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Abstract

This study examines procurement auctions in which a price-per-quality-ratio (PQR) scoring rule is used to evaluate bids consisting of price and quality. Equilibrium bidding behavior is characterized for first-score (FS) and second-score (SS) auctions in which bidder cost consists of unidimensional type and quality. In contrast to well-known quasilinear scoring rules, we show that the SS auction yields a lower expected score, and we provide a set of conditions under which expected quality and price are higher in the FS auction. We also find that buyers can achieve a lower PQR ratio by distorting the scoring rule.

Keywords: multidimensional bidding, scoring auctions, procurement, price-per-quality-ratio scoring rule

JEL codes: D44, H57, L13

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1 Introduction

The rapid accumulation of public debt in recent years has intensified the need for governments worldwide to ensure value for money in their procurement processes. Traditionally, low-price auctions have been the cornerstone of public procurement, valued for their transparency and competitive nature. However, there is a marked shift towards evaluating bids not solely on price but also on non-monetary factors such as delivery speed, design, and overall quality. *Scoring auctions* have emerged as a prominent mechanism designed to balance the dual objectives of price competition and value maximization.

In a scoring auction, bidders submit multidimensional bids consisting of both price and quality attributes. These bids are evaluated using a pre-announced scoring rule, which assigns scores to rank the bidders. The seminal work by Che (1993) established that under a symmetric independent private value setting, scoring auctions with quasilinear (QL) scoring rules—where scores are linear in price and additively separable from quality—can be reduced to the classic unidimensional auction model.[1] This reduction brings scoring auctions under the umbrella of the revenue equivalence theorem, ensuring that price and quality outcomes are consistent across various auction formats. The QL framework's simplicity has since attracted significant theoretical and empirical interest, reinforcing its status as a cornerstone of auction theory.

In real-world procurement auctions, however, a variety of scoring rules are used that are not quasilinear. A typical example is the "price-per-quality-ratio" (PQR) scoring rule in which a score is given by the price bid divided by the quality bid. Many state Departments of Transportation (DOTs) in the United States, including those in Alaska, Colorado, Florida, Michigan, North Carolina, and South Dakota, have adopted the PQR-equivalent "adjusted bid," and the Department of Health and Aging in Australia also employs a PQR awarding rule to achieve better returns

[1] In this paper, we use the abbreviation QL only for *quasilinear scoring rules* but not for other situations such as the quasilinearity of the payoff function.

on public investment (The Department of Health and Ageing, Australia, 2011). In addition, most public procurement contracts in Japan are allocated to the bidder with the lowest price-per-quality bid ratio. However, despite the frequent use of PQR scoring rules in the real world, little is known about their properties.[2]

In this paper, we provide a theoretical analysis of PQR scoring auctions in order to acquire a deeper understanding of bidding behavior and outcomes of non-QL scoring auctions. We follow Che (1993) by focusing on settings with a unidimensional private signal and unidimensional quality level, and we characterize bidding behavior and compare auction outcomes for the following two auction formats: first-score (FS) and second-score (SS) auctions. In an FS auction, the winner delivers quality at the price specified in its bid, and in an SS auction, the winner is free to choose any price–quality pair as long as its score matches the score of the most-competitive rival. In our model, the winner of both auctions is the lowest-score bidder.

We first demonstrate that under a PQR scoring rule a symmetric Bayesian Nash equilibrium exists in both FS and SS auctions for a broad class of cost functions and we then show that SS auctions yield lower expected scores than FS auctions. Based on the existing literature such as Che (1993) and Asker and Cantillon (2008), multidimensional-bid scoring auctions are transformed into a unidimensional score-bid auction game. Under the PQR scoring rule, bidders choose higher quality as the score increases, leading to the profit from winning bids being a convex function in score. This means that bidders are "risk-loving" in score, taking on greater risk for potentially larger winning profits in FS auctions. Consequently, the well-established "revenue equivalence theorem" fails to hold and the SS auction yields a lower expected score than the FS auction in equilibrium, which is similar to Maskin and Riley (1984) with non-risk-neutral bidders. This suggests that SS auctions are more favorable than FS auctions to buyers seeking to decrease price per quality.

[2]While bid ranking is preserved in any monotonic transformation of the scoring rule, this transformation does not generally convert a non-QL scoring rule into a QL rule. If, for instance, we take a logarithm of the price-per-quality-ratio scoring rule, score is not linear in price anymore and so a necessary condition for quasilinearity is violated.

To present more comprehensive arguments regarding the differences between FS and SS auctions, we analyze the equilibrium quality and price and then establish sets of conditions under which FS auctions produce a higher expected quality and price than in SS auctions. We observe that under these conditions both the quality and price increase with score, which leads to expected quality and price being higher in the FS auction than in the SS auction. These findings suggest that when the bidders' scores are sorted by price-per-quality, the contracted quality and prices are more likely to be skewed upward in the FS auction than in the SS auction.

Furthermore, we discuss the design of scoring rules when the buyer's objective is to minimize price per quality ratio. We confirm that the PQR scoring auctions achieve ex post Pareto efficient contracts. However, it is hard to characterize the optimal mechanism or optimal scoring rule for the buyer because the standard mechanism design by Myerson (1981) cannot be applied to nonquasilinear objective functions. Nevertheless, we show that a buyer can lower the price per quality ratio by using a distorted scoring rule that leads to a downward distortion of quality from the original PQR scoring rule. This is because by adopting a scoring rule in which bidders propose a lower quality, the buyer can reduce the bidder's information rent and thus, the distorted scoring rule is beneficial for the buyer. This result is consistent with an optimal distortion in the traditional mechanism design of Mussa and Rosen (1978) and Myerson (1981) and an optimal scoring rule in Che (1993).

Lastly, we discuss how the equilibrium properties under the PQR scoring rule can be generalized to other non-QL scoring rules. We characterize the equilibrium of the SS and FS auctions and show that given a scoring rule, the expected score in an FS auction is higher (lower) than in an SS auction if the bidders' indirect payoff function is convex (concave) in score.

Related Literature This paper contributes to the theoretical literature on scoring auctions pioneered by Che (1993) which to date has focused on QL scoring auctions. In Che (1993)'s original approach, scoring auctions are modelled in such a way that the price and quality bids reduce to a model of auctions in which bidders submit only scores as if it were a price-only auction. This research has been extended to cases of interdependent cost (Branco, 1997), multidimensional signals (Asker and Cantillon, 2008) and multidimensional quality (Nishimura, 2015). Furthermore, Asker and Cantillon (2008, 2010), Awaya, Fujiwara and Szabo (2025) and Sano (2023) compare the performance of QL scoring auctions and alternative mechanisms. While these previous studies focus on the properties of QL scoring auctions, we extend these studies by comparing the performance of FS and SS auctions under the non-QL PQR scoring rule.

In contrast to the research on scoring auctions using a QL scoring rule, there are few papers to date that study non-QL scoring rules,one of them being Wang and Liu (2014), who examine a non-QL scoring rule where price and quality are additively separable. Meanwhile, Dastidar (2014) analyzes scoring auctions with a general scoring rule and finds that the equilibrium bidding function of the FS auction is explicitly obtained if the bidder's cost function is additively separable in quality and their private information.

In another study, Hanazono, Hirose, Nakabayashi and Tsuruoka (2020), henceforth HHNT, discuss the equilibrium existence and the structural estimation of FS auctions incorporating general scoring rules and multidimensional private signals. This paper complements HHNT in two ways. First, the cost structure does not fall into that of HHNT despite multidimensionality: to ensure equilibrium existence, HHNT require that the cost function have a private fixed cost component. This paper covers the case of single-dimensional private signal on variable costs which is out of scope in HHNT. Second, the results of comparing expected price, quality, and score between different auction formats fails to obtain in HHNT because the monotonicity of equilibrium on a single-dimensional signal space is intrinsically different from that on a multidimensional signal space.

In addition to theoretical studies, empirical research on scoring auctions is growing as well (e.g., Lewis and Bajari, 2011; Koning and van de Meerendonk, 2014; Iimi, 2016; Andreyanov, 2018; Takahashi, 2018; Huang, 2019; Krasnokutskaya, Song and Tang, 2020; Ryan, 2020; Kong, Perrigne and Vuong, 2022; Allen, Clark, Hickman and Richert, 2023 and Andreyanov, Decarolis, Pacini and Spagnolo, 2024). Building on the literature on scoring auctions, Bajari, Houghton and Tadelis (2014) and Bolotnyy and Vasserman (2023) develop structural auction models in which firms post unit price bids for each item required to complete a construction project. Among these studies, Takahashi (2018) examine scoring auctions with the PQR scoring rule and quantify the impact of uncertainty on reviewers' evaluations of quality. Ortner, Chassang, Kawai and Nakabayashi (2025) present theoretical predictions based on repeated procurement auctions and examine bidder collusion in PQR scoring auctions. These empirical studies motivate us to deepen a theoretical understanding of properties of non-QL scoring auctions.

