The Equilibrium Existence Duality: Equilibrium with Indivisibilities & Income Effects

Elizabeth Baldwin
Oxford

Omer Edhan
Manchester

Ravi Jagadeesan
Stanford

Paul Klemperer
Oxford

Alex Teytelboym
Oxford

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Markets for indivisible goods

- markets in which indivisibilities are important include
  - exchange: housing markets, markets for used cars, ...
  - auctions: spectrum auctions, ad auctions, ...
  - labour markets: specialized jobs ...
- most previous work assumed transferable utility (TU) for tractability
  - Kelso and Crawford (1982), Bikhchandani and Mamer (1997), Ma (1998), Gul and Stacchetti (1999), Sun and Yang (2006), Milgrom and Strulovici (2009), Hatfield et al. (2013), Baldwin and Klemperer (2019), ...

Markets for indivisible goods

- income effects or financing constraints are often important
  - if indivisibles are "large"—exactly when indivisibilities are important
  - e.g, houses, large spectrum auctions...
  - but existence of equilibrium with indivisibilities and income effects is tricky!

- this paper: analyzes markets for indivisible goods with income effects by isolating the roles of income and substitution effects

This paper

0. separate income and substitution effects by using **Hicksian demand**
1. combine Hicksian demands to form **Hicksian economies**
   - hold utility levels fixed instead of endowments; turn off income effects
2. derive **Equilibrium Existence Duality**: equilibrium always exists in the original economy iff it always exists in each Hicksian economy

key consequences of Equilibrium Existence Duality:

- substitution effects fundamentally determine whether equilibria exist, i.e., any condition for existence can be written in terms of substitution effects alone.
- get new domains for equilibrium existence from previous TU results
  - interpret Hicksian demand as quasilinear utility maximization

New domains for existence

$$
\begin{tikzcd}
\text{equil. existence under TU} \arrow[r, dashed, leftrightarrow, "\text{equil. existence}", "\text{duality}"'] & \color{red}\text{equil. existence w/income effs.} \\
\text{(gross) substitutability} \arrow[u, Rightarrow, "\text{old}"] \arrow[r, dashed, leftrightarrow, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & \color{red}\text{net substitutability} \arrow[u, Rightarrow, red, "\text{new}"']
\end{tikzcd}
$$

- **net substitutability**, not gross substitutability, defines a maximal domain
- each TU existence result extends to settings with income effects

Related literature

- separable preferences
  - Kaneko and Yamamoto (1986), vd Laan et al. (1997, 2002), Yang (2000)
- gross substitutability with income effects
  - Kelso and Crawford (1982), Fleiner et al. (2019)
- housing markets with endowments
  - Quinzii (1984), Gale (1984), Svensson (1984)
- unimodularity
  - Danilov, Koshevoy, and Murota (2001)

Outline

1. model
2. equilibrium existence duality
3. application to substitutes
4. further applications

Model

- finite set I of indivisible goods; money (numéraire)
- finite set J of agents

for each agent j:

- finite set $X_I^j \subseteq \mathbb{Z}^I$ of feasible consumption bundles of indivisibles
- minimum level $\underline{x}_0^j$ of consumption of numéraire
- hence, feasible consumption bundles are

$$\mathbf{x} = (x_0, \mathbf{x}_I) \in (\underline{x}_0^j, \infty) \times X_I^j =: X^j$$

Utility Function

- minimum level $\underline{x}_0^j = 0$ of consumption of numéraire

utility function $U^j : X^j \to (\underline{u}^j, \overline{u}^j)$.

- continuous and strictly increasing in x_0, and
- for each $\mathbf{x}_I \in X_I^j$: $U^j(\mathbf{x}) \to \underline{u}^j$ as $x_0 \to (\underline{x}_0^j)^+$ and $U^j(\mathbf{x}) \to \overline{u}^j$ as $x_0^j \to \infty$
- in particular, we do not allow agents to run out of money (cf., Ravi's Lecture 4)

Example: quasilinear preferences

- minimum level $\underline{x}_0^j = -\infty$ of consumption of numéraire

utility function $U^j(\mathbf{x}) = x_0 + V^j(\mathbf{x}_I)$ for some **valuation** $V^j : X_I^j \to \mathbb{R}$.

