The Geometry of Preferences: "Demand Types", Equilibrium with Indivisibilities, and Bidding Languages

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Oxford University: Hertford College and Nuffield College

Including work with Martin Bichler, Maximilian Fichtl, Paul Goldberg, Ravi Jagadeesan, Edwin Lock and Alex Teytelboym

September 2021

This work was supported by ESRC grant ES/L003058/1.

Overview: Preferences and Equilibrium via Geometry

- Address real-world situations in which new auction designs needed
- Use geometric approaches to represent bidders' preferences
  - Build them up of **simple** pieces.
  - Easy to understand and work with.
  - Aggregating these pieces can give wide classes of preferences.
- Develop new bidding languages

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  - Strong Substitutes Language
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  - Icelandic Auction Language

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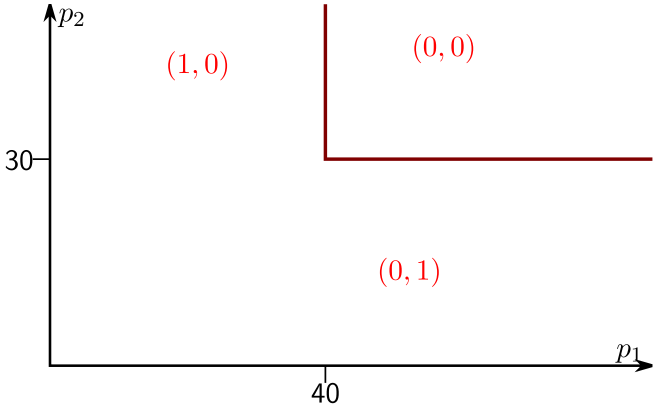

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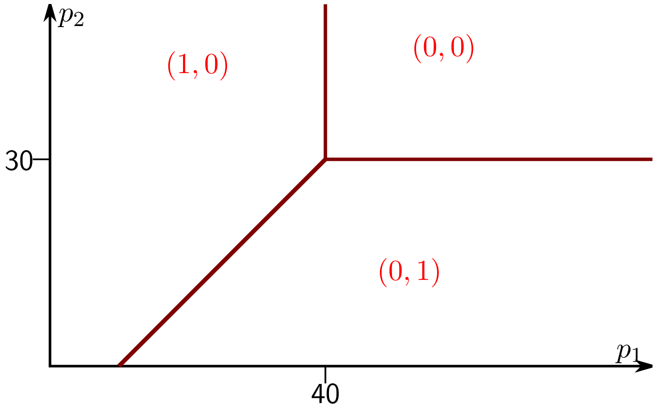
Introduction: The Hotel with 2 Rooms



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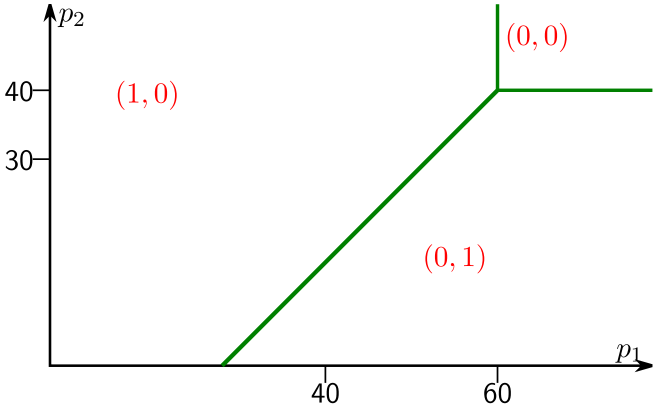
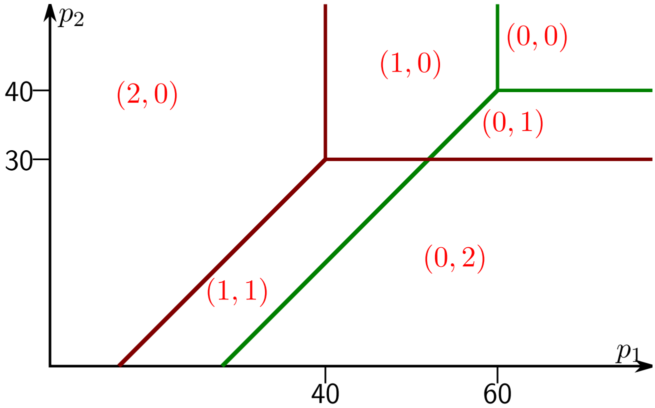


Alex is willing to pay at most £60 room 1, £40 for room 2.



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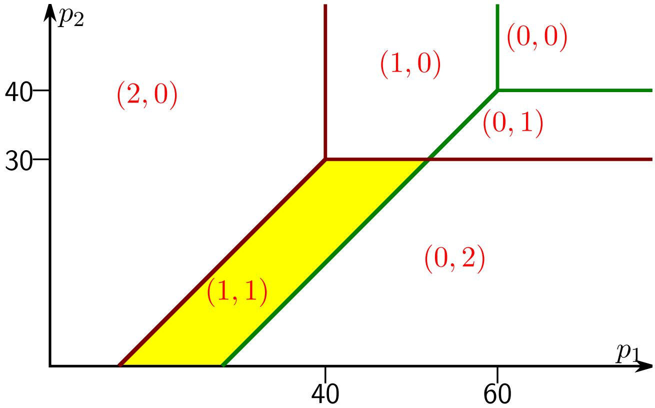

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There exist prices giving a competitive equilibrium.

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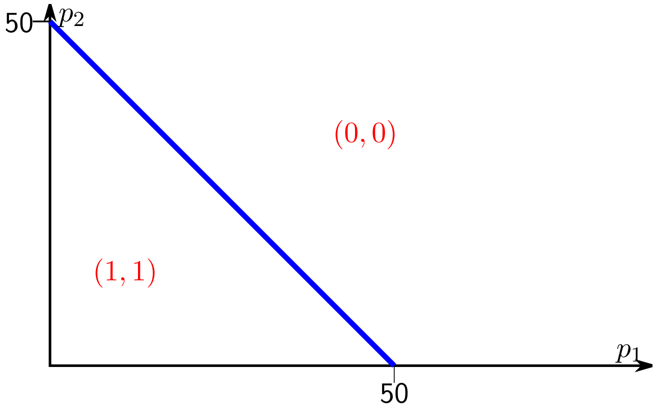

Paul is willing to pay at most £50 for **both** rooms.

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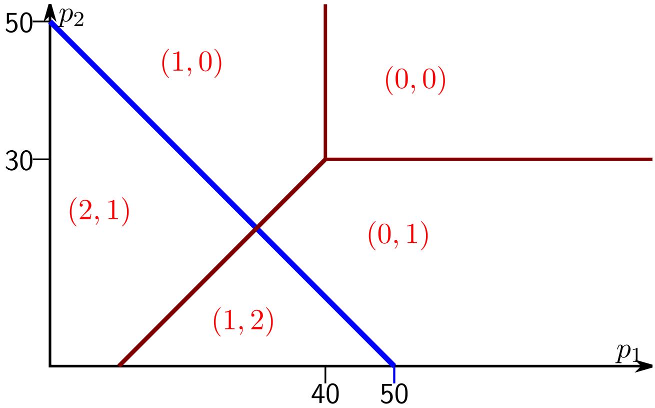

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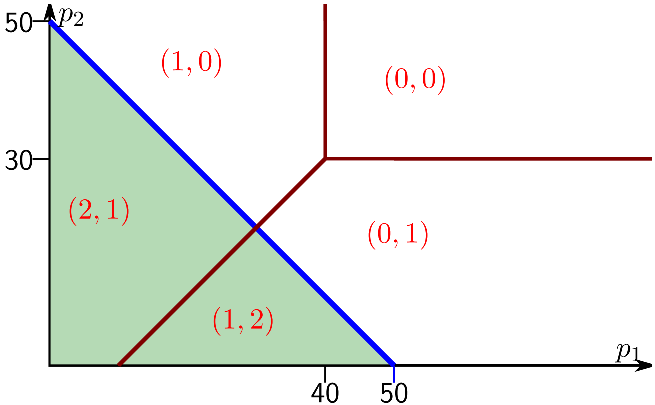

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If $p_1 + p_2 < 50$ then there is excess demand for hotel rooms.

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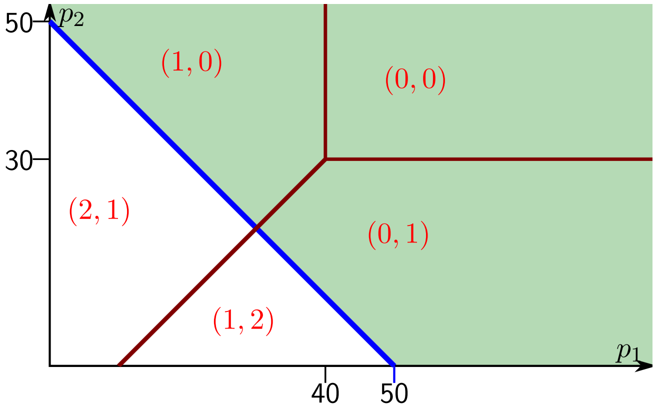

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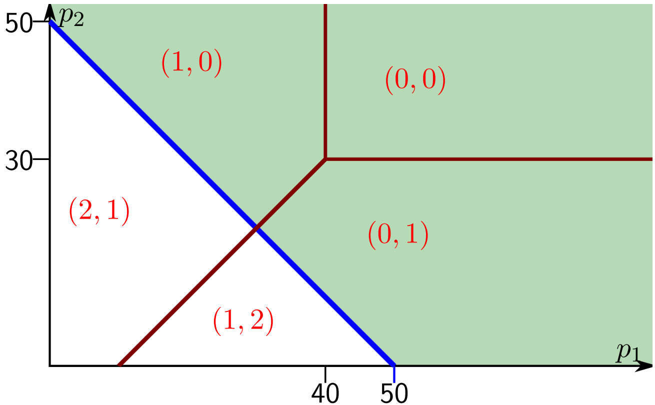

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If $p_1 + p_2 = 50$, Paul chooses between those situations.

Competitive equilibrium does not exist!

The Unimodularity Theorem

Competitive equilibrium:

- always exist if valuations 'look like' Elizabeth and Alex;
- sometimes fails if valuations 'look like' Elizabeth and Paul.
- Interpret what valuations 'look like' in properties that are:
  - economically meaningful;
  - mathematically useful

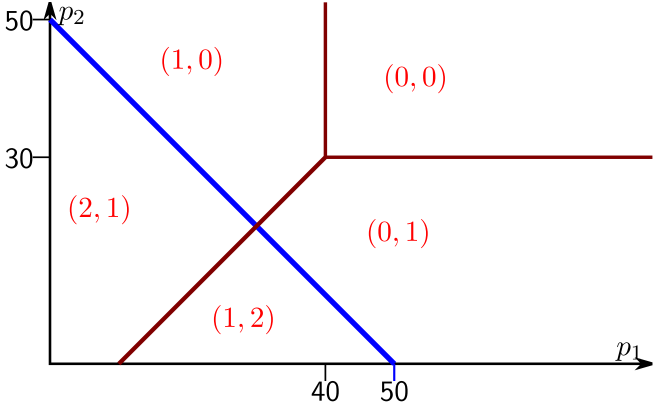
First Key Result

The Unimodularity Theorem

Competitive equilibrium:

- always exist if valuations ‘look like’ Elizabeth and Alex;
- sometimes fails if valuations ‘look like’ Elizabeth and Paul.
- Interpret what valuations ‘look like’ in properties that are:
  - economically meaningful;
  - mathematically useful
- Necessary and sufficient characterisation of such “properties” to guarantee existence of equilibrium:
  - easy to test;
  - exhibits entirely new classes.

Back to the hotel



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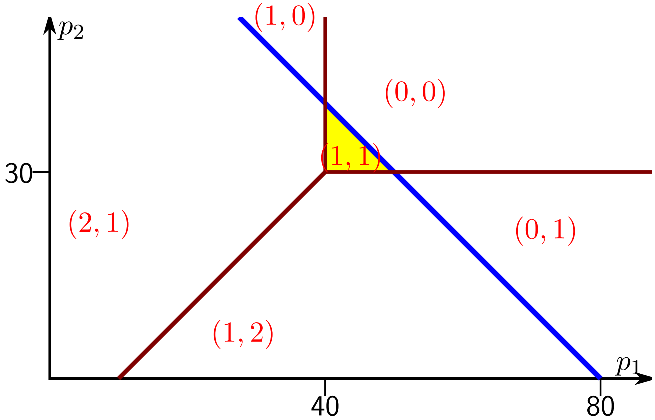

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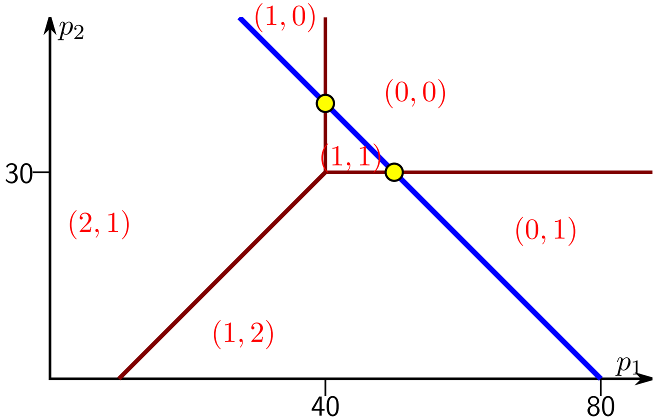

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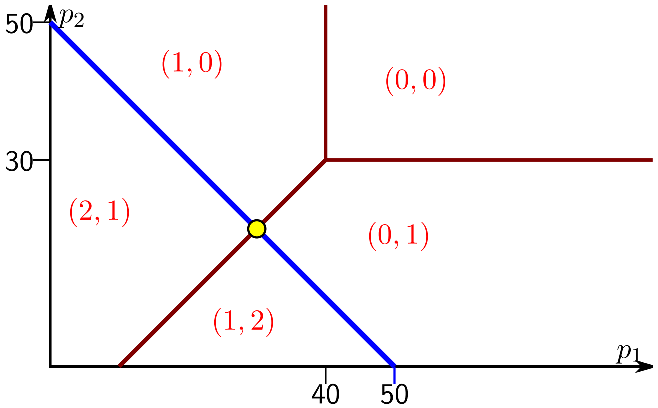

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There are two intersections between the figures drawn.
Previously there was only one intersection.