The remainder of the paper is organized as follows. Section 2 describes the canonical model of scoring auctions in which they can be transformed into a unidimensional score-bid auction game. In Section 3, we focus on the PQR scoring rule and analyze symmetric equilibria in FS and SS auctions, comparing the expected winning score, quality, and price of the two auction formats. Section 4 discusses the efficiency of the PQR scoring rule and how to design a scoring rule to improve the price per quality ratio. Section 5 analyzes general scoring rules and characterizes the expected score rankings for FS and SS auctions, and the final section concludes the paper.

2 Model

Consider that a procurement buyer auctions off a procurement contract to $n \geq 2$ risk-neutral bidders who are all ex ante symmetric. Bidder i's private type is denoted by θ_i and is independently and identically drawn from a cumulative distribution F over $\Theta \equiv [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}_+$ with a continuous density $f(\theta) > 0$ for every $\theta \in \Theta$. Let $q \in \mathbb{R}_+$ be a non-monetary attribute (quality) so that each bidder's production cost is given by $C(q, \theta_i)$. We assume that the cost function C is:

- thrice differentiable and strictly increasing in both q and θ (C_q, $C_\theta > 0$);
- strictly convex in quality ($C_{qq} > 0$);
- exhibits non-decreasing differences ($C_{q\theta} \geq 0$); and
- there exists a sufficiently large number $B > 0$, and for all θ, $C_q(q, \theta) \geq B$ for some $q > \underline{q}$.

Note that $C_\theta = \partial C/\partial\theta$ and that the other subscripts are defined in the same manner. The production cost is increasing in quality and type so that a bidder of a lower type is more efficient. The third assumption means that a bidder of a lower type has a smaller marginal cost, and the last assumption guarantees that that an optimal quality exists.

When bidder i wins the auction and signs a contract with a price p and a quality q, their payoff is given by

$$p - C(q, \theta_i),$$

and we suppose that every losing bidder's payoff is zero.

In a scoring auction, each bidder submits a proposal (p, q), where $p \leq \bar{p}$ is a price bid and $q \geq \underline{q}$ is a quality bid, with reserve price and minimum quality denoted by $\bar{p} > 0$ and $\underline{q} > 0$. Each proposal is evaluated by a pre-announced scoring rule $S : [0, \bar{p}] \times [\underline{q}, \infty) \to \mathbb{R}$ which maps a multidimensional bid into a unidimensional score $s = S(p, q)$. The lowest-score bidder wins. We assume that the scoring rule is sufficiently smooth and satisfies $S_p > 0$ and $S_q < 0$.

We focus here on first-score (FS) and second-score (SS) auctions. In both types of auction, each bidder submits (p, q), and the bidder with the lowest score wins. In an FS auction, the winner's proposal is finalized as a contract whereas in an SS auction, the winner is required to match the highest rejected (i.e. the second lowest) score. To meet the score, the winner is free to choose any quality-price pair, so the finalized contract of an SS auction generally differs from both the winning bid (p, q) and the lowest losing bid.

Although our model allows general scoring rules, in most of the paper we focus on the *price-per-quality ratio* (PQR) scoring rule:

$$S(p, q) = \frac{p}{q}, \tag{1}$$

with $p \leq \bar{p}$ and $q \geq \underline{q}$.[3] When focusing on the PQR scoring rule, the buyer aims to lower price per quality ratio, and we suppose that the buyer's utility function is given by $v(p, q) = -p/q$.[4]

Remark 1 Score ranking is preserved in any monotone transformation of the scoring rule so that most properties of scoring auctions, equilibrium price and quality in particular, do not change in a monotone transformation of the scoring rule. However, in the following sections, we evaluate the expected scores of different auction formats which generally do change in monotone transformation. For example, the quality-per-price-ratio rule $S(p, q) = -q/p$ is a monotone (but not affine) transformation of the PQR scoring rule, so the equilibrium price and quality under such a scoring rule are the same as those presented in the next section, though the expected score ranking may differ. Note that the expected score ranking is preserved in any affine transformation.

2.1 Score-bid Auctions

The equilibrium of scoring auctions is derived in a manner similar to Che (1993). Given an arbitrary score s, every bidder will choose the optimal contract (p, q) that induces score s so that an auction with a multidimensional bid is reduced to a unidimensional auction in terms of the score bid.

[3] Quality q here is measured in terms of "quality score." One might consider a scoring rule $S(p, q) = p/V(q)$, where V is an increasing function. This is equivalent to the case in which a quality is defined as $\tilde{q} = V(q)$.

[4] We assume that the buyer's payoff of not contracting is sufficiently small and that the buyer necessarily procures from some bidder.

Suppose that the winner of type θ needs to enforce a contract that fulfills score s. The winner determines a contract (p, q) that solves

$$\begin{aligned} &\max_{(p,q)} p - C(q,\theta) \\ &\text{s.t. } S(p,q) = s, \\ &p \leq \bar{p},\ q \geq \underline{q}. \end{aligned} \tag{2}$$

Throughout the analysis, we assume that the reserve price and the minimum quality are not binding at (2). By substituting the score constraint into the objective function, payoff maximization is written as

$$\max_{q} P(s,q) - C(q,\theta), \tag{3}$$

where P is the inverse function of S with respect to p. When the objective function in (3) is strictly concave in q, the maximization problem has a unique solution, with the optimal quality denoted by

$$q^*(s,\theta) \in \arg\max_{q} P(s,q) - C(q,\theta) \tag{4}$$

and the indirect payoff function denoted by

$$u(s,\theta) \equiv P\left(s, q^*(s,\theta)\right) - C\left(q^*(s,\theta),\theta\right). \tag{5}$$

Note that as $S_p > 0$, we have $P_s > 0$. By the envelope theorem, the indirect payoff u is strictly increasing in s and strictly decreasing in θ. The equilibrium of the scoring auction is derived by solving standard auctions in terms of score bid s, where each bidder has the winning profit $u(s,\theta)$.

Let $z(\theta)$ be *the break-even score* for type θ, which is determined by the unique solution of

$$u(z(\theta),\theta) = 0;$$

that is, z is the score bid such that the winner's indirect payoff is zero, which is the minimum willingness to accept for a bidder of type θ in the auction.

Lemma 1 *Suppose that $P(s,q) - C(q,\theta)$ is strictly concave in q. Then, $z(\theta)$ is well-defined and strictly increasing in θ.*

Proof See Appendix.

2.2 QL Scoring Rule

The seminal paper Che (1993) examines the *quasilinear (QL)* scoring rule $S(p,q) = p - q$ whereby optimal quality is given by the profit maximization problem

$$\max_q s + q - C(q,\theta).$$

When the optimal quality is determined by the first-order condition $1 - C_q(q^*, \theta) = 0$, q^* depends only on θ and is independent of s. The indirect payoff is reduced to a function

$$u(s,\theta) = s - k(\theta)$$

that is quasilinear in score, where

$$k(\theta) = -\max_q \{q - C(q,\theta)\}$$

is called the *productive potential* (Che, 1993) or *pseudotype* (Asker and Cantillon, 2008). Thus, in this framework, the QL scoring auction is reduced to a score-bid auction with a quasilinear payoff. Because k is increasing in θ, the bidder of the lowest type wins in both the FS and SS auctions and so the exercised quality is ex post equivalent between the two formats. The revenue equivalence theorem applies and thus, the expected scores of the FS and SS auctions are equivalent in equilibrium. As there is score equivalence and $p = s + q$, equivalence holds for the expected price too.

2.3 PQR Scoring Rule

Next we examine a scoring rule that is not QL. Consider the PQR scoring rule $S(p,q) = p/q$. The inverse function of S with respect to p is given by $P(s,q) = sq$, and the optimal quality is derived by the profit maximization problem

$$\max_{q \geq \underline{q}} sq - C(q,\theta). \tag{6}$$

It is clear that the objective function is strictly concave in q, and we assume that the optimal quality q^* always lies in the interior in equilibrium. This is satisfied if the optimal quality at $(z(\theta), \theta)$ is not binding.

Assumption 1 In the PQR scoring rule, for all θ, the optimal quality satisfies

$$q^*\left(z(\theta), \theta\right) > \underline{q}.$$

When the optimal quality q^* lies in the interior, it is determined by the first-order condition

$$s - C_q(q^*, \theta) = 0. \tag{7}$$

By the implicit function theorem, we have $q_s^* = 1/C_{qq} > 0$ and $q_\theta^* = -C_{q\theta}/C_{qq} \leq 0$, so the optimal quality is increasing in score s and non-increasing in type θ. It is immediately clear that the indirect payoff function is convex in score s.

