} \{\mathscr{P}_\omega \times \{\omega\}\}$.

Nicolò and Rodríguez-Álvarez (2017) show that on the common ranking domain, no ℓ-feasible ($3 \leq \ell \leq n-1$) mechanism satisfies *individual rationality*, (*constrained*) *efficiency*, and *strategy-proofness*.[11] Additionally, a positive result emerges when only pairwise exchange is admitted: the "natural priority mechanism" is the only pairwise exchange mechanism that satisfies the three properties.[12] The following result is in sharp contrast with this Nicolò and Rodríguez-Álvarez's

[11] The notion of *efficiency* requires that no agent can be made better off without making someone else worse off. This notion is formally defined as follows: for each $e = (\succ, \omega) \in \mathscr{E}$, there is no $x \in X$ such that for each $i \in N$, $x_i \succsim_i f_i(e)$, and for some $j \in N$, $x_j \succ_j f_j(e)$. Similarly, the notion of *constrained efficiency*, which incorporates exchange constraints into the definition of *efficiency*, is formally defined as follows: for each $e = (\succ, \omega) \in \mathscr{E}$, there is no $x \in X_\ell(\omega)$ such that for each $i \in N$, $x_i \succsim_i f_i(e)$, and for some $j \in N$, $x_j \succ_j f_j(e)$.

[12] The natural priority mechanism allocates objects via an algorithm that prioritizes agents that own objects with lower index numbers. In the algorithm, we start with a set of *individually rational* pairwise assignments, and each agent sequentially refines the set of assignments to his best assignments according to priority ordering. See Nicolò and Rodríguez-Álvarez (2017) for a formal definition of this mechanism.

result, as a negative result holds even for pairwise exchanges if *efficiency* and *strategy-proofness* are replaced with *endowments-swapping-proofness*.

Theorem 3. *Let $\ell \in \{1, 2, \ldots, n-1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^{\mathrm{cm}}$ satisfies individual rationality and endowments-swapping-proofness.*

Outline of the proof. We can prove this theorem using the same argument as in Theorem 2. The only concern is whether the economy e^k ($k \in N_\ell$) constructed in the proof of Theorem 2 (Step 1) is in the common ranking domain, $\mathscr{E}^{\mathrm{cm}}$. In fact, we can show that for each $k \in N_\ell$, $e^k \in \mathscr{E}^{\mathrm{cm}}$. Therefore, the proof of Theorem 2 can be applied to the proof of this theorem. See Appendix A for a formal proof of Theorem 3. □

Lastly, we verify that the two properties in Theorem 3 are independent. The no-trade mechanism is an ℓ-feasible ($\ell \leq n-1$) mechanism that is *individually rational* but not *endowments-swapping-proof*. The restriction of f^{nir} to $\mathscr{E}^{\mathrm{cm}}$ is a pairwise exchange mechanism that is *endowments-swapping-proof* but not *individually rational* in the three-agent case.[13]

4.2 Single-peaked preferences

Here, we consider another restricted domain of preferences, called "single-peaked" preferences. Recently, object reallocation mechanisms on the single-peaked domain have gained wide attention (Bade, 2019; Liu, 2022; Tamura, 2022; Tamura and Hosseini, 2022; Huang and Tian, 2023). For the object reallocation problem with single-peaked preferences and no exchange constraints, while TTC satisfies *efficiency*, *individual rationality*, and *strategy-proofness*, there are many non-TTC mechanisms that satisfy these three properties, such as the "crawler" (Bade, 2019), the "neighborhood top trading cycles mechanisms" (Liu, 2022), and the "r-neighborhood mechanisms" (Huang and Tian, 2023). Moreover, in that setting, TTC still satisfies *endowments-swapping-proofness*.[14] However, these mechanisms are not ℓ-feasible ($\ell \leq n-1$). Therefore, we look for ℓ-feasible ($\ell \leq n-1$) mechanisms that satisfy *endowments-swapping-proofness* when each agent has single-peaked preferences.

[13] Given a mechanism f on $\mathscr{E}^{\mathrm{st}}$, a restriction of f to $\mathscr{E} \subset \mathscr{E}^{\mathrm{st}}$ is a mechanism on $\mathscr{E}$, $f|_{\mathscr{E}} \colon \mathscr{E} \to X$, such that for each $e \in \mathscr{E}$, $f|_{\mathscr{E}}(e) = f(e)$.

[14] The crawler surprisingly violates *endowments-swapping-proofness* when there are more than three agents. The detailed proof of this fact is available upon request. It is an interesting open question to identify which non-TTC mechanisms satisfy *endowments-swapping-proofness* when exchange constraints are not imposed.

We now describe a formal definition of single-peaked preferences. To do this, we need to consider a linear order $\lhd$ on H. Without loss of generality, assume that

$$h_1 \lhd h_2 \lhd \cdots \lhd h_n. \tag{2}$$

Given $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-peaked** (with respect to $\lhd$) if there is an object, $p(\succ_i) \in H$, such that

(i) for each $h \in H \setminus \{p(\succ_i)\}$, $p(\succ_i) \succ_i h$;

(ii) for each $\{h, h'\} \subseteq H \setminus \{p(\succ_i)\}$, if either $h' \lhd h \lhd p(\succ_i)$ or $p(\succ_i) \lhd h \lhd h'$, then $h \succ_i h'$.

We denote the set of single-peaked preferences by $\mathscr{S}_\wedge \subset \mathscr{P}$. Let $\mathscr{E}^\wedge = \mathscr{S}_\wedge^N \times X$ be the **single-peaked domain**.

When there are constraints on the length of an exchange cycle, we face a similar negative result even on the single-peaked domain. This negative result is somewhat surprising because there are many desirable mechanisms on the single-peaked domain in the absence of exchange constraints.

Theorem 4. *Let $\ell \in \{1, 2, \ldots, n-1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^\wedge$ satisfies individual rationality and endowments-swapping-proofness.*

Outline of the proof. The basic structure of the proof of this theorem is similar to that of Theorem 2. However, each economy e^k constructed in the proof of Theorem 2 (Step 1) is not in the single-peaked domain, and hence, we cannot directly apply the proof of Theorem 2. To overcome this difficulty, we construct new economies $\tilde{e}^k$ ($k \in N_\ell$) that are in the single-peaked domain, so that our methods in the proof of Theorem 2 can be used (with some modifications) to prove this theorem. See Appendix A for a formal proof of Theorem 4. □

Lastly, we verify the independence of properties of Theorem 4. The no-trade mechanism is an ℓ-feasible ($\ell \leq n-1$) mechanism that is *individually rational* but not *endowments-swapping-proof*. Furthermore, the restriction of f^{nir} to $\mathscr{E}^\wedge$ is a pairwise exchange mechanism that is *endowments-swapping-proof* but not *individually rational* in the three-agent case.

4.3 Single-dipped preferences

In this subsection, we consider "single-dipped" preferences, which are preferences defined on a fixed order of objects similar to single-peaked preferences considered in the previous subsection. Recently, Theorem 1 has been found to hold true for the single-dipped domain in the absence of exchange constraints (Tamura, 2023). As such, in contrast to the single-peaked domain, TTC is the central mechanism on the single-dipped domain. However, given the negative results achieved above, one might think that a similar negative result holds on the single-dipped domain once exchange constraints are imposed. Here, we will examine whether this conjecture is true.

We first describe a formal definition of single-dipped preferences. As in the previous subsection, we consider the linear order $\lhd$ on H such that (2) holds. Given $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped** (with respect to $\lhd$) if there is an object, $d(\succ_i) \in H$, such that

(i) for each $h \in H \setminus \{d(\succ_i)\}$, $h \succ_i d(\succ_i)$;

(ii) for each $\{h, h'\} \subseteq H \setminus \{d(\succ_i)\}$, if either $h' \lhd h \lhd d(\succ_i)$ or $d(\succ_i) \lhd h \lhd h'$, then $h' \succ_i h$.

We denote the set of single-dipped preferences by $\mathscr{S}_\vee \subset \mathscr{P}$. We call $\mathscr{E}^\vee = \mathscr{S}_\vee^N \times X$ the **single-dipped domain**.

Interestingly, TTC on the single-dipped domain is a pairwise exchange mechanism. Before we provide a proof of this fact, we introduce some notation that will be useful. Fix any economy $e = (\succ, \omega) \in \mathscr{E}^\vee$ and any integer $t \geq 1$. We write $\mathbb{S}_t(e) \subset 2^N$ for the set of groups of agents that form trading cycles in Round t of the TTC algorithm at e. We denote the set of agents who are assigned objects in Round t of the TTC algorithm by

$$N_t(e) = \bigcup_{S \in \mathbb{S}_t(e)} \{S\}.$$

We denote the set of objects that are assigned to agents in Round t of the TTC algorithm by

$$H_t(e) = \{h \in H \colon \exists\, i \in N_t(e), h = \omega_i\}.$$

Define $N^t(e)$ and $H^t(e)$ as follows:

$$N^t(e) = \bigcup_{z=1}^{t} N_z(e) \quad \text{and} \quad H^t(e) = \bigcup_{z=1}^{t} H_z(e).$$

For the sake of convenience, let $N^0(e) = H^0(e) = \emptyset$. With a slight abuse of notation, each $S \in \mathbb{S}_t(e)$ also represents a trading cycle, that is, "$S = \{i_1(=$

$i_{K+1}), i_2, \ldots, i_K\} \in \mathbb{S}_t(e)$" means that (i) for each $k \in \{1, 2, \ldots, K\}$, $i_k \in N \setminus N^{t-1}(e)$ and $\omega_{i_k} \in H \setminus H^{t-1}(e)$, and (ii) for each $h \in H \setminus (H^{t-1}(e) \cup \{\omega_{i_{k+1}}\})$, $\omega_{i_{k+1}} \succ_{i_k} h$. We denote by $\underline{i}(e,t)$ (resp. $\bar{i}(e,t)$) the agent whose endowment has the lowest (resp. highest) index among the set of remaining objects in Round t of the TTC algorithm at e. That is,

- $\omega_{\underline{i}(e,t)} = h_{\underline{t}} \in H \setminus H^{t-1}(e)$ and for each $h_m \in H \setminus (H^{t-1}(e) \cup \{h_{\underline{t}}\})$, $\underline{t} < m$.
- $\omega_{\bar{i}(e,t)} = h_{\bar{t}} \in H \setminus H^{t-1}(e)$ and for each $h_m \in H \setminus (H^{t-1}(e) \cup \{h_{\bar{t}}\})$, $\bar{t} > m$.