Second Result

The Intersection Count Theorem

- Given sets of bundles considered, predict max. number of intersections.
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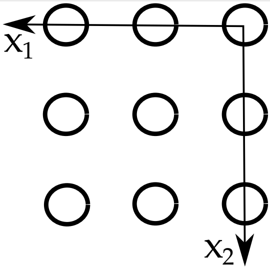
Outline of Talk

- **Individual** valuations and trade-offs
  - Understand economic properties, geometrically
  - Classify according to "type" of trade-offs.
- **Aggregations** of individual valuations
  - Understand easily, geometrically
  - Individual classifications extend.
- **Competitive equilibrium** between agents.
  - When guaranteed? **Why**?
  - How to efficiently check for even if not guaranteed?
- **Application: the Product-Mix Auction**



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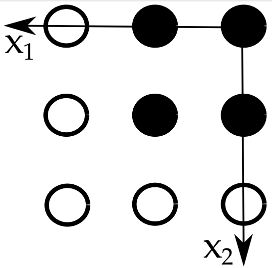

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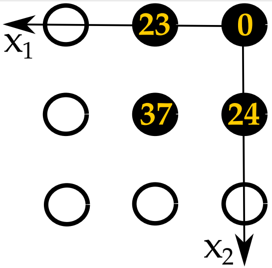
Geometric Analysis of Demand: Model

Example of $V^j(\mathbf{x})$



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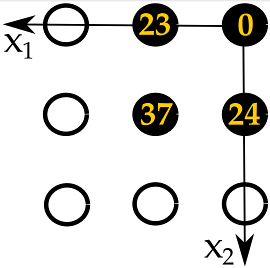

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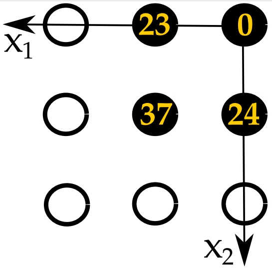

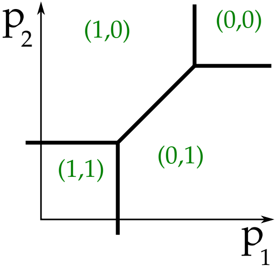

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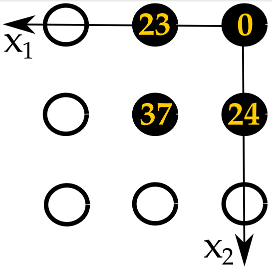

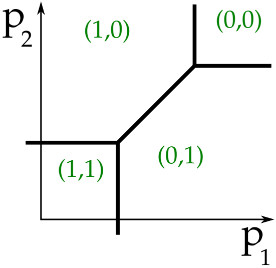

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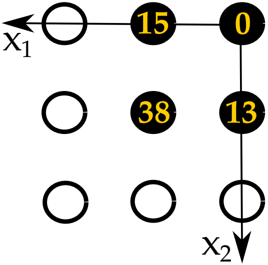

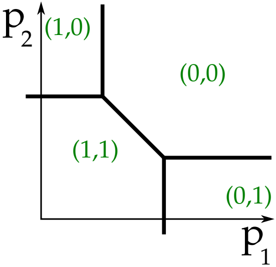

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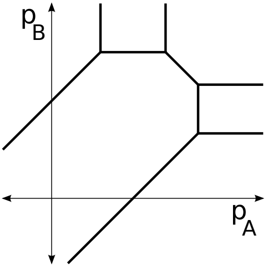

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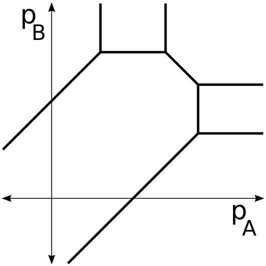

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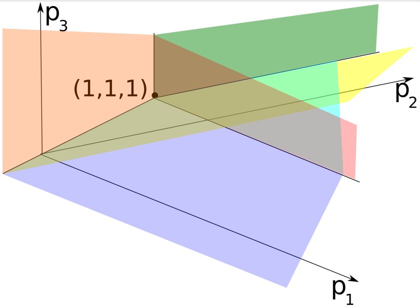

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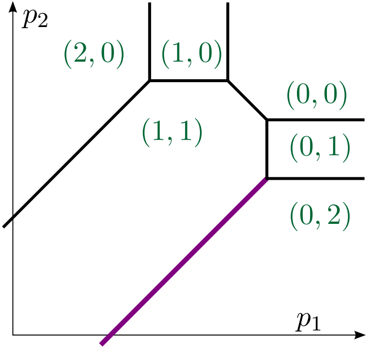
These meet in $(n-2)$-diml linear pieces, which meet in $(n-3)$-diml pieces.... the linear pieces of dimension k are the "k-cells".

How does demand change as you cross a facet?



If $\mathbf{p}$ is in a facet then the agent is indifferent between two bundles:

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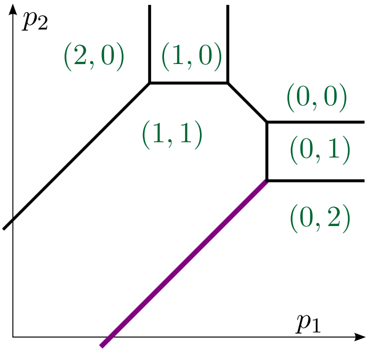

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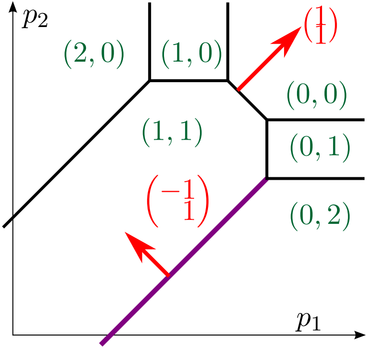

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The precise change in bundle is minus this direction times the ‘weight’.

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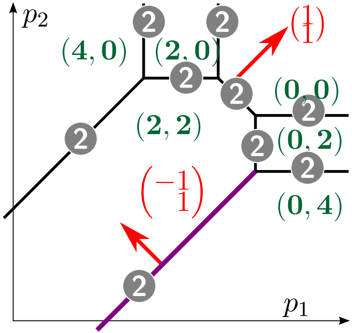

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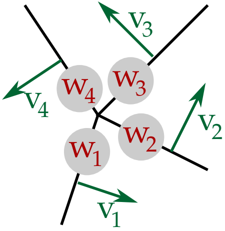

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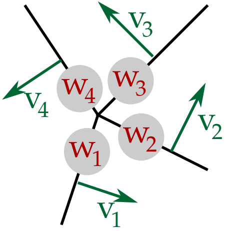
Economics from Geometry



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"Valuation-Complex Equivalence Theorem" (Mikhalkin, 2004)

A "weighted rational polyhedral complex of pure dimension $(n-1)$" forms a LIP of a valuation **iff** it is **balanced**.

- We need not write down valuations of discrete bundles.
- We can simply draw LIPs.

Project Aim understand economics via geometry.

Classifying valuations

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Let $\mathbf{x}^*(\mathbf{p})$ be optimal demands of each good at a given price.

- $\frac{\partial x_i^*}{\partial p_j} > 0$ means goods are 'substitutes' (tea, coffee).
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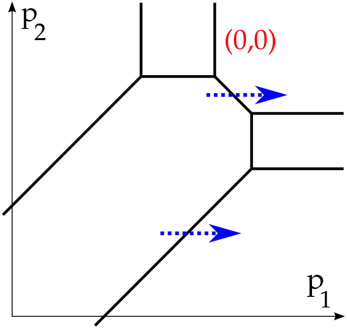
With LIPs, look first at discrete price changes that cross one facet.



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More general price changes?

Lemma ("The Law of Demand")

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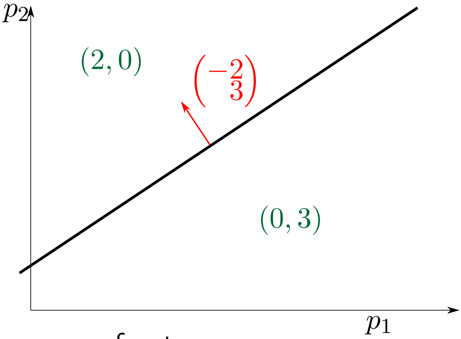

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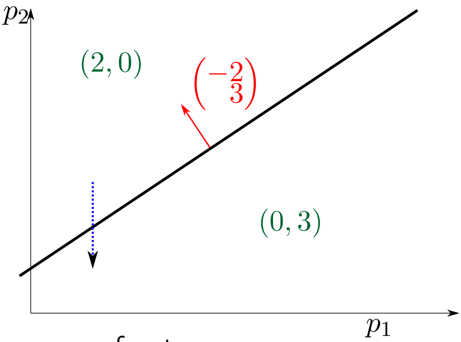

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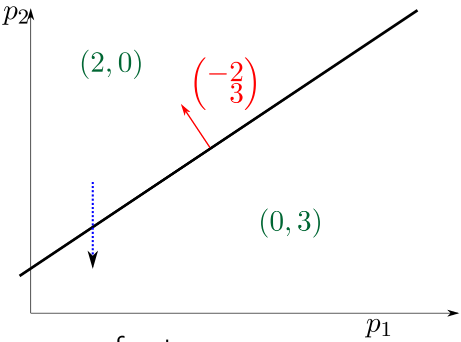

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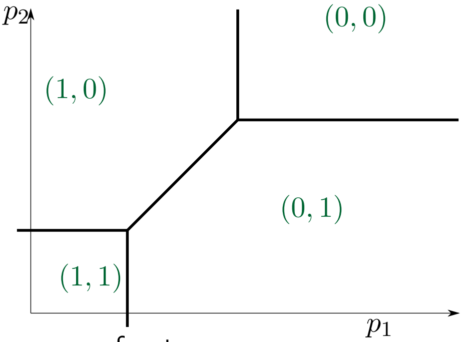

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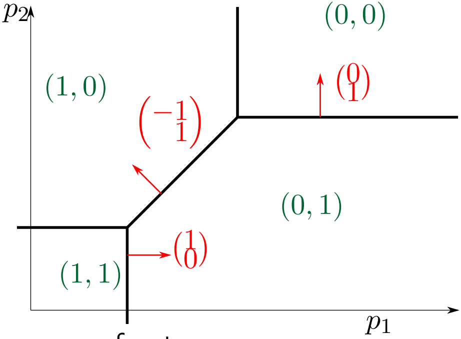

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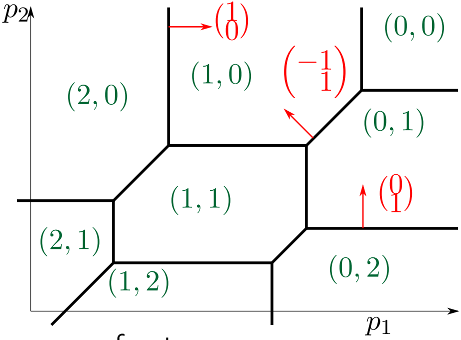

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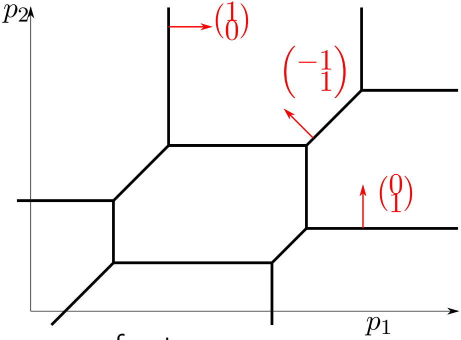

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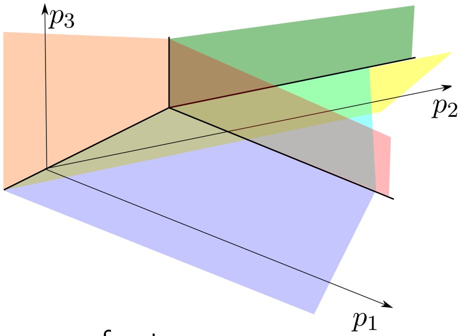

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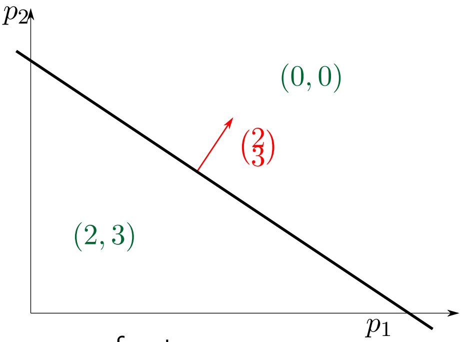

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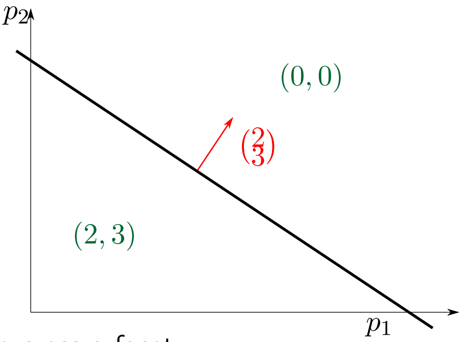

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COMPLEMENTS

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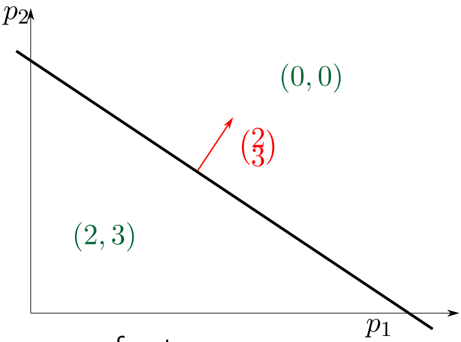

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Definition: "Demand Type"

V^j is **of demand type** $\mathcal{D}$ if every facet of $\mathcal{L}^j$ has normal in $\mathcal{D}$.