Lemma 2 *Under the PQR scoring rule and Assumption 1, the indirect payoff function u is strictly convex in s.*

Proof By the envelope theorem, we have $u_s(s, \theta) = q^*(s, \theta) > 0$ and $u_{ss}(s, \theta) = q_s^*(s, \theta) > 0$. $\square$

Quality choice and the indirect payoff function under the PQR scoring rule are both closely related to standard producer theory whereby the maximization problem (6) is equivalent to the profit maximization problem of a firm in a competitive market when s is the price per unit of quality. The optimal quality supplied is thus determined by "price equals marginal cost" (7) and the supply function q^* is increasing in price s. Since the suppliers optimally adjust their quality supplied in response to price, the profit function u is convex in s. The break-even score $z(\theta)$ here corresponds to the break-even price for the firm.

From this interpretation, Assumption 1 requires that there exists a non-sunk fixed cost. As is well known, average cost is minimized and generally equals marginal cost at the break-even price. With the presence of fixed costs, average cost is U-shaped and minimized in the interior. When there are no fixed costs, the average cost is always smaller than the marginal cost. Hence, by ignoring the quality constraint $q \geq \underline{q}$, suppliers could always earn a positive profit by providing a small quality, and the quality supplied at the break-even point is zero. Thus, a non-sunk fixed cost is necessary to satisfy Assumption 1.

The PQR scoring rule is distinct from the QL scoring rule in two respects. First, the optimal quality under the PQR rule depends not only on bidder type but also on the required score s. Second, the indirect payoff function is not quasilinear, so the revenue equivalence theorem does not apply to the PQR rule.

3 Equilibrium Analysis of PQR Scoring Auctions

3.1 Equilibrium

We first characterize the equilibria of the SS and FS auctions, showing that in both auctions the bidder with the lowest type is selected as the winner.

In the SS auction, it is a weakly dominant strategy to bid $z(\theta)$ as in the standard second-price auction. The following proposition is shown in a standard manner and is similar to Maskin and Riley (1984), Saitoh and Serizawa (2008), and Sakai (2008), so the proof is omitted.

Proposition 1 *In the SS auction, it is a weakly dominant strategy for each bidder to submit* $s^{SS}(\theta) = z(\theta)$.

Under the PQR scoring rule, the score-bid auction game can be interpreted as competition among suppliers in terms of unit price per quality, and the supplier who submits the lowest price per quality ratio wins. From the perspective of standard producer theory, the break-even score is equal to the supplier's minimum average cost: $z(\theta) = \min_q C(q, \theta)/q$. In the SS auction, the unit price per quality for the winner is determined by the best rival offer, so suppliers are price takers. They are competitive and submit their minimum average cost in equilibrium. The supplier with the lowest minimum average cost wins and supplies quality at the unit price equal to the second-lowest minimum average cost.

As for the FS auction, Maskin and Riley (1984) and Athey (2001) show that it has a symmetric, monotone Bayesian Nash equilibrium if the payoff function u is log-supermodular:

$$\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} > 0. \tag{8}$$

To meet this log-supermodularity condition, we additionally impose the technical conditions below.

Assumption 2 At least one of the following conditions holds.

1. C_q/C_θ is non-decreasing in q, or

2. $qC_{qq}/C_{q\theta}$ is increasing in q.

A wide range of cost functions satisfy either of the above. The first case is equivalent to $C_\theta C_{qq} - C_q C_{q\theta} \geq 0$ and, roughly speaking, this condition is met when the marginal cost is more sensitive to a change in quality than to a change in type; that is, when C_{qq} is large and $C_{q\theta}$ is small. A special case is $C_{q\theta} = 0$ in which bidder marginal cost is independent of θ whereby bidder variable costs for quality are identical but fixed costs are heterogeneous.[5] The second case is likely satisfied when the cost function is polynomial in q and type θ does not depend on the coefficient of the maximum degree of q. For example, this condition is met if $C(q,\theta) = q^2 + \theta q + \kappa(\theta)$. Note that these two conditions are not disjoint. For example, a cost function $C(q,\theta) = c(q+\theta)$ in which c is a convex function satisfies both conditions.

Under the log-supermodularity condition, the equilibrium bidding function is characterized by the first-order condition. Let $G(\theta) = 1 - (1 - F(\theta))^{n-1}$ be the distribution of the lowest order statistic of $n-1$ independent draws from F. In addition, let $g = G'$ be its density.

[5] Dastidar (2014) focuses on this type of cost function and examines the equilibrium of non-QL scoring auctions.

Proposition 2 *If Assumptions 1 and 2 hold, there exists a symmetric Bayesian Nash equilibrium in the FS auction. Equilibrium score-bidding function s^{FS} is characterized by*

$$(s^{FS})'(\theta) = \frac{u\left(s^{FS}(\theta), \theta\right)}{u_s\left(s^{FS}(\theta), \theta\right)} \cdot \frac{g(\theta)}{1 - G(\theta)}, \tag{9}$$

or equivalently,

$$s^{FS}(\theta) = \int_{\theta}^{\bar{\theta}} \frac{g(\tau)}{1 - G(\theta)} \cdot \frac{C\left(q^*\left(s^{FS}(\tau), \tau\right), \tau\right)}{q^*\left(s^{FS}(\tau), \tau\right)} d\tau. \tag{10}$$

with $s^{FS}(\bar{\theta}) = z(\bar{\theta})$.

Proof See Appendix.

Recall that a PQR scoring auction is a competition among suppliers in terms of unit price per quality. In an SS auction, the unit price of the winner is determined by the best rival offer, and suppliers submit their minimum average cost in equilibrium. In an FS auction, by contrast, unit price per quality is determined by the supplier's own offer, so bidding one's minimum average cost is not a best response for suppliers. Instead, suppliers submit a unit price higher than their minimum average cost, $s^{FS}(\theta) > z(\theta)$, and the equilibrium score bid is expressed by the expected average cost of the most-competitive rival bid (10).

3.2 Comparison of FS and SS Auctions

We now compare the equilibrium performance of FS and SS auctions under the PQR scoring rule. In contrast to a QL scoring rule, equivalence between the two formats does not hold, so we evaluate the two formats with respect to expected score, quality and price.

3.2.1 Score Ranking

Because the buyer aims to minimize price per quality ratio, they prefer an auction format that yields a lower (expected) score. The expected score rankings of the FS and SS auctions depends on the curvature of the bidder's indirect payoff. Maskin and Riley (1984) show that if u is *concave* in payment, the expected revenue from the first-price auction is higher than that of the second-price auction. Here, by Lemma 2, u is *convex* in score in a PQR scoring auction, so we have a similar but reverse expected score ranking, which is shown in an analogous manner to Maskin and Riley (1984). The following theorem states that the buyer prefers the SS to the FS auction.

Theorem 1 *Suppose that Assumptions 1 and 2 hold. The expected score of the SS auction is lower than that of the FS auction. Moreover, for every winner's type θ, we have*

$$E[s^{SS}(\tau) \mid \tau > \theta] \leq s^{FS}(\theta), \tag{11}$$

where τ is the lowest order statistic of $n-1$ independent draws from F.

Proof This is shown in a manner parallel to Theorem 4 of Maskin and Riley (1984). Although Maskin and Riley (1984) consider a concave payoff function, it is not necessary to assume concavity to ensure the existence of a symmetric equilibrium. □

Theorem 1 is also proved by using expression (10). Note that the equilibrium score bid in the FS auction is the conditional expected average cost of the most competitive rival. Given that the winner's type is θ, (10) yields

$$\begin{aligned} s^{FS}(\theta) &= E\left[\frac{C\left(q^*(s^{FS}(\tau),\tau),\tau\right)}{q^*(s^{FS}(\tau),\tau)} \middle| \, \tau > \theta \right] \\ &\geq E\left[\min_q \frac{C(q,\tau)}{q} \middle| \, \tau > \theta \right] \\ &= E\left[z(\tau) \mid \tau > \theta \right], \end{aligned}$$

where τ is the lowest order statistic of $n-1$ independent draws from F. Because bidders submit a higher score than the break-even score in the FS auction, the associated average cost is not minimized. Thus, the equilibrium score in the FS auction is higher than the expected break-even score of the most competitive rival.

3.2.2 Quality Ranking

Because optimal quality depends on score s, and score equivalence does not hold for the PQR scoring rule, the equilibrium quality obtained by the two auction formats also differs. Note that the optimal quality function q^* is increasing in score. Therefore, as the FS auction yields a higher expected score, it is thus likely to provide a higher quality than the SS auction.

The expected quality is ranked under additional conditions. Note that in an FS auction, the winner's quality is deterministic at the bidding stage because the winner's quality bid is enforced. In contrast, in an SS auction, the winner's quality is stochastic because the optimal quality depends on the second-lowest score which is uncertain for the winner. Hence, to obtain the expected quality ranking, we need a condition on the curvature of the optimal quality function q^*. The following theorem states that the FS auction provides a higher expected quality than the SS auction when the optimal quality q^* is weakly concave in score.

Theorem 2 *Suppose that Assumptions 1 and 2 hold and that $C_{qqq} \geq 0$. Then, the expected quality in the FS auction is higher than that in the SS auction.*

Proof See Appendix.

