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School choice with preference rank classes

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We introduce and study a large family of rules for many-to-one matching problems, the Preference Rank Partitioned (PRP) rules. PRP rules are student-proposing Deferred Acceptance rules, where the schools select among applicants in each round taking into account both the students' preferences and the schools' priorities. In a PRP rule each school first seeks to select students based on priority rank classes, and subsequently based on preference rank classes. PRP rules include many well-known matching rules, such as the classic Deferred Acceptance rule, the Boston rule, the Chinese Application-Rejection rules of Chen and Kesten (2017), and the French Priority rules of Bonkoungou (2020), in addition to matching rules that have not been studied yet. We analyze the stability, efficiency and incentive properties of PRP rules in this unified framework.

1. Introduction

We study a family of matching rules, which includes many already well-known rules, for many-to-one matching problems that match heterogeneous objects to agents, where objects have strict priorities over agents. Since this model is known as the school choice model due to Abdulkadiroğlu and Sönmez (2003), we call the objects schools and refer to the agents as students. However, our theoretical approach and results pertain to a broad range of applications, not just to school choice, such as centralized university admissions and refugee resettlement, among others. Balinski and Sönmez (1999) introduced this model first, which only differs from the college admissions model of Gale and Shapley (1962) in that school priorities are mandated by policies or by the law and thus school seats can be viewed as objects to be allocated, while in college admissions schools have preferences and are considered to be strategic agents. In the school choice model only the students' welfare and incentives are considered, and stability translates into fairness in this model, since the exogenously given school priorities are taken into account from the students' perspective.

We call the family of rules that we introduce and study Preference Rank Partitioned (PRP) rules, since these matching rules are student-proposing Deferred Acceptance (DA) rules in which schools use a choice function to select among applicants, which relies not only on the school priorities, but also on a partitioning of student preferences. Choice-based DA mechanisms are studied and characterized by Kojima and Manea (2010) and Ehlers and Klaus (2016). Choice functions are employed in matching with diversity constraints (e.g., Ehlers et al., 2014) or with distributional constraints (e.g., Kamada and Kojima, 2018), among others, which typically do not consider choice functions that depend on preferences.

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PRP rules are determined by a partition of each school's priority ranks of students and by a partition of each student's preference ranks of schools, which lead to priority and preference rank classes respectively. Students who are in a higher priority rank class are selected by the school's choice function first, followed by a comparison of the preference rank classes in which applying students place the school in question, in order to make further selections. If ties remain then the school-specific strict priorities over students are used for tie-breaking. Since the given priorities are assumed to be strict, a PRP rule specifies, as the first selection criterion, priority rank classes which lead to coarse (i.e., weak) priorities that are consistent with the given strict priorities. As the second selection criterion, a PRP rule bases the selection of students on their preference rank classes in the instances where some students applying to the school are in the same priority rank class and selecting all of them would result in exceeding the school's capacity. The strict priorities within the priority rank classes specified by the PRP rule are used only for tie-breaking, as a last resort, when neither the priority rank classes nor the preference rank classes can determine the selection of students by a school in a particular round of the iterated DA procedure.

When the given priorities are coarse, the DA and most other prominent matching rules are not well-defined, and the ties need to be resolved. The most straightforward way is to simply use lotteries; for example, Abdulkadiroğlu et al. (2009) propose to use a single tie-breaking procedure in the DA, which is a tie-breaking lottery that applies to each school. Further interesting papers on tie-breaking in conjunction with efficiency and strategyproofness are Ehlers and Erdil (2010) and Ehlers and Westkamp (2018). PRP rules can also be interpreted and used as matching rules that break the ties in coarse priorities by relying on the submitted preferences.

1.1. PRP rules in theory and practice

The class of PRP rules includes many well-known matching rules, such as the classic Deferred Acceptance (Gale and Shapley, 1962) and Boston rules (Abdulkadiroğlu and Sönmez, 2003), as well as the family of Application-Rejection mechanisms of Chen and Kesten (2017), the French Priority mechanisms of Bonkoungou (2020), and the Secure Boston mechanism proposed by Dur et al. (2019) to replace the Boston mechanism, among others. We analyze the entire family of PRP rules and a subfamily, the Equitable PRP rules, which treat students symmetrically due to their homogeneous preference rank classes across students. All previously studied rules and families of rules that are PRP rules are Equitable PRP rules. We also study new PRP rules which are not Equitable PRP rules, for example, the class of Favored Students rules, which treat students in one of two ways: each student has either the coarsest or the finest preference rank partition. We also identify some further classes of PRP rules which are Equitable PRP rules and include the DA and Boston rules, but are distinct from families of rules already studied in the literature, such as the Application-Rejection rules and the generalized class of Secure Boston rules, proposed by Dur et al. (2019). One such class is what we call the Deferred Boston rules, which are PRP rules that have homogeneous priority rank partitions combined with the finest preference rank partitions. We also identify the class of Homogeneous PRP rules, characterized by having both homogeneous priority and preference rank partitions, which makes Homogeneous PRP rules a superset of almost all previously studied PRP rules (except for some French Priority rules such as First Preference First and Secure Boston rules) and a subset of Equitable PRP rules.

Several PRP rules have been used worldwide in school choice, university admissions, and hospital-intern matching, apart from the widely used and celebrated Deferred Acceptance rule. Thus, PRP rules and their properties are not just of theoretical interest, but also have practical relevance. The Boston rule was in use in Boston for school choice until 2005 and is still a popular procedure for student placement. The Boston rule is a special case of the priority matching mechanisms of Roth (1991), which were used in several UK cities starting in the 1960s for allocating hospital positions to graduating medical students, and it is the only PRP rule which is also a priority matching mechanism. The First Preference First rule used in England for school choice was banned in 2007 (Pathak and Sönmez, 2013). The French Priority rules are employed in centralized university admissions in France (Bonkoungou, 2020). The Parallel Mechanisms (more generally, Application-Rejection mechanisms) are in use for college admissions in most provinces of China and are also widely used in China for school choice (Chen and Kesten, 2017, 2019).

1.2. Our contribution

We introduce the large class of PRP matching rules in a common setting and use this unified approach to establish stability, efficiency, and incentive properties of PRP rules. We first demonstrate that PRP rules choose the optimal stable matching that is consistent with the specific school choice functions at each preference profile (Proposition 1). We also characterize the subclass of PRP rules which treat students symmetrically, the Equitable PRP rules, by applying a natural weak stability property that relies on preference ranks in addition to the priorities to justify assignments (Proposition 2). This characterization generalizes to the larger class of rules which are not necessarily optimal but share the choice-function-specific stability properties of PRP rules (Theorem 1). We also show that the only Pareto-efficient PRP rules are what we call the Near-Boston rules (an implication of Theorem 2). Surprisingly, the set of Near-Boston rules includes some PRP rules other than the Boston rule, which is well-known to be efficient (Abdulkadiroğlu and Sönmez, 2003; Kojima and Ünver, 2014). However, the class of Pareto-efficient PRP rules is still quite restricted, since for these matching rules only one student's preference rank partition may differ from that of the Boston rule, which itself calls for the finest preference rank partition for each student.

However, the Boston rule is well-known to be highly manipulable, and these concerns carry over to other Near-Boston rules as well. If many students misrepresent their true preferences, the matching outcome is unlikely to be Pareto-efficient, and in fact the welfare loss can be significant (Ergin and Sönmez, 2006; Pathak and Sönmez, 2008). We prove that the only strategyproof PRP rule is the Deferred Acceptance rule (an implication of Theorem 3). While PRP rules are not strategyproof, except for the DA, they can be shown to be less manipulable than their non-optimal counterparts, based on the criteria and general results of Pathak and Sönmez (2013) and Chen et al. (2016). Our main result on incentives (Theorem 4) sheds new light on the incentive properties of well-known PRP rules and offers insight into their manipulability properties in general. It shows that students cannot manipulate PRP rules to obtain a school that was unattainable when reporting their preferences truthfully by placing this school in the same or in a lower preference rank class. This implies that manipulation is only possible by placing a desired but unattainable school in a higher preference rank class in the reported preferences than where it is truthfully. Although intuitive, this is a general result with significant and not necessarily obvious implications. It clarifies the different extents of possible manipulation for different PRP rules, specifically whether and how a school can potentially be obtained by manipulation, depending on the preference rank classes. It also helps us understand the scope of manipulation better, which may differ widely among students, even for Equitable PRP rules.