The demand type is the set of all such valuations.

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Work in preparation with Ravi Jagadeesan and Alex Teytelboym.

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Suppose $X^j \subseteq \{0,1\}^I$. *Valuation* V^j *is of demand type* $\mathcal{D}$ *iff:*
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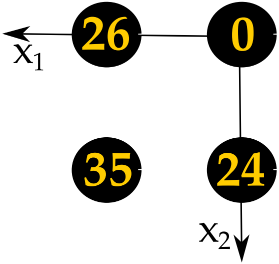

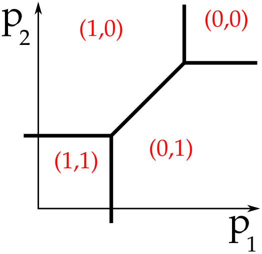

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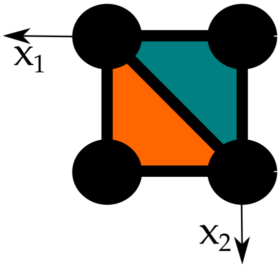

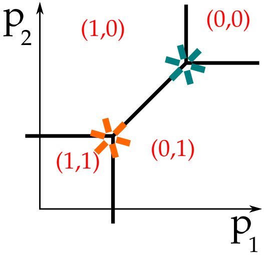

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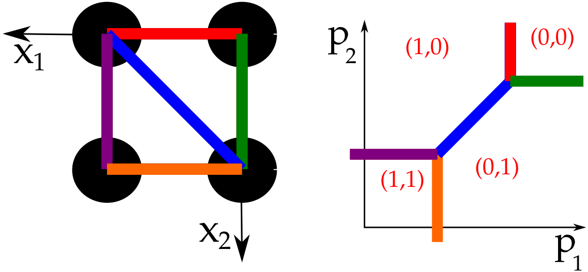

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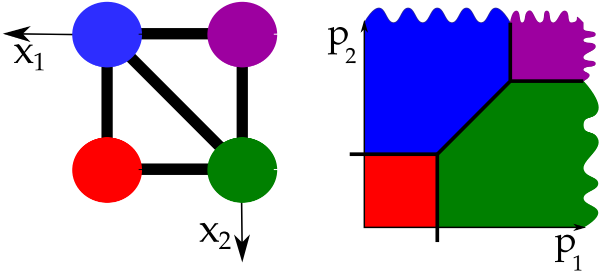

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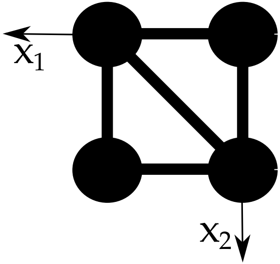

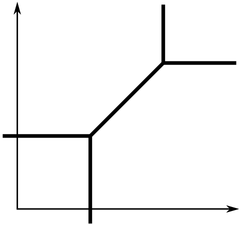

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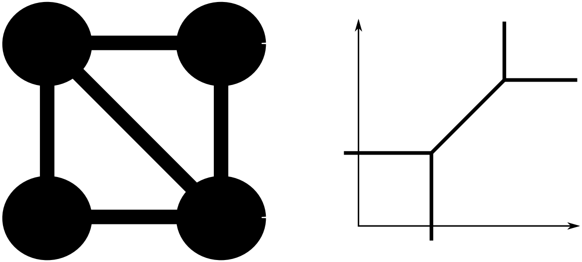

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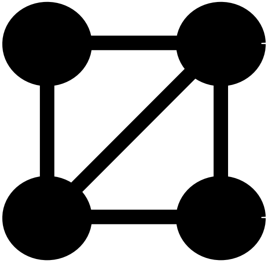

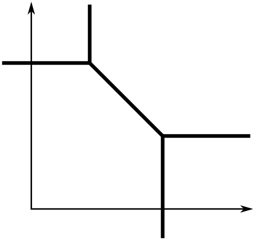

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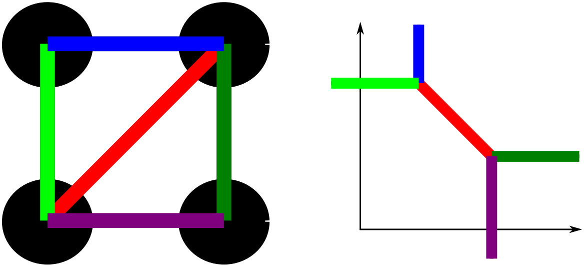

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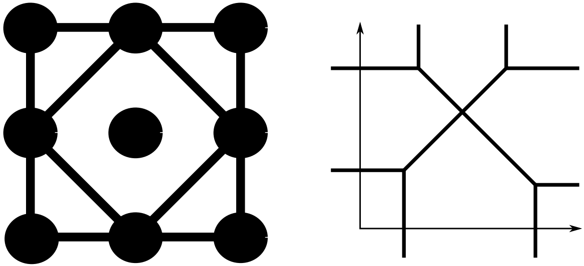

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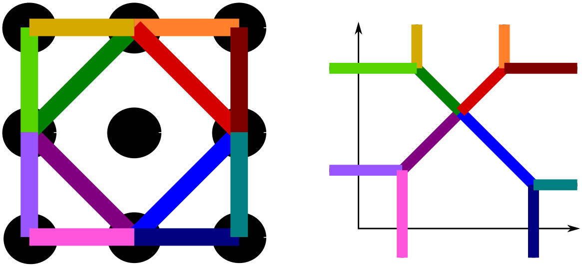

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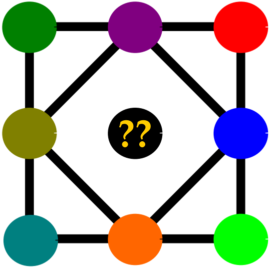

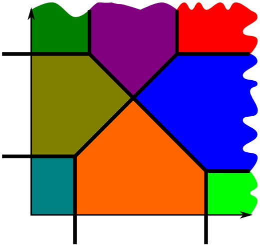

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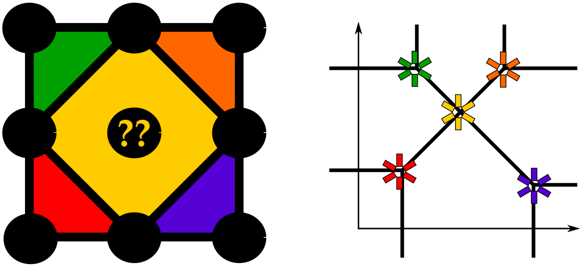

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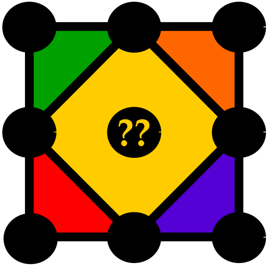

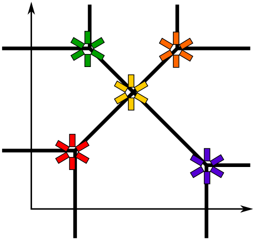

Lemma

Bundles are only (possibly) demanded at prices corresp. to the demand complex cell they're in.

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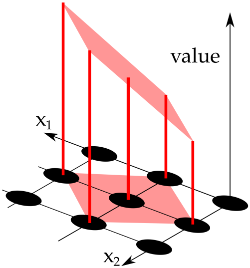

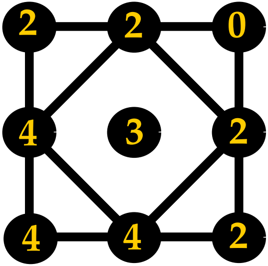

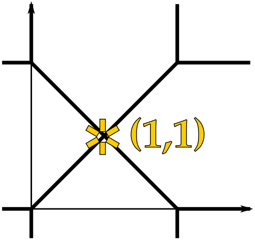

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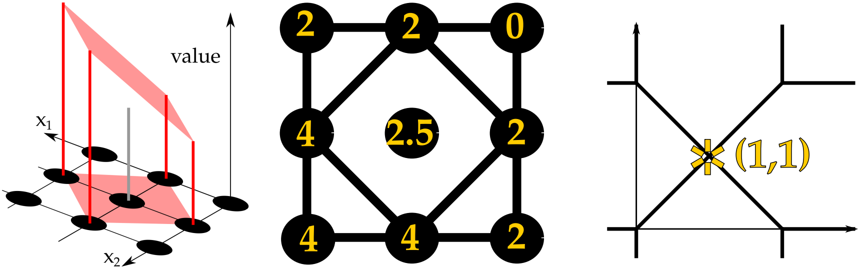

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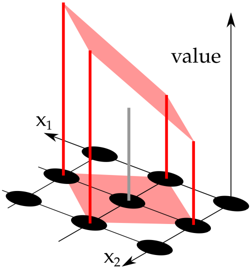

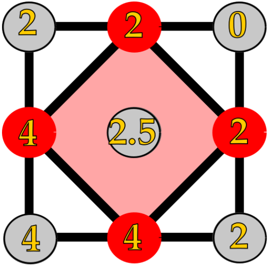

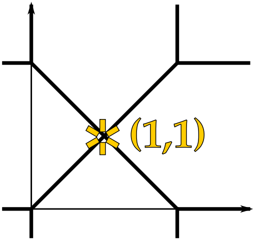

$D^j(1,1)$ not discrete-convex

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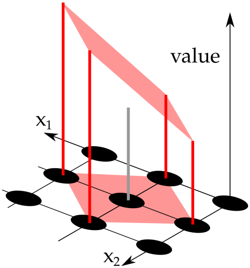
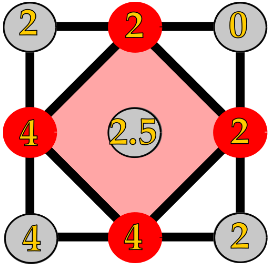


V^j not concave



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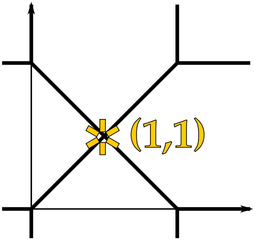

- $X \subset \mathbb{Z}^n$ is *discrete-convex* if $\mathrm{Conv}(X) \cap \mathbb{Z}^n = X$.
- $V^j : X^j \to \mathbb{R}$ is *concave* if X^j discrete-convex and can extend V^j to weakly-concave $\mathrm{Conv}(V^j) : \mathrm{Conv}(X^j) \to \mathbb{R}$.

Duality: The Demand Complex

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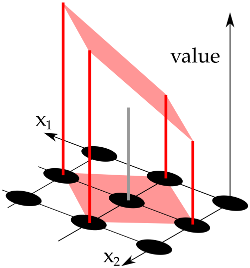

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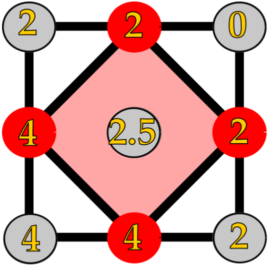

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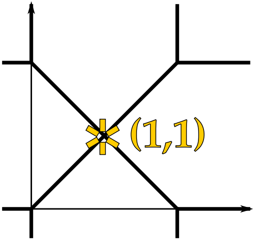

Lemma (Standard)

Every $\mathbf{x} \in \mathrm{Conv}(X^j) \cap \mathbb{Z}^n$ *demanded iff* V^j *is concave.*

iff $D^j(\mathbf{p})$ *discrete-convex for all* $\mathbf{p}$*.*

Aggregate demand and equilibrium

Agents $j \in J$. Valuations $u^j : X^j \to \mathbb{R}$. Domains $X^J = \sum_{j \in J} X^j$.

Aggregate demand set $\sum_{j \in J} D^j(\mathbf{p})$

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Translation to competitive equilibrium from Alex's talk:

The same as competitive equilibrium in an exchange economy if $\mathbf{x} = \mathbf{0}$.

Otherwise, can include an additional agent whose domain is $\{-\mathbf{x}\}$.

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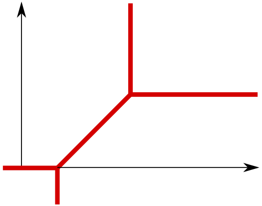
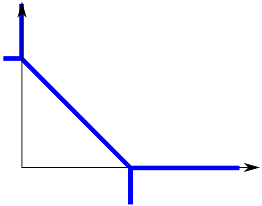
Call supplies in $\mathrm{Conv}(X^J) \cap \mathbb{Z}^n$ “relevant”.

$$D^J(\mathbf{p}) = \sum_{j \in J} D^j(\mathbf{p})$$

Easy to draw $\mathcal{L}^J$,

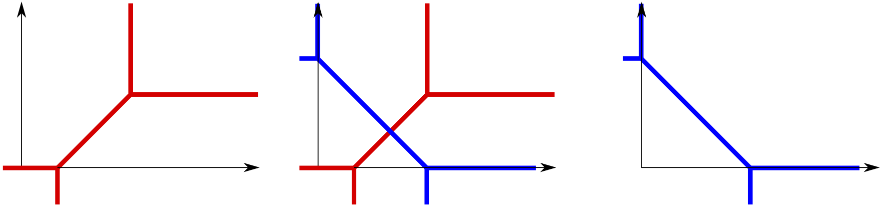




LIP of aggregate demand

$$D^J(\mathbf{p}) = \sum_{j \in J} D^j(\mathbf{p})$$

Easy to draw $\mathcal{L}^J$, just superimpose individual LIPs.



Corollary

If V^j are of demand type $\mathcal{D}$ for all $j \in J$ then so is V^J.

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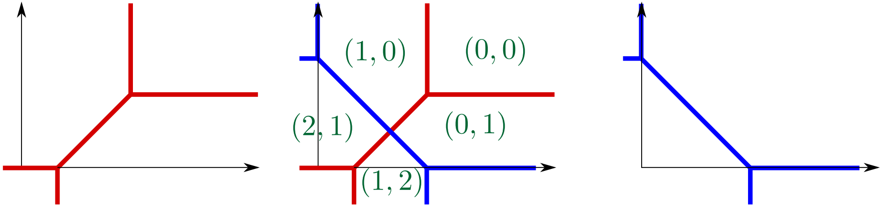

Then what is $D^J(\mathbf{p})$?