The condition $C_{qqq} \geq 0$ means that marginal cost is weakly convex, which implies that the optimal quality function q^* is weakly concave in s. The optimal quality is determined by (7) whereby the unit price per quality equals the marginal cost. When marginal cost is convex, it rapidly increases as q increases. Hence, the optimal quality does not increase very much when the score or unit price per quality is increased, meaning that it is weakly concave.

3.2.3 Price Ranking

Given that an FS auction yields a higher expected score and quality when marginal cost is convex, it is natural to conjecture that expected price would also be higher for an FS auction. However, the expected price ranking is more ambiguous than the quality ranking. Under the PQR scoring rule, the equilibrium price is given by

$$\pi(s, \theta) \equiv sq^*(s, \theta).$$

Analogous to the quality ranking, we have an expected price ranking if the optimal price π is weakly concave in score. However, because q^* is increasing in s, π is more sensitive to a change in s and is likely to be convex. Thus, the concavity of π is more stringent than the concavity of q^*.

We provide two sufficient conditions for ranking the expected prices of FS and SS auctions. The first one is when the optimal price is weakly concave in score.

Theorem 3 *Suppose that Assumptions 1 and 2 hold and*

$$C_q C_{qqq} \geq 2(C_{qq})^2. \tag{12}$$

Then, the expected price in the FS auction is higher than that in the SS auction.

Proof See Appendix.

The price function π is weakly concave under (12) above. An example of such a cost function is

$$C(q, \theta) = \log \frac{a}{a - \theta - q},$$

where $a > \bar{\theta}$ is constant. This cost function satisfies all the basic assumptions and Assumption 2.

We provide another condition under which the expected price can be ranked even when the equilibrium price π is convex, assuming that the bidders' type represents their fixed costs, or $C_{q\theta} = 0$. In this case, the optimal quality q^* is independent of type; that is, $q^*(s, \theta) = q^*(s)$, so the optimal price is also independent of type and $\pi(s) = sq^*(s)$. Because the quality function q^* is increasing in s, the optimal price $\pi(s) = sq^*(s)$ is also increasing so there is a one-to-one correspondence between score and optimal price. Thus we transform the indirect payoff function $u(s, \theta)$ in terms of s into one in terms of price p; with $\hat{u}(p, \theta) \equiv u(\pi^{-1}(p), \theta)$. The payoff function $\hat{u}(p, \theta)$ is the winner's payoff when they sign a contract under which they optimally choose the price as p. As the bidder of the lowest score bid also makes the lowest price bid, the score-bid auction is transformed into a unidimensional price-bid auction. The equilibrium price of the two auction formats can be ranked when the bidder payoff $\hat{u}$ is convex (or concave) for the associated price-bid auction.

Theorem 4 *Consider the PQR scoring rule. Suppose that Assumption 1 holds and $C_{q\theta} = 0$. The expected price in the FS auction is at least as high as that in the SS auction if qC_{qq}/C_q is nondecreasing in q, or equivalently,*

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0 \tag{13}$$

holds for all $q \geq \underline{q}$. The expected price in the SS auction is at least as high as that in the FS auction if qC_{qq}/C_q is nonincreasing in q, or equivalently,

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \leq 0 \tag{14}$$

holds for all $q \geq \underline{q}$.

Proof See Appendix.

Given $C_{q\theta} = 0$, condition (13) is weaker than the concavity of π, (12). Indeed, when (12) holds, we have

$$\begin{aligned} C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 &= C_q C_{qq} + q\left(C_q C_{qqq} - 2(C_{qq})^2 + (C_{qq})^2\right) \\ &\geq (C_q + qC_{qq})C_{qq} \\ &> 0. \end{aligned}$$

Note that (13) for the price ranking is relatively stronger than that for the quality ranking because the price function π is more likely to be convex than the quality function. Thus, although the FS auction yields a higher expected score than the SS auction, the convex price function could lead to a higher expected price in the SS auction than the FS auction. In sum, while the expected quality is higher for the FS auction than the SS auction, the expected price of the FS auction may be equal to or even lower than that of the SS auction.

To see this, consider a specific cost function $C(q,\theta) = q^a + bq + \theta$ with $a \geq 2$ and $b \in \mathbb{R}$.[6] Since $C_{qqq} \geq 0$, expected quality is higher in the FS auction than in the SS auction. Also, since

$$C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 = a(a-1)^2 bq^{a-2},$$

the expected price is higher in the FS auction than in the SS auction if $b > 0$ and, conversely, is lower in the FS auction if $b < 0$. When $b = 0$, the optimal quality and price are explicitly given by $q^*(s) = a^{-\frac{1}{a-1}} s^{\frac{1}{a-1}}$ and $\pi(s) = sq^*(s) = a^{-\frac{1}{a-1}} s^{\frac{a}{a-1}}$, respectively. The indirect payoff function is

$$u(s,\theta) = \left(1 - a^{-1}\right) a^{-\frac{1}{a-1}} s^{\frac{a}{a-1}} - \theta$$

which can be transformed into

$$\hat{u}(p,\theta) = \frac{a-1}{a}\left(p - \frac{a\theta}{a-1}\right),$$

where $p = \pi(s)$. That is, the score-bid auction is transformed into a price-bid auction with a quasilinear payoff function and a pseudotype $a\theta/(a-1)$. Thus, we can apply the revenue equivalence theorem, and so the equilibrium price is the same in the FS and SS auctions.

Corollary 1 *Consider the PQR scoring rule, and suppose that Assumption 1 and $C_{q\theta} = 0$ hold. If $C_{qqq} \geq 0$ and qC_{qq}/C_q is nonincreasing in $q \geq \underline{q}$, then the expected quality is higher in the FS auction than in the SS auction, and the expected price in the FS auction is at most as high as in the SS auction. Thus, the FS auction achieves a higher expected quality with a weakly lower expected price.*

At first glance, Corollary 1 seems inconsistent with Theorem 1 which shows that the SS auction yields a lower expected score than the FS auction. However, even though the expected price per quality ratio is higher, the FS auction can lead to a higher expected quality and lower expected price than the SS auction. Thus, if

[6] We focus on the region where the cost is increasing in q when $b < 0$.

the buyer's true objective is to achieve a higher expected quality at a lower expense rather than to minimize the price per quality ratio, the FS auction can be more beneficial for the buyer than the SS auction.

4 Designing Scoring Rules

As shown by Che (1993), when the buyer has a quasilinear preference $v(p,q) = V(q) - p$, the "truthful" QL scoring rule $S(p,q) = p - V(q)$ achieves efficiency and maximizes the social surplus. This is because, under the truthful scoring rule, every bidder voluntarily chooses the optimal quality that solves

$$\max_{(p,q)} p - C(q,\theta) \quad \text{s.t. } p - V(q) = s \quad \Rightarrow \quad \max_{q} V(q) - C(q,\theta) + s,$$

which is equivalent to the social surplus maximization.

This efficiency property holds even when the buyer has a nonquasilinear preference such as price per quality ratio. Note that in both FS and SS auctions, each bidder chooses a contract that maximizes (2). This problem is the supplier's ex post profit maximization given a buyer's payoff $v(p,q) = -s$. Hence, the chosen contract is Pareto efficient between the buyer and the bidder. Because both FS and SS auction chooses the most efficient (lowest-type) bidder as the winner, the equilibrium outcome of PQR scoring auctions is ex post Pareto efficient when the buyer's preference is represented by $v(p,q) = -p/q$.

Proposition 3 *Suppose that the buyer's utility function is given by $v(p,q) = -p/q$. The equilibirum outcomes of both FS and SS auctions with PQR scoring rule are ex post Pareto efficient.*[7]

This property supports the adoption of PQR scoring auctions when the buyer has the objective of minimizing price per quality ratio.

[7] Note that when the buyer has non-quasilinear preferences, the ex post Pareto efficiency does not imply ex ante Pareto efficiency.

Another interesting question is what is the optimal mechanism or optimal scoring rule for the buyer. Che (1993) shows that when the buyer's utility function is quasilinear, the buyer-optimal mechanism is obtained using Myerson (1981). Furthermore, Che (1993) shows that there exists a QL scoring rule that implements the optimal allocations. In the optimal scoring rule, the buyer sets a scoring rule which differs from the true utility function $(V(q) - p)$ and results in downward distortion of quality relative to the true value for quality $V(q)$.

Unfortunately, however, when the buyer's utility function is not quasilinear, the standard approach by Myerson (1981) is not applicable because the expected price does not capture the buyer's expected utility. Therefore, it is hard to obtain the optimal mechanism or optimal scoring rule that minimizes the price per quality ratio.

Nevertheless, we provide a qualitative result that the buyer can achieve a lower price per quality ratio by distorting the PQR scoring rules. We show that it is beneficial for the buyer to skew the quality lower than that achieved under the PQR scoring rule.