Overall, we find that the priority and preference rank partitions in PRP rules indicate trade-offs. Our results imply that there is a trade-off between respecting priorities and ensuring student welfare: fine priority rank partitions lead to fewer priority violations but allow less room for efficiency, while efficiency improvements require coarser priority rank partitions and finer preference rank partitions. The incentives also change along these trade-offs, since efficiency improvements in PRP rules increase the opportunity for manipulation, and thus strategic considerations may lead the market designer to choose finer priority rank partitions and coarser preference rank partitions, both of which lead to fewer priority violations.[1]

The closest papers to ours are Chen and Kesten (2017), which introduces and examines the by now well-known class of PRP rules known as Chinese Parallel Mechanisms, and Bonkoungou (2020), which studies another important subclass of PRP rules, the French Priority mechanisms, which are distinct from the Parallel Mechanisms. Chen and Kesten (2017) study the subclass of PRP rules for which the preference rank classes are homogeneous while the priority rank classes are the coarsest. We generalize the findings of Chen and Kesten (2017) about the extreme members of the class of Application-Rejection rules (Theorem 2 and Theorem 3), while their results on manipulability comparisons of different matching rules are only intuitively related to our main theorem on incentives (Theorem 4). Bonkoungou (2020) has coarse priorities as primitives of his model, and thus the French Priority rules, which always have the finest preference rank partition for students (just like the Boston rule does), collapse to one single rule in his approach. He explores the incentive properties of the French Priority rule from both an ex-ante and an ex-post perspective and introduces a notion called strategic accessibility, which serves as a basis for further manipulability comparisons (see also Bonkoungou and Nesterov, 2021). Specifically, Bonkoungou (2020) makes comparisons based on how fine the given coarse priorities are, while our main theorem on incentives pertains to the preference rank classes and complements the ex-post perspective results of Bonkoungou (2020).

2. Model

Let S be the set of n students and C the set of m schools. Each school $c \in C$ has capacity $q_c \geq 1$, that is, q_c is the number of seats available at school c. In order to simplify the exposition, we assume that $m \geq 3$ and that there exist schools $a, b, c \in C$ such that $q_a + q_b + q_c < n$, which implies that $n \geq 4$. This assumption ensures that there is scarcity for at least the three schools with the least capacity, so we don't need to take care of special cases where this minimum condition doesn't hold. Each school $c \in C$ has a strict priority ordering $\succ_c$ of students in S. Let π be the set of all priority orderings (i.e., permutations) of the n students in S. Then for all $c \in C$, $\succ_c \in \pi$. Let $\Pi = \pi \times \ldots \times \pi$ be the set of priority profiles (the m-fold Cartesian product of π). A priority profile is $\succ = (\succ_{c_1}, \ldots, \succ_{c_m})$, where $\succ \in \Pi$. We also use the notation $\succ_{-c}$ to denote $(\succ_a)_{a \in C \setminus \{c\}}$.

Each student $s \in S$ has a preference relation P_s, a strict ordering over $C \cup \{0\}$, where assigning 0 to student s represents staying unmatched (or being matched to an outside option). If $0 \; P_s \; c$ then school c is *unacceptable* to student s, and otherwise the school is *acceptable* to s. For $c, c' \in C \cup \{0\}$, we write $c \; P_s \; c'$ if student s strictly prefers c to c', and $c \; R_s \; c'$ if either $c \; P_s \; c'$ or $c = c'$. We also use the notation $r_s(c)$ for student s's rank of school $c \in C$ for each acceptable school c, where $r_s(c) = k$ indicates that c is ranked in the kth position by s. Note that if $r_s(c) < r_{\hat{s}}(c)$ then s ranks school c higher than $\hat{s}$ does, since a lower rank number indicates higher preference. Let $\mathcal{P}$ denote the set of all preference relations for a student, and let $\mathbb{P} = \mathcal{P} \times \ldots \times \mathcal{P}$ denote the set of preference profiles (the n-fold Cartesian product of $\mathcal{P}$). A preference profile is $P = (P_{s_1}, \ldots, P_{s_n})$, where $P \in \mathbb{P}$. We also use the notation P_{-s} to denote $(P_{s'})_{s' \in S \setminus \{s\}}$.

A **problem** is given by $(S, C, (q_c)_{c \in C}, (\succ, P))$, consisting of the fixed set of students S, the fixed set of schools C, the fixed school capacities $(q_c)_{c \in C}$, and a priority and preference profile pair $(\succ, P) \in \Pi \times \mathbb{P}$ (henceforth a *profile,* for short). Since only the profile is allowed to vary, a problem is specified by a profile. A **matching** is a function $\mu : S \to C \cup \{0\}$, where $\mu(s) \in C$ is the school to which student s is assigned. If a student s is unmatched in matching μ, we write $\mu(s) = 0$. For ease of notation, let μ_s denote $\mu(s)$. We will refer to μ_s as the *assignment* of student s in matching μ. Also, let μ_c denote $\mu^{-1}(c)$, the set of students assigned to c. For all $c \in C$, $|\mu_c| \leq q_c$, that is, the school capacity q_c cannot be exceeded.

[1] See also Abdulkadiroğlu et al. (2011) for a related analysis on the trade-offs between the Boston and DA rules.

Let the set of matchings be denoted by M. A **matching rule** φ assigns a matching to each profile $(\succ, P) \in \Pi \times \mathbb{P}$. Thus, $\varphi : \Pi \times \mathbb{P} \to M$.

A matching μ is individually rational for student $s \in S$ at profile $(\succ, P)$ if s weakly prefers μ_s to being unmatched, that is, $\mu_s \, R_s \, 0$. A matching is **individually rational** at profile $(\succ, P)$ if it is individually rational for all students $s \in S$ at $(\succ, P)$. A matching μ is **non-wasteful** at $(\succ, P)$ if no student s prefers a school to μ_s which has empty seats in matching μ, that is, for all $s \in S$ and $c \in C$, if $c \, P_s \, \mu_s$ then $|\mu_c| = q_c$.

Student s has **justified envy** in matching μ at profile $(\succ, P)$ if there exist school $c \in C$ and student $\hat{s} \in S$ such that $c \, P_s \, \mu_s$, $s \succ_c \hat{s}$ and $\mu_{\hat{s}} = c$. That is, student s's envy is justified, given that $\hat{s}$ is matched to c and s has higher priority for c than $\hat{s}$. A matching μ is **stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has justified envy in μ at $(\succ, P)$. A matching rule is stable if it assigns a stable matching to each profile $(\succ, P)$.

A matching $\eta \in M$ **Pareto-dominates** $\mu \in M$ at profile $(\succ, P)$ if, for all $s \in S$, $\eta_s \, R_s \, \mu_s$ and, for some $\hat{s} \in S$, $\eta_{\hat{s}} \, P_{\hat{s}} \, \mu_{\hat{s}}$. A matching $\mu \in M$ is **Pareto-efficient** at $(\succ, P)$ if there is no matching in M which Pareto-dominates μ at $(\succ, P)$. A matching rule is Pareto-efficient if it assigns a Pareto-efficient matching at $(\succ, P)$ to each profile $(\succ, P) \in \Pi \times \mathbb{P}$. A matching is **optimal stable** at $(\succ, P)$ if it is stable at $(\succ, P)$ and Pareto-dominates all other stable matchings at $(\succ, P)$. It is shown by Gale and Shapley (1962) that there is a unique optimal stable matching at each profile. A matching rule is optimal stable if it assigns the unique optimal stable matching at $(\succ, P)$ to each profile $(\succ, P) \in \Pi \times \mathbb{P}$.