- If $\mathbf{p} \notin \mathcal{L}^J$, easy: use "facet normal $\times$ weight $=$ change in demand".
- If $\mathbf{p} \in \mathcal{L}^j$, only one j, and individual valuations concave, also easy.
- Interesting case: $\mathbf{p} \in \mathcal{L}^j, \mathcal{L}^k$ for $j \neq k$.

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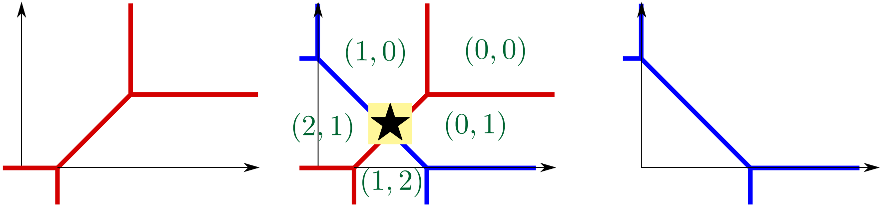

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Lemma

If individual valuations concave, equilibrium fails iff $D^J(\mathbf{p})$ not discrete-convex at some $\mathbf{p}$ in the intersection.

Classic theorems of competitive equilibrium

Theorem (Kelso and Crawford 1982)

Suppose

- *domain* $X^j = \{0,1\}^n$ *for all agents* j.
- $V^j : X^j \to \mathbb{R}$ *is a concave* substitute *valuation for all agents.*
- *Supply* $\mathbf{x} \in \{0,1\}^n$.

Then competitive equilibrium exists.

Classic theorems of competitive equilibrium

Theorem (Milgrom and Strulovici 2009)

Suppose

- *domain* $X^j = X$*, a fixed product of intervals*, *for all agents* j.
- $V^j : X^j \to \mathbb{R}$ *is a concave* *strong substitute* *valuation for all agents.*
- *Supply* $\mathbf{x} \in X$.

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Classic theorems of competitive equilibrium

Theorem (Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp 2013)

Suppose

- *domain* $X^j \subset \{-1, 0, 1\}^n$ *for all agents* j.
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Seek generalised result of this form:

Suppose we fix a 'description'.

- Agents all have concave valuations of this description.
- Supply is in the domain of their aggregate demands.

Ask: does competitive equilibrium always exist?

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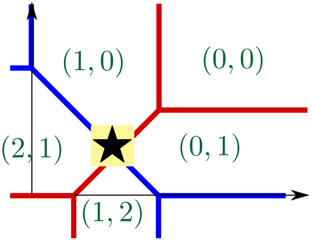
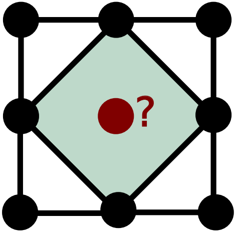
Yes, iff $\mathcal{D}$ has a certain property. . .

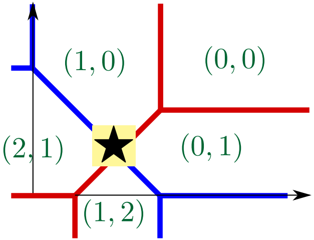
Aggregate demand and equilibrium

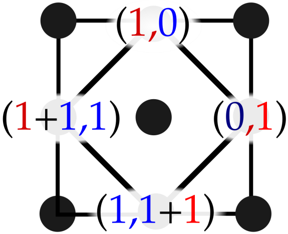




Is $D^J(\bigstar)$ discrete-convex?

Aggregate demand and equilibrium





Is $D^J(\bigstar)$ discrete-convex?

- At price ★,
  - Red demands $(1, 0)$ or $(0, 1)$
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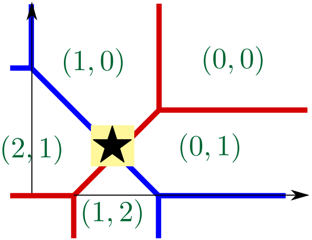
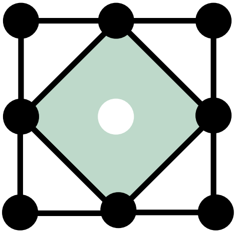
- There is no way to demand the bundle in the middle.

Aggregate demand and equilibrium





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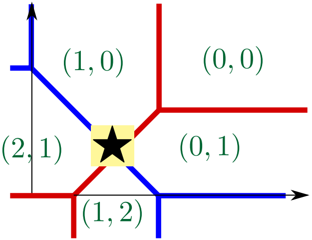
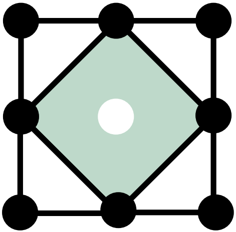
NO!

Aggregate demand and equilibrium





Area=2.

- There *exists* a non-vertex bundle because the square's *area* is > 1.
- The *area* is (abs. value of) the determinant of vectors along its edges.

$$\det \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = 2$$

- Avoid problems iff all sets of n demand type vectors have det ± 1 or 0.

$\Rightarrow$ **"unimodularity"***.

*When vectors in $\mathcal{D}$ span $\mathbb{R}^n$, unimodularity $\Leftrightarrow$ all sets of n vectors have det ± 1 or 0.

The Unimodularity Theorem

"Unimodularity Theorem"

Fix a set $\mathcal{D} \subsetneq \mathbb{Z}^n$. A competitive equilibrium exists for

- every finite set of agents with concave valuations of type $\mathcal{D}$
- all relevant supply bundles

iff $\mathcal{D}$ is unimodular.

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Can also show (with Omer Edhan, Ravi Jagadeesan and Alex Teytelboym) that if $\mathcal{D}$ is a **maximal unimodular set of vectors** then it defines a **maximal domain of valuations** such that equilibrium exists.

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From this, follows existence of equilibrium in:

- Gross substitutes (Kelso and Crawford, 1982, ECMA).
- Step-wise / Strong substitutes (Danilov et al., 2003, Discrete Applied Math., Milgrom and Strulovici, 2009, JET).
- Gross substitutes and complements (Sun and Yang, 2006, ECMA).
- Full substitutability on a trading network (Hatfield et al. 2013, JPE).

Cf. Danilov et al. (2001), Danilov and Koshevoy (2004) for sufficiency.

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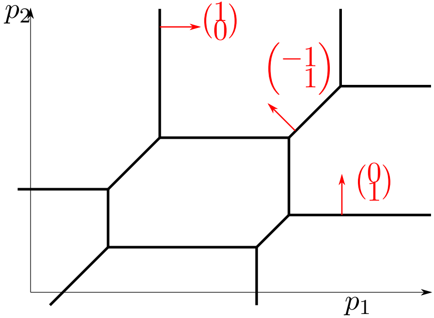
$\mathcal{D}^n_{ss} \subset \mathbb{Z}^n$ vectors have at most one $+1$, at most one -1, otherwise 0s. Substitutes where trade-offs are 1-1.

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$$



- Unimodular set (classic result).
- Equilibrium always exists
- Model of Kelso and Crawford (1982), Danilov et al. (2003), Milgrom and Strulovici (2009), Hatfield et al. (2013).
- The model of Sun and Yang (2006) is a basis change.

Unimodular examples: Strong / step-wise substitutes

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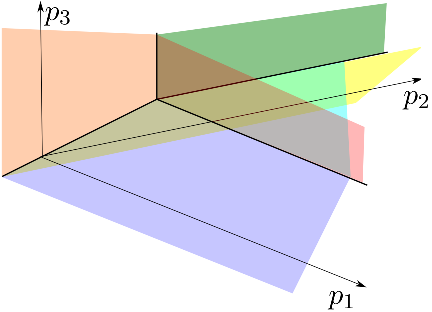

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Beyond strong substitutes

But (strong) substitutes are *not* necessary for equilibrium when $n \geq 4$:

- Have unimodular demand types, **not** a basis change of substitutes.
- All unimodular demand types **are** a basis change of complements!

Beyond strong substitutes

But (strong) substitutes are *not* necessary for equilibrium when $n \geq 4$:

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Smallest example: let $\mathcal{D}$ be the columns of:

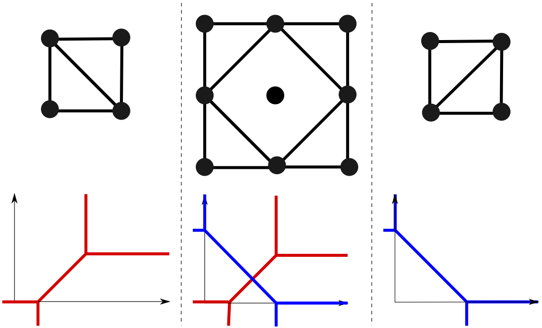
$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} \text{front-line workers} \\ \left.\vphantom{1}\right\} \text{manager} \end{array}$$

Interpretation:

- The first three goods (rows) represent front-line workers.
- The final good (row) is a manager.
- 'Bundles', i.e. teams, worth bidding for, are:
  - a worker on their own (*not* a manager on their own);
  - a worker and a manager;
  - two workers and a manager.

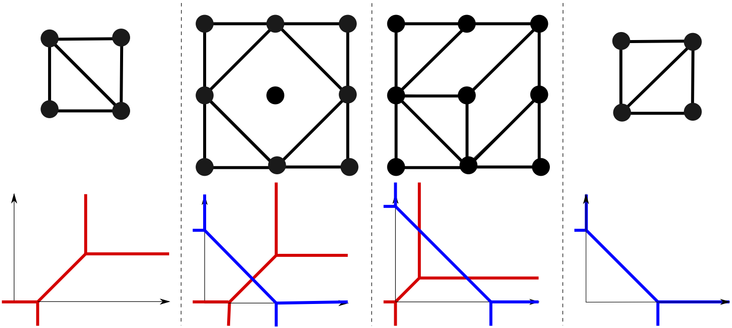
Interpret as coalitions: **model matching with transferable utility.**

The Intersection Count Theorem



Return to substitutes / complements example.

The Intersection Count Theorem



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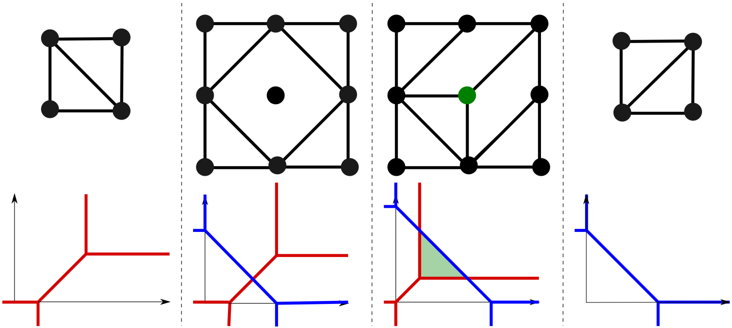
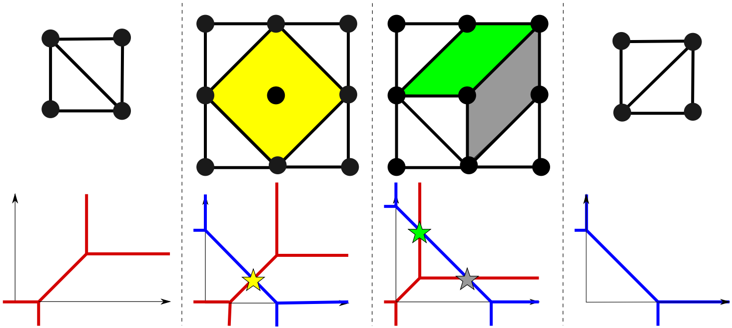


Return to substitutes / complements example. Modify the valuations. Now:

- Bundle $(1, 1)$ **is** demanded for some prices.
- Every bundle is demanded for some prices.

The Intersection Count Theorem



Before the shift

- One intersection.
- Demand complex cell area 2.

After the shift

- Two intersections.
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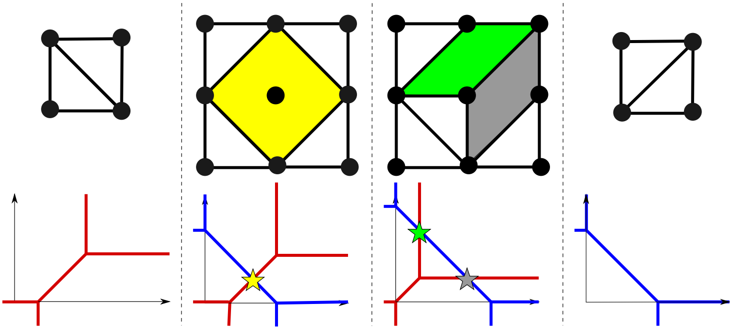
The Intersection Count Theorem



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Up to multiplicity, # of intersections is constant

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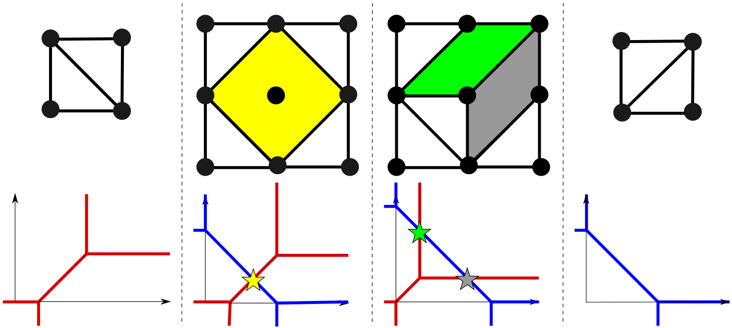

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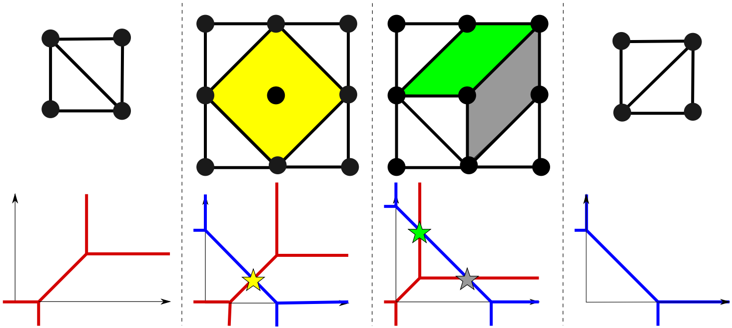

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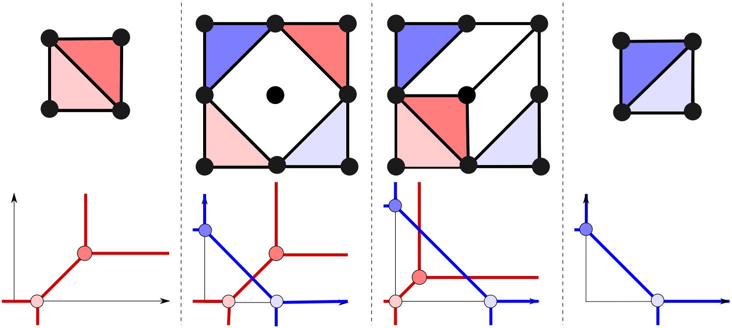

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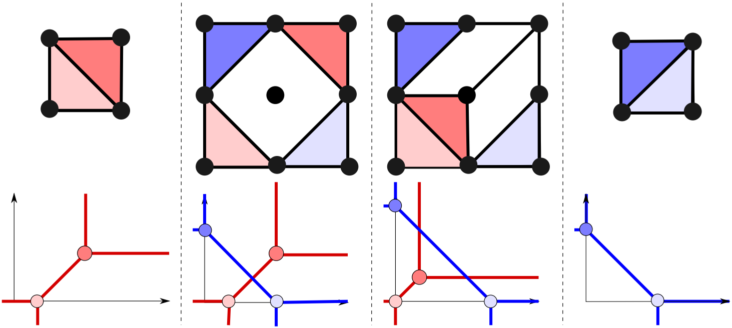

Theorem

When $n = 2$, and intersection is 'transverse', then equilibrium exists for all relevant supply bundles iff $\#$ intersections, weighted by product of facet weights, equals $\Gamma^2(X^1, X^2)$.