Consider a scoring rule

$$S^t(p, q) = \frac{p - t}{q}, \tag{15}$$

where t is a parameter representing the extent of distortion from the PQR scoring rule. When the scoring rule S^t is interpreted as a utility function on (p, q), the relative value of quality over price is captured by the marginal rate of substitution of q for p, which is given by

$$-\frac{S^t_q}{S^t_p} = \frac{p - t}{q}.$$

This is decreasing in t, and thus, a scoring rule S^t with $t > 0$ evaluates quality lower than the original PQR scoring rule. The following theorem states that the scoring rule S^t with a small $t > 0$ achieves a lower price per quality ratio than the PQR scoring rule.

Theorem 5 *Suppose $C_{q\theta} > 0$. For any type profile of bidders in which the lowest* *and second lowest types differ, an SS auction employing the scoring rule S^t, where $t > 0$ is sufficiently small, results in a lower price per quality ratio than an SS auction employing the PQR scoring rule.*

This theorem is consistent with the standard mechanism design of Mussa and Rosen (1978) and Myerson (1981). Since the cost function satisfies increasing differences $C_{q\theta} > 0$, the higher the quality, the more sensitive the cost is to type, resulting in an increase in the supplier's information rent. Thus, the buyer has an incentive to procure at a lower quality than a (Pareto) efficient quality level and reduce the supplier's information rent. This result is also consistent with an optimal scoring rule in Che (1993).

The distorted scoring rule S^t can be interpreted as a mechanism where the buyer pays a fixed subsidy t to the winner in a PQR scoring auction. This subsidy reduces the bidder's (non-sunk) fixed cost and lowers the break-even score, implying lower bidding in the SS auction. Thus, the subsidy decreases the quality bid, and the price per quality ratio improves. Note that the distortion improves the price per quality ratio in an ex post sense. Hence, distorting scoring rule is beneficial for the buyer even if their true preference is a nonlinear transformation of the price per quality ratio, such as the quality-per-price-ratio maximizer.

5 General Scoring Rules

The analysis thus far can be applied to more general scoring rules. Unfortunately, however, it is difficult to obtain sharp theoretical results for general non-QL scoring rules. Suppose that a scoring rule S is increasing in p and decreasing in q. The inverse function in terms of p is denoted by $P(s, q)$, which is the price function given score s and quality q, with $P_s > 0$ and $P_q > 0$. The bidder indirect payoff function is given by

$$u(s, \theta) \equiv \max_q P(s, q) - C(q, \theta).$$

We assume that the payoff function $P - C$ is strictly concave in q and that the payoff maximization problem always has a (unique) interior solution. That is, the optimal quality q^* is determined by the first-order condition

$$P_q(s, q^*) - C_q(q^*, \theta) = 0. \tag{16}$$

We further assume that the indirect payoff function u satisfies the log-supermodularity condition $\partial^2 \log u / \partial s \partial \theta > 0$. The equilibrium of the SS and FS auctions is characterized in the same manner with the PQR scoring rule.

Proposition 4 *Suppose that $P(s, q) - C(q, \theta)$ is strictly concave in q and that the optimal quality q^* is determined by the first-order condition (16). In the SS auction, it is a weakly dominant strategy for each bidder to submit $s^{SS}(\theta) = z(\theta)$. In the FS auction, the symmetric equilibrium score-bidding function s^{FS} is characterized by (9) with $s^{FS}(\bar{\theta}) = z(\bar{\theta})$ if u is log-supermodular.*

Proof The proof is the same as Propositions 1 and 2. □

Suppose that the buyer's utility is identical to the scoring rule; $v(p, q) = -S(p, q)$. The argument of Section 3.2.1 can be directly applied to general scoring rules. Namely, the expected score is lower (higher) in the SS than in the FS auction if $u(s, \theta)$ is convex (concave) in s. The following proposition is shown in the same manner as Theorem 1.

Proposition 5 *Suppose that the FS auction has a symmetric Bayesian Nash equilibrium. Then, the expected score in the SS auction is weakly lower (higher) than in the FS auction if $u(s, \theta)$ is convex (concave) in s for all θ.*

The curvature of the bidder's indirect payoff function depends on both scoring rule and cost function. The indirect payoff function is relatively likely to be convex, but can be concave. It is even more difficult to obtain clear properties with respect to equilibrium quality and price. We will discuss the details of the indirect payoff function and equilibrium quality and price in Appendix B.

6 Concluding Remarks

This study has examined scoring auctions using PQR and more general non-QL scoring rules. For the PQR scoring rule, we have characterized the equilibrium bidding strategies in FS and SS auctions and have found that the expected score is lower in SS auctions and that under a set of conditions expected quality and price are also lower. We also provided an example in which the expected quality in an FS auction is higher than in an SS auction while the expected price is equivalent or lower. These results suggest that if the price per quality ratio is the procurement buyer's true objective function, an SS auction is better for the buyer than an FS auction. However, the results also imply that the FS auction may perform better than the SS auction with respect to expected quality and price. Moreover, for the buyer with the objective of minimizing price per quality ratio, it is beneficial for them to adopt a scoring rule that skews quality downwardly relative to the original PQR scoring rule. Finally, we characterized the expected score ranking via the curvature of the indirect payoff function.

There are several potential extensions for further research. One important extension would be a theoretical consideration of a scoring auction with an interdependent scoring rule. In this study, we have restricted our attention to scoring rules in which each bidder's score depends only on its own price and quality. However, in practice, the buyer sometimes uses an interdependent scoring rule in which the score depends not only on the bidder's own price and quality bid but also on some or all competitors' price and quality bids. Another would be to incorporate the uncertainty of buyer's quality bid evaluation. Our model, following Che (1993), assumes that bidders do not face uncertainty in how their quality bids are evaluated by the buyer but, in practice, the bids are evaluated by reviewers and hence the scores of quality bids include noise (Takahashi, 2018). These theoretical analyses are left to future research.

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A Proofs

A.1 Proof of Lemma 1

Consider the following minimization problem:

$$\min_{q \geq \underline{q}} S(C(q,\theta), q).$$

Given an arbitrary q_0, set $s_0 = S(C(q_0,\theta), q_0)$. We can restrict the constraint set to $\left\{q \geq \underline{q} | S(C(q,\theta), q) \leq s_0\right\}$ without affecting the solution. We show that the restricted set is compact: Suppose not. Since the set is closed, it must be unbounded. Then we can take an arbitrarily large q_1 such that $S(C(q_1,\theta), q_1) \leq s_0$, which implies that $P(s_0, q_1) \geq C(q_1, \theta)$. Thus

$$\int_{q_0}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi = P(s_0,q_1) - C(q_1,\theta) - \{\overbrace{P(s_0,q_0) - C(q_0,\theta)}^{0}\} \geq 0.$$

Because $P(s,q) - C(q,\theta)$ is strictly concave in q, $P_q(s_0,q) < C_q(q,\theta)$ for all $q > q_2$, where $P_q(s_0,q_2) = C_q(q_2,\theta)$. Therefore

$$\int_{q_0}^{q_2} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi + \int_{q_2}^{q_1} \{P_q(s_0,\xi) - C_q(\xi,\theta)\} d\xi \geq 0. \tag{17}$$

The second term of the left-hand side is negative and has a sufficiently large absolute value as $q_1 \to \infty$, which is a contradiction to inequality (17). By the Weierstrass Theorem, a solution to the score minimization exists, and the value is the break-even score.

To show that $z(\cdot)$ is strictly increasing, let $q^z(\theta)$ denote a solution to the above score-minimization problem. Then $z(\theta) = S(C(q^z(\theta), \theta), q^z(\theta))$. Note that $P(z(\theta), q) \leq C(q, \theta)$ for all q (with equality at $q = q^z(\theta)$). Consider $\tilde{\theta} > \theta$. Since $C(q, \theta) < C(q, \tilde{\theta})$, we must have $P(z(\theta), q) < C(q, \tilde{\theta})$ for all q, implying that there is no intersection between $P(z(\theta), \cdot)$ and $C(\cdot, \tilde{\theta})$. Since $P_s(s, q) > 0$ and $P(z(\tilde{\theta}), q^z(\tilde{\theta})) = C(q^z(\tilde{\theta}), \tilde{\theta})$, $z(\tilde{\theta}) > z(\theta)$. $\square$

A.2 Proof of Proposition 2

Note that $q_s^* = 1/C_{qq}$ and $q_\theta^* = -C_{q\theta}/C_{qq}$. By differentiation, we have

$$\frac{\partial \log u(s, \theta)}{\partial s} = \frac{q^*(s, \theta)}{u(s, \theta)}$$

and

$$\begin{aligned} \frac{\partial^2 \log u(s, \theta)}{\partial s \partial \theta} &= \frac{1}{u(s, \theta)^2} \left(q_\theta^*(s, \theta) u(s, \theta) + q^*(s, \theta) C_\theta(q^*(s, \theta), \theta)\right) \\ &= \frac{1}{u(s, \theta)^2} \left(-q_s^*(s, \theta) C_{q\theta}(q^*, \theta) u(s, \theta) + q^*(s, \theta) C_\theta(q^*, \theta)\right). \end{aligned}$$

It is immediately clear that log-supermodularity holds if $C_{q\theta} \leq 0$. In what follows, we assume $C_{q\theta} > 0$ and provide two sufficient conditions under which the log-supermodularity condition holds.