3. Preference Rank Partitioned rules

We now describe the family of matching rules that we study in this paper, called Preference Rank Partitioned (or PRP) rules. PRP rules are choice-based Deferred Acceptance rules, that is, each school uses a choice function to select among its applicants in each round of the DA procedure. For each school $c \in C$, we define a choice function Ch_c such that for all $S' \subseteq S$, $Ch_c(S') \subseteq S'$ with the following properties. If $|S'| \leq q_c$ then $Ch_c(S') = S'$ and if $|S'| > q_c$ then $|Ch_c(S')| = q_c$.[2] For PRP rules the choice function for school c depends not only on the priority ordering $\succ_c$ but also on the preference profile, which is the main distinguishing feature of these matching rules.

Choice-based Deferred Acceptance rules

The choice-based DA rules are iterative. We describe the first round and a general round k.

Round 1: Each student applies to her highest-ranked acceptable school. Each school tentatively assigns its seats to its applicants *according to its choice function*. Any remaining applicants are rejected.

Round *k*: Each student who was rejected in round $k - 1$ applies to her next highest-ranked acceptable school, if any remains. Each school considers the students who are tentatively assigned to the school, if any, together with its new applicants (henceforth the "applicant pool"), and tentatively assigns its seats *according to its choice function*. Any remaining applicants are rejected.

The algorithm terminates when each student is either tentatively assigned to some school or has been rejected by each school that is acceptable to the student. The last tentative assignments constitute the final matching, while students without a tentative assignment in the last round remain unmatched.

Each PRP rule is determined by a profile of "partitions" of both the priority ranks of schools and the preference ranks of students. Each school's priority ranks are partitioned by specifying the number of consecutively ranked students in each member of the partition, starting from the top of the rankings.[3] Each student's preference ranks are also partitioned similarly by specifying the number of consecutively ranked schools in each member of the partition. The priority and preference rank partitions are used by the choice function Ch_c of each school c which selects among students in the applicant pool as a function of the priority and preference rank partitions specified by the PRP rule.

Given a partition of the priority ranks for school c, the **priority rank classes**, and given a partition of the preference ranks for each student, the **preference rank classes**, school c first selects students from its applicant pool in its highest priority rank class(es). If this does not determine which students are selected, where the maximum number of selected students is the capacity q_c of the school, which is possible because priority rank classes may contain more than one student, the choice function then considers the partitioned preference ranks, and selects students who have school c in their highest possible preference rank class(es) relative to each other. If the preference rank partitions still do not determine the selected set of students for school c, then the choice is resolved according to the exogenously given strict priority ordering $\succ_c$, which can be seen as a tie-breaking step. This defines a choice function for each school $c \in C$, as it determines the set of selected students unambiguously from any given applicant pool $S' \subseteq S$ for each $\succ_c$ and preference profile P.

In sum, each school's choice function selects students from the applicant pool lexicographically in the following order:

[2] This property of choice functions is known as *acceptant* (Kojima and Manea, 2010).

[3] A partition of priority ranks may arise naturally when priorities are coarse, as is often the case in school choice, but here we treat these partitions as part of the matching rule, while strict priorities are fixed exogenously for each school. We discuss how to start from coarse priorities in Section 9.

1. based on the **priority rank classes** of the school;
2. based on the **preference rank classes** that this school is placed in;
3. based on **tie-breaking** according to the given strict priority ordering of the school.

A collection of choice functions $(Ch_c)_{c\in C}$, which are determined by the priority and preference rank classes, identifies a PRP rule. Given their central role in the definition of PRP rules, we now define priority and preference rank classes formally. For all $c \in C$, let the cardinalities of the priority rank classes be denoted by $v_c^1, v_c^2, \ldots$, starting with the top-ranked students, such that $\sum_t v_c^t = n$. Let $v_c = (v_c^1, \ldots)$ denote the list of these cardinalities for each school $c \in C$, which we refer to as the priority rank partition for school c, and let $v = (v_c)_{c\in C}$ be the **priority rank partition profile.**

The **coarsest priority rank partition** is when $v_c = (n)$ and hence the partition has one member only. The **finest priority rank partition** is given by $(v_c^1, \ldots, v_c^n) = (1, \ldots, 1)$ with n members, where each priority rank class contains one student.[4] We also define **homogeneous priority rank partition profiles** as follows: a priority rank partition profile v is homogeneous if for all $c, c' \in C,\ v_c = v_{c'}$, that is, if the priority rank partition is the same for each school.[5]

Given a priority profile $\succ \in \Pi$, for all $c \in C$, let $V^1(\succ_c)$ be the set of students whose rank in $\succ_c$ is between 1 and v_c^1 and, for all $t \geq 2$, let $V^t(\succ_c)$ be the set of students whose rank in $\succ_c$ is between $\sum_{l=1}^{t-1} v_c^l + 1$ and $\sum_{l=1}^{t} v_c^l$. Note that for all $t \geq 1,\ |V^t(\succ_c)| = v_c^t$.

We define preference rank partitions similarly to priority rank partitions. For all $s \in S$, let the cardinalities of the preference rank classes be denoted by $x_s^1, x_s^2, \ldots$, starting with the top-ranked schools, such that $\sum_t x_s^t = m$. Let $x_s = (x_s^1, \ldots)$ denote the list of these cardinalities for each student $s \in S$, which we refer to as the preference rank partition for student s, and let $x = (x_s)_{s\in S}$ be the **preference rank partition profile.**

The **coarsest preference rank partition** is when $x_s = (m)$ and hence the partition has one member only.[6] The **finest preference rank partition** is given by $(x_s^1, \ldots, x_s^m) = (1, \ldots, 1)$ with m members, where each preference rank class contains one school. We also define **homogeneous preference rank partition profiles** as follows: a preference rank partition profile x is homogeneous if for all $s, s' \in S,\ x_s = x_{s'}$, that is, if the preference rank partition is the same for each student.

Given a preference profile $P \in \mathbb{P}$, for all $s \in S$, let $X^1(P_s)$ be the set of *acceptable* schools whose rank in P_s is between 1 and x_s^1 and, for all $t \geq 2$, let $X^t(P_s)$ be the set of *acceptable* schools whose rank in P_s is between $\sum_{l=1}^{t-1} x_s^l + 1$ and $\sum_{l=1}^{t} x_s^l$. Note that $|X^t(P_s)| = x_s^t$ for all $t \geq 1$ only if all schools are acceptable to student s (i.e., when assignment 0 is ranked last) and otherwise there exists $\hat{t} \geq 1$ such that $|X^{\hat{t}}(P_s)| < x_s^{\hat{t}}$, and for all $t < \hat{t}$, if there are any, $|X^t(P_s)| = x_s^t$, while for all the remaining preference rank classes of s with $t > \hat{t}$, if there are any, $X^t(P_s) = \emptyset$, where $\hat{t}$ depends on the number of acceptable schools to student s. Note that assignment 0 is not a member of a preference rank class, so $\sum_t |X^t(P_s)| \leq m$ is the number of acceptable schools according to P_s.

Each PRP rule is determined by a pair of a priority rank partition profile and a preference rank partition profile (v, x). This is not to be confused with the profile $(\succ, P)$, a pair of a strict priority profile and a strict preference profile. The latter determines a specific problem and is a primitive of our model, while (v, x) is part of the specification of the PRP rule. We will indicate explicitly the priority and preference rank partition profiles for a PRP rule and denote it by $f^{v,x}$.

We can now define the choice function Ch_c for school c in a PRP rule $f^{v,x}$. Fix $c \in C$ and let $S' \subseteq S$. The set of students is selected from applicant pool S' based on $\succ_c, v_c, (P_s)_{s\in S'}$, and $(x_s)_{s\in S'}$, as follows. If $|S'| \leq q_c$ then $Ch_c(S') = S'$, and if $|S'| > q_c$ then $Ch_c(S') = T$ such that $T \subset S'$, $|T| = q_c$ and the following selection rules are satisfied.