Note that for each $e = (\succ, \omega) \in \mathscr{E}^{\vee}$, $\omega_{\underline{i}(e,1)} = h_1$ and $\omega_{\bar{i}(e,1)} = h_n$

To familiarize ourselves with these definitions, Figure 1 uses them to illustrate how the TTC algorithm works for an economy with five agents whose preferences are single-dipped. We see from Figure 1 that the TTC algorithm on the single-dipped domain involves self-pointing trading cycles and pairwise trading cycles in each round. This is because in each round of the TTC algorithm, best objects always appear at both or one of the two ends of the "remaining" line, and hence, only one or both of the agents at the two ends form a trading cycle. The following result tells us that these observations hold for any number of agents.

Proposition 1. *For each $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ and each integer $t \geq 1$,*

$$\mathbb{S}_t(e) \in \Big\{ \{\{\underline{i}(e,t), \bar{i}(e,t)\}\}, \{\{\underline{i}(e,t)\}, \{\bar{i}(e,t)\}\}, \{\{\underline{i}(e,t)\}\}, \{\{\bar{i}(e,t)\}\} \Big\}.$$

Proof. Let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ and $t \geq 1$. We write $\underline{i}(t)$ (resp. $\bar{i}(t)$) for $\underline{i}(e,t)$ (resp. $\bar{i}(e,t)$). Then, we show that $N_t(e) \subseteq \{\underline{i}(t), \bar{i}(t)\}$. We consider Round t of the TTC algorithm. Let $i \in N \setminus N^{t-1}(e)$. By $\succ_i \in \mathscr{S}_{\vee}$ and the definitions of $\underline{i}(t)$ and $\bar{i}(t)$, we have either

(a) for each $h \in H \Big\backslash \Big(H^{t-1}(e) \cup \{\omega_{\underline{i}(t)}\}\Big)$, $\omega_{\underline{i}(t)} \succ_i h$, or

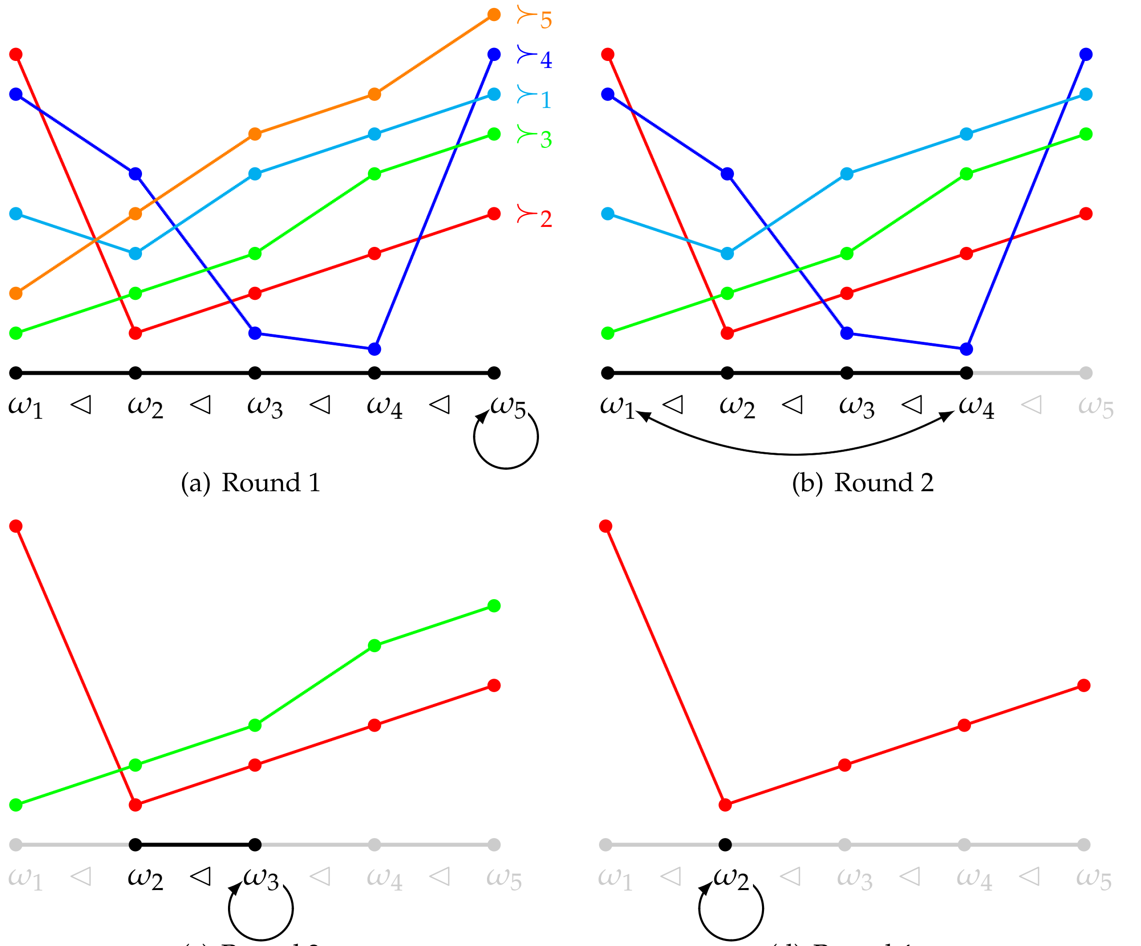
(b) for each $h \in H \Big\backslash \Big(H^{t-1}(e) \cup \{\omega_{\bar{i}(t)}\}\Big)$, $\omega_{\bar{i}(t)} \succ_i h$.

Thus,

$$\mathbb{S}_t(e) \in \Big\{ \{\{\underline{i}(t), \bar{i}(t)\}\}, \{\{\underline{i}(t)\}, \{\bar{i}(t)\}\}, \{\{\underline{i}(t)\}\}, \{\{\bar{i}(t)\}\} \Big\},$$

the desired conclusion. □

Proposition 1 immediately implies that TTC on the single-dipped domain is a pairwise exchange mechanism.



(a) Round 1 (b) Round 2

(c) Round 3 (d) Round 4

Figure 1: An illustration of the TTC algorithm on the single-dipped domain. Let $n = 5$. Consider economy $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ where $\succ$ is depicted in Panel (a) and for each $i \in N$, $\omega_i = h_i$. Panel (a) represents Round 1 of the TTC algorithm. Then, $\underline{i}(e,1) = 1$ and $\bar{i}(e,1) = 5$. In this round, agents 1, 3, 4, and 5 point to agent $\bar{i}(e,1) = 5$, and agent 2 points to agent $\underline{i}(e,1) = 1$. Then, there is only one cycle: the self-pointing cycle of agent 5. Therefore, $S_1(e) = \{\{\bar{i}(e,1)\}\} = \{\{5\}\}$. Then, agent 5 is assigned object ω_5 and removed from the economy. Panel (b) represents Round 2 of the TTC algorithm. Then, $\underline{i}(e,2) = 1$ and $\bar{i}(e,2) = 4$. In this round, agents 1 and 3 point to agent $\bar{i}(e,2) = 4$, and agents 2 and 4 point to agent $\underline{i}(e,1) = 1$. Then, there is only one trading cycle $(\underline{i}(e,2), \bar{i}(e,2)) = (1,4)$ and hence, $S_2(e) = \{\{\underline{i}(e,2), \bar{i}(e,2)\}\} = \{\{1,4\}\}$. Then, agents 1 and 4 are assigned object ω_4 and object ω_1, respectively, and removed from the economy. Panel (c) represents Round 3 of the TTC algorithm. Then, $\underline{i}(e,3) = 2$ and $\bar{i}(e,3) = 3$. In this round, agents 2 and 3 point to agent $\bar{i}(e,2) = 3$. Then, there is only one cycle: the self-pointing cycle of agent 3. Therefore, $S_3(e) = \{\{\bar{i}(e,3)\}\} = \{\{3\}\}$. Then, agent 3 is assigned object ω_3 and removed from the economy. Panel (d) represents Round 4 of the TTC algorithm. Then, $\underline{i}(e,4) = \bar{i}(e,4) = 2$. In this round, agent 2 points to himself and thus, $S_4(e) = \{\{\underline{i}(e,4)(= \bar{i}(e,4))\}\} = \{\{2\}\}$. Then, agent 2 is assigned object ω_2 and removed from the economy. The algorithm terminates at this round.

Corollary 1. *TTC on $\mathscr{E}^\vee$ is a pairwise exchange mechanism.*

When we focus on pairwise exchanges, we can characterize TTC by dropping *strategy-proofness* from the list of axioms in Tamura's characterization.

Theorem 5. *A pairwise exchange mechanism on $\mathscr{E}^\vee$ satisfies individual rationality and endowments-swapping-proofness if and only if it is TTC.*

Outline of the proof. The "if" part immediately follows from Tamura (2023). Thus, we here provide an outline of the proof of the "only if" part. Suppose a pairwise exchange mechanism f satisfies the two properties. Let $e = (\succ, \omega) \in \mathscr{E}^\vee$. We show by induction that for each $t \geq 1$ and each $i \in N_t(e)$, $f_i(e) = TTC_i(e)$. Let $S \in \mathbb{S}_t(e)$. By Corollary 1, we have either $|S| = 1$ or $|S| = 2$. The desired conclusion simply follows from *individual rationality* in the case $|S| = 1$. In the case $|S| = 2$, that is, $S = \{\{\underline{i}(e,t), \bar{i}(e,t)\}\}$, 2-feasibility of f plays a crucial role. Suppose on the contrary that $f_{\underline{i}(e,t)}(e) \neq \omega_{\bar{i}(e,t)}(= TTC_{\underline{i}(e,t)}(e))$. Then, the 2-feasibility implies $f_{\bar{i}(e,t)}(e) \neq \omega_{\underline{i}(e,t)}(= TTC_{\bar{i}(e,t)}(e))$, thereby enabling the two agents to benefit from swapping their endowments. This contradicts *endowments-swapping-proofness* of f. See Appendix A for a formal proof of Theorem 5. □

We now verify the independence of properties listed in Theorem 5. The no-trade mechanism is an ℓ-feasible ($\ell \leq n-1$) mechanism that is *individually rational* but not *endowments-swapping-proof*. The following example illustrates a pairwise exchange mechanism that is *endowments-swapping-proof* but not *individually rational* in the three-agent case.