Example: “quasilogarithmic” utility

- minimum level $\underline{x}_0^j = 0$ of consumption of numéraire

utility function $U^j(\mathbf{x}) = \log x_0 + f(\mathbf{x}_I)$ for some $f : X_I^j \to \mathbb{R}$

- for each $\mathbf{x}_I \in X_I^j$, have $U^j \to -\infty$ as $\underline{x}_0^j \to 0^+$ and $U^j \to \infty$ as $x_0 \to \infty$

Marshallian and Hicksian demand

- Marshallian: given endowment $\mathbf{w} \in X^j$ and price vector $\mathbf{p}_I \in \mathbb{R}^I$, let

$$D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = \left\{ \mathbf{x}^*_I \;\middle|\; \mathbf{x}^* \in \underset{\mathbf{x} \in X^j | \mathbf{p} \cdot \mathbf{x} \leq \mathbf{p} \cdot \mathbf{w}}{\arg\max} \; U^j(\mathbf{x}) \right\}$$

- Hicksian: given utility level u and price vector $\mathbf{p}_I \in \mathbb{R}^I$, let

$$D^j_{\mathrm{H}}(\mathbf{p}_I; u) = \left\{ \mathbf{x}^*_I \;\middle|\; \mathbf{x}^* \in \underset{\mathbf{x} \in X^j | U^j(\mathbf{x}) \geq u}{\arg\min} \; \mathbf{p} \cdot \mathbf{x} \right\}$$

- A bundle of goods is expenditure-minimizing if and only if it is utility-maximizing.
- for quasilinear preferences: $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = D^j_{\mathrm{H}}(\mathbf{p}; u)$, so we write

$$D^j(\mathbf{p}) = \underset{\mathbf{x}_I \in X^j_I}{\arg\max} \left\{ V^j(\mathbf{x}_I) - \mathbf{p}_I \cdot \mathbf{x}_I \right\}$$

Quasilinear interpretation of Hicksian demand

definition

for a utility level u, the **Hicksian valuation** of agent j is

$$V_{\mathrm{H}}^{j}(\cdot; u) = -U(\cdot, \mathbf{x}_I)^{-1}(u)$$

Hicksian valuation is (negative of) the money to get utility u given $\mathbf{x}_I$.

lemma

for all price vector $\mathbf{p}_I$ and utility levels u, we have

$$D_{\mathrm{H}}^{j}(\mathbf{p}_I; u) = \arg\max_{\mathbf{x}_I \in X_I^j} \{V_{\mathrm{H}}(\mathbf{x}_I; u) - \mathbf{p}_I \cdot \mathbf{x}_I\}.$$

- the Hicksian valuations at fixed u captures substitution effects, while variation in the Hicksian valuations across u captures income effects

The Hicksian economies

definition

- for a utility level u, the **Hicksian valuation** of agent j is $V_{\mathrm{H}}^{j}(\cdot; u)$
- for a profile $(u^j)_{j \in J}$ of utility levels, the **Hicksian economy** is the TU economy in which agent j's valuation is her Hicksian valuation for u^j

- lemma $\implies$ demand in Hicksian econ. is Hicksian demand in original

- by construction, no income effects in the Hicksian economies
  - price effects in each Hicksian economy are substitution effects
- under quasilinearity, each Hicksian economy is ordinally equivalent to the original economy

Example: housing market with endowments

- housing market: exchanging houses among unit-demand agents with endowments + income effects (Quinzii, 1984; Gale, 1984; Svensson, 1984)
- assignment game: assigning objects to unit-demand agents with quasilinear preferences (Koopmans and Beckmann, 1957; Shapley and Shubik, 1971)

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$$\begin{tikzcd} \text{assignment game} \arrow[r, leftrightarrow, dashed, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & \text{housing market} \end{tikzcd}$$

Outline

1. model
2. equilibrium existence duality
3. application to substitutes
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Endowment allocations and competitive equilibrium

- fix a total endowment $\mathbf{y}_I \in \mathbb{Z}^I$ of goods in the economy

definition

an **endowment allocation** consists of an endowment $\mathbf{w}^j \in X^j$ for each agent j, such that $\sum_{j \in J} \mathbf{w}^j_I = \mathbf{y}_I$

definition

given an endowment allocation $(\mathbf{w}^j)_{j \in J}$, a **competitive equilibrium** is a price vector $\mathbf{p}_I$ and bundles $\mathbf{x}^j_I \in D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}^j)$ for agents j with $\sum_{j \in J} \mathbf{x}^j_I = \mathbf{y}_I$

- with TU, equilibrium does not depend on the endowment allocation

Equilibrium existence duality

theorem

if an endowment allocation exists, then

$$\begin{array}{c}\text{competitive equilibria exist}\\ \text{in each Hicksian economy}\\ \text{for all utility levels}\end{array} \Longleftrightarrow \begin{array}{c}\text{competitive equilibria exist}\\ \text{in the original economy}\\ \text{for all endowment allocations}\end{array}$$

- intuitively, substitution effects fundamentally determine existence
  - each Hicksian economy (LHS) only contains substitution effects

- $\Longleftarrow$: essentially Maskin and Roberts (1980/2008)
  - equilibria in the Hicksian economy $\sim$ quasiequilibria with transfers
- $\Longrightarrow$: our main technical contribution