1. Selection based on the **priority rank classes**: there exists $k \geq 1$ such that for all $s \in T, s \in \cup_{t=1}^{k} V^t(\succ_c)$, and for all $\hat{s} \in S' \setminus T$ such that $\hat{s} \in \cup_{t=1}^{k} V^t(\succ_c)$, $\hat{s} \in V^k(\succ_c)$;
2. Selection based on the **preference rank classes**: there exists $k' \geq 1$ such that for all $s \in (T \cap V^k(\succ_c))$, $c \in \cup_{t=1}^{k'} X^t(P_s)$, and for all $\hat{s} \in S' \setminus T$ such that $\hat{s} \in V^k(\succ_c)$ and $c \in \cup_{t=1}^{k'} X^t(P_{\hat{s}})$, $c \in X^{k'}(P_{\hat{s}})$;
3. Selection based on **tie-breaking** according to the given strict priority ordering: for all $s \in T$ and $\hat{s} \in S' \setminus T$ such that $s, \hat{s} \in V^k(\succ_c)$, $c \in X^{k'}(P_s)$, and $c \in X^{k'}(P_{\hat{s}})$, $s \succ_c \hat{s}$.

Example 1 *(PRP choice functions).* Consider a matching problem with four students and four schools ($n = m = 4$). The preferences are given in the table below, in which the bars indicate the preference rank classes for the PRP rule. That is, $x_1 = (1, 2, 1), x_2 = (1, 3), x_3 = (3, 1), x_4 = (1, 3)$.

[4] Technically, the priority rank partition is considered to be the finest whenever the resulting matching rule is outcome-equivalent with the above. Given that a school's capacity may be greater than one, this implies that a priority rank partition for school c is still the finest if the students in priority ranks $q_c + 1, q_c + 2, \ldots, n$ are in their own singleton priority rank classes, and the priority rank classes of ranks 1 to q_c are arbitrary. However, in the standard representation of a PRP rule presented in Appendix A, if school c has the finest priority rank partition then $(v_c^1, \ldots, v_c^n) = (1, \ldots, 1)$.

[5] Since schools may have different capacities, a weaker notion of homogeneous priority rank partition profiles can also be defined as follows: for all $c, c' \in C$ such that $q_c = q_{c'},\ v_c = v_{c'}$. In a simpler model where each school has unit capacity, the two notions of homogeneity are the same.

[6] Technically, the preference rank partition is also the coarsest if having more than one preference rank class always leads to the same matching as having just one. For example, if $n \leq \sum_{c\in C} q_c$ then no student is rejected by her mth-ranked school, and thus $x_i^1 = m - 1$ also yields a coarsest partition. However, in the standard representation of a PRP rule presented in Appendix A, if student $s \in S$ has the coarsest priority rank partition then $x_s = (m)$.

P_1	P_2	P_3	P_4
b	b	d	b
c	a	b	a
a	d	a	d
d	0	c	c

Let $\succ_a = (1, 2, 3, 4)$ indicate the strict priorities in descending order for school a, and let $v_a = (3, 1)$. Assume that school a has capacity one ($q_a = 1$). Let the applicant pool for school a be $\{1, 3, 4\}$. Student 4 is eliminated based on the priority rank classes of school a, since $1, 3 \in V^1(\succ_a)$ and $4 \in V^2(\succ_a)$. This leaves students 1 and 3. Student 3 is selected based on the preference rank classes, since 1 ranks a in the second highest preference rank class and 3 ranks a in the highest preference rank class: $a \in X^2(P_1)$ and $a \in X^1(P_3)$.

Now consider the same matching problem and a slightly different PRP rule, where student 3's preference rank classes are given by $x_3 = (2, 1, 1)$. Given the same applicant pool $\{1, 3, 4\}$ for school a, student 4 is eliminated based on the priority rank classes of school a, as before, and given the new preference rank classes for student 3, now the selection cannot be made between 1 and 3 based on the preference rank classes, since both rank a in their second highest preference rank class: $a \in X^2(P_1)$ and $a \in X^2(P_3)$. Thus, we break the tie according to the strict priority order $\succ_a$ and hence, since $1 \succ_a 3$, student 1 is selected. $\diamond$

If both the priority rank partition for each school and the preference rank partition for each student are the coarsest, then the PRP rule relies only on the strict priorities as tie-breakers, and therefore this rule is the classic DA rule, which simply selects the highest-priority students from each applicant pool. Equivalently, we can let the priority rank partition be arbitrary. As long as the preference rank partition is the coarsest, the PRP rule is the DA. Hence, there may be multiple representations (v, x) of the same PRP rule, and thus we use the convention that the role of tie-breaking should be minimized as much as possible, by making the priority rank partition finer. Given the finest possible priority rank partition, the preference rank partition should be left as coarse as possible, which clarifies the actual impact of the preferences. This is the most salient feature of PRP rules, namely, that the choice functions of the schools depend on the students' preferences, and specifically on the students' preference rank partitions, which is why we call these matching rules Preference Rank Partitioned rules. This specification of a PRP rule delineates which information is used by the choice function, whether it is the priority rank partition or the preference rank partition, and specifically which priority or preference rank classes play a role in the schools' selections. In the case of the classic DA rule, this means that we let the priority rank partition be the finest for each school, so as to entirely eliminate tie-breaking. This also reveals immediately that the preference rank partitions don't play any role in the choice function, as all selections can be made based on the priority rank partitions, and therefore we can let each student's preference rank partition be the coarsest for the DA rule. This convention not only lets us have a specific unique representation of each PRP rule, but it also provides an informative representation. We show the details of the formal construction of the priority and preference rank classes that follow this convention in Appendix A, and refer to this unique specification (v^*, x^*) as the *standard representation* of a PRP rule. In the following we assume that PRP rules are specified using this standard representation.

4. Special subclasses of PRP rules

As already noted, the classic DA rule is a PRP rule, which is described by the finest priority rank partition profile and the coarsest preference rank partition profile. Another well-studied PRP rule besides the DA rule is the Boston (Immediate Acceptance) rule. The Boston rule is a PRP rule with the coarsest priority rank partition and the finest preference rank partition, hence tie-breaking is only necessary when students have the same rank for a school: if students s and $\hat{s}$ are competing for school c then s is chosen over $\hat{s}$ if s ranks c higher than $\hat{s}$ (i.e., $r_s(c) < r_{\hat{s}}(c)$) or if s and $\hat{s}$ rank c equally (i.e., $r_s(c) = r_{\hat{s}}(c)$) and $s \succ_c \hat{s}$. Ergin and Sönmez (2006) pointed out first that the Boston rule can be viewed as a lexicographic rule which first considers the student preferences and then the strict school priorities.

Previously studied classes of matching rules that belong to the set of PRP rules include the First Preference First rules (Pathak and Sönmez, 2013), the Secure Boston rules and their generalizations (Dur et al., 2019), the French Priority rules studied by Bonkoungou (2020) which correspond to a broad class of PRP rules in our setting and includes all of the above rules, as well as the Application-Rejection rules of Chen and Kesten (2017) which are distinct from the French Priority rules (except for the classic DA and Boston rules, the two extreme members of all of the above classes of rules). We list these in Table 1, together with some further notable subfamilies of PRP rules which have not been studied before. The Deferred Boston rules include both the DA rule and the Boston rule and allow for any homogeneous priority rank partition profile, while the preference rank partitions are the finest. If both the preference and priority rank partition profiles are homogeneous then we have a Homogeneous PRP rule. All Deferred Boston rules are Homogeneous PRP rules, but the French Priority rules in general are not Homogeneous PRP rules, and specifically the First Preference First rule and the Secure Boston rule are not Homogeneous PRP rules,[7] since they don't have homogeneous priority rank partition profiles. On the

[7] However, the Secure Boston rule has a priority rank partition profile that satisfies the weaker notion of homogeneity defined in footnote 5.

Table 1
Special members and subclasses of PRP rules.