Example 2. Let $n = 3$ and let $\ddot{e} = (\ddot{\succ}, \ddot{\omega}) \in \mathscr{E}^\vee$ be such that

$\ddot{\succ}_1$	$\ddot{\succ}_2$	$\ddot{\succ}_3$
h_1	h_1	h_1
h_2	h_2	h_2
h_3	h_3	h_3

and $\ddot{\omega} = (h_1, h_3, h_2)$. Consider the following pairwise exchange mechanism $\ddot{f}$ defined by, for each $e = (\succ, \omega) \in \mathscr{E}^\vee$,

$$\ddot{f}(e) = \begin{cases} (h_1, h_2, h_3) & \text{if } e = \ddot{e} \\ TTC(e) & \text{otherwise.} \end{cases}$$

Note that $TTC(\ddot{e}) = (h_1, h_3, h_2) \neq \ddot{f}(\ddot{e})$. One can easily verify that $\ddot{f}$ is not *individually rational*. For the proof of *endowments-swapping-proofness* of this mechanism, see Online Appendix B. ∎

It is worth mentioning that Theorem 5 no longer holds if we consider the size of exchanges larger than pairwise exchanges. That is, we can construct a non-TTC mechanism that satisfies *individual rationality* and *endowments-swapping-proofness*. The following example illustrates such a mechanism.

Example 3. Let $n = 4$ and $e' = (\succ', \omega') \in \mathscr{E}^{\vee}$ be such that

$\succ'_1$	$\succ'_2$	$\succ'_3$	$\succ'_4$
h_4	h_1	h_1	h_1
h_3	h_2	h_2	h_2
h_2	h_3	h_3	h_3
h_1	h_4	h_4	h_4

and $\omega' = (h_1, h_2, h_3, h_4)$. Consider the following 3-feasible mechanism $f^{\vee}$ defined by, for each $e \in \mathscr{E}^{\vee}$,

$$f^{\vee}(e) = \begin{cases} (h_4, h_1, h_3, h_2) & \text{if } e = e' \\ TTC(e) & \text{otherwise} \end{cases}$$

Note that $TTC(e') = (h_4, h_2, h_3, h_1) \neq f^{\vee}(e')$. It is easy to see that $f^{\vee}$ is *individually rational*. For the proof of *endowments-swapping-proofness* of this mechanism, see Online Appendix B. ∎

From Example 3, one might think that *strategy-proofness* is indispensable for characterizing TTC using *individual rationality* and *endowments-swapping-proofness* for the general exchange constraints, as in Tamura (2023). The following theorem shows that this is indeed the case and it follows from Tamura (2023) and Corollary 1.

Theorem 6. *Let $\ell \in \{3, 4, \ldots, n-1\}$. An ℓ-feasible exchange mechanism on $\mathscr{E}^{\vee}$ satisfies individual rationality, strategy-proofness, and endowments-swapping-proofness if and only if it is TTC.*

Proof. We know from Corollary 1 that TTC is a pairwise exchange mechanism, and hence, is ℓ-feasible. Then, the "if" part follows from the fact that TTC satisfies the three properties. Also, we know from Tamura (2023) that TTC is the only mechanism on $\mathscr{E}^{\vee}$ that satisfies the three properties without exchange constraints. Therefore, if an ℓ-feasible mechanism f on $\mathscr{E}^{\vee}$ satisfies the three properties, $f = TTC$, which completes the proof of the "only if" part. $\square$

Remark 2. Single-dipped preferences considered in this section are often called single-dipped preferences on a "line." We can define single-dipped preferences to a more general structure called a "tree." These generalized single-dipped preferences are often called single-dipped preferences on a tree. Without constraints on the size of exchanges, the characterization of TTC holds on the domain of single-dipped preferences on a tree (Tamura, 2023). However, unlike when dealing with a line, Theorem 5 no longer holds on the domain of single-dipped preferences on a tree if exchange constraints are stringent. We discuss it in detail in Online Appendix C. $\lozenge$

Remark 3. TTC violates a strict version of *endowments-swapping-proofness*, called *strict endowments-swapping-proofness*, even on the single-dipped domain. The notion of *strict endowments-swapping-proofness* is formally defined as follows: there are no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset N$ such that (i) $e^{i,j} \in \mathscr{E}$, and (ii) $f_i(e^{i,j}) \succsim_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$. To see that TTC on the single-dipped domain violates *strict endowments-swapping-proofness*, let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ be such that

$\succ_1$	$\succ_{i \geq 2}$
h_n	h_1
h_{n-1}	h_2
$\vdots$	$\vdots$
h_2	h_{n-1}
h_1	h_n

and $\omega = (h_1, h_2, \ldots, h_n)$. Then, $TTC(e) = (h_n, h_2, h_3, \ldots, h_{n-1}, h_1)$ and $TTC(e^{1,2}) = (h_n, h_1, h_3, \ldots, h_{n-1}, h_2)$. Hence, $e^{1,2} \in \mathscr{E}^{\vee}$, and

$$TTC_1(e^{1,2}) = h_n = TTC_1(e);$$
$$TTC_2(e^{1,2}) = h_1 \succ_2 h_2 = TTC_2(e),$$

in violation of *strict endowments-swapping-proofness*. $\lozenge$

5 Concluding remarks

This paper explored *endowments-swapping-proof* mechanisms in object reallocation problems (also known as housing markets) that incorporate constraints on trading cycles. We found that when preferences are strict, the introduction of exchange constraints renders *individual rationality* and *endowments-swapping-proofness* incompatible. Unfortunately, this incompatibility persists even when preferences are restricted to either common ranking preferences or single-peaked preferences. However, we established a positive result for single-dipped preferences: when preferences are single-dipped, TTC emerges as the only pairwise exchange mechanism satisfying the two properties. It stands as the only ℓ-feasible ($\ell \geq 3$) exchange mechanism that satisfies *strategy-proofness* in addition to the two properties.

We conclude our discussion by mentioning two potential extensions of the model. First, our setting does not accommodate the possibility of agents having indifferences. Nicolò and Rodríguez-Álvarez (2017) and Rodríguez-Álvarez (2023) extend the common ranking domain to domains where agents' preferences may be weak. They term these "age-based domains" and propose a pairwise exchange mechanism that satisfies *individual rationality*, (*constrained*) *efficiency*, and *strategy-proofness*. It remains an open question whether there is an exchange-constrained mechanism satisfying *individual rationality* and *endowments-swapping-proofness* on age-based domains.

Second, this paper does not delve into probabilistic mechanisms. Recently, Balbuzanov (2020) succeeds in identifying an *efficient* and "anonymous" pairwise exchange mechanism on the strict domain by introducing randomness, whereas no deterministic mechanism satisfies both properties. However, he demonstrates that under certain mild conditions, no exchange-constrained mechanism on the strict domain satisfies *individual rationality*, *efficiency*, and *strategy-proofness* even when randomness is allowed. Hence, it remains an open question whether there exists an exchange-constrained probabilistic mechanism on the strict domain that meets the criteria of *individual rationality* and *endowments-swapping-proofness*.

A Appendix: Omitted proofs

A.1 Proof of Theorem 3

Suppose on the contrary that there is an ℓ-feasible mechanism f on $\mathscr{E}^{\text{cm}}$ satisfying the two properties. Let $N_\ell = \{1, 2, \ldots, \ell+1\}$. We derive a contradiction. The proof consists of four steps.

Step 1: Constructing economies. We construct the same economies $e^k = (\succ, \omega^k)$ as those in the proof of Theorem 2.

Step 2: For each $k \in N_\ell$, $e^k \in \mathscr{E}^{\text{cm}}$. Let $i \in N$. There are four cases.

- **Case 1: $i \leq k-1$.** Then, by the definition of $\succ_i$ (Step 1), $A(\succ_i, \omega_i^k = h_{i+1}) = \{h_i\}$. Hence, $\succ_i \in \mathscr{P}_{\omega_i^k}$.
- **Case 2: $i = k$.** Then, by the definition of $\succ_k$ (Step 1), $A(\succ_k, \omega_k^k = h_1) = \{h_k, h_{k+1}, \ldots, h_{\ell+1}\}$.[15] Recall here Condition (ii) of $\succ_k$ defined in the proof of Theorem 2; that is, for each $\{j, j'\} \subset N$, if $j < j'$ and $\{j, j'\} \subseteq \{k, k+1, \ldots, \ell+1\}$, then $h_j \succ_k h_{j'}$. This yields $\succ_k \in \mathscr{P}_{\omega_k^k}$.
- **Case 3: $k+1 \leq i \leq \ell+1$.** Then, by the definition of $\succ_i$ (Step 1), $A(\succ_i, \omega_i^k = h_i) = \emptyset$. Hence, $\succ_i \in \mathscr{P}_{\omega_i^k}$.
- **Case 4: $i \geq \ell+2$.** Then, by the definition of $\succ_i$ (Step 1), $A(\succ_i, \omega_i^k = h_i) = \{h_1, h_2, \ldots, h_{i-1}\}$. Recall here Condition (iii) of $\succ_i$ defined in the proof of Theorem 2; that is, for each $\{j, j'\} \subset N$, if $j < j'$, then $h_j \succ_k h_{j'}$. This yields $\succ_i \in \mathscr{P}_{\omega_i^k}$.

From Cases 1–4, $\succ \in \mathscr{P}_{\omega^k}$, which implies $e^k \in \mathscr{E}^{\text{cm}}$.

Step 3: For each $k \in N_\ell$ and each $i \in N_\ell$, $f_i(e^k) = h_i$. By Step 2, for each $k \in N_\ell$, $e^k \in \mathscr{E}^{\text{cm}}$, and thus, $f(e^k)$ is well-defined. By the similar argument in the proof of Theorem 2, we obtain the desired conclusion.

Step 4: Concluding. Step 3 implies that for each $i \in N_\ell$, $f_i(e^{\ell+1}) = h_i$; that is, $f_1(e^{\ell+1}) = h_1 = \omega_{\ell+1}^{\ell+1}$ and for each $i \in N_\ell \setminus \{1\}$, $f_i(e^{\ell+1}) = h_i = \omega_{i-1}^{\ell+1}$. Since $(1, 2, \ldots, \ell+1)$ is a trading cycle at $(f(e^{\ell+1}), \omega^{\ell+1})$, $f(e^{\ell+1}) \notin X_\ell(\omega^{\ell+1})$, which is a contradiction. $\square$

[15] Note that when $k = i = 1$, $A(\succ_1, \omega_1^1 = h_1) = \emptyset$.