Proof that existence in Hicksian economy $\implies$ existence

- by Marshallian–Hicksian duality, need to find utility levels $(u^j)_{j\in J}$ and a competitive equilibrium in the Hicksian economy for $(u^j)_{j\in J}$ such that expenditure $=$ value of the endowment for all agents
  - such $(u^j)_{j\in J}$ is an equilibrium utility level profile
- apply "Walrasian auctioneer" on $(u^j)_{j\in J}$ to balance agents' budgets
  - lower u^j (to a low level) if j is overspending
  - raise u^j (to a high level) if j is underspending
  - "high level" determined by giving total surplus to j (Fleiner et al., 2019)
- key technical point: set of possible equilibrium expenditure/payoff levels in the Hicksian economy for $(u^j)_{j\in J}$ is compact and convex for each $(u^j)_{j\in J}$ and varies upper hemicontinuously in $(u^j)_{j\in J}$
  - relies crucially on transferability of utility in the Hicksian economies

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$$\begin{tikzcd}
\text{equil. existence under TU} \arrow[r, dashed, leftrightarrow, "\text{equil. existence}", "\text{duality}"'] & \text{equil. existence w/income effs.} \\
\text{assignment game} \arrow[u, Rightarrow] \arrow[r, dashed, leftrightarrow, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & \text{housing market} \arrow[u, Rightarrow]
\end{tikzcd}$$

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Gross substitutability

- for this part of the talk: assume that $X^j_I \subseteq \{0,1\}^I$ (relax in the paper)

definition (∼Kelso and Crawford, 1982; Fleiner et al., 2019)

utility function U^j is **gross substitutable at endowment** $\mathbf{w}$ if for all money endowments w_0, price vectors $\mathbf{p}_I$, and price increments $\lambda > 0$,
if $D^j_{\mathrm{M}}(\mathbf{p}_I, \mathbf{w}) = \{\mathbf{x}_I\}$ and $D^j_{\mathrm{M}}(\mathbf{p}_I + \lambda \mathbf{e}^i, \mathbf{w}) = \{\mathbf{x}'_I\}$, then $x'_k \geq x_k$ for all $k \neq i$

- for valuations, there is no distinction between gross and net, so call the condition "substitutability"
  - in this case, the condition also doesn't depend on the endowment

Net substitutability and the existence of equilibria

definition

utility function U^j is **net substitutable** if for all u, $\mathbf{p}_I$, and $\lambda > 0$,
if $D^j_{\mathrm{H}}(\mathbf{p}_I; u) = \{\mathbf{x}_I\}$ and $D^j_{\mathrm{H}}(\mathbf{p}_I + \lambda \mathbf{e}^i; u) = \{\mathbf{x}'_I\}$, then $x'_k \geq x_k$ for all $k \neq i$

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$$\text{substitutability} \xleftrightarrow[\text{of Hicksian demand}]{\text{quasilinear interpretation}} \text{net substitutability}$$

Net substitutability and the existence of equilibria

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$$\begin{tikzcd} \text{equil. existence under TU} \arrow[rr, leftrightarrow, dashed, "\text{equil. existence}", "\text{duality}"'] & & \text{equil. existence w/income effs.} \\ \text{substitutability} \arrow[u, Rightarrow] \arrow[rr, leftrightarrow, dashed, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & & \text{net substitutability} \arrow[u, Rightarrow] \end{tikzcd}$$

theorem

under net substitutability, equilibria exist for all endowment allocations

Net substitutability versus gross substitutability

proposition

if there is an endowment $\mathbf{w}_I$ of goods for which U^j is gross substitutable at $\mathbf{w}$ for all money endowments w_0, then U^j is net substitutable

- converse false: housing example has net substitutability but not gross
  - suppose Martine owns a house and is considering selling her house and buying either a fancy house or a mediocre house
  - if she only wants to buy the fancy house if she will have enough money left over, then increases in the price of her house can make Martine stop demanding the mediocre house.
- intuitively: gross substitutability constrains income and substitution effects, while net substitutability only constrains substitution effects

Net substitutability as a maximal domain

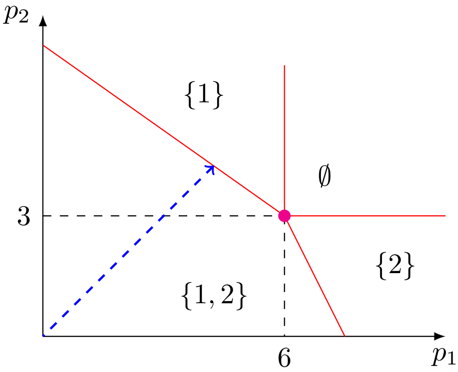
- with TU, substitutability defines a maximal domain for the existence of equilibrium (Gul and Stacchetti, 1999; Hatfield et al., 2013)
- using $\Longleftarrow$ direction of equilibrium existence duality, it follows that:

proposition

suppose that $y_i = 1$ for all goods i and that $|J| \geq 2$.
if U^j is not net substitutable, then there are substitutable valuations for the other agents and an endowment allocation such that no equilibrium exists