PRP rules	Priority rank partition	Preference rank partition
Deferred Acceptance (DA)	Finest	Coarsest
Boston	Coarsest	Finest
Deferred Boston	Homogeneous	Finest
First Preference First	Equal-preference schools: finest Preference-first schools: coarsest	Finest
Secure Boston	For each school c: finest for the top q_c ranks, then coarsest	Finest
French Priority	Arbitrary	Finest
Application-Rejection	Coarsest	Homogeneous
Homogeneous PRP	Homogeneous	Homogeneous
Equitable PRP	Arbitrary	Homogeneous
Favored Students	Coarsest	Favored students: coarsest Non-favored students: finest

other hand, the Application-Rejection rules are Homogeneous PRP rules. The class of Equitable PRP rules, characterized by a homogeneous preference rank partition profile, is even larger than the class of Homogeneous PRP rules, and contains all of the above mentioned matching rules. Lastly, to identify a specific class of PRP rules which does not belong to the class of Equitable PRP rules, we included in the table the family of Favored Students rules, which allow for different treatments of students. Specifically, Favored Students rules have the coarsest priority rank partition profile, and each student is either favored or not. Favored students have the coarsest preference rank partition, while all the other students have the finest. Favored Students and other PRP rules which distinguish among students may be desirable if one of the objectives of the matching is to prioritize certain classes of students, for example when affirmative action or equal opportunity policies are employed. These policies are often carried out by reserving seats for disadvantaged groups,[8] and the Favored Students and similar but less extreme PRP rules that are not Equitable PRP rules provide a novel approach to implementing affirmative action and other priority policies.

Although we defined PRP rules by using first the priority rank classes when selecting students, note that for the PRP rules that have the coarsest priority rank partitions the priority ranks do not play any role in the selection of students up front, and we understand intuitively that some PRP rules, such as the Boston rule and more generally Application-Rejection rules with small preference rank classes, make selections based on the preference rank classes primarily, and the strict priorities are used for tie-breaking only when needed. However, for PRP rules which don't have the coarsest priority rank partition profile, such as the Deferred Boston rules or the First Preference First rules, the priority rank classes play a role in student selection.

5. Stability and optimality of PRP rules

The dependence of PRP choice functions on student preferences, the most notable general feature of PRP rules, accounts for violating typical stability conditions that are independent of the preferences. Given that when preference rank partitions are the coarsest the preferences play no role in choosing among applicants, and given that the only such PRP rule is the classic DA rule, this is the only rule which satisfies the standard stability axiom in the class of PRP rules.

Therefore, we introduce a stability concept inspired by PRP rules, which we call *rank-partition stability.* Given a profile of rank partitions (v, x), for each profile $(\succ, P)$ we construct a modified strict priority profile $\bar{\succ}((\succ, P), (v, x))$ (or simply $\bar{\succ}$ to simplify notation) as follows. For each school $c \in C$ the ordering of students across priority rank classes based on $\succ_c$ and v_c remain the same in $\bar{\succ}_c$, and within priority rank classes we order students according to the preference rank partitions of P based on x. If ties remain due to students being in the same priority rank class and ranking c in the same preference rank class then we use the given strict priority ordering $\succ_c$ for tie-breaking.

More formally, for all $c \in C$, we construct $\bar{\succ}_c((\succ, P), (v, x))$ as follows. Let $s, \hat{s} \in S$ and let $k, k' \geq 1$ such that $s \in V^k(\succ_c)$ and $\hat{s} \in V^{k'}(\succ_c)$. If $k \neq k'$ then s and $\hat{s}$ are in different priority rank classes and $s \bar{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. If $k = k'$ then s and $\hat{s}$ are in the same priority rank class. By convention, students who find a school c unacceptable are ranked last within the priority rank class that they belong to in $\bar{\succ}_c$, although such students could be ranked anywhere in $\bar{\succ}_c$ as long as the basic property of individual rationality is required of matching rules. If $k = k'$ and both $c\ P_s\ 0$ and $c\ P_{\hat{s}}\ 0$ then let $t, t' \geq 1$ such that $c \in X^t(P_s)$ and $c \in X^{t'}(P_{\hat{s}})$. Then $t < t'$ implies $s \bar{\succ}_c \hat{s}$ and $t > t'$ implies $\hat{s} \bar{\succ}_c s$. Finally, if $k = k'$ and $t = t'$ then

8 See, for example, Hafalir et al. (2013) and Doğan (2016).

$s \bar{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. From now on we will refer to $\bar{\succ}((\succ, P), (v, x))$ as the **modified priority profile**. Very importantly, the modified priority profile is a function of the preference profile, in addition to the priority profile, and thus it changes as preferences vary.

A matching rule φ is **rank-partition stable** if there exists (v, x) such that

1. for all $(\succ, P) \in \Pi \times \mathbb{P}$, $\varphi(\succ, P)$ is stable at $(\bar{\succ}((\succ, P), (v, x)), P)$;
2. for all $\succ, \succ' \in \Pi$ and for all $P \in \mathbb{P}$, if $\bar{\succ}((\succ, P), (v, x)) = \bar{\succ}((\succ', P), (v, x))$ then $\varphi(\succ, P) = \varphi(\succ', P)$.

That is, given (v, x), if a matching rule assigns a matching to each profile $(\succ, P)$ that is stable with respect to the modified priority profile $\bar{\succ}((\succ, P), (v, x))$ at P and if the selection of a matching only depends on this modified priority profile at each preference profile, then the matching rule is rank-partition stable. We also say that a matching rule φ is rank-partition stable with respect to (v, x).

Example 2 *(Construction of the modified priority profile).* Consider a matching problem with five students and four schools ($n = 5, m = 4$). The priority and preference profiles are given below, with the profile of rank partitions (v, x) specified by the bars in the two profiles.

School priorities $\succ$

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	4	4
1	1	1	5
2	5	5	3
3	2	2	1
5	4	3	2

Student preferences P

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	a
a	a	d	c	0
d	d	c	d	

Modified priority profile $\bar{\succ}$ at P

$\bar{\succ}_a$	$\bar{\succ}_b$	$\bar{\succ}_c$	$\bar{\succ}_d$
1	3	1	5
2	1	4	4
4	2	2	3
5	4	3	1
3	5	5	2

The modified priority profile $\bar{\succ}$ shows how the priority profile $\succ$ is modified for this PRP rule $f^{v,x}$ at the specified preference profile P. We will explain the construction of $\bar{\succ}_a$ in detail. The priority rank partition of $\succ_a$ implies that students 4, 1 and 2 are ranked above students 3 and 5 in the modified priority ordering $\bar{\succ}_a$ for school a. Given the preference rank partitions of students 4, 1 and 2, a is in the first preference rank class for students 1 and 2, but only in the second preference rank class for student 4 at preference profile P. Therefore, $\bar{\succ}_a$ ranks 1 and 2 above 4. In order to break the tie between 1 and 2, we use the given strict priority ordering: since $1 \succ_a 2$, we get $1 \bar{\succ}_a 2$. Therefore, $1 \bar{\succ}_a 2 \bar{\succ}_a 4$. Moreover, considering the second priority rank class of $\succ_a$, student 5 has a in her first preference rank class, whereas 3 has a in his second preference rank class, and hence $5 \bar{\succ}_a 3$. ◇

The classic results of Gale and Shapley (1962) imply in our setting that for each profile $(\succ, P)$ and priority and preference rank partition profile pair (v, x) there exists a matching which is stable at profile $(\bar{\succ}((\succ, P), (v, x)), P)$, and each student weakly prefers this matching to any other matching which is stable at $(\bar{\succ}((\succ, P), (v, x)), P)$. We call this unique matching the **$(\boldsymbol{v}, \boldsymbol{x})$-optimal rank-partition stable** matching at $(\succ, P)$ since it is optimal stable at profile $(\bar{\succ}((\succ, P), (v, x)), P)$. In light of the definition of the modified priority profile, a straightforward finding is stated below. Note that all the proofs are collected in Appendix B.

Proposition 1. *Each PRP rule $f^{v,x}$ is rank-partition stable and assigns the unique (v, x)-optimal rank-partition stable matching at $(\succ, P)$ to each profile $(\succ, P) \in \Pi \times \mathbb{P}$.*

Rank-partition stability can be seen as a natural stability property of PRP rules, based on stability with respect to the appropriately modified priority profile at each preference profile. A similar property is used by Bonkoungou (2020) for French Priority rules. This representation of PRP rules and the underlying stability concept serve as a foundation for later results, since they highlight the parallel features between PRP rules and the classic DA rule, and allow us to see the PRP rules as optimal rules within the set of rank-partition stable rules, which are stable with respect to the modified priority profile at each preference profile, where the modifications of the priorities correspond to the selections made by PRP choice functions. The proposition also implies that PRP rules are individually rational and non-wasteful, although these properties can easily be verified directly as well.