A.2 Proof of Theorem 4

Suppose on the contrary that there is an ℓ-feasible mechanism f on $\mathscr{E}^{\wedge}$ satisfying the two properties. Let $N_\ell = \{1, 2, \ldots, \ell+1\}$.[16] We derive a contradiction. The proof consists of three steps.

Step 1: Constructing economies. Let $\widehat{\succ} \in \mathscr{S}_{\wedge}^N$ be such that:

(i) for each $i \in N_\ell$, each $j \in \{i, i+1, \ldots, \ell+1\}$, each $j' \in \{1, 2, \ldots, i-1\}$, and each $j'' \in N \setminus N_\ell$, $h_j \mathrel{\widehat{\succ}_i} h_{j'} \mathrel{\widehat{\succ}_i} h_{j''}$;

(ii) for each $i \in N_\ell$ and each $\{j, j'\} \subset N$, if either $[j < j'$ and $\{j, j'\} \subseteq \{i, i+1, \ldots, n\}]$ or $[j > j'$ and $\{j, j'\} \subseteq \{1, 2, \ldots, i-1\}]$, then $h_j \mathrel{\widehat{\succ}_i} h_{j'}$;

(iii) for each $i \in N \setminus N_\ell$ and each $\{j, j'\} \subset N$, if $j < j'$, then $h_j \mathrel{\widehat{\succ}_i} h_{j'}$;

The preference profile $\widehat{\succ}$ can be represented as follows:

$\widehat{\succ}_1$	$\widehat{\succ}_2$	$\widehat{\succ}_3$	$\cdots$	$\widehat{\succ}_{\ell-1}$	$\widehat{\succ}_\ell$	$\widehat{\succ}_{\ell+1}$	$\widehat{\succ}_{j\geq\ell+2}$
h_1	h_2	h_3	$\cdots$	$h_{\ell-1}$	h_ℓ	$h_{\ell+1}$	h_1
h_2	h_3	h_4	$\cdots$	h_ℓ	$h_{\ell+1}$	h_ℓ	h_2
h_3	h_4	h_5	$\cdots$	$h_{\ell+1}$	$h_{\ell-1}$	$h_{\ell-1}$	h_3
h_4	h_5	h_6	$\cdots$	$h_{\ell-2}$	$h_{\ell-2}$	$h_{\ell-2}$	h_4
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
$h_{\ell-1}$	h_ℓ	$h_{\ell+1}$	$\cdots$	h_3	h_3	h_3	$h_{\ell-1}$
h_ℓ	$h_{\ell+1}$	h_2	$\cdots$	h_2	h_2	h_2	h_ℓ
$h_{\ell+1}$	h_1	h_1	$\cdots$	h_1	h_1	h_1	$h_{\ell+1}$
$h_{\ell+2}$	$h_{\ell+2}$	$h_{\ell+2}$	$\cdots$	$h_{\ell+2}$	$h_{\ell+2}$	$h_{\ell+2}$	$h_{\ell+2}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$
h_n	h_n	h_n	$\cdots$	h_n	h_n	h_n	h_n

For each $k \in N_\ell$, consider the same endowments ω^k as those in the proof of Theorem 2. That is, ω^k is defined by (1). For each $k \in N_\ell$, let $\widehat{e}^k = (\widehat{\succ}, \omega^k)$. Note that for each $k \in N_\ell$, $\widehat{e}^k \in \mathscr{E}^{\wedge}$.

Step 2: For each $k \in N_\ell$ and each $i \in N_\ell$, $f_i(\widehat{e}^k) = h_i$. We use the induction to prove this step.

Base step. Let $K = 1$ and $i \in N_\ell$. Then, by the definition of $\widehat{\succ}_i$ (Step 1), $A(\widehat{\succ}_i, \omega_i^1 = h_i) = \emptyset$. Thus, *individual rationality* implies $f_i(\widehat{e}^1) = h_i$.

[16] Since $\ell \leq n-1$, agent $\ell+1$ exists.

Induction hypothesis. Let $K \in N_\ell \setminus \{1\}$. For each $k \in \{1, 2, \ldots, K-1\}$ and each $i \in N_\ell$, $f_i(\widehat{e}^k) = h_i$.

Induction step. Let $K \in N_\ell \setminus \{1\}$.

▶ **Substep 2-1: For each $i \in \{K+1, K+2, \ldots, \ell+1\}$, $f_i(\widehat{e}^K) = h_i$.** This substep can be proved in the same way as the base step.

▶ **Substep 2-2: For each $i \in \{1, 2, \ldots, K\}$, $f_i(\widehat{e}^K) = h_i$.** Suppose on the contrary that there is $k \in \{1, 2, \ldots, K\}$ such that $f_k(\widehat{e}^K) \neq h_k$. Without loss of generality, we assume that

$$\forall\, i \in \{1, 2, \ldots, k-1\},\ f_i(\widehat{e}^K) = h_i. \tag{3}$$

There are two cases.

• **Case 1: $k \leq K-1$.** By $k \leq K-1 \leq \ell$ and the definition of $\widehat{\succ}_k$ (Step 1), $A(\widehat{\succ}_k, \omega_k^K = h_{k+1}) = \{h_k\}$. Thus, $f_k(\widehat{e}^K) \neq h_k$ and *individual rationality* together imply $f_k(\widehat{e}^K) = h_{k+1}$. Also, by the definition of $\widehat{\succ}_{k+1}$ (Step 1), $A(\widehat{\succ}_{k+1}, \omega_{k+1}^K = h_{k+2}) = \{h_{k+1}\}$. Thus, $f_{k+1}(\widehat{e}^K) \neq h_{k+1}$ and *individual rationality* together implies $f_{k+1}(\widehat{e}^K) = h_{k+2}$. By repeating this argument, we finally obtain that

$$\forall\, i \in \{k, k+1, \ldots, K-1\},\ f_i(\widehat{e}^K) = h_{i+1}. \tag{4}$$

In particular, $f_{K-1}(\widehat{e}^K) = h_K$. Note that by $K \geq 2$ and the definition of $\succ_K$ (Step 1),

$$A(\succ_K, \omega_K^K = h_1) = N_\ell \setminus \{h_1\}. \tag{5}$$

By Substep 2-1, (3), and (4),

$$\forall\, i \in N_\ell \setminus \{k\},\ f_K(\widehat{e}^K) \neq h_i. \tag{6}$$

By (5) and (6), *individual rationality* implies $f_K(\widehat{e}^K) = h_k$. Since $\omega^{K-1} = (\omega^K)^{K-1,K}$ and $(f_{K-1}(\widehat{e}^{K-1}), f_K(\widehat{e}^{K-1})) = (h_{K-1}, h_K)$ by the induction hypothesis,

$$f_{K-1}(\widehat{e}^{K-1}) = h_{K-1}\ \widehat{\succ}_{K-1}\ h_K = f_{K-1}(\widehat{e}^K)$$
$$f_K(\widehat{e}^{K-1}) = h_K\ \widehat{\succ}_K\ h_k = f_K(\widehat{e}^K),$$

in violation of *endowments-swapping-proofness*.

• **Case 2: $k = K$.** Then, by $K \geq 2$ and the definition of $\succ_K$ (Step 1),

$$A(\succ_K, \omega_K^K = h_1) = N_\ell \setminus \{h_1\}. \tag{7}$$

By Substep 2-1 and (3),

$$\forall\, i \in N_\ell \setminus \{k = K\},\ f_K(\hat{e}^K) \neq h_i. \tag{8}$$

By (7) and (8), *individual rationality* implies that $f_K(\hat{e}^K) = h_K$, which is a contradiction.

From Substep 2-1 and 2-2, we have that for each $i \in N_\ell$, $f_i(\hat{e}^K) = h_i$.

Step 3: Concluding. Step 2 implies that for each $i \in N_\ell$, $f_i(\hat{e}^{\ell+1}) = h_i$; that is, $f_1(\hat{e}^{\ell+1}) = h_1 = \omega_{\ell+1}^{\ell+1}$ and for each $i \in N_\ell \setminus \{1\}$, $f_i(\hat{e}^{\ell+1}) = h_i = \omega_{i-1}^{\ell+1}$. Since $(1, 2, \ldots, \ell+1)$ is a trading cycle at $(f(\hat{e}^{\ell+1}), \omega^{\ell+1})$, $f(\hat{e}^{\ell+1}) \notin X_\ell(\omega^{\ell+1})$, which is a contradiction. $\square$

A.3 Proof of Theorem 5

The "if" part immediately follows from Tamura (2023) because the size of cycles formed in the TTC algorithm is either one or two even without exchange constraints. Thus, it suffices to show the "only if" part.

We now prove that for each $e = (\succ, \omega) \in \mathscr{E}^\vee$, each integer $t \geq 1$ and each $i \in N_t(e)$, $f_i(e) = TTC_i(e)$. Let $e = (\succ, \omega) \in \mathscr{E}^\vee$. For each integer $t \geq 1$, we simply write $\underline{i}(t)$ (resp. $\bar{i}(t)$) for $\underline{i}(e,t)$ (resp. $\bar{i}(e,t)$). We use induction on t.

Base step. Let $t = 1$. Let $S \in \mathbb{S}_1(e)$. By Proposition 1, we know that

$$S \in \{\{\underline{i}(1), \bar{i}(1)\}, \{\underline{i}(1)\}, \{\bar{i}(1)\}\}.$$

There are two cases.