- net substitutability is the most general way to incorporate income effects into (quasilinear) substitutability that guarantees existences

Example: net substitutability versus gross substitutability



- two buyers have \$10 each and utility:

$$U^b(\mathbf{x}) = U^{b'}(\mathbf{x}) = \log x_0 - \log(10 - 6x_1 - 3x_2).$$

- seller owns both goods, valuation 0.
- there is a unique competitive equilibrium price vector: $(6,3)\ldots$
- $\ldots$ but ascending auctions generally don't find equilibrium!
- happens because buyers' utility function is not gross substitutable:
  - as $\mathbf{p}_I$ goes from $(4,4)$ to $(5,4)$, buyers' demand for good 2 falls!
- disconnect between existence and tâtonnement—even for substitutes!

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Further applications

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c}\text{equil. existence}\\ \text{under TU}\end{array} \arrow[r, leftrightarrow, dashed, "\text{equil. existence}", "\text{duality}"'] & \begin{array}{c}\text{equil. existence}\\ \text{w/income effs.}\end{array} \\
\begin{array}{c}\text{condition on}\\ \text{demand/valuations}\end{array} \arrow[u, Rightarrow, "\text{old}"] \arrow[r, leftrightarrow, dashed, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & \begin{array}{c}\text{condition on Hicksian}\\ \text{demand/valuations}\end{array} \arrow[u, Rightarrow, red, "\text{new!}"']
\end{tikzcd}
$$

- Can apply, e.g., to Candogan et al. (2015), Rostek and Yoder (2020)...

Further applications: demand types

$$\begin{tikzcd}
\text{equil. existence under TU} \arrow[rr, leftrightarrow, dashed, "\text{equil. existence}", "\text{duality}"'] & & \text{equil. existence w/income effs.} \\
\text{concave valuation of a unimodular demand type} \arrow[u, Rightarrow, "\text{BK'19}"] \arrow[rr, leftrightarrow, dashed, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & & \text{quasiconcave utility fn of a unimodular demand type} \arrow[u, Rightarrow, "\sim\text{DKM'01}"']
\end{tikzcd}$$

- clear economic interpretation: demand types capture comparative statics of Hicksian demand.
- use also can the $\Leftarrow$ direction of the Equilibrium Existence Duality to give a maximal domain result for unimodularity in setting with income effects.

Further applications

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{equil. existence under TU} \arrow[r, leftrightarrow, dashed, "\text{equil. existence}", "\text{duality}"'] \arrow[d, Leftrightarrow, "\text{old}"'] & \text{equil. existence w/income effs.} \arrow[d, Leftrightarrow, "\text{new!}"] \\
\text{condition on demand/valuations} \arrow[r, leftrightarrow, dashed, "\text{quasilinear interpretation}", "\text{of Hicksian demand}"'] & \text{condition on Hicksian demand/valuations}
\end{tikzcd}$$

Further applications: integer programming

- using Bikhchandani and Mamer's (1997) necessary and sufficient condition for the existence of equilibrium with TU, it follows that

corollary

competitive equilibria exist for all endowment allocations if and only if, for each profile $(u^j)_{j\in J}$ of utility levels, the linear program

$$\max_{\left(\alpha^j \in \mathbb{R}^{X_I^j}_{\geq 0}\right)_{j\in J}} \sum_{j\in J} \sum_{\mathbf{x}_I \in X_I^j} \alpha^j_{\mathbf{x}_I} V^j_{\mathrm{H}}(\mathbf{x}_I; u^j)$$

subject to $\displaystyle\sum_{\mathbf{x}_I \in X_I^j} \alpha^j_{\mathbf{x}_I} = 1$ for all $j \in J$ and $\displaystyle\sum_{j\in J} \sum_{\mathbf{x}_I \in X_I^j} \alpha^j_{\mathbf{x}_I} \mathbf{x}_I = \mathbf{y}_I$

has an integer optimum.

Conclusion

- by using duality to separate income and substitution effects, we analyze equilibrium in markets with income effects
- approach allows us to port equilibrium existence results from TU to settings with income effects via Equilibrium Existence Duality

- substitution effects fundamentally determine whether equilibria exist
- but income effects matter for finding equilibrium
  - e.g., net vs. gross substitutability and ascending vs. sealed-bid auctions

- potential applications: matching markets (Ravi's Lecture 4); package auctions with financing constraints. . .
- open question: algorithms to find equilibria?

Thank you!