A choice function Ch_c for school c is called *acceptant responsive* if, for all $S' \subseteq S$, it selects the set of $\min(q_c, |S'|)$ top-priority students in S' (Kojima and Manea, 2010). This property is satisfied by the choice functions of the standard DA rule. For the broad class of PRP rules, given a priority profile, the modified priority ordering of each school is allowed to depend on the preference profile, and thus PRP choice functions only satisfy a weak preference-profile-specific version of this property. Essentially, Proposition 1 tells us that a PRP rule $f^{v,x}$ can be seen as a DA rule with preference-profile-specific (v, x)-modified acceptant responsive priorities.

6. Equitable PRP rules

Not all PRP choice functions treat students the same way when considering their reported preferences. For example, if student s has a coarser preference rank partition than student $\hat{s}$ then s gets a preferential treatment compared to $\hat{s}$. We call the subfamily of PRP rules specified with a homogeneous preference rank partition profile **Equitable PRP rules**, since these PRP rules treat students symmetrically in terms of their preference rank partitions. Given that the priority rank partition profile and the preference rank partition profile that describe a PRP rule are not always unique, we define Equitable PRP rules by having a homogeneous preference rank partition x^* in the standard representation (v^*, x^*) of the PRP rule, which means all the PRP rules that have any representation with a homogeneous preference rank partition profile (see Appendix A for details).

To aid our analysis, we propose a general stability property of matching rules, called PP-stability, which lets the matching to be based on either a comparison of students' preference ranks of schools that they compete for or on the priority ordering of students by this school. Characterizations of the Boston rule given by Kojima and Ünver (2014) and Doğan and Klaus (2018) rely on axioms that compare the preference ranks of alternatives, since the Boston rule makes matches primarily based on the preference ranks. Our concept says that envy is only justified if neither the priorities nor the preference ranks can explain the selection of one student over another at a school to which both students have applied, and thus ruling out only these demanding cases of justified envy relaxes the standard stability axiom which is based only on the priority ranking of students. Afacan (2013) introduces a related property that combines priority and preference rankings, but his axiom is not implied by standard stability and it makes explicit use of the school capacity.

Preference and Priority Rank Stability (PP-Stability): Student s has **PP-justified envy** in matching μ at $(\succ, P)$ if there exist school $c \in C$ and student $\hat{s} \in S$ such that $c\ P_s\ \mu_s$, $s \succ_c \hat{s}$, $r_s(c) \le r_{\hat{s}}(c)$ at P, and $\mu_{\hat{s}} = c$. A matching μ is **PP-stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has PP-justified envy in μ at $(\succ, P)$.

Student s has PP-justified envy if s prefers some school c to her assignment and some other student $\hat{s}$ is matched to c such that s has both higher priority for c than $\hat{s}$ and ranks c the same or higher than $\hat{s}$. The sole difference between this definition and the standard definition of justified envy is that envy is only justified if $r_s(c) \le r_{\hat{s}}(c)$ also holds. Unlike rank-partition stability, PP-stability is a property of matchings primarily (just like standard stability), and we define a matching rule to be PP-stable if it assigns a PP-stable matching to each profile $(\succ, P)$.

The next example illustrates that not all PRP rules are PP-stable.

Example 3 *(A Favored Students PRP rule which is not PP-stable).* Consider the following matching problem with five students and four schools $(n = 5, m = 4)$. School capacities are given by $q_a = q_b = q_c = 1$ and $q_d = 2$. The priority and preference profiles are given below. The school priorities are the coarsest for each school, while the bars indicate the preference rank partitions in the preference profile.

School priorities $\succ$

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	1	4
1	1	2	1
2	4	3	5
3	2	4	2
5	5	5	3

Student preferences P

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	b
a	a	d	c	a
d	d	c	d	c

In this example students 1 and 2 are so-called favored students, since there is only one preference rank class for these students that includes all the schools (the coarsest preference rank partition), while the non-favored students, 3, 4, and 5, have the finest preference rank partition: $x_1 = x_2 = (4)$ and $x_3 = x_4 = x_5 = (1, 1, 1, 1)$.

The rounds of the specified Favored Students rule at the given profile are summarized in the table below, with the selected students in each round underlined.

Round	a	b	c	d
1		$1, 2, \underline{3}, 4$		$\underline{5}$
2	$\underline{4}$	$\underline{3}$	$\underline{1}, 2$	$\underline{5}$
3	$\underline{2}, 4$	$\underline{3}$	$\underline{1}$	$\underline{5}$
4	$\underline{2}$	$\underline{3}$	$\underline{1}, 4$	$\underline{5}$
5	$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{4}, \underline{5}$

In this example $r_4(a) = 2 < 3 = r_2(a)$ and $4 \succ_a 2$. Since $a\, P_4\, d$ and 2 is assigned to a, student 4, a non-favored student, has PP-justified envy and therefore this rule is not PP-stable. Note that student 2, whose assignment to school a is the cause of PP-justified envy, is a favored student. $\diamond$

We remark that both the DA and Boston rules are PP-stable. Given that stability implies PP-stability, the DA rule is PP-stable. The Boston rule is also PP-stable, since if either $r_s(c) < r_{\hat{s}}(c)$ holds by itself or both $r_s(c) = r_{\hat{s}}(c)$ and $s \succ_c \hat{s}$ hold then $\hat{s}$ cannot be assigned to c unless s is assigned to school $\tilde{c}$, where $\tilde{c}\, R_s\, c$. Thus, the Boston rule is preference-rank stable in the sense that whenever $r_s(c) < r_{\hat{s}}(c)$, s does not envy $\hat{s}$ when $\hat{s}$ is assigned to c. PRP rules in general do not satisfy this stronger property, and it is easy to check that the Boston rule is the only such preference-rank stable rule within the class of PRP rules. On the other hand, the DA rule is the only priority-rank stable (i.e., stable) rule in the class of PRP rules.

While not all PRP rules satisfy PP-stability, as demonstrated by Example 3, the subfamily of PRP rules which satisfy PP-stability is much larger than just the DA and Boston rules, and we will show in the next proposition that it exactly corresponds to the Equitable PRP rules. We remark that all previously studied PRP rules in the literature are Equitable PRP rules. Proposition 2 below provides an explanation for this, since it demonstrates that all the studied rules are PP-stable, which is an intuitive feature of matching rules and an attractive attribute in school choice, assuming that we want to treat all students the same way.

Proposition 2. *A PRP rule is PP-stable if and only if it is an Equitable PRP rule.*

We say that a matching rule is **equitable-rank-partition stable** if it is rank-partition stable with respect to (v, x), where x is homogeneous. Clearly, equitable-rank-partition stability implies rank-partition stability of a matching rule. The next result further clarifies the relationship between rank-partition stability and equitable-rank-partition stability.

Theorem 1. *A rank-partition stable matching rule is PP-stable if and only if it is equitable-rank-partition stable.*

As in the case of PRP rules, we can easily verify that an Equitable PRP rule $f^{v,x}$ is equitable-rank-partition stable, and we know from Proposition 1 that $f^{v,x}$ selects the (v, x)-optimal rank-partition stable matching at each profile. Thus, we can state a similar result to Proposition 1 for Equitable PRP rules: each Equitable PRP rule $f^{v,x}$ is equitable-rank-partition stable and assigns the unique (v, x)-optimal equitable-rank-partition stable matching at $(\succ, P)$ to each profile $(\succ, P) \in \Pi \times \mathbb{P}$. Therefore, Theorem 1 can be seen as a generalization of Proposition 2.