Higher Dimensions

Summary so far

- **Individual** valuations and trade-offs
  - Understand geometrically
  - Classify according to "type" of trade-offs.
- **Aggregations** of individual valuations
  - Understand easily, geometrically
  - Individual classifications extend.
- **Competitive equilibrium** between agents.
  - When guaranteed? **Why**?
  - How to efficiently check for even if not guaranteed?

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    Implementing Walrasian Equilibrium: The Language of Product-Mix Auctions

Product-Mix Auctions

- Address real-world situations in which new auction designs needed
- Use geometric approaches to represent bidders' preferences
  - Build them up of **simple** pieces.
  - Easy to understand and work with.
  - Aggregating these pieces can give wide classes of preferences.
- Develop new bidding languages
  - Bank of England Language
  - Strong Substitutes Language
  - All Substitutes Language

  } "Tropical Languages" (Bank of England, Strong Substitutes and All Substitutes Languages)
  - Icelandic Auction Language — "Arctic Language"

Bank of England Problem

After Northern Rock bank run, Bank of England urgently wants to loan funds to banks, etc., – willing to take weaker-than-usual collateral, but only in return for higher interest rate.

i.e., wanted to sell related goods to banks (loans against different kinds of collateral: “strong” (UK / US sovereign debt), “weak” (mortgage-backed securities?!), etc.

Central Bank of Iceland's Problem

After financial crisis Iceland imposed capital controls. How to exit?

Central Bank of Iceland planned to buy back the "offshore" accounts they had blocked. Offer owners three choices of bonds or cash.

June 2015: CBI announces it will use a Product-Mix Auction

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April 2016: "Panama papers" reveal Prime Minister's wife has money in such an account herself. Plan is abandoned.

General Problem

Supplier wants to sell multiple versions of a product: multiple "goods".

Seller costs depend on bundle of goods sold. So their preferred bundle to sell depends on prices on **all** goods.

Bidders' demand depends on prices on **all** goods.

Reason to prefer a sealed bid mechanism.

Product-Mix Auctions: Basic Steps and Issues

1. Gather bid data

2. Find prices and allocations

Product-Mix Auctions: Basic Steps and Issues

1. **Gather bid data**
    - What form of preferences are relevant and allowed?
    - How should preferences be communicated?
    - How can bidders think about and derive their own preferences?
    - Does simplicity of bid data restrict the class of preferences?
    - Is that bid data in a reasonable form to aggregate?

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2. Find prices and allocations

- What is the objective - profit maximisation or equilibrium?
- How can we include seller preferences?
- Do allowed preferences ensure competitive equilibrium exists?
- Can we find that equilibrium in reasonable time?

Discrete Convex Analysis approaches, and related work

Kelso and Crawford (1982), Murota and co-authors (long literature); Milgrom (2000), Ausubel (2006); Paes Leme and Wong (2015)

- Focus on **finding Walrasian equilibrium**
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Existing Approaches

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"Bidding language" approaches

Milgrom (2009); Nisan (2006); Klemperer (2008, 2010)

- Focus on **gathering bid data**
- Limitations on the form of preferences that may be communicated
- Limitations on tractability of algorithms described.

All in context of "strong substitute" ($M^\natural$-concave) preferences

Bank of England Bidding Language Assumptions

Goods

- Divisible or indivisible

Bidder valuations

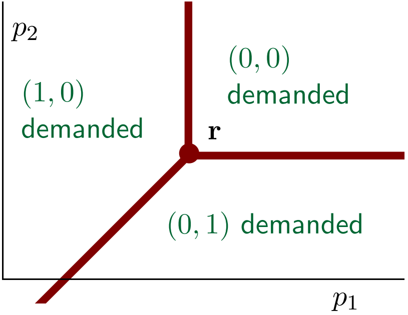
- Associated integer valuation is for strong substitutes
- Valuations break down as simple "either/or" trade-offs.

Sellers

- Maximise efficiency
- Considerable flexibility in preferences

A single dot bid at $\mathbf{r}$ represents valuation $V^{\mathbf{r}}$

$$V^{\mathbf{r}}(\mathbf{0}) = 0, \quad V^{\mathbf{r}}(\mathbf{e}^i) = r_i$$



- Bid for at most one unit. Gul and Stacchetti (1999) "unit demand"
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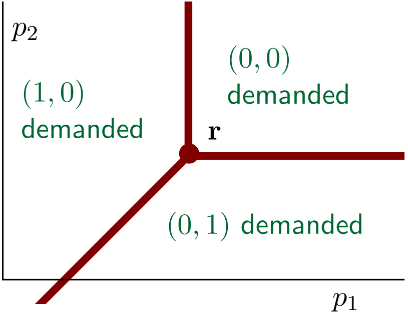

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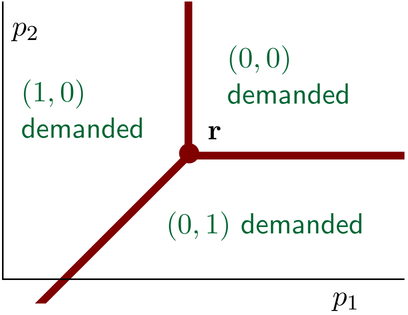

Associate with $V^{\mathbf{r}}$ simple LIP $\mathcal{L}^{\mathbf{r}}$, with facets:

- Where bidder indifferent between nothing and unit of good i
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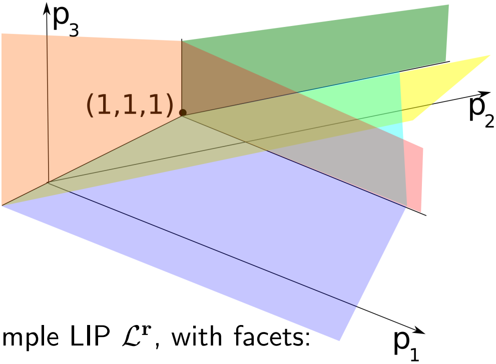

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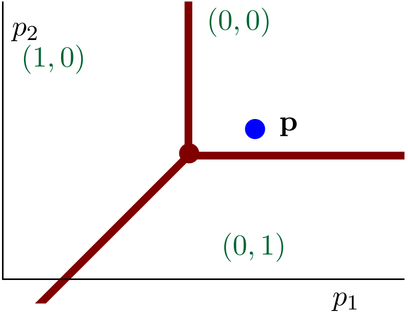

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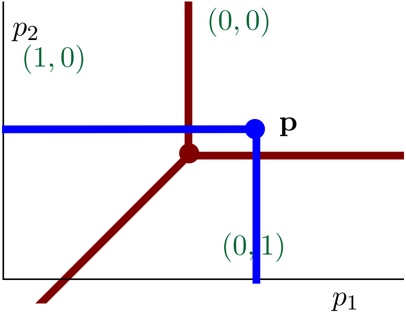

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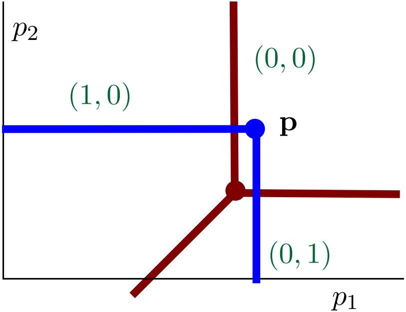

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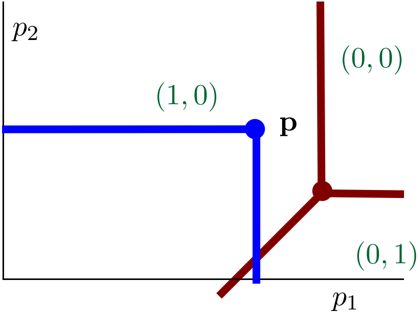

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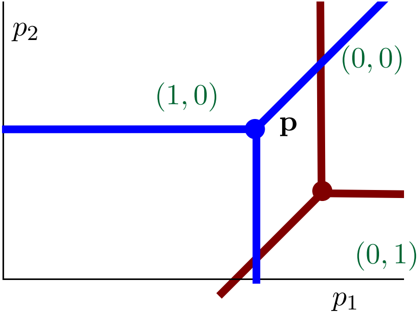

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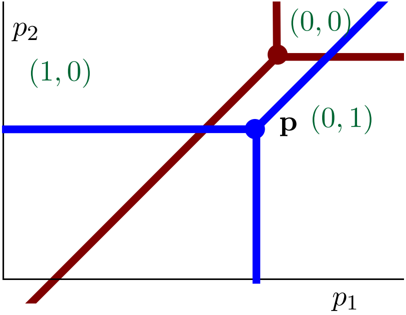

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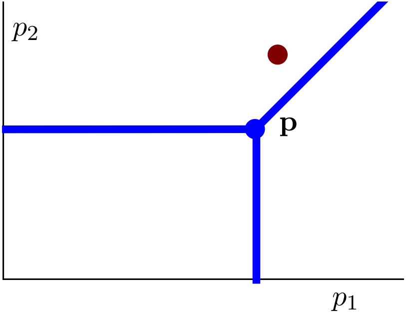

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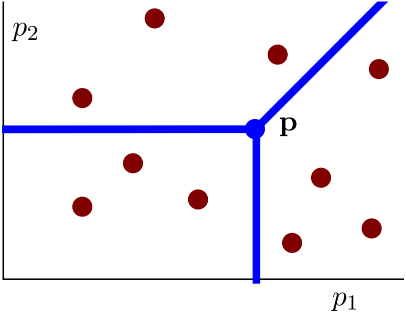

$D^j(\mathbf{p}) = (4, 3)$

Given a price:

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- Or accept on the most favourable good.
- Aggregate demand is easy to find.

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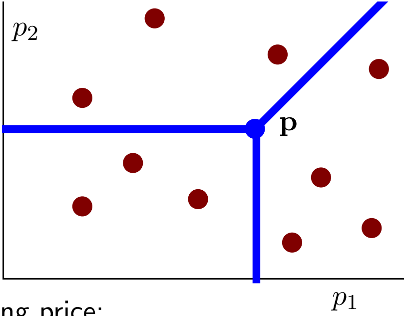

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Finding market clearing price:

- Optimise individual bids via linear / integer programming
- Aggregate these linear programs by adding them up

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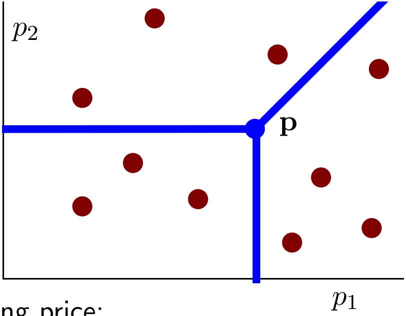
Easy to optimise



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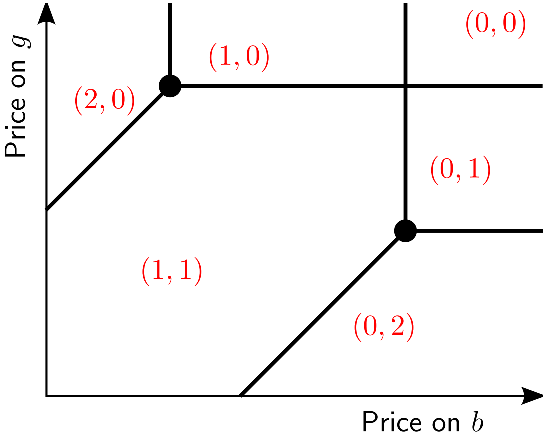
Need for the Strong Substitute Bidding Language

So we can depict any valuation like this, in any dimension.