- **Case 1: $S \in \{\{\underline{i}(1)\}, \{\bar{i}(1)\}\}$.** Without loss of generality, we assume $S = \{\underline{i}(1)\}$. Then, $A(\succ_{\underline{i}(1)}, \omega_{\underline{i}(1)}) = \emptyset$. Hence, *individual rationality* implies $f_{\underline{i}(1)}(e) = \omega_{\underline{i}(1)} = TTC_{\underline{i}(1)}(e)$.
- **Case 2: $S = \{\underline{i}(1), \bar{i}(1)\}$.** Then, $(\succ_{\underline{i}(1)}, \succ_{\bar{i}(1)})$ is represented as

$$\begin{array}{cc} \succ_{\underline{i}(1)} & \succ_{\bar{i}(1)} \\ \hline \omega_{\bar{i}(1)} & \omega_{\underline{i}(1)} \\ \vdots & \vdots \end{array}$$

Suppose on the contrary that

$$(f_{\underline{i}(1)}(e), f_{\bar{i}(1)}(e)) \neq (TTC_{\underline{i}(1)}(e), TTC_{\bar{i}(1)}(e)) = (\omega_{\bar{i}(1)}, \omega_{\underline{i}(1)}).$$

Without loss of generality, we assume $f_{\underline{i}(1)}(e) \neq \omega_{\bar{i}(1)}$. Since f is a pairwise exchange mechanism, $f_{\bar{i}(1)}(e) \neq \omega_{\underline{i}(1)}$. Consider $e^{\underline{i}(1),\bar{i}(1)}$. Then, $e^{\underline{i}(1),\bar{i}(1)} \in \mathscr{E}^{\vee}$, and by *individual rationality*,

$$f_{\underline{i}(1)}(e^{\underline{i}(1),\bar{i}(1)}) = \omega_{\underline{i}(1)}^{\underline{i}(1),\bar{i}(1)} = \omega_{\bar{i}(1)} \succ_{\underline{i}(1)} f_{\underline{i}(1)}(e);$$
$$f_{\bar{i}(1)}(e^{\underline{i}(1),\bar{i}(1)}) = \omega_{\bar{i}(1)}^{\underline{i}(1),\bar{i}(1)} = \omega_{\underline{i}(1)} \succ_{\bar{i}(1)} f_{\bar{i}(1)}(e),$$

in violation of *endowments-swapping-proofness*.

From Cases 1 and 2, we have that for each $i \in N_1(e)$, $f_i(e) = TTC_i(e)$.

INDUCTION HYPOTHESIS. For each $t \in \{1, 2, \ldots, r-1\}$ and each $i \in N_t(e)$, $f_i(e) = TTC_i(e)$.

INDUCTION STEP. Let $t = r$. By the induction hypothesis,

$$H^{r-1}(e) = \left\{ h \in H \colon \exists\, i \in N^{r-1}(e), h = f_i(e) \right\}. \tag{9}$$

Let $S \in \mathbb{S}_r(e)$. By Proposition 1, we know that

$$S \in \left\{ \{\underline{i}(r), \bar{i}(r)\}, \{\underline{i}(r)\}, \{\bar{i}(r)\} \right\}.$$

To simplify notation, let $\underline{i} = \underline{i}(r)$ and $\bar{i} = \bar{i}(r)$. There are two cases.

- **Case 1:** $S \in \{\{\underline{i}\}, \{\bar{i}\}\}$. Without loss of generality, suppose $S = \{\underline{i}\}$. Then, agent $\{\underline{i}\}$ forms a self-pointing cycle in Round r of the TTC algorithm at e. Thus, for each $h \in H \setminus (H^{r-1}(e) \cup \{\omega_{\underline{i}}\})$, $\omega_{\underline{i}} \succ_{\underline{i}} h$. By (9), $f_{\underline{i}}(e) \in H \setminus H^{r-1}(e)$. Hence, by *individual rationality*, $f_{\underline{i}}(e) = \omega_{\underline{i}} = TTC_{\underline{i}}(e)$.
- **Case 2:** $S = \{\underline{i}, \bar{i}\}$. Then, agents $\{\underline{i}, \bar{i}\}$ form a cycle in Round r of the TTC algorithm at e. Thus,

$$\forall\, h \in H \Big\backslash \left(H^{r-1}(e) \cup \{\omega_{\bar{i}}\} \right), \ \omega_{\bar{i}} \succ_{\underline{i}} h; \tag{10}$$
$$\forall\, h \in H \Big\backslash \left(H^{r-1}(e) \cup \{\omega_{\underline{i}}\} \right), \ \omega_{\underline{i}} \succ_{\bar{i}} h. \tag{11}$$

Suppose on the contrary that

$$(f_{\underline{i}}(e), f_{\bar{i}}(e)) \neq (TTC_{\underline{i}}(e), TTC_{\bar{i}}(e)) = (\omega_{\bar{i}}, \omega_{\underline{i}}).$$

Without loss of generality, suppose $f_{\underline{i}}(e) \neq \omega_{\bar{i}}$. Since f is a pairwise exchange mechanism, $f_{\bar{i}}(e) \neq \omega_{\underline{i}}$. By (9), $\{f_{\underline{i}}(e), f_{\bar{i}}(e)\} \subset H \setminus H^{r-1}(e)$. Thus, by (10) and (11)

$$\omega_{\bar{i}} \succ_{\underline{i}} f_{\underline{i}}(e) \quad \text{and} \quad \omega_{\underline{i}} \succ_{\bar{i}} f_{\bar{i}}(e).$$

Consider $e^{\underline{i},\bar{i}}$. Then, $e^{\underline{i},\bar{i}} \in \mathscr{E}^{\vee}$, and by *individual rationality*,

$$\begin{aligned} f_{\underline{i}}(e^{\underline{i},\bar{i}}) &\succsim_{\underline{i}} \omega_{\underline{i}}^{\underline{i},\bar{i}} = \omega_{\bar{i}} \succ_{\underline{i}} f_{\underline{i}}(e); \\ f_{\bar{i}}(e^{\underline{i},\bar{i}}) &\succsim_{\bar{i}} \omega_{\bar{i}}^{\underline{i},\bar{i}} = \omega_{\underline{i}} \succ_{\bar{i}} f_{\bar{i}}(e), \end{aligned}$$

in violation of *endowments-swapping-proofness*.

From Cases 1 and 2, for each $i \in N_r(e)$, $f_i(e) = TTC_i(e)$. □

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Online Appendix to "Endowments-swapping-proofness in housing markets with exchange constraints" by Fujinaka and Wakayama (June 7, 2024)

B Omitted proofs in the main text

B.1 Example 1

Here, we show that f^{nir} is *endowments-swapping-proof*. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$. There are three cases.

• **Case 1:** $\succ\ \in \mathscr{P}^N \setminus (\mathscr{P}_{23} \cup \mathscr{P}_{32})$. It immediately follows from *endowments-swapping-proofness* of *TTC*.

• **Case 2:** $\succ\ \in \mathscr{P}_{23}$. Note that for each $\omega' \in X$, $f_k^{\text{nir}}(\succ, \omega') = h_1$, which implies that agent k cannot benefit from swapping his endowment with that of another agent at e. Thus, it suffices to consider the pair of agent i and agent j, $\{i, j\}$. Note that $\{f_i^{\text{nir}}(e), f_j^{\text{nir}}(e)\} = \{f_i^{\text{nir}}(e^{i,j}), f_j^{\text{nir}}(e^{i,j})\} = \{h_2, h_3\}$, and $h_2 \succ_i h_3$ and $h_2 \succ_j h_3$. Then, it never happens that both agents prefer the objects assigned at $e^{i,j}$ to those assigned at e.

• **Case 3:** $\succ\ \in \mathscr{P}_{32}$. Note that for each $\omega' \in X$, $f_j^{\text{nir}}(\succ, \omega') = h_1$, which implies that agent j cannot benefit from swapping his endowment with that of another agent at e. Thus, it suffices to consider the pair of agent i and agent k, $\{i, k\}$. Note that $\{f_i^{\text{nir}}(e), f_k^{\text{nir}}(e)\} = \{f_i^{\text{nir}}(e^{i,k}), f_k^{\text{nir}}(e^{i,k})\} = \{h_2, h_3\}$, and $h_3 \succ_i h_2$ and $h_3 \succ_k h_2$. Then, it never happens that both agents prefer the objects assigned at $e^{i,k}$ to those assigned at e. □

B.2 Example 2

Here, we show that $\ddot{f}$ is *endowments-swapping-proof*. Let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ and $\{i, j\} \subset N$. There are three cases.

• **Case 1:** $\{e, e^{i,j}\} \subset \mathscr{E}^{\vee} \setminus \{\ddot{e}\}$**.** It immediately follows from *endowments-swapping-proofness* of *TTC*.

• **Case 2:** $e = \ddot{e}$ **and** $e^{i,j} \neq \ddot{e}$**.** Then, $\omega = (h_1, h_3, h_2)$. Since agent 1 receives his best object h_1 according to $\succ_1 = \ddot{\succ}_1$ under $f(e)$, he has no incentive to collude with another agent at e. Thus, it suffices to consider the case $\{i, j\} = \{2, 3\}$. Then, $\ddot{f}(e^{2,3}) = TTC(e^{2,3}) = (h_1, h_2, h_3) = \ddot{f}(e)$. This implies that both agents cannot benefit from swapping their endowments at e.

• **Case 3: $e \neq \ddot{e}$ and $e^{i,j} = \ddot{e}$.** Note that by $e^{i,j} = (\succ, \omega^{i,j}) = \ddot{e} = (\ddot{\succ}, (h_1, h_3, h_2))$, $\succ = \ddot{\succ}$. Let $k \in N$ be an agent such that $\omega_k = h_1$. Then, $\ddot{f}_k(e) = TTC_k(e) = h_1$, and thus, agent k receives his best object h_1 according to $\succ_k = \ddot{\succ}_k$. Hence, agent k has no incentive to collude with another agent at e. We next consider the case where $\{\omega_i, \omega_j\} = \{h_2, h_3\}$. By $\omega^{i,j} = (h_1, h_3, h_2)$, $\omega = (h_1, h_2, h_3)$. Since $\ddot{f}(e) = TTC(e) = (h_1, h_2, h_3) = \ddot{f}(e^{i,j})$, both agents cannot benefit from swapping their endowments. $\square$

B.3 Example 3

Here, we show that $f^{\vee}$ is *endowments-swapping-proof*. Let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ and $\{i, j\} \subset N$. There are three cases.

• **Case 1: $\{e, e^{i,j}\} \subset \mathscr{E}^{\vee} \setminus \{e'\}$.** It immediately follows from *endowments-swapping-proofness* of TTC.

• **Case 2: $e = e'$ and $e^{i,j} \neq e'$.** Since each agent $i \in \{1, 2\}$ receives his best object according to $\succ_i = \succ'_i$, he has no incentive to collude with another agent at e. Next, consider the case where $\{i, j\} = \{3, 4\}$. Then, $f^{\vee}_4(e)(= f^{\vee}_4(e')) = h_2 \succ'_4 h_3 = f^{\vee}_4(e^{3,4}) = TTC_4(e^{3,4})$. Hence, agent 4 has no incentive to collude with agent 3 at e.