Finally, let us mention that rank-partition stability and PP-stability are logically independent. Proposition 1 and Example 3 together demonstrate that rank-partition stability does not imply PP-stability. Alternatively, this also follows from Theorem 1. The other direction can be inferred from the definition of rank-partition stability, as it requires that the same (v, x) is applied to all profiles. To see that PP-stability does not imply rank-partition stability we can specify, for example, a matching rule which picks the DA matching at some profiles and the Boston matching at others. Such a rule is PP-stable (since both the DA and Boston matchings are PP-stable), but we can easily construct examples where a combination of DA and Boston matchings leads to different priority and preference rank partitions at different profiles.

7. Efficiency of PRP rules

PRP rules select the optimal rank-partition stable matching at each profile, as shown in Section 5. This implies that if the matching selected by a PRP rule is not Pareto-efficient then it can only be Pareto-dominated by a matching that is not stable with respect to the modified priority profile at the preference profile in question. We know that the DA matching is not necessarily Pareto-efficient when considering students only (Balinski and Sönmez, 1999), so PRP rules, which choose the DA matching at the modified priority profile, are generally not Pareto-efficient. One notable exception is the Boston rule, which is Pareto-efficient due to the fact that it assigns school seats permanently at the time of the application based on the students' preference rankings and uses the school priorities only for tie-breaking, and thus the selected matching cannot be Pareto-dominated. One may conjecture that the only Pareto-efficient PRP rule is the Boston rule, but it turns out that a somewhat larger set of PRP rules are Pareto-efficient. Namely, there may be one student whose preference rank partition is not necessarily the finest and may be chosen arbitrarily, but all other students' preference rank partitions have to be the finest. We call this class of rules, including the Boston rule, Near-Boston rules.

Near-Boston rules are PRP rules such that:

i) each school has the coarsest priority rank partition;
ii) there exists $j \in S$ such that each student $s \in S \setminus \{j\}$ has the finest preference rank partition (while student j has an arbitrary preference rank partition).

Theorem 2. *A rank-partition stable matching rule is Pareto-efficient if and only if it is a Near-Boston rule.*

Given Proposition 1, Theorem 2 implies that the only Pareto-efficient PRP rules are the Near-Boston rules.

Corollary to Theorem 2. *An equitable-rank-partition stable matching rule is Pareto-efficient if and only if it is the Boston rule.*

This is an immediate corollary to Theorem 2, since if the rule is equitable-rank-partition stable then it uses a homogeneous preference rank partition profile, and thus Theorem 2 implies that all students have the finest preference rank partition. Given Proposition 1, it follows from the corollary that the only Pareto-efficient Equitable PRP rule is the Boston rule. Therefore, this corollary generalizes the result of Chen and Kesten (2017) which shows that only the Boston rule is Pareto-efficient in the class of Application-Rejection rules.

8. Incentive properties of PRP rules

Since PRP rules are generally not strategyproof, we need to analyze their incentive properties. We start with some relevant definitions. Given a profile $(\succ, P) \in \Pi \times \mathbb{P}$, if there is a student $s \in S$ and an alternative preference ordering $P'_s \in \mathcal{P}$ such that $\varphi_s(\succ, (P'_s, P_{-s}))\ P_s\ \varphi_s(\succ, P)$ then s can **manipulate** rule φ at $(\succ, P)$ via P'_s and φ is **manipulable** (at $(\succ, P)$). We also say that student s can manipulate φ at $(\succ, P)$ to obtain a seat at school $\varphi_s(\succ, (P'_s, P_{-s}))$. If a rule is not manipulable (at any profile) then the rule is **strategyproof.**

The classic DA rule is well-known to be strategyproof when only the students' incentives are taken into account (Dubins and Freedman, 1981; Roth, 1982). However, we can find examples of preference profiles at which a PRP rule (other than the DA) is manipulable, and we get a negative result for all PRP rules excluding the DA rule. We prove this result not only for the class of PRP rules but also, more generally, for all rank-partition stable rules. Chen et al. (2016) shows that if a stable rule φ weakly Pareto-dominates another stable rule φ' then φ is less manipulable than φ', using a criterion proposed by Pathak and Sönmez (2013). This result is related to ours, given the optimality of PRP rules among all rank-partition stable rules, but does not have any immediate implication since rank-partition stability is weaker than standard stability. The findings of Alva and Manjunath (2019) are also related, as they show that in many settings there is at most one strategyproof rule that weakly Pareto-dominates an individually rational and Pareto-constrained participation-maximal benchmark rule.

Theorem 3. *A rank-partition stable matching rule is strategyproof if and only if it is the Deferred Acceptance rule.*

Given Proposition 1, this theorem implies that the DA rule is the only strategyproof rule in the class of PRP rules, which in turn implies a similar result by Chen and Kesten (2017) for Application-Rejection rules. This makes intuitive sense, as the DA rule is the only PRP rule for which the schools' choice functions are independent of the preferences. Moreover, the characterization of the DA rule by stability and strategyproofness, first implied by Alcalde and Barberà (1994) and also established by Alva and Manjunath (2019), is another corollary of Theorem 3, given that standard stability implies rank-partition stability for a matching rule.

The next result is our main theorem on the incentive properties of PRP rules. The theorem says that a student cannot manipulate a PRP rule to obtain a seat at school c by placing c in the same or a lower preference rank class than the preference rank class where c belongs truthfully. Remarkably, this holds regardless of the rest of the preference ordering reported by the student who attempts to manipulate. This theorem has interesting and wide-ranging implications for the strategic manipulability of PRP rules, as we will explain below.

Theorem 4. *Let $f^{v,x}$ be a PRP rule and let $(\succ, P) \in \Pi \times \mathbb{P}$. Let $s \in S$ and $c \in C$ such that $c\ P_s\ f^{v,x}_s(\succ, P)$. Let $\check{P}_s \in \mathcal{P}$ such that c is in the same or a lower preference rank class in $\check{P}_s$ than in P_s, given x. Then $f^{v,x}_s(\succ, (\check{P}_s, P_{-s})) \neq c$.*

Based on this theorem, a seat at a school may only be obtained by manipulation when the school is reported to be in a higher preference rank class than it truthfully is, regardless of what the reported preference ordering is otherwise, and leaving the envied school in the same preference rank class would be futile. This gives a good idea about how the PRP rules are manipulable in general. The theorem also offers an intuitive explanation for two well-known results: why the DA rule is not manipulable, and why the Boston rule is markedly manipulable (see, for example, Troyan and Morrill (2020)). Notably, given that the classic DA rule is the PRP rule with the coarsest preference rank partition for each student, this theorem implies that the DA rule is strategyproof, since there is only one preference rank class for each student, and thus no school seat can be obtained by manipulation at any profile. At the other extreme, the Boston rule is the PRP rule with not only the finest preference rank partition for each student, but also with the coarsest priority rank partition for each school, which means that the priorities are only used to break ties, and thus the theorem sheds light on why the Boston rule is so manipulable: each change in the reported preferences results in placing at least one school in a higher preference rank class, and the top-ranked school is the only one which can never be obtained by manipulation when the Boston rule is used.

PRP rules between the DA and Boston rules are more or less manipulable, and the extent of manipulability depends on how coarse their priority rank partitions and how fine their preference rank partitions are. For example, the Application-Rejection rules have the coarsest priority rank partition for each school and homogeneous preference rank partitions for students, and our theorem suggests that larger preference rank classes would imply that there is generally less room for manipulation. Indeed, it is shown by Chen and Kesten (2017) that larger equal-sized homogeneous preference rank classes for Application-Rejection rules lead to less manipulability, but this comparison result is not a direct implication of Theorem 4. When the preference rank partition profile is not homogeneous, that is, when the PRP rule is not an Equitable PRP rule, the extent to which the PRP rule is vulnerable to manipulation varies with the student. In the extreme case of Favored Students rules, where each student has either the coarsest or the finest preference rank partition, combined with the coarsest priority rank partitions for schools, it follows from the theorem that the favored students (students with the coarsest preference rank partition) cannot manipulate at all. However, the non-favored students have lots of room to manipulate due to having the finest preference rank partition.

The following corollary displays some useful direct implications of Theorem 4.