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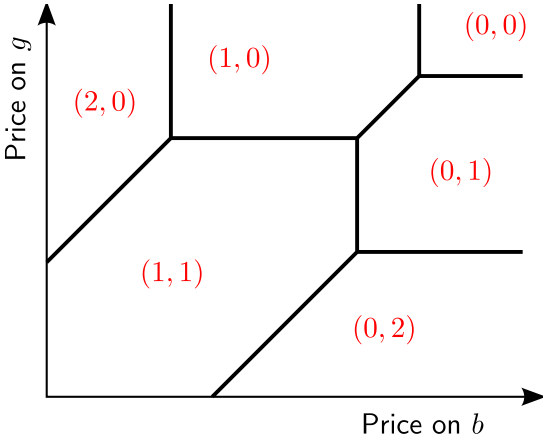

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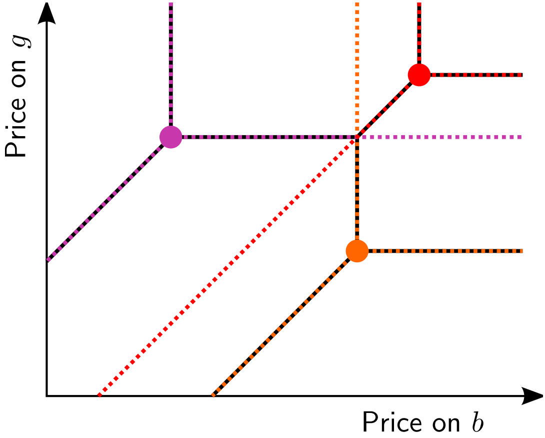

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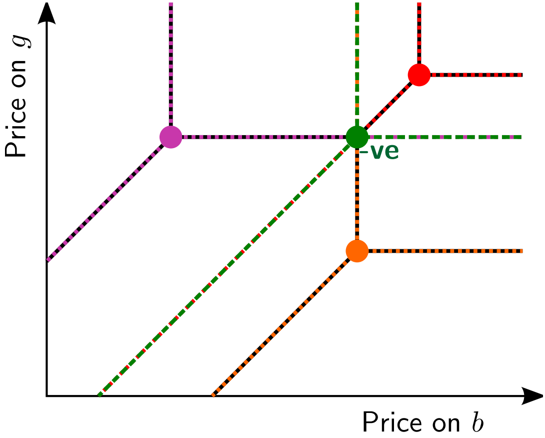

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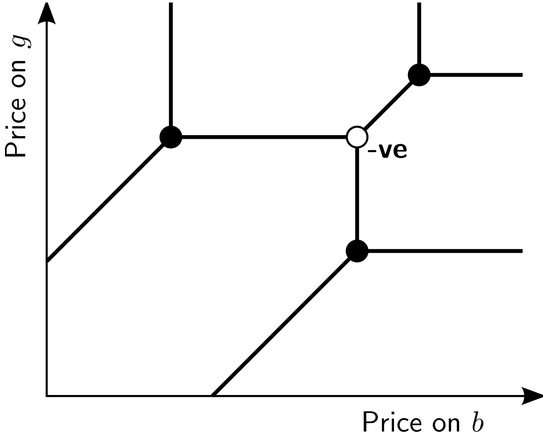

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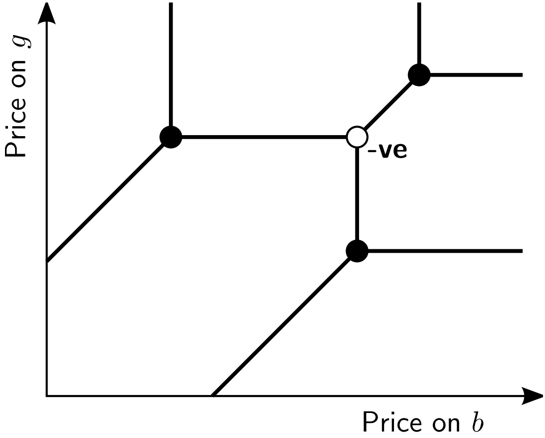
Works if we "subtract a bit"
But what does that mean?



Strong Substitute Bidding Language: The Objective

Want to break the figure down with the dots:

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Pay-off: **depict all preferences for strong substitutes.**

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Bidder valuations

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Sellers

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Multiple Dot Bids: "Addition" of LIPs

A collection of positive dot bids $\mathbf{r} \in \mathcal{R}$

- $\Leftrightarrow$ Aggregate valuation of $\{V^{\mathbf{r}}, \mathbf{r} \in \mathcal{R}\}$.
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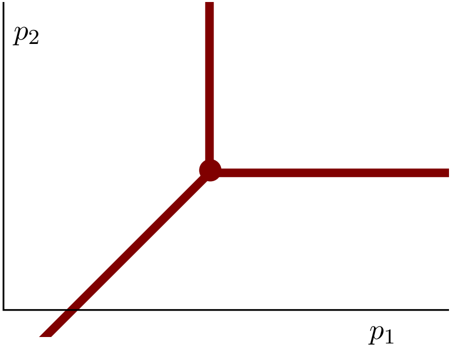

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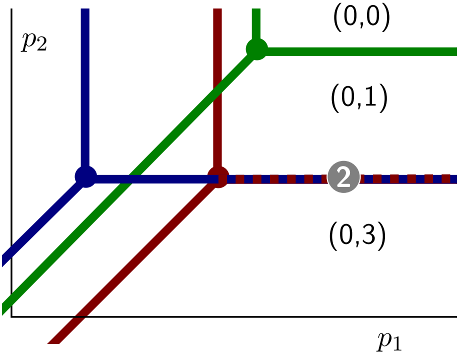

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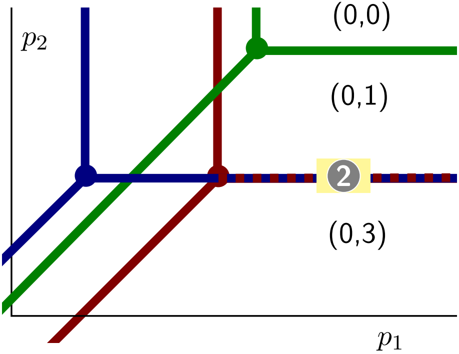
- $\Leftrightarrow$ LIP $\mathcal{L}^{\mathcal{R}} = \bigcup_{\mathbf{r} \in \mathcal{R}} \mathcal{L}^{\mathbf{r}}$
  the weights are the number of dot bids associated with each facet



Multiple Dot Bids: "Addition" of LIPs

A collection of positive dot bids $\mathbf{r} \in \mathcal{R}$

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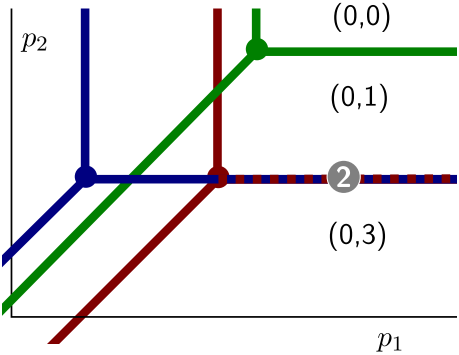
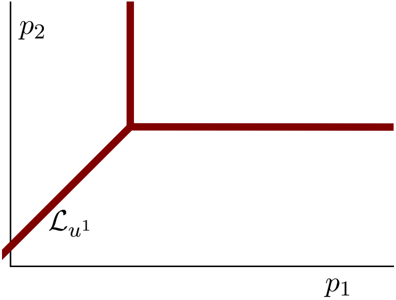


Write $(\mathcal{L}^{\mathcal{R}}, \mathbf{w}) = \boxplus_{\mathbf{r} \in \mathcal{R}} (\mathcal{L}^{\mathbf{r}}, \mathbf{1})$. "Addition" of LIPs

Similarly, we can formally subtract

$$(\mathcal{L}, \mathbf{w}) = (\mathcal{L}^1, \mathbf{w}^1) \boxminus (\mathcal{L}^2, \mathbf{w}^2)$$



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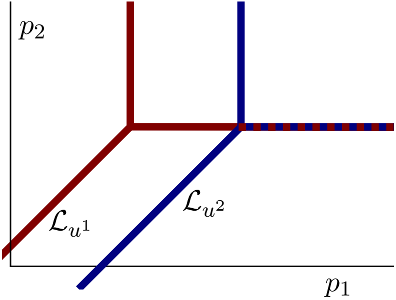

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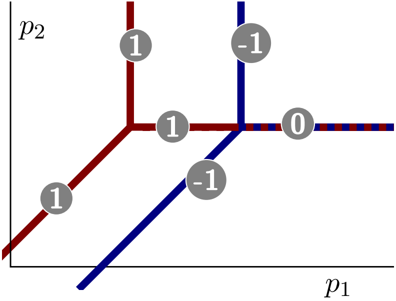

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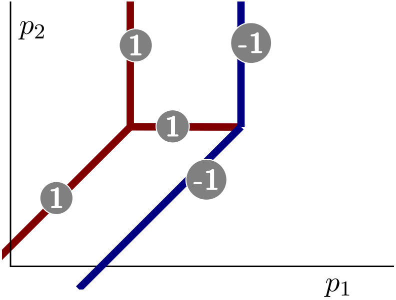

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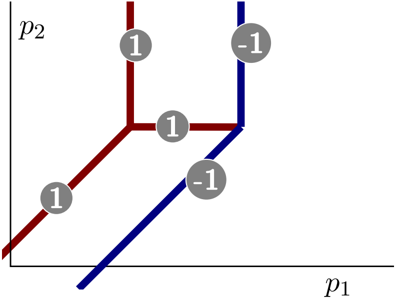

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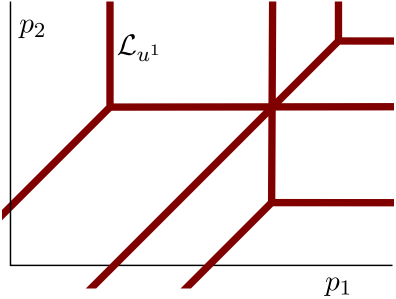

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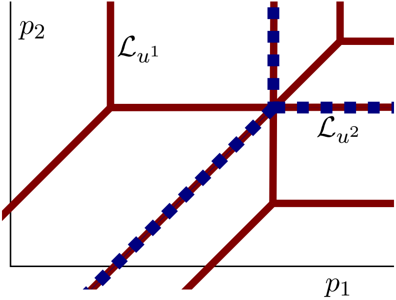

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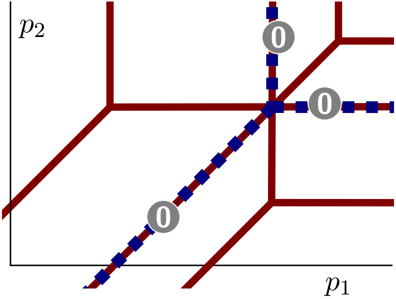

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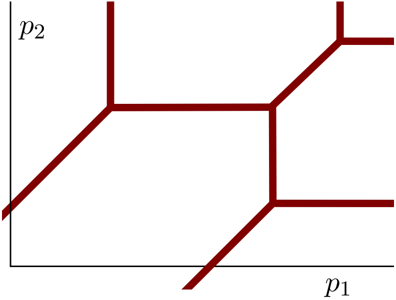

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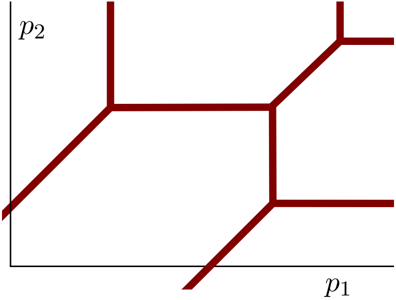

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$$(\mathcal{L}^{\mathcal{R}-\mathcal{S}}, \mathbf{w}^{\mathcal{R}-\mathcal{S}}) := (\mathcal{L}^{\mathcal{R}}, \mathbf{w}^{\mathcal{R}}) \boxminus (\mathcal{L}^{\mathcal{S}}, \mathbf{w}^{\mathcal{S}}).$$

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Definition

A collection of positive **and negative** dot bids are **valid** if $\mathbf{w}^{\mathcal{R}-\mathcal{S}} \geq 0$.

$\mathcal{L}^{\mathbf{r}}$ is strong subs, so by valuation-complex equivalence theorem:

If bids are valid, they generate a strong substitute valuation.

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Translating $\mathcal{R}, \mathcal{S}$ to $V^{\mathcal{R}-\mathcal{S}}$ is convoluted.

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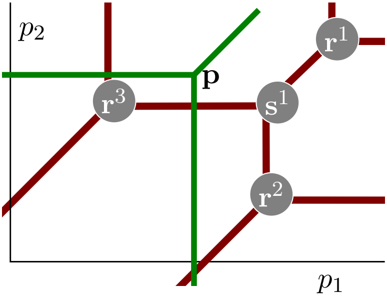

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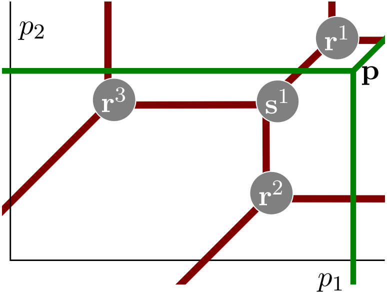

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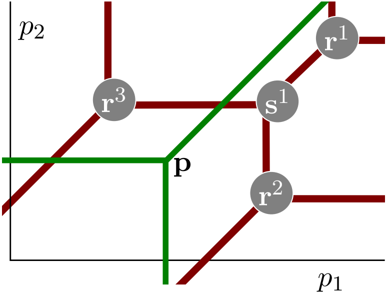

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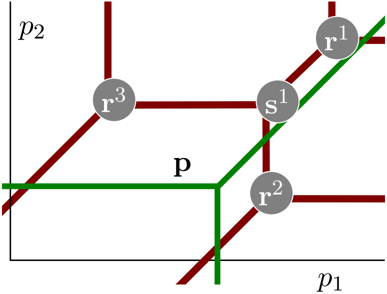

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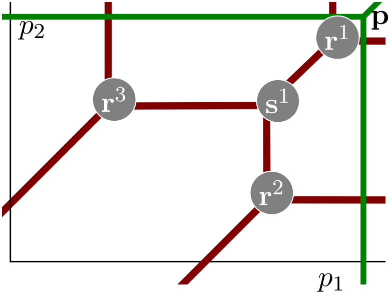

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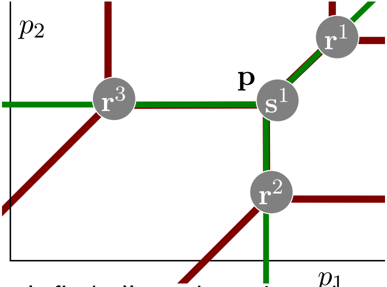

$D^{\mathcal{R}-\mathcal{S}}(\mathbf{p})$

$= \{(1,0), (0,1), (1,1)\}$

In general, find all nearby unique demands and take discrete convex hull.
Use this principle to implement the auction.