• **Case 3: $e \neq e'$ and $e^{i,j} = e'$.** Note that by $e^{i,j} = (\succ, \omega^{i,j}) = e' = (\succ', \omega')$, $\succ = \succ'$. Since $f^{\vee}(e) = TTC(e)$ is *efficient* at e, $f^{\vee}_1(e) = h_4$; that is, agent 1 receives his best object h_4 according to $\succ_1 = \succ'_1$. Thus, he has no incentive to collude with another agent at e. Here we consider the case where $\{i, j\} \subset \{2, 3, 4\}$. By $1 \notin \{i, j\}$, $\omega_1 = \omega^{i,j}_1 = \omega'_1 = h_1$. Let $k \in \{2, 3, 4\}$ be an agent such that $\omega_k = h_4$. Then, by $(\omega_1, \omega_k) = (h_1, h_4)$ and the definition of $\succ'$, $f^{\vee}_k(e) = TTC_k(e) = h_1$. Since agent k receives his best object h_1 according to $\succ_k = \succ'_k$, he has no incentive to collude with another agent at e. Hence, we only consider the case where $\{\omega_i, \omega_j\} = \{h_2, h_3\}$. By $\omega^{i,j} = \omega' = (h_1, h_2, h_3, h_4)$, $\omega = (h_1, h_3, h_2, h_4)$ and $\{i, j\} = \{2, 3\}$. Then, $f^{\vee}_3(e) = TTC_3(e) = h_2 \succ'_3 h_3 = f^{\vee}_3(e^{2,3})(= f_3(e'))$. Hence, agent 3 has no incentive to collude with agent 2 at e. $\square$

C Single-dipped preferences on a tree

In Section 4 of the main text, we considered single-dipped preferences on a line. This class of preferences can be extended to a more general structure called a "tree." Tamura (2023) has characterized TTC as the only mechanism that satisfies *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness* on this extended single-dipped domain without restrictions on the size of possible exchanges. Here, we ask whether Tamura's characterization holds even when there is a restriction on the size of possible exchanges.

C.1 Definitions and preliminary results

To formally define single-dipped preferences on a tree, we begin by introducing some graph theoretical notions. An **(indirected) graph** is a pair $G = (H, E)$, where $E \subset \{\{h', h''\} \subset H : h' \neq h''\}$ is the set of **edges**. The **degree** of object $h \in H$ in a graph $G = (H, E)$ is the number of edges that contain h; that is,

$$\deg^G(h) = \left|\{\{h', h''\} \in E : h \in \{h', h''\}\}\right|.$$

Given an object $h \in H$, we say that h is a **leaf** in G if $\deg^G(h) = 1$. We denote the set of leaves in G by $\mathbb{L}$.[17] Given $\{h', h''\} \subset H$ with $h' \neq h''$, a **path from h' to h''** in $G = (H, E)$ is a sequence $(h^1, h^2, \ldots, h^K)$ such that $h^1 = h'$, $h^K = h''$, $|\{h^1, h^2, \ldots, h^K\}| = K$, and for each $k \in \{1, 2, \ldots, K-1\}$, $\{h^k, h^{k+1}\} \in E$. A graph $G = (H, E)$ is a **tree** if

(i) it is connected (i.e., for each $\{h', h''\} \subset H$ with $h' \neq h''$, there is a path from h' to h'' in G), and

(ii) it has no cycle (i.e., there is no sequence $(h^1, h^2, \ldots, h^K)$ such that $K \geq 3$, $h^1 = h^K$, for each $k \in \{1, 2, \ldots, K-1\}$, $\{h^k, h^{k+1}\} \in E$, and for each $\{k', k''\} \subset \{1, 2, \ldots, K\}$ such that $k' \neq k''$ and $\{k', k''\} \neq \{1, K\}$, $h^{k'} \neq h^{k''}$).

It is well-known that if a graph G is a tree, then, for each $\{h', h''\} \subset H$ with $h' \neq h''$, there is a unique path from h' to h'' in G (see, for example, Theorem 2.1.4 in West (2001)). We often denote the path from h' to h'' by $[h', h'']$. For each $\{h, h', h''\} \subset H$, we write $h \in [h', h'']$ if h is on the path from h' to h''; that is, when $[h', h''] = (h^1 = h', h^2, \ldots, h^K = h'')$, there is $k \in \{1, 2, \ldots, K\}$ such that $h^k = h$.

[17] Formally, it should be $\mathbb{L}(G)$, but unless otherwise specified, we omit G for simplicity.

Given a tree $G = (H, E)$ and an agent $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped on the tree G** if there is an object, $d(\succ_i) \in H$, such that

(i) for each $h \in H \setminus \{d(\succ_i)\}$, $h \succ_i d(\succ_i)$;

(ii) for each $\{h, h'\} \subset H \setminus \{d(\succ_i)\}$ with $h \neq h'$, if $h \in [d(\succ_i), h']$, then $h' \succ_i h$.

Given a tree G, we denote the set of single-dipped preferences on the tree G by $\mathscr{P}_G \subset \mathscr{P}$. Given a tree G, let $\mathscr{E}^G = \mathscr{P}_G^N \times X$.

Remark 4. Note that for each $i \in N$ and each $\succ_i \in \mathscr{P}_G$, i's best object according to $\succ_i$ is a leaf in G. To observe this, let $h \in H \setminus \mathbb{L}$. We only consider the case where $h \neq d(\succ_i)$; if $h = d(\succ_i)$, it is obvious that h is not his best object according to $\succ_i$. By $h \neq d(\succ_i)$, there is the unique path from $d(\succ_i)$ to h in the tree G, $[d(\succ_i), h] = (h^1 = d(\succ_i), h^2, \ldots, h^K = h)$. By $h \notin \mathbb{L}$, $\deg^G(h) > 1$. Thus, there is $h' \in H$ such that $h' \neq h^{K-1}$ and $\{h, h'\} \in E$. Since G has no cycle, for each $k \in \{1, 2, \ldots, K\}$, $h' \neq h^k$. Hence, $[d(\succ_i), h'] = (h^1 = d(\succ_i), h^2, \ldots, h^K = h, h')$. Since $h \in [d(\succ_i), h']$ and $\succ_i$ is single-dipped on G, $h' \succ_i h$, which implies that h is not i's best object according to $\succ_i$. Thus, i's best object according to $\succ_i$ must be in $\mathbb{L}$. $\Diamond$

It is noteworthy that TTC on the domain of single-dipped preferences on a tree is an $|\mathbb{L}|$-feasible mechanism. In addition, we observe that the maximal size of possible exchanges under TTC is $|\mathbb{L}|$.

Proposition 2. *Suppose that G is a tree. Then, TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$-feasible.*

Proof. Let $e = (\succ, \omega) \in \mathscr{E}^G$. Recall that for each integer $t \geq 1$, $N_t(e)$ is the set of agents that form trading cycles in Round t of the TTC algorithm at e and $H_t(e)$ is the set of objects that are assigned to agents in $N_t(e)$. We now introduce additional notation:

- $\overline{N}^1 = N$ and for each integer $t \geq 2$, $\overline{N}^t = \overline{N}^{t-1} \setminus N_{t-1}(e)$;
- $G^1 = (\overline{H}^1, \overline{E}^1) = (H, E)$ and for each integer $t \geq 2$, $G^t = (\overline{H}^t, \overline{E}^t)$, where $\overline{H}^t = \overline{H}^{t-1} \setminus H_{t-1}(e)$ and $\overline{E}^t = \{\{h', h''\} \in \overline{E}^{t-1} \colon \{h', h''\} \subset \overline{H}^t\}$;
- for each $i \in \overline{N}^1$, $d^1(\succ_i) = d(\succ_i)$ and for each integer $t \geq 2$ and each $i \in \overline{N}^t$, $d^t(\succ_i)$ denotes i's worst object among $\overline{H}^t$ according to $\succ_i$ (i.e., $d^t(\succ_i) \in \overline{H}^t$ and for each $h \in \overline{H}^t \setminus \{d^t(\succ_i)\}$, $h \succ_i d^t(\succ_i)$).

We will observe below that for each integer $t \geq 2$, $G^t = (\overline{H}^t, \overline{E}^t)$ is a tree. We denote the set of leaves in G^t by $\mathbb{L}^t$. Note that $\mathbb{L}^1 = \mathbb{L}$. Moreover, for each integer $t \geq 1$ and each $\{h', h''\} \subset \overline{H}^t$ with $h' \neq h''$, we denote the unique path from h' to h'' in G^t by $[h', h'']^t$. We now consider each round of the TTC algorithm.

ROUND 1. As stated in Remark 4, for each $i \in \overline{N}^1 = N$, i's best object among $\overline{H}^1 = H$ according to $\succ_i$ is in $\mathbb{L}^1 = \mathbb{L}$. Hence, $N_1(e) \subset \{i \in \overline{N}^1 : \omega_i \in \mathbb{L}^1\}$ and $H_1(e) \subset \mathbb{L}^1$. This implies that the size of each trading cycle formed in Round 1 is less than or equal to $|\mathbb{L}^1| = |\mathbb{L}|$.

ROUND 2. Note that the set of remaining agents (resp. objects) is $\overline{N}^2 = \overline{N}^1 \setminus N_1(e)$ (resp. $\overline{H}^2 = \overline{H}^1 \setminus H_1(e)$). We present a series of claims before completing the proof.

Claim 1. *G^2 is a tree.*

Proof of Claim 1. Since $\overline{H}^2 = \overline{H}^1 \setminus H_1(e)$ and $H_1(e) \subset \mathbb{L}^1$, by Lemma 2.1.3 in West (2001), G^2 is a tree. □

Claim 2. $|\mathbb{L}^2| \leq |\mathbb{L}^1|$.

Proof of Claim 2. Note that by $H_1(e) \subset \mathbb{L}^1$,

$$\begin{aligned}
|\mathbb{L}^1| &= |\mathbb{L}^1 \cap H_1(e)| + |\mathbb{L}^1 \setminus H_1(e)| = |H_1(e)| + |\mathbb{L}^1 \setminus H_1(e)|; \\
|\mathbb{L}^2| &= |\mathbb{L}^2 \cap \mathbb{L}^1| + |\mathbb{L}^2 \setminus \mathbb{L}^1|.
\end{aligned}$$

In what follows, we show that (i) $|\mathbb{L}^2 \cap \mathbb{L}^1| \leq |\mathbb{L}^1 \setminus H_1(e)|$ and (ii) $|\mathbb{L}^2 \setminus \mathbb{L}^1| \leq |H_1(e)|$, which together imply $|\mathbb{L}^2| \leq |\mathbb{L}^1|$.