Condition 1. Suppose that C_q/C_θ is non-decreasing in q. That is, we have

$$C_{q\theta} C_q - C_\theta C_{qq} \leq 0 \Leftrightarrow -\frac{C_{q\theta}}{C_{qq}} \geq -\frac{C_\theta}{C_q}$$

for all q and θ. By evaluating this at $q = q^*(s, \theta)$, we have

$$q_\theta^*(s, \theta) > -\frac{C_\theta(q^*, \theta)}{C_q(q^*, \theta)}. \tag{18}$$

Because $u(s,\theta) \geq 0$ for $s \geq z(\theta)$, we have

$$\begin{aligned}
\frac{\partial^2 \log u(s,\theta)}{\partial s \partial \theta} &= \frac{1}{u(s)^2}\left(q^*_\theta(s)u(s) + q^*(s)C_\theta(q^*(s))\right) \\
&\geq \frac{1}{u(s)^2}\left(-\frac{C_\theta(q^*(s))}{C_q(q^*(s))}u(s) + q^*(s)C_\theta(q^*(s))\right) \\
&= \frac{C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}\left(q^*(s)C_q(q^*(s)) - (sq^*(s) - C(q^*(s)))\right) \\
&> \frac{q^*(s)C_\theta(q^*(s))}{C_q(q^*(s))u(s)^2}(C_q(q^*(s)) - s) \\
&= 0.
\end{aligned} \tag{19}$$

Note that we omit the parameter θ from the presentation. The second line is derived from (18). The third line comes from the definition of the indirect payoff $u(s,\theta)$. The strict inequality is due to $C(q^*,\theta) > 0$ under Assumption 1. Finally, the last line comes from the first-order condition for the optimal quality $s - C_q(q^*,\theta) = 0$. Thus, log-supermodularity condition holds.

Condition 2. Fix an arbitrary θ and define a function V of score s by[8]

$$V(s) \equiv -q^*_s(s)C_{q\theta}\left(q^*(s)\right)u(s) + q^*(s)C_\theta\left(q^*(s)\right).$$

What we want to show is that $V(s) > 0$ for all $s \geq z(\theta)$. Note that $V(z(\theta)) = q^*C_\theta > 0$ by $u(z(\theta)) = 0$. Hence, it suffices to show that $V(s) = 0 \Rightarrow V'(s) > 0$ for every $s > z(\theta)$.

By differentiation, we have

$$\begin{aligned}
V'(s) &= -q^*_{ss}C_{q\theta}u - (q^*_s)^2 C_{qq\theta}u - q^*_s C_{q\theta}q^* + q^*_s C_\theta + q^* q^*_s C_{q\theta} \\
&= q^*_s C_\theta - q^*_{ss}C_{q\theta}u - (q^*_s)^2 C_{qq\theta}u.
\end{aligned} \tag{20}$$

Suppose $V(s) = 0 \Leftrightarrow u = q^*C_\theta / q^*_s C_{q\theta}$. By substituting this into (20), we have

$$V'(s)\big|_{V(s)=0} = q^*_s C_\theta - \frac{q^*_{ss}q^*C_\theta}{q^*_s} - \frac{q^*_s C_{qq\theta}q^*C_\theta}{C_{q\theta}}. \tag{21}$$

Note that $q^*_s = 1/C_{qq}$ and $q^*_{ss} = -C_{qqq}/(C_{qq})^3$. By substituting them into (21), we

[8] We omit the fixed parameter θ from the presentation.

have

$$
\begin{aligned}
V'(s)\big|_{V(s)=0} &= \frac{C_\theta}{C_{qq}} + \frac{C_{qqq}q^* C_\theta}{(C_{qq})^2} - \frac{q^* C_{qq\theta} C_\theta}{C_{qq} C_{q\theta}} \\
&= \frac{C_\theta}{C_{qq}} \left(1 + q^* \left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}}\right)\right).
\end{aligned} \tag{22}
$$

Note that C_θ, $C_{qq} > 0$, and

$$
\frac{\partial}{\partial q}\left(\frac{qC_{qq}}{C_{q\theta}}\right) = \frac{C_{qq}}{C_{q\theta}} \left(1 + q \left(\frac{C_{qqq}}{C_{qq}} - \frac{C_{qq\theta}}{C_{q\theta}}\right)\right).
$$

Hence, $V'(s)\big|_{V(s)=0} > 0$ and the log-supermodularity holds if $qC_{qq}/C_{q\theta}$ is increasing in q.

If the log-supermodularity condition (8) holds, there exists a monotone pure-strategy Bayesian Nash equilibrium in FPA (Athey, 2001). The equilibrium strategy is symmetric and characterized by the first-order condition as shown by Maskin and Riley (1984, Theorem 2).[9] Suppose that the equilibrium is symmetric and let s^{FS} be the symmetric equilibrium strategy. Suppose that every bidder other than i follows s^{FS}. The interim expected payoff when bidder i makes an equilibrium bid of type τ is

$$
(1 - G(\tau))\, u\left(s^{FS}(\tau), \theta\right).
$$

The first-order condition for the payoff maximization is

$$
-g(\tau)u\left(s^{FS}(\tau), \theta\right) + (s^{FS})'(\tau)\,(1 - G(\tau))\, u_s\left(s^{FS}(\tau), \theta\right) = 0.
$$

Because the first-order condition should hold with $\tau = \theta$, we have

$$
-g(\theta)u\left(s^{FS}(\theta), \theta\right) + (s^{FS})'(\theta)((1 - G(\theta))\, u_s\left(s^{FS}(\theta), \theta\right) = 0, \tag{23}
$$

which is (9). The terminal condition for the differential equation is $u(s^{FS}(\bar{\theta}), \bar{\theta}) = 0$. Thus, $s^{FS}(\bar{\theta}) = z(\bar{\theta})$. Under the log-supermodularity condition (8), the monotonicity of a strategy and the first-order condition are sufficient for the best response. Hence,

[9] Although Maskin and Riley (1984) assume that U is concave, this is not used nor is it necessary to obtain the FSA equilibrium. For instance, Board (2007, Lemma 3) is an example of a convex payoff function.

the strategy s^{FS} characterized by (9) is the symmetric equilibrium. In addition, note that

$$\frac{u(s,\theta)}{u_s(s,\theta)} = s - \frac{C(q^*(s,\theta),\theta)}{q^*(s,\theta)},$$

thus that the first-order condition (23) yields

$$-\left(s^{FS}(\theta) - \frac{C(q^*(s^{FS}(\theta),\theta),\theta)}{q^*(s^{FS}(\theta),\theta)}\right) g(\theta) + (s^{FS})'(\theta)(1-G(\theta)) = 0.$$

Solving the differential equation gives (10). □

A.3 Proof of Theorem 2

By the first-order condition for the optimal quality $s = C_q(q^*,\theta)$, we have

$$q^*_{ss} = -\frac{C_{qqq}}{(C_{qq})^3}.$$

Hence, the optimal quality function q^* is weakly concave if $C_{qqq} \geq 0$. Let $\theta_{(1)}$ and $\theta_{(2)}$ be the lowest and second lowest order statistics of bidder types. When q^* is weakly concave in s, we have

$$\begin{aligned} E\left[q^*\left(s^{SS}(\theta_{(2)}),\theta_{(1)}\right)\right] &= E_{\theta_{(1)}}\left[E_{\theta_{(2)}}\left[q^*\left(s^{SS}(\theta_{(2)}),\theta_{(1)}\right) \mid \theta_{(2)} > \theta_{(1)}\right]\right] \\ &\leq E_{\theta_{(1)}}\left[q^*\left(E_{\theta_{(2)}}\left[s^{SS}(\theta_{(2)}) \mid \theta_{(2)} > \theta_{(1)}\right], \theta_{(1)}\right)\right] \\ &\leq E\left[q^*\left(s^{FS}(\theta_{(1)}),\theta_{(1)}\right)\right]. \end{aligned}$$

Note that E_X means that we take an expectation regarding X. The first inequality is Jensen's inequality. The second inequality comes from Theorem 1. □

A.4 Proof of Theorem 3

Let $\pi(s,\theta) = sq^*(s,\theta)$ be the optimal price given score s and type θ. Then, by differentiation, we have

$$\pi_{ss}(s,\theta) = sq^*_{ss} + 2q^*_s.$$

Bu substituting $q^*_s = 1/C_{qq}$, $q^*_{ss} = -C_{qqq}/(C_{qq})^3$, and the first-order condition $s = C_q$, we have

$$\pi_{ss} = \frac{2(C_{qq})^2 - C_q C_{qqq}}{(C_{qq})^3}.$$

Thus, the optimal price is weakly concave if (12) holds. When π is weakly concave in s, we have the expected price ranking in the same manner with the quality ranking Theorem 2. $\square$