Details

Representation of Strong Substitute Valuations

Suppose A is a "d-simplex", i.e. $A = \{\mathbf{x} \in \mathbb{Z}^n_+ : \sum_i x_i \leq d\}$ for some d. For A not of this form, we can extend to the minimal d-simplex domain containing it, giving the valuation arbitrarily low / high values.

Theorem (Characterisation of Strong Substitutes)

A valuation $V^j : X^j \to \mathbb{R}$ is a strong substitute valuation iff it can be presented using a valid finite collection of positive and negative dot bids.

Sketch Proof

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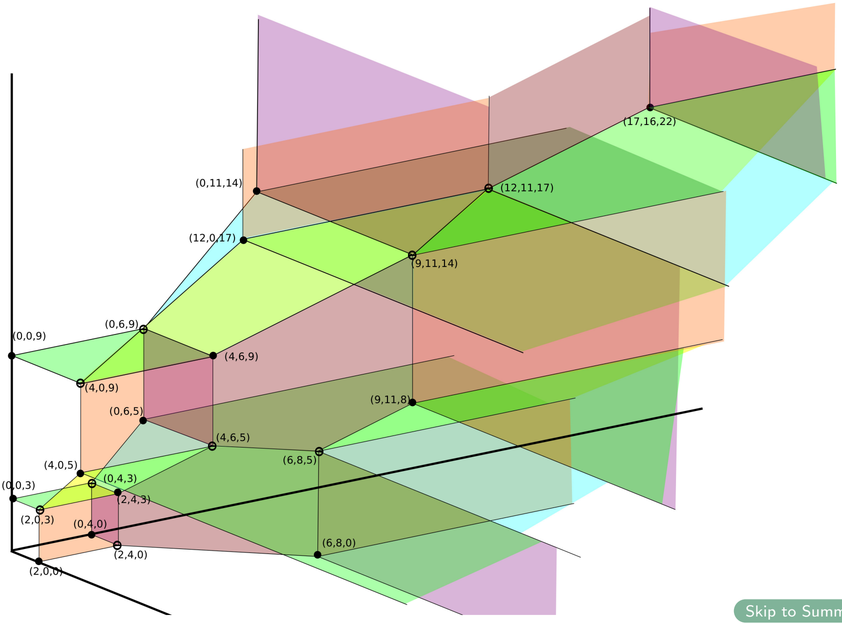
3-dimensional example

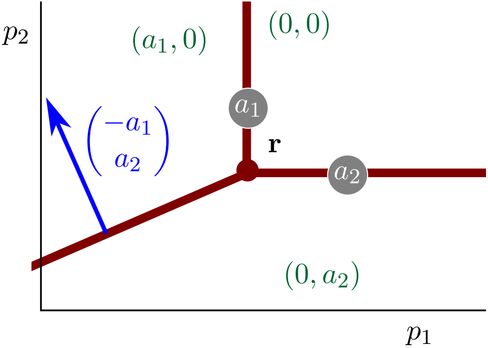


Skip to Summary

Weighted Dots

A single dot bid at $\mathbf{r}$ with weight $\mathbf{a}$ represents valuation $V^{(\mathbf{r},\mathbf{a})}$

$$V^{(\mathbf{r},\mathbf{a})}(\mathbf{0}) = 0, \quad V^{(\mathbf{r},\mathbf{a})}(a_i\mathbf{e}^i) = a_i r_i$$



At price $\mathbf{r}$, indifferent between:

- a_1 units of good 1;
- a_2 units of good 2;
- $\mathbf{0}$.

All Substitutes Bidding Language

Assumptions as for Strong Substitutes, but

- All "ordinary" substitute preferences can be communicated
- Goods must be divisible to guarantee equilibrium. Why?

Language: consists of weighted positive and negative dot bids.

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Let $A \subset \mathbb{Z}^n_{\geq 0}$ satisfy:

- $\mathbf{0} \in A$
- $\mathrm{argmax}\{x_i \mid \mathbf{x} \in A\} = \{W_i \mathbf{e}^i\}$ for some $W_i \in \mathbb{Z}_{>0}$, for all $i \in I$

Theorem

A valuation $V^j : X^j \to \mathbb{R}$ is a substitute valuation iff it can be presented using a valid finite collection of weighted positive and negative dot bids.

Example

“Arctic” Languages

Recall the Central Bank of Iceland’s problem:

After financial crisis Iceland imposed capital controls. Needed to exit.

Planned to buy back the “offshore” accounts they had blocked.

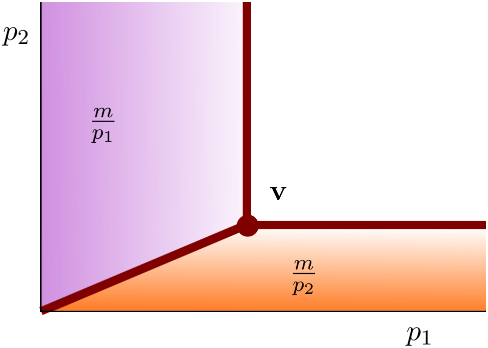
Offer owners three choices of bonds or cash.

"Arctic" Language

- Goods are divisible.
- Each buyer has a fixed budget.
- Constant value for each good.

Intuition: fixed sum for currency transaction. Now a bid of $\mathbf{v}$ means:

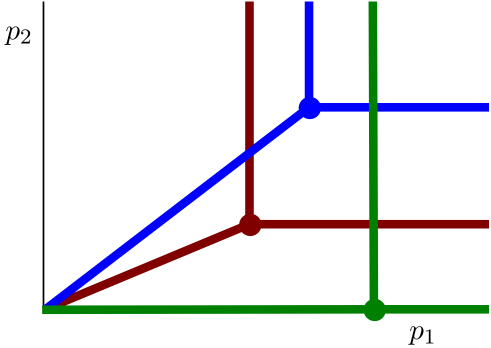
- With budget m could buy $\frac{m}{p_i}$ units of good i, worth $\frac{v_i m}{p_i}$.
- So choose good maximising $\frac{v_i}{p_i}$ s.t. $v_i > p_i$.



Unlike in tropical languages, we assume seller is **profit-maximising**.

Optimal point for a seller will always be at an intersection of bidders' LIPs.

Find these intersections. Maximise objective over finite set of points.



Summary

- Need for sealed-bid auctions simultaneously selling multiple goods
- We can approach auction design using "bidding languages"
- We can design bidding languages using geometry
- We have theoretically analysed and practically implemented three languages
  - Bank of England Bidding Language
  - Strong Substitutes Bidding Language
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  Third is very different from the other two
- We can depict all substitute valuations with our languages (no implementation as yet).

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  Third is very different from the other two
- We can depict all substitute valuations with our languages (no implementation as yet).
- This is an important application of our earlier work on the geometry of preferences, which developed our understanding of individual and aggregate valuations, and of competitive equilibrium between agents.

Proof of Representation of Strong Substitutes
Implementation of Strong Substitutes Auction

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General "Intersection Count Theorem"

Given concave valuations $V^j : X^j \to \mathbb{R}$, $j = 1, 2$.

- An "intersection 0-cell" for $\mathcal{L}^1, \mathcal{L}^2$ is a 0-cell of their aggregate that lies in their intersection.
- Generalise "facet weight" to lower-dimensional "cells" of LIP.
- "Naïve multiplicities" at intersection 0-cells: in simple ("transverse") cases, this is product of the weights of cells intersecting there.
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$$\Gamma^n(X^1, X^2) = \sum_{k=1}^{n-1} \sum_{r=0}^{k} \sum_{s=0}^{n-k} (-1)^{n-r-s} \binom{k}{r} \binom{n-k}{s} \mathrm{Vol}_n \, \mathrm{Conv}(rX^1 + sX^2).$$

is a sum of "mixed volumes".

is much easier to calculate in special cases!

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1. *Eqm exists for all relevant supplies iff exists at all intersection 0-cells.*
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4. *If $n \leq 3$, intersection is "transverse", and bound is not tight, then equilibrium fails for some relevant supply.*

Back

Implementation Phase 1: Find an Equilibrium Price

Wish to sell bundle $\mathbf{y}$.

Method 1: (with Paul Goldberg and Edwin Lock)

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- Minimise the *difference* between objectives of these lin progs, subject to the difference between bids accepted being the target $\mathbf{y}$

Finding an equilibrium allocation

with Paul Goldberg and Edwin Lock

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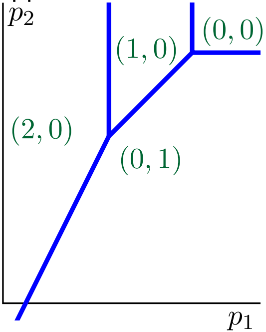
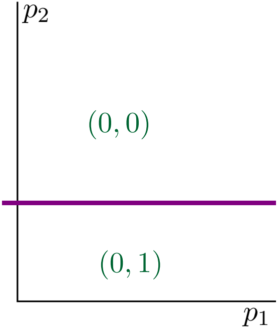
- Iteratively eliminate leaves
- Break cycles labelled by more than one bidder by 'tweaking' bids up slightly (requires defined order of priority).

Demand in Strong Substitutes Bidding Language

Summary

Suppose trade-offs are not all 1-1.





Allowing More General Substitute Trade-offs?

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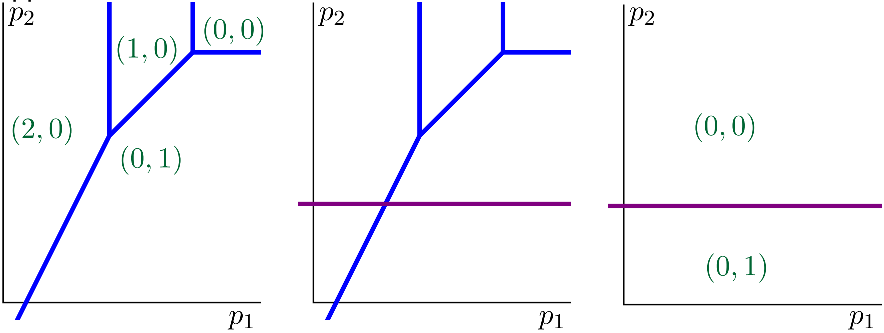

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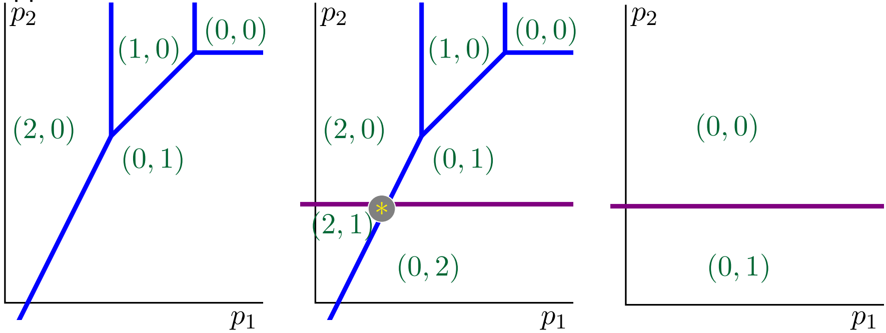
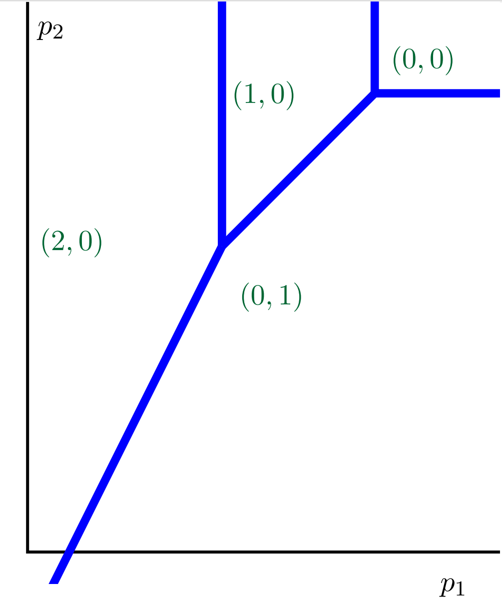
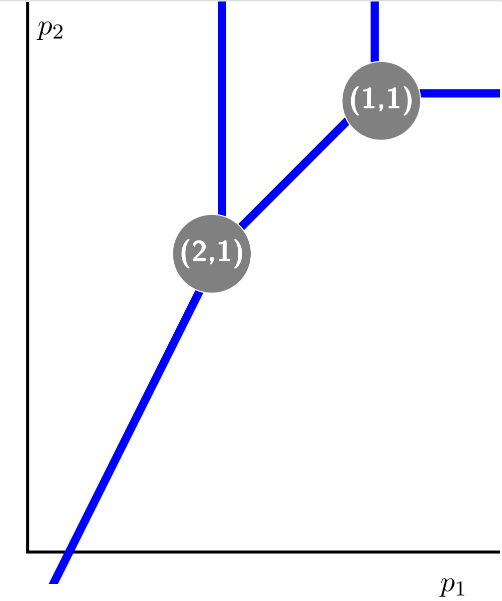


Equilibrium not guaranteed with indivisible goods:
Bundle $(1, 1)$ "should" be demanded at price ✱.
Weaken again to divisible goods.