(i) Let $h \in \mathbb{L}^2 \cap \mathbb{L}^1$. By $h \in \mathbb{L}^2 \subset \overline{H}^2$, $h \notin H_1(e)$, which implies $h \in \mathbb{L}^1 \setminus H_1(e)$. Hence, $\mathbb{L}^2 \cap \mathbb{L}^1 \subseteq \mathbb{L}^1 \setminus H_1(e)$ and $|\mathbb{L}^2 \cap \mathbb{L}^1| \leq |\mathbb{L}^1 \setminus H_1(e)|$.[18]

(ii) Let $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$. Note that $\deg^{G^2}(h) = 1$ and $\deg^{G^1}(h) > 1$. Then, there is $\hat{h} \in H_1(e) (\subset \mathbb{L}^1)$ such that $\{h, \hat{h}\} \in \overline{E}^1$.[19] Thus, we can construct a mapping $\alpha: \mathbb{L}^2 \setminus \mathbb{L}^1 \to H_1(e)$ such that for each $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$, $\alpha(h) \in H_1(e)$ with $\{h, \alpha(h)\} \in \overline{E}^1$. We now show that α is injective, which immediately implies $|\mathbb{L}^2 \setminus \mathbb{L}^1| \leq |H_1(e)|$.

[18] In fact, $\mathbb{L}^2 \cap \mathbb{L}^1 = \mathbb{L}^1 \setminus H_1(e)$ also holds. It suffices to prove $\mathbb{L}^1 \setminus H_1(e) \subseteq \mathbb{L}^2 \cap \mathbb{L}^1$. Let $h \in \mathbb{L}^1 \setminus H_1(e)$. Suppose on the contrary that $h \notin \mathbb{L}^2 \cap \mathbb{L}^1$. Since $h \in \mathbb{L}^1$, $h \notin \mathbb{L}^2$, which implies $\deg^{G^2}(h) > 1$. Then, for some $\{h', h''\} \in \overline{H}^2$ with $h' \neq h''$, $\{\{h, h'\}, \{h, h''\}\} \subset \overline{E}^2 \subset \overline{E}^1$, which implies $\deg^{G^1}(h) > 1$, a contradiction to $h \in \mathbb{L}^1$. Hence, $h \in \mathbb{L}^2 \cap \mathbb{L}^1$.

[19] Otherwise, for each $\hat{h} \in \overline{H}^1$ with $\{h, \hat{h}\} \in \overline{E}^1$, $\hat{h} \notin H_1(e)$. Then, $\deg^{G^2}(h) > 1$, a contradiction.

Suppose on the contrary that there is $\{h', h''\} \subset \mathbb{L}^2 \setminus \mathbb{L}^1$ such that $h' \neq h''$ but $\alpha(h') = \alpha(h'')$. Then, by $\{\{h', \alpha(h')\}, \{h'', \alpha(h'') = \alpha(h')\}\} \subset \overline{E}^1$, $\deg^{G^1}(\alpha(h')) = \deg^{G^1}(\alpha(h'')) > 1$, which is a contradiction to $\alpha(h') = \alpha(h'') \in \mathbb{L}^1$. $\square$

Claim 3. *For each $i \in \overline{N}^2$, $\succ_i$ is single-dipped on G^2.*[20]

Proof of Claim 3. By the definition of $d^2(\succ_i)$, $d^2(\succ_i) \in \overline{H}^2$ and for each $h \in \overline{H}^2 \setminus \{d^2(\succ_i)\}$, $h \succ_i d^2(\succ_i)$. Next, let $\{h', h''\} \subset H^2 \setminus \{d^2(\succ_i)\}$ be such that $h' \neq h''$, $h' \in [d^2(\succ_i), h'']^2 = (h^1 = d^2(\succ_i), h^2, \ldots, h^K = h'')$. Note that for each $k \in \{1, 2, \ldots, K-1\}$, by $\{h^k, h^{k+1}\} \in \overline{E}^2$, $\{h^k, h^{k+1}\} \in \overline{E}^1$. Hence, $[d^2(\succ_i), h'']^1 = (h^1 = d^2(\succ_i), h^2, \ldots, h^K = h'') = [d^2(\succ_i), h'']^2$. There are two cases.

- **Case 1: $\boldsymbol{d^1(\succ_i) \in \overline{H}^2}$.** It is obvious that $d^2(\succ_i) = d^1(\succ_i)$. Since $\succ_i$ is single-dipped on G^1 and $h' \in [d^2(\succ_i) = d^1(\succ_i), h'']^1$, $h'' \succ_i h'$.

- **Case 2: $\boldsymbol{d^1(\succ_i) \notin \overline{H}^2}$.** Then, $d^1(\succ_i) \in H_1(e) \subset \mathbb{L}^1$. It thus follows that $\deg^{G^1}(d^1(\succ_i)) = 1$. Let $h^* \in \overline{H}^1$ be the unique object such that $\{d^1(\succ_i), h^*\} \in \overline{E}^1$. Then, $h^* \in \overline{H}^2$.[21] We now show that $h^* = d^2(\succ_i)$; that is, for each $h \in \overline{H}^2 \setminus \{h^*\}$, $h \succ_i h^*$. Let $h \in \overline{H}^2 \setminus \{h^*\}$. By $h \in \overline{H}^1$, we can find $[d^1(\succ_i), h]^1 = (\bar{h}^1 = d^1(\succ_i), \bar{h}^2, \ldots, \bar{h}^{\bar{K}} = h)$. Since h^* is the unique object such that $\{d^1(\succ_i), h^*\} \in \overline{E}^1$, $\bar{h}^2 = h^*$, and thus, $h^* \in [d^1(\succ_i), h]^1$. Since $\succ_i$ is single-dipped on G^1, $h \succ_i h^*$. It remains to show $h'' \succ_i h'$. Since $[d^2(\succ_i) = h^*, h'']^1 = (h^1 = d^2(\succ_i) = h^*, h^2, \ldots, h^K = h'')$ and $\{d^1(\succ_i), h^*\} \in \overline{E}^1$, $[d^1(\succ_i), h'']^1 = (d^1(\succ_i), h^1 = d^2(\succ_i) = h^*, h^2, \ldots, h^K = h'')$. By $h' \in [d^2(\succ_i), h'']^2 = [d^2(\succ_i), h'']^1$, $h' \in [d^1(\succ_i), h'']^1$. Since $\succ_i$ is single-dipped on G^1, $h'' \succ_i h'$. $\square$

Since G^2 is a tree (Claim 1) and for each $i \in \overline{N}^2$, $\succ_i$ is single-dipped on G^2 (Claim 3), by the similar argument as in Remark 4, we have that for each $i \in \overline{N}^2$, i's best object among $\overline{H}^2$ according to $\succ_i$ is in $\mathbb{L}^2$. Hence, $N_2(e) \subset \{i \in \overline{N}^2 : \omega_i \in \mathbb{L}^2\}$ and $H_2(e) \subset \mathbb{L}^2$. This together with Claim 2 implies that the size of each trading cycle formed in Round 2 is less than or equal to $|\mathbb{L}^1| = |\mathbb{L}|$.

By repeating this argument, we observe that the size of each trading cycle formed in each round of TTC is less than or equal to $|\mathbb{L}^1| = |\mathbb{L}|$. Then, we can conclude that TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$-feasible. $\square$

[20] With a slight abuse of notation, we use $\succ_i$ to denote the restricted preference relation over $\overline{H}^2$.

[21] If $h^* \notin \overline{H}^2$, then $h^* \in H_1(e)$. Then, $h^* \in \mathbb{L}^1$ and $\deg^{G^1}(h^*) = 1$. This implies that $\overline{H}^1 = \{d^1(\succ_i), h^*\}$, $\overline{E}^1 = \{\{d^1(\succ_i), h^*\}\}$, and $\overline{H}^2 = \overline{H}^1 \setminus H_1(e) = \emptyset$; that is, the TTC algorithm terminates in Round 1, a contradiction.

Proposition 3. *Suppose that G is a tree. Then, the maximal size of possible trading cycles under TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$.*

Proof. Without loss of generality, assume $\mathbb{L} = \{h_1, h_2, \ldots, h_m\}$. Let $e = (\succ, \omega) \in \mathscr{E}^G$ be such that

$\succ_1$	$\succ_2$	$\cdots$	$\succ_k$	$\cdots$	$\succ_{m-1}$	$\succ_m$
h_2	h_3	$\cdots$	h_{k+1}	$\cdots$	h_m	h_1
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$

and for each $i \in N$, $\omega_i = h_i$. Then, for each $i \in \{1, 2, \ldots, m-1\}$, $TTC_i(e) = h_{i+1}$ and $TTC_m(e) = h_1$ that is, the size of this trading cycle is m. By Proposition 2, since for each $e \in \mathscr{E}^G$, each integer $t \geq 1$, and each $S \in \mathbb{S}_t(e)$, $|S| \leq m$, the maximal size of possible trading cycles under TTC is $m = |\mathbb{L}|$. □

C.2 Stringent exchange constraints

Theorem 5 characterized TTC as the only *individually rational* and *endowments-swapping-proof* pairwise exchange mechanism on the domain of single-dipped preferences on a line. However, we cannot directly extend this characterization of TTC to the domain of single-dipped preferences on a tree when there are three or more leaves and possible exchanges restrict attention to pairwise ones. This is because TTC defined on the domain of single-dipped preferences on a tree is no longer a pairwise mechanism when there are three or more leaves (Proposition 3). Furthermore, we can show that when there are three or more leaves, no pairwise exchange mechanism satisfies *individual rationality* and *endowmens-swapping-proofness*. More generally, as shown below, this negative result holds as long as the possible exchanges are less than the number of leaves. Our negative result implies that Tamura's characterization no longer holds under such a "stringent" constraint on the size of possible exchanges.