A.5 Proof of Theorem 4

Suppose that $C_{q\theta} = 0$. Then, it is clear that the optimal quality q^* is independent of θ and is denoted by $q^*(s)$. Let π be the optimal price function $\pi(s) = sq^*(s)$. Because q^* is increasing in s, π is also increasing in s. Thus, each price bid corresponds to a score bid in the one-to-one sense. That is, for every score s, we have a unique associated price $p = \pi(s)$. We define a payoff function in terms of the price bid $\hat{u}$ as

$$\hat{u}(p, \theta) \equiv u(\pi^{-1}(p), \theta).$$

Abusing notation, the cost function is denoted by $C = C(q) + \theta$, where $C(q)$ is variable cost and θ is the fixed cost.[10] Then, we have

$$\hat{u}(p, \theta) = p - C\left(\frac{p}{\pi^{-1}(p)}\right) - \theta.$$

By differentiation, we have

$$\hat{u}_p = 1 - \left(\frac{p}{\pi^{-1}(p)}\right)' C'\left(\frac{p}{\pi^{-1}(p)}\right)$$

and

$$\hat{u}_{pp} = -\left(\frac{p}{\pi^{-1}(p)}\right)'' C'\left(\frac{p}{\pi^{-1}(p)}\right) - \left\{\left(\frac{p}{\pi^{-1}(p)}\right)'\right\}^2 C''\left(\frac{p}{\pi^{-1}(p)}\right).$$

By differentiation, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{\pi^{-1}(p) - p(\pi^{-1})'(p)}{(\pi^{-1}(p))^2} = \frac{\pi'(\pi^{-1}(p))\pi^{-1}(p) - p}{\pi'(\pi^{-1}(p))(\pi^{-1}(p))^2}$$

and

$$\begin{aligned}
\left(\frac{p}{\pi^{-1}(p)}\right)'' &= \frac{1}{(\pi^{-1})^3}\left[-p(\pi^{-1})''\pi^{-1} - 2(\pi^{-1})'\pi^{-1} + 2p((\pi^{-1})')^2\right] \\
&= \frac{1}{(\pi^{-1})^3}\left[\frac{p\pi^{-1}\pi''}{(\pi')^3} - \frac{2\pi^{-1}}{\pi'} + \frac{2p}{(\pi')^2}\right] \\
&= \frac{1}{(\pi')^3(\pi^{-1})^3}\left[p\pi^{-1}\pi'' + 2p\pi' - 2\pi^{-1}(\pi')^2\right].
\end{aligned}$$

[10] Because the cost function is increasing in θ (by assumption), θ can be defined by fixed cost without any loss of generality.

Note that by definition, we have $p = \pi(s) = sq^*(s)$, $\pi^{-1}(p) = s$, $\pi'(s) = q^* + sq_s^*$, and $\pi''(s) = sq_{ss}^* + 2q_s^*$. By substituting them into the above, we have

$$\left(\frac{p}{\pi^{-1}(p)}\right)' = \frac{(q^* + sq_s^*)s - sq^*}{(q^* + sq_s^*)s^2} = \frac{q_s^*}{q^* + sq_s^*}$$

and

$$\begin{aligned}\left(\frac{p}{\pi^{-1}(p)}\right)'' &= \frac{1}{(q^* + sq_s^*)^3 s^3}\left[s^2 q^*(sq_{ss}^* + 2q_s^*) + 2sq^*(q^* + sq_s^*) - 2s(q^* + sq_s^*)^2\right] \\ &= \frac{q^* q_{ss}^* - 2(q_s^*)^2}{(q^* + sq_s^*)^3}.\end{aligned}$$

By the first-order condition for the optimal quality $s = C_q$, we have

$$\hat{u}_p = 1 - \frac{q_s^* C_q}{q^* + sq_s^*} = \frac{q^* C_{qq}}{q^* C_{qq} + C_q} > 0.$$

Also, we have

$$\begin{aligned}\hat{u}_{pp} &= \frac{2(q_s^*)^2 - q^* q_{ss}^*}{(q^* + sq_s^*)^3} C_q(q^*) - \frac{(q_s^*)^2}{(q^* + sq_s^*)^2} C_{qq}(q^*) \\ &= \frac{1}{(q^* + sq_s^*)^3}\left[2(q_s^*)^2 C_q - q^* q_{ss}^* C_q - (q_s^*)^2 C_{qq}(q^* + sq_s^*)\right] \\ &= \frac{1}{(q^* + sq_s^*)^3}\left[\frac{2C_q}{(C_{qq})^2} + \frac{q^* C_q C_{qqq}}{(C_{qq})^3} - \frac{q^* C_{qq} + C_q}{(C_{qq})^2}\right] \\ &= \frac{C_q C_{qq} + q^* C_q C_{qqq} - q^*(C_{qq})^2}{(q^* + sq_s^*)^3 (C_{qq})^3}.\end{aligned}$$

The third line comes from $q_s^* = 1/C_{qq}$ and $q_{ss}^* = -C_{qqq}/(C_{qq})^3$. Hence, $\hat{u}$ is convex in p if $C_q C_{qq} + qC_q C_{qqq} - q(C_{qq})^2 \geq 0$. Then, the expected price in the FS auction is higher than in the SS auction, which is analogous to Theorem 1 and Maskin and Riley (1984). $\square$

A.6 Proof of Theorem 5

Consider a scoring rule $S^t(p, q) = (p - t)/q$. Under the scoring rule S^t, bidders' indirect payoff function is given by

$$u^t(s, \theta) \equiv \max_q sq + t - C(q, \theta)$$

by $s = (p - t)/q \Leftrightarrow p = sq + t$. Thus, the optimal quality is independent of t and denoted by $q^*(s, \theta)$. The indirect payoff function satisfies

$$u^t(s, \theta) = u(s, \theta) + t,$$

where u is the indirect payoff function under the PQR scoring rule S^0. The break-even score for the scoring rule S^t is denoted by $z(\theta, t)$, which is determined by

$$u^t(z(\theta, t), \theta) = u(z(\theta, t), \theta) + t = 0.$$

By the implicit function theorem and the previous analysis, we have

$$z_t(\theta, t) = -\frac{1}{u_s(q^*, \theta)} = -\frac{1}{q^*(z(\theta, t), \theta)}. \tag{24}$$

Consider the SS auction with the scoring rule S^t. It is weakly dominant for each bidder to submit their break-even score $z(\theta, t)$. Let θ_i and θ_j be the lowest and the second lowest types among all bidders, respectively. Bidder i of type θ_i wins and chooses the contract with quality $q_i = q^*(z(\theta_j, t), \theta_i)$ and price $p_i = z(\theta_j, t)q^*(z(\theta_j, t), \theta_i) + t$. Thus, the associated price per quality ratio is

$$PQR \equiv z(\theta_j, t) + \frac{t}{q^*(z(\theta_j, t), \theta_i)}. \tag{25}$$

By differentiation, we have

$$\begin{aligned} PQR_t &= z_t(\theta_j, t) + \frac{q^*(z(\theta_j, t), \theta_i) - t z_t(\theta_j, t) q^*_s(z(\theta_j, t), \theta_i)}{(q^*(z(\theta_j, t), \theta_i))^2} \\ &= \frac{1}{q^*(z(\theta_j, t), \theta_i)} - \frac{1}{q^*(z(\theta_j, t), \theta_j)} + \frac{q^*_s(z(\theta_j, t), \theta_i)}{(q^*(z(\theta_j, t), \theta_i))^2 q^*(z(\theta_j, t), \theta_j)} t. \end{aligned}$$

Note that $\theta_i < \theta_j$ and q^* is decreasing in θ by $C_{q\theta} > 0$. Hence, we have $q^*(z(\theta_j, t), \theta_j) < q^*(z(\theta_j, t), \theta_i)$ and

$$PQR_t|_{t=0} < 0.$$

Thus, the equilibrium price per quality ratio decreases by slightly adding a distortion $t > 0$ to the PQR scoring rule. $\square$

B Properties of General Scoring Rules

In this appendix, we explore a set of conditions on primitives that guarantees the log-supermodularity condition for general scoring rules. We can restrict the domain to $\{(s,\theta)|u(s,\theta)>0\}$, since otherwise, the score bid is clearly suboptimal for a type θ bidder. Suppose that $u_{s\theta}$ exists. We suppose that the payoff function $P(s,q)-C(q,\theta)$ is strictly concave in q and that the optimal quality q^* always lies in the interior $q^*(s,\theta)>\underline{q}$.

Proposition 6 *The log-supermodularity condition holds if the optimal quality (and price) are not binding for all* (s,θ) *and*

1. $P_{sq}C_{q\theta}\leq 0$, *or*
2. $P_{sq}>0$, $C_{q\theta}\geq 0$, P/P_s *weakly increasing in* q,*and* $C_{q\theta}/(C_{qq}-P_{qq})<C_\theta/C_q$.