Back


p_2
$(0,0)$
$(1,0)$
$(2,0)$
$(0,1)$
p_1

Ordinary Substitutes via Weighted Dot Bids



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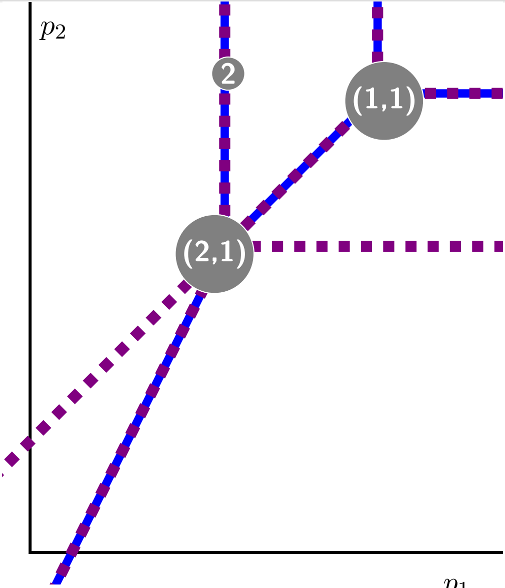

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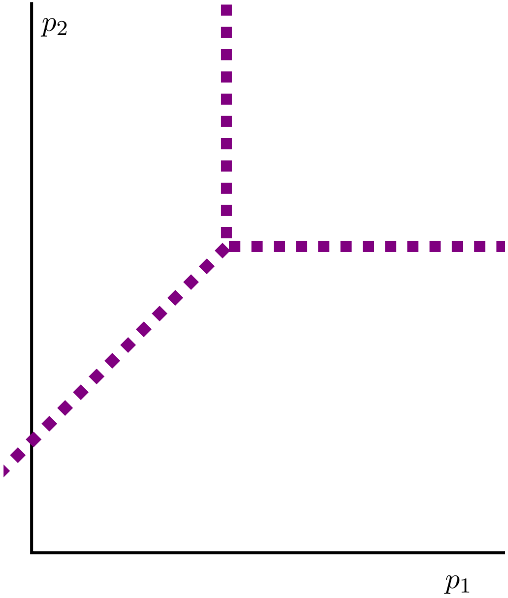

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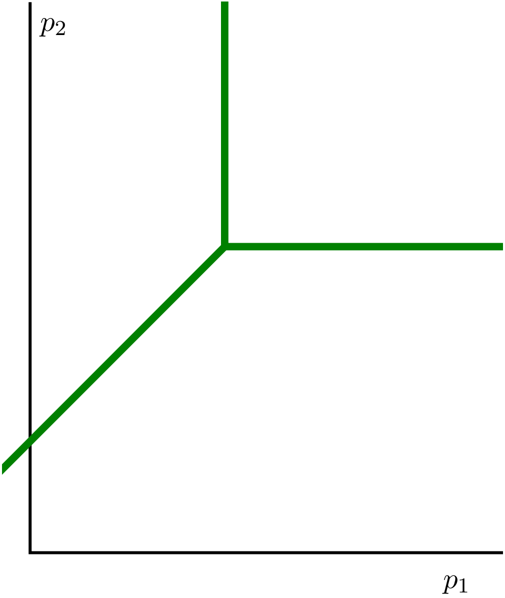

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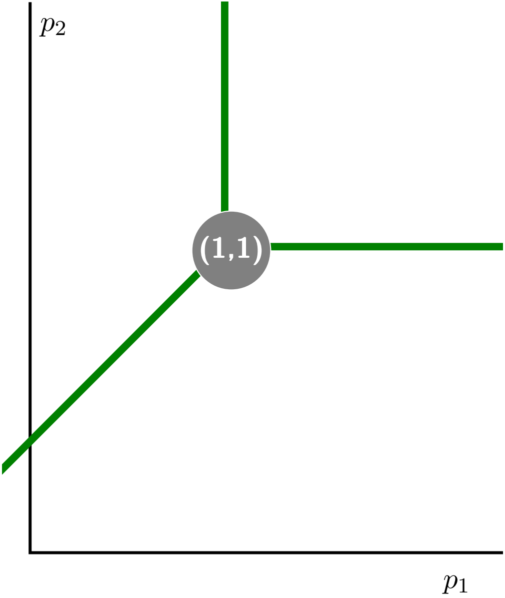

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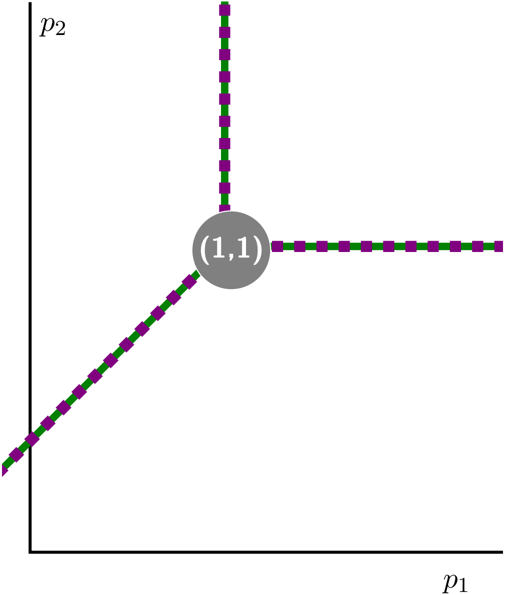

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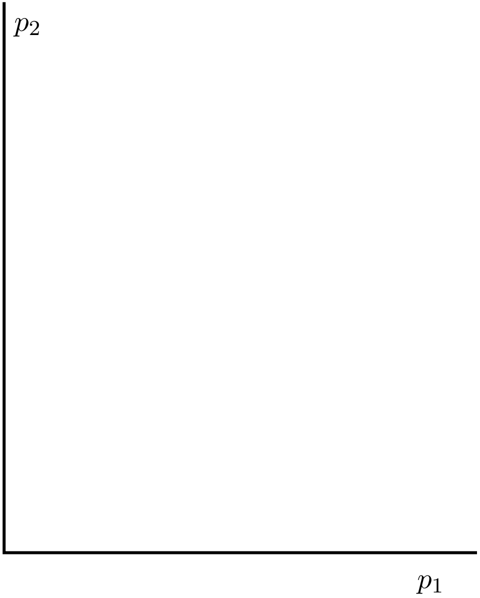

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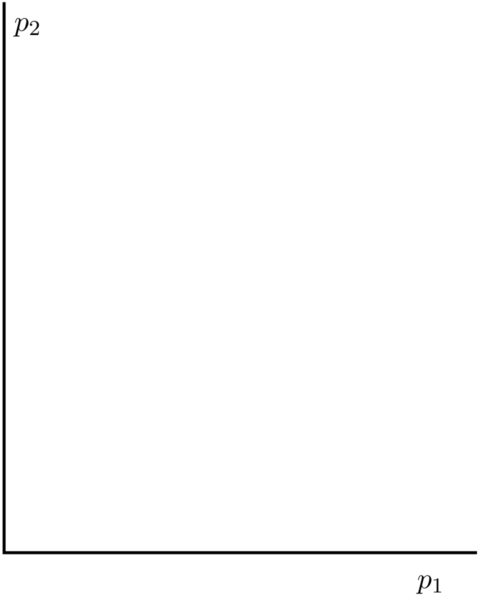
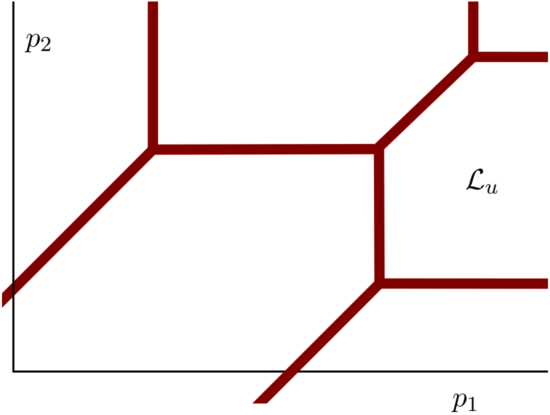


Back

Illustration of the construction of the dot bid set



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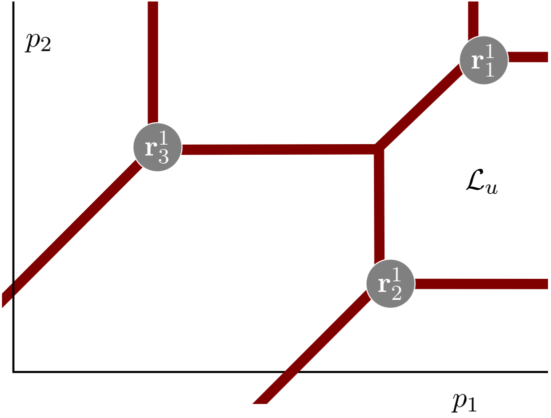

Identify minimal points on horizontal and vertical facets.

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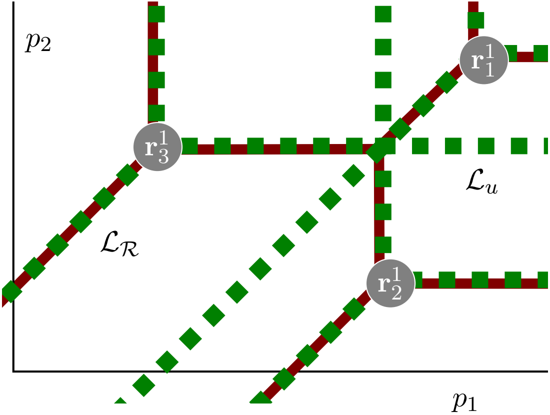

Putting bids at these points gives $\mathcal{L}^{\mathcal{R}}$ 'covering' $\mathcal{L}^j$.

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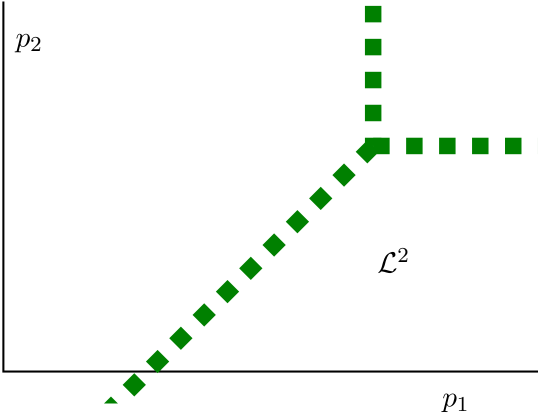

Subtract the original LIP.

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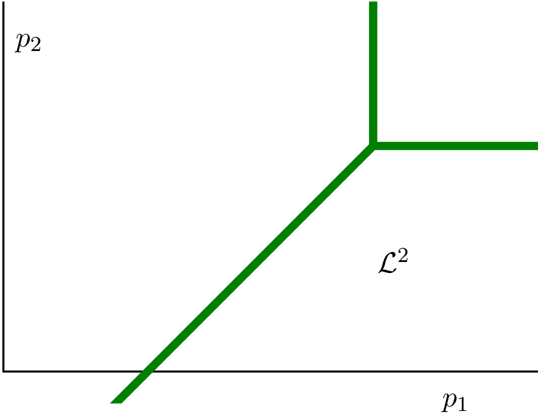

The remainder **is** the LIP of a strong substitutes valuation.

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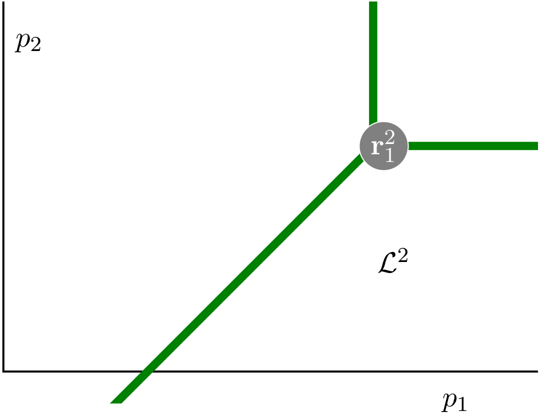
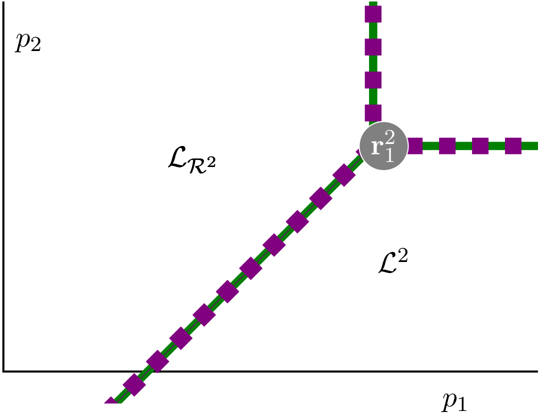


So we can go again.



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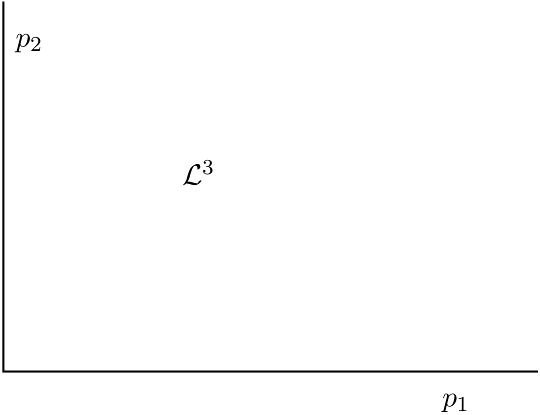

So we can go again. Eventually this terminates.

- Identify finite set of points at which dot bids might ever be placed: Intersections of affine spans of facets in $\mathcal{L}^j$ normal to $\mathbf{e}^i$ for all i.
- The minimal point we might use strictly increases at each stage.

Then

$$\begin{aligned}(\mathcal{L}^j, \mathbf{w}^u) &= (\mathcal{L}^{\mathcal{R}^1}, \mathbf{w}^{\mathcal{R}^1}) \boxminus (\mathcal{L}^{\mathcal{R}^2}, \mathbf{w}^{\mathcal{R}^2}) \boxplus \cdots \boxplus (-1)^{l-1}(\mathcal{L}^{\mathcal{R}^l}, \mathbf{w}^{\mathcal{R}^l}) \\ &= (\mathcal{L}^{\mathcal{R}}, \mathbf{w}^{\mathcal{R}}) \boxminus (\mathcal{L}^{\mathcal{S}}, \mathbf{w}^{\mathcal{S}})\end{aligned}$$

where $\mathcal{R} = \mathcal{R}^1 \cup \mathcal{R}^3 \cup \cdots$ and $\mathcal{S} = \mathcal{R}^2 \cup \mathcal{R}^4 \cup \cdots$.

This completes the proof.

Theorem Statement

Summary