Theorem 7. *Suppose that G is a tree. Let $\ell \in \{1, 2, \ldots, |\mathbb{L}| - 1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^G$ satisfies individual rationality and endowments-swapping-proofness.*

Proof. Without loss of generality, we assume $\mathbb{L} = \{h_1, h_2, \ldots, h_m\}$. Let $M = \{1, 2, \ldots, m\} \subset N$. Note that $\ell < m$. Suppose on the contrary that there is an ℓ-feasible mechanism f on $\mathscr{E}^G$ satisfying the two properties. We derive a contradiction. The proof consists of three steps.

Step 1: Constructing an economy. Let $\succ^* \in \mathscr{P}_G^N$ be such that:

(i) for each $i \in M$, each $j \in \{i, i+1, \ldots, m\}$, each $j' \in \{1, 2, \ldots, i-1\}$, and each $j'' \in N \setminus M$, $h_j \succ_i^* h_{j'} \succ_i^* h_{j''}$;

(ii) for each $i \in M$ and each $\{j, j'\} \subset N$, if $j < j'$ and either $\{j, j'\} \subseteq \{i, i+1, \ldots, m\}$ or $\{j, j'\} \subseteq \{1, 2, \ldots, i-1\}$, then $h_j \succ_i^* h_{j'}$;

(iii) for each $i \in N \setminus M$ and each $\{j, j'\} \subseteq M$, if $j < j'$, then $h_j \succ_i^* h_{j'}$

The preference profile $\succ^*$ can be represented as follows:

$\succ_1^*$	$\succ_2^*$	$\succ_3^*$	$\cdots$	$\succ_{m-1}^*$	$\succ_m^*$	$\succ_{j \geq m+1}^*$
h_1	h_2	h_3	$\cdots$	h_{m-1}	h_m	h_1
h_2	h_3	h_4	$\cdots$	h_m	h_1	h_2
h_3	h_4	h_5	$\cdots$	h_1	h_2	h_3
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
h_{m-2}	h_{m-1}	h_m	$\cdots$	h_{m-4}	h_{m-3}	h_{m-2}
h_{m-1}	h_m	h_1	$\cdots$	h_{m-3}	h_{m-2}	h_{m-1}
h_m	h_1	h_2	$\cdots$	h_{m-2}	h_{m-1}	h_m
$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$

For each $k \in M$, consider the endowments ω^k similar to as those in the proof of Theorem 2. That is, ω^k is defined by (1). For each $k \in M$, let $e^{*k} = (\succ^*, \omega^k)$. Note that for each $k \in M$, $e^{*k} \in \mathscr{E}^G$.

Step 2: For each $k \in M$ and each $i \in M$, $f_i(e^{*k}) = h_i$. The profile $(\succ_i^*)_{i=1}^m$ constructed in Step 1 is similar to the profile $(\succ_i)_{i=1}^{\ell+1}$ constructed in the proof of Theorem 2. Then, by the similar argument in the proof of Theorem 2, we obtain the desired conclusion.[22]

Step 3: Concluding. Step 2 implies that for each $i \in M$, $f_i(e^{*m}) = h_i$; that is, $f_1(e^{*m}) = h_1 = \omega_m^m$ and for each $i \in M \setminus \{1\}$, $f_i(e^{*m}) = h_i = \omega_{i-1}^m$. Since $(1, 2, \ldots, m)$ is a trading cycle at $(f(e^{*m}), \omega^m)$ and $\ell < m$, $f(e^{*m}) \notin X_\ell(\omega^m)$, which is a contradiction. $\square$

As a corollary to Theorem 7, we obtain a negative result for pairwise exchange: *individual rationality* and *endowments-swapping-proofness* are incompatible when there are three or more leaves.

[22]Note that the profile $(\succ_i^*)_{i=m+1}^n$ constructed in Step 1 may differ from the profile $(\succ_i)_{i=\ell+2}^n$ constructed in the proof of Theorem 2. However, these profiles are only relevant to whether the economies, such as e^k or e^{*k}, belong to the domains that are being considered. Thus, we can apply the proof of Theorem 2 to Step 2 of this proof.

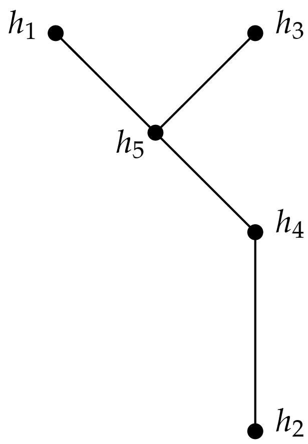

Figure 2: Tree in Example 4.

Corollary 2. *Suppose that G is a tree and $|\mathbb{L}| \geq 3$. Then, no pairwise exchange mechanism on $\mathscr{E}^G$ satisfies individual rationality and endowments-swapping-proofness.*

C.3 Lenient exchange constraints

Based on Proposition 2, one might think that TTC on the domain of single-dipped preferences on a tree can be characterized by means of *individual rationality* and *endowments-swapping-proofness* if $|\mathbb{L}|$-feasible exchanges are allowed. However, this conjecture is not true whenever $|\mathbb{L}| \geq 3$. In fact, if $|\mathbb{L}| \geq 3$, we can construct a non-TTC $|\mathbb{L}|$-feasible mechanism that is *individually rational*, and *endowments-swapping-proof*. The following is an example of such a mechanism.

Example 4. Let $n = 5$. Suppose that a tree G is represented as in Figure 2. Then, $\mathbb{L} = \{h_1, h_2, h_3\}$. Let $\check{e} = (\check{\succ}, \check{\omega}) \in \mathscr{E}^G$ be such that

$\check{\succ}_1$	$\check{\succ}_2$	$\check{\succ}_3$	$\check{\succ}_4$	$\check{\succ}_5$
h_2	h_1	h_3	h_2	h_1
h_4	h_3	h_1	h_4	h_2
h_1	h_2	h_2	h_1	h_3
h_3	h_4	h_4	h_3	h_4
h_5	h_5	h_5	h_5	h_5

and $\check{\omega} = (h_1, h_2, h_3, h_4, h_5)$. Let $f^{\nabla} : \mathscr{E}^G \to X$ be a 3-feasible mechanism such that for each $e \in \mathscr{E}^G$,

$$f^{\nabla}(e) = \begin{cases} (h_4, h_1, h_3, h_2, h_5) & \text{if } e = \check{e} \\ TTC(e) & \text{otherwise.} \end{cases}$$

Note that $TTC(\check{e}) = (h_2, h_1, h_3, h_4, h_5) \neq f^{\nabla}(\check{e})$ and by Proposition 2, TTC is a 3-feasible mechanism. It is obvious that this mechanism is *individually rational*. To see why f^{∇} is *endowments-swapping-proof*, let $e = (\succ, \omega) \in \mathscr{E}^G$ and $\{i, j\} \subset N$ with $i \neq j$. If $\{e, e^{i,j}\} \subset \mathscr{E}^G \setminus \{\check{e}\}$, then *endowments-swapping-proofness* of f^{∇} immediately follows from *endowments-swapping-proofness* of TTC. Thus, we consider the following two cases.

• **Case 1: $e = \check{e}$ and $e^{i,j} \neq \check{e}$.** Since each agent $i \in \{2, 3, 4\}$ receives his best object according to $\succ_i = \check{\succ}_i$, he has no incentive to collude with another agent at e. Thus, we only consider the case where $\{i, j\} = \{1, 5\}$. Then, $f^{\nabla}_1(e^{1,5}) = TTC_1(e^{1,5}) = h_5$. That is, agent 1 continues to receive his worst object h_5 according to $\succ_1 = \check{\succ}_1$ even if he swaps his endowment with that of agent 5. Hence, agent 1 has no incentive to collude with agent 5 at e.

• **Case 2: $e \neq \check{e}$ and $e^{i,j} = \check{e}$.** Note that by $e^{i,j} = (\succ, \omega^{i,j}) = \check{e} = (\check{\succ}, \check{\omega})$, $\succ = \check{\succ}$. If $5 \in \{i, j\}$, by $f^{\nabla}_5(e^{i,j}) (= f^{\nabla}_5(\check{e})) = h_5$, then agent 5 receives his worst object h_5 according to $\succ_5 = \check{\succ}_5$ even if he swaps his endowment with that of any agent. Hence, agent 5 has no incentive to collude with another agent at e. Here we consider the case where $\{i, j\} \subset \{1, 2, 3, 4\}$. Note that by $5 \notin \{i, j\}$, $\omega_5 = \omega^{i,j}_5 (= \check{\omega}_5) = h_5$. Then, $f^{\nabla}_5(e) = TTC_5(e) = h_5$. In addition, since $f^{\nabla}(e) = TTC(e)$ is *efficient* at e, $f^{\nabla}(e) \in \{(h_2, h_1, h_3, h_4, h_5), (h_4, h_1, h_3, h_2, h_5)\}$. Then, in either case, three agents out of $\{1, 2, 3, 4\}$ receive their best objects according to their preferences. Hence, any pair $\{i, j\} \subset \{1, 2, 3, 4\}$ of agents have no incentive to swap their endowments at e. ■

Note that mechanism f^{∇} defined in Example 4 violates *strategy-proofness*. To see this, let $\check{\succ}'_1 \in \mathscr{P}_G$ be such that $A(\check{\succ}'_1, \check{\omega}_1 = h_1) = \{h_2\}$. Then,

$$f^{\nabla}_1((\check{\succ}'_1, \check{\succ}_{-1}), \check{\omega}) = TTC_1((\check{\succ}'_1, \check{\succ}_{-1}), \check{\omega}) = h_2 \check{\succ}_1 h_4 = f^{\nabla}_1(\check{e}).$$

Thus, agent 1 with preferences $\check{\succ}_1$ can benefit from announcing the false preference relation $\check{\succ}'_1$. This suggests that, by adding *strategy-proofness*, one could obtain a characterization of TTC. Recall here that when the size of possible exchanges is more than or equal to three, Tamura (2023) and Theorem 6 propose a characterization of TTC by means of *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness*. In fact, this characterization holds true even when the size of possible exchanges is greater than or equal to the number of leaves. This is simply because the $|\mathbb{L}|$-feasibility of TTC (Proposition 2) makes it possible for TTC to satisfy such a "lenient" exchange constraint. Since, as mentioned above, TTC satisfies the lenient exchange constraint on the size of possible exchange, we obtain the following result:

Theorem 8. *Suppose that G is a tree. Let $\ell \geq |\mathbb{L}|$. Then, an ℓ-feasible mechanism on $\mathscr{E}^G$ satisfies individual rationality, strategy-proofness, and endowments-swapping-proofness if and only if it is TTC.*

References

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