Proof The log-supermodular condition holds if and only if

$$\frac{u(s,\theta)}{u_s(s,\theta)}u_{s\theta}(s,\theta)-u_\theta(s,\theta)>0. \tag{26}$$

Note that by the envelope theorem, we have $u_s(s,\theta)=P_s(s,q^*)$, $u_\theta(s,\theta)=-C_\theta(q^*,\theta)$, and $u_{s\theta}(s,\theta)=P_{sq}(s,q^*)q^*_\theta(s,\theta)$. Thus, (26) holds if

$$\frac{u(s,\theta)}{u_s(s,\theta)}P_{sq}(s,q^*(s,\theta))q^*_\theta(s,\theta)+C_\theta(q^*(s,\theta),\theta)>0. \tag{27}$$

Because $q^*_\theta(s,\theta)=-C_{q\theta}/[C_{qq}(q^*(s,\theta),\theta)-P_{qq}(s,q^*(s,\theta))]$, we have condition 1 by the concavity of $P(s,q)-C(q,\theta)$ in q.

In what follows, we provide the proof for condition 2. We assume that $C_{q\theta}(q,\theta)\geq 0$ and that $P(s,q)/P_s(s,q)$ is weakly increasing in q. Let us further assume that

$$C_q(\cdot)\left[-\frac{C_{q\theta}(\cdot)}{C_{qq}(\cdot)-P_{qq}(\cdot)}\right]+C_\theta(\cdot)>0$$

holds for all q and θ. Then we evaluate this inequality at $q=q^*(s,\theta)$. Recall that the square-bracket term equals $q^*_\theta(s,\theta)$ if $q=q^*(s,\theta)$. Hence, we obtain

$$C_q(q^*(s,\theta),\theta)q^*_\theta(s,\theta)+C_\theta(q^*(s,\theta),\theta)>0. \tag{28}$$

Next, we show that if $P(s,q)/P_s(s,q)$ is weakly increasing in q for all s and q and $P_{sq}(\cdot) \geq 0$, then $[u(s,\theta)/u_s(s,\theta)]P_{sq}(\cdot) \leq C_q$. First, the condition that P/P_s is weakly increasing in q implies that

$$\frac{d}{dq}\frac{P(s,q)}{P_s(s,q)} = \frac{1}{(P_s(s,q))^2}\left[P_q(s,q)P_s(s,q) - P(s,q)P_{sq}(s,q)\right] \geq 0$$

for all s and q. Given the fact that $P_s > 0$, this inequality is equivalent to

$$\frac{P(s,q)}{P_s(s,q)}P_{sq}(s,q) \leq P_q(s,q) \quad \text{for all } s \text{ and } q.$$

Then we consider this (weak) inequality, replacing $P(s,q)$ with $P(s,q) - C(q,\theta)$ on the left-hand side. Given that $P_{sq} \geq 0$ and that $C(q,\theta)$ is nonnegative, the inequality implies that

$$\frac{[P(s,q) - C(q,\theta)]P_{sq}(s,q)}{P_s(s,q)} \leq P_q(s,q) \tag{29}$$

for all s and q.

By substituting $q = q^*(s,\theta)$ into (29), we have

$$\begin{aligned}\frac{u(s,\theta)}{u_s(s,\theta)}P_{sq}(s,q^*(s,\theta)) &= \frac{P(s,q^*) - C(q^*,\theta)}{P_s(s,q^*)}P_{sq}(s,q^*) \\ &\leq P_q(s,q^*) \\ &= C_q(q^*,\theta). \end{aligned} \tag{30}$$

The last equality comes from the first-order condition for the optimal quality q^*. Expressions (28) and (30) imply

$$\begin{aligned}&\frac{u(s,\theta)}{u_s(s,\theta)}P_{sq}(s,q^*(s,\theta))q^*_\theta(s,q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta) \\ &\geq C_q(q^*(s,\theta),\theta)q^*_\theta(s,q^*(s,\theta)) + C_\theta(q^*(s,\theta),\theta) \\ &> 0\end{aligned}$$

by $P_{sq} > 0$ and $q^*_\theta \leq 0$. Thus, log-supermodularity holds. □

Next, we explore the curvature of the bidder's indirect payoff function.Two factors affect the curvature of the indirect payoff function. Note that, by differentiation, we have

$$u_{ss}(s,\theta) = P_{ss}(s, q^*(s,\theta)) + P_{sq}(s, q^*(s,\theta))q_s^*(s,\theta). \tag{31}$$

The first term on the right-hand side of (31) captures the direct effect on u_{ss} of a change in the marginal payments with respect to s given q, while the second term in (31) captures the indirect effect of the change in the marginal payments with respect to s through the change in q.

Regarding the direct effect, the curvature of the scoring rule directly affects the bidder's induced utility function. Since $P_{ss}(s,q) = -S_{pp}/(S_p)^3$, as the scoring function becomes more concave (convex) in p, $u(s,\theta)$ becomes more (less) convex in s, ceteris paribus. Note that this direct effect is independent of the properties of the cost function.

On the other hand, the indirect effect, $P_{sq}(s, q^*(s,\theta))q_s^*(s,\theta)$, is always nonnegative. Indeed, by the first-order condition (16) for optimal quality and the implicit function theorem, we have

$$q_s^* = -\frac{P_{sq}}{P_{qq} - C_{qq}}. \tag{32}$$

Hence, $P_{sq}q_s^*$ is always nonnegative because of the strict concavity of the payoff function in q. Intuitively, with a scoring rule in which the associated P_s falls (rises) as q rises, the bidder will optimally choose a smaller (larger) q as s becomes larger. Moreover, as the indirect effect increases, $u(s,\theta)$ becomes more convex in s, ceteris paribus. Thus, given that the indirect effect is always nonnegative, $u(s,\theta)$ is convex if $S_{pp} \leq 0$.

We then discuss the properties of expected price and quality. An interesting feature of the PQR scoring rule is that the optimal quality q^* is increasing in score s. This suggests that under a PQR scoring auction, the lower-type bidders compete on price at the expense of quality. Note that the lower-type bidder submits a lower-score bid in equilibrium, so these bidders propose a lower quality with a much lower price as they become more efficient. This property may not be desirable for the procurer unless the scoring function represents their true preferences over a price-quality choice.

Note that by (32), the signs of q_s^* and P_{sq} coincide. Also, we have

$$P_{sq} = -\frac{S_{pp}P_q + S_{pq}}{(S_p)^2} = \frac{S_{pp}S_q - S_pS_{pq}}{(S_p)^3}. \tag{33}$$

The sign of S_{pq} is crucial for the slope of the optimal quality in s. In particular, if the scoring rule is linear in p (i.e., $S_{pp} = 0$), then the sign of q_s^* is determined by $-S_{pq}$. In the PQR scoring rule, $S_{pq} < 0$ and the optimal quality is increasing in score s.[11]

A scoring rule with $S_{pq} < 0$ implies that S_p, the marginal score with respect to price, increases as quality decreases. That is, when quality is already relatively low, a lower price lowers the score even more. In other words, the lower the quality, the more price competition is encouraged. Thus, even though lower-type bidders choose higher quality, scoring rules such as PQR are prone to price competition at the expense of quality.

Additionally, the quality ranking between FS and SS auctions depends on the curvature of the quality function q^*. When the indirect payoff u is convex, the FS auction yields a higher expected score than the SS auction. Similar to the discussion in the previous section, the expected quality is higher in the FS than in the SS auction if q^* is increasing and weakly concave in s but is higher in the SS auction if q^* is decreasing and weakly convex in s. However, because the condition for the concavity or convexity of q^* is complicated, it is difficult to obtain a clear comparison of quality between FS and SS auctions.

Moreover, the price ranking between FS and SS auctions is more ambiguous than quality. Let $\pi(s, \theta) \equiv P(s, q^*(s, \theta))$ be the price associated with score s and the optimal quality q^*. Then,

$$\pi_s = P_s + q_s^* P_q$$

[11] Note that the optimal quality is not affected by any monotone transformation of scoring rule S. Hence, we can focus on scoring rules with $S_{pp} = 0$ because every reasonable scoring rule can be transformed into this.

and

$$\pi_{ss} = P_{ss} + q^*_{ss} P_q - \frac{(P_{sq})^2}{P_{qq} - C_{qq}} \left(2 - \frac{P_{qq}}{P_{qq} - C_{qq}} \right)$$

by (32). The last term of π_{ss} is positive if $P_{qq} \geq 0$. Hence, the optimal price π is likely to be convex and so the expected price ranking becomes ambiguous when u is convex and q^* is increasing in s. This is analogous to Theorem 3 for the PQR scoring rule.

Further, for a scoring rule in which the associated optimal quality q^* is decreasing in s, the price function π may no longer be monotone, which makes the price ranking more ambiguous. Thus, with respect to general scoring rules, whether the expected price (quality) in the FS auction is lower relative to that in the SS auction is an empirical question.