Corollary to Theorem 4. *Let $f^{v,x}$ be a PRP rule. Let $(\succ, P) \in \Pi \times \mathbb{P}$ and let $s \in S$.*

1. *If $x_s = (n)$ then student s cannot manipulate $f^{v,x}$ at $(\succ, P)$.*
2. *If school $c \in C$ is ranked in s's top preference rank class at P, given x, then student s cannot manipulate $f^{v,x}$ at $(\succ, P)$ to obtain a seat at school c.*
3. *Let $P'_s \in \mathcal{P}$ such that, for all t, $X^t(P_s) = X^t(P'_s)$. Then s cannot manipulate $f^{v,x}$ at $(\succ, P)$ via P'_s.*

The first statement in the corollary points out that if a student has the coarsest preference rank partition then this student can never manipulate. This follows from Theorem 4 since such a student cannot report any school in a higher preference rank class. This is the case for all students in the classic DA rule and this also holds, for instance, for all favored students in Favored Students rules. The second statement in the corollary states that a school which is ranked in the top preference rank class of a student cannot be obtained as a result of manipulation by this student. This also follows from Theorem 4 since it is not possible to place such a school in a higher preference rank class. This implies that a student cannot get into his first-ranked school by manipulating any French Priority rule, and specifically the Boston rule. Our corollary is more general, since PRP rules allow for coarse preference rank partitions and thus multiple schools may be in the top preference rank class, and hence none of these schools can be profitably obtained by misrepresenting the preferences. In fact, this statement implies the first statement, since a student with the coarsest preference rank partition has every school in her top preference rank class. The third statement in the corollary says that if a student misrepresents her preferences by reporting a preference ordering that is a reshuffle of the schools within each of her preference rank classes then this student cannot obtain a more preferred school. Note that if $x_s = (n)$ then all different preference orderings are such reshuffles, so this statement also implies the first statement.

9. PRP rules with coarse priorities

PRP rules can be interpreted as rules that first coarsen the given strict priorities and then refine them using preference rank classes. An alternative interpretation is that they start from given coarse (i.e., weak) priorities and use the preference rank classes to refine the priorities. Thus, PRP rules can accommodate coarse priorities which are typical in school choice, such as for New York City high schools (Abdulkadiroğlu et al., 2009) or for Boston's public schools (Abdulkadiroğlu et al., 2005). In international refugee assignment countries may have mandated priorities based on the level of emergency, or specific training or skills may be prioritized by different countries, which would lead to coarse priorities (Jones and Teytelboym, 2017; Sayedahmed, 2022). If coarse priorities are given exogenously instead of strict priorities, we can modify the PRP rules by adopting the equivalence classes of the given coarse priorities as the priority rank classes of the PRP rule. This would determine the priority rank partition profile v, which therefore would no longer be part of the specification of the PRP rule. However, since strict priorities have a tie-breaking role in PRP choice functions, strict priority orderings for schools would need to be specified by the PRP rule. Therefore, in this setup the given coarse priorities for schools are primitives of the model, instead of being part of the specification of the matching rule, and each member of the family of PRP rules is given by $(x, \succ)$, where x is a preference rank partition profile and $\succ$ is a fixed strict priority profile to be used for tie-breaking within the equivalence classes of the given coarse priorities, whenever the selection cannot be done based on the preference rank classes. Thus, each school's choice function would select first based on the exogenously given coarse priorities of the school, then based on the preference rank classes determined by the preference profile as well as the preference rank partition profile x specified by the matching rule and, finally, if ties still remain, based on $\succ$, also specified by the matching rule. This corresponds to and extends the setup of Bonkoungou (2020) due to the use of arbitrary preference rank classes, since the French Priority rules only allow for the finest preference rank partition.

This extension lends additional applicability to PRP rules and makes the current study relevant to situations where coarse priorities arise naturally instead of strict priorities. We could further enhance the applicability of PRP rules by combining the two different interpretations of PRP rules (i.e., strict versus weak priorities as primitives) in a natural manner, which would allow for coarse exogenous priorities but let the matching rule further partition the priority ranks within the priority rank classes that are determined by the given coarse priorities based on the priority rank partition profile v, while breaking the ties over the remaining weak priorities based on the preference rank partition profile x and the strict priority profile $\succ$ specified by the matching rule.

10. DA rules with preference-based priorities: extensions

An extension of PRP rules is given by using preference-profile-based modified priority profiles for the DA rule which don't necessarily rely only on priority and preference rank partitions. We can specify such a matching rule formally by a function $\psi : \Pi \times \mathbb{P} \to \Pi$ that assigns a priority profile to each profile, which would define a DA$^{\psi}$ rule by assigning to each profile $(\succ, P)$ the DA matching at profile $(\psi(\succ, P), P)$. PRP rules are a subclass of this more general class of rules, and so are the Taiwan Deduction rules of Dur et al. (2022), which can be represented as DA rules with modified priority profiles that are a function of the preferences.[9] It would be interesting to study this general class of DA$^{\psi}$ rules with preference-based priorities. Some of our results on PRP rules may generalize to this class; for instance, one may conjecture that DA^{ψ} is only strategyproof if ψ is independent of the preference profile, generalizing Theorem 3.

A further extension of PRP rules would result if we allowed for a wider class of choice functions than the acceptant responsive choice functions used by PRP rules. By Proposition 1, PRP choice functions are responsive with respect to the modified preference-profile-specific priority orderings at each preference profile, and it may be of interest to expand the set of choice functions to include, for example, arbitrary substitutable choice functions with respect to the modified priority profile. This would allow for having reserved seats set aside for groups of students to be prioritized, among other things. These generalizations that lead to very broad classes of matching rules may be best described by directly specifying the set of modified priority profiles together with the properties of the choice functions applied to the modified priority profiles. We leave the exploration of these matching rules to future research.

11. Conclusion

In this study we introduce and examine PRP rules, which are generalized Deferred Acceptance rules that allow students' preference rankings to play a role in the schools' choice functions which make the selections among applicants. We unify a large class of matching rules that, in addition to many previously known real-world mechanisms, includes interesting new matching rules that are studied here for the first time, and we use this unified approach to establish results that apply to all of them. We identify an important subclass of PRP rules which had not been studied previously, namely, PRP rules which treat students asymmetrically, allowing for a new way of applying preferential treatment. Overall, our results underline the difficulty of obtaining matching rules with both good efficiency and incentive properties, since the efficiency improvements compared to the DA rule are due to allowing the preferences of students to directly affect their chances of being accepted by the schools they are applying to, which however makes these matching rules more vulnerable to manipulation.

There is typically a trade-off among the stability, efficiency, and incentive properties of matching rules, which has been studied by a vast array of papers in various matching models and settings. Our main contribution to this literature is to show the specifics of this tension within the class of PRP rules, for which the selection of applicants by the schools takes into account the preference rankings. When the school priorities have a small impact (mainly used for tie-breaking), we get more efficiency, since the matching rule relies primarily on the preference rankings, as in the Boston rule and other PRP rules that significantly coarsen the exogenously given fine priority rank classes. When the school priorities have a large impact, preference rankings play only a small role in the schools' selection, and stability improves due to fewer justified envy instances. Incentive properties are also improved in this case, but only in the extreme case of the DA rule, when the preferences have no impact on the schools' choice functions, can strategyproofness be achieved. Thus, the main trade-off is between efficiency and incentives, and within the class of PRP rules better incentives also come with respecting the priorities more. Corresponding to this trade-off, the Boston rule and the DA rule can be seen as the two extreme members of the family of PRP rules. Intuitively, as we place more emphasis on the preferences, we attain more efficiency and get closer to the Boston rule, and as we place more emphasis on the priorities, we get better incentives and get closer to the DA rule. This gives a general idea to the market designer about how to select among PRP rules, depending on the relative importance of the objectives: is efficiency more desirable or correct incentives and stability? If efficiency comes first, choose coarser priority rank partitions and finer preference rank partitions, and if incentives and respecting the priorities are more important then choose finer priority rank partitions and coarser preference rank partitions.

One may argue that incentives should always come first, since if preferences are reported inaccurately then efficiency cannot be measured accurately and thus ensured, given that the fulfillment of any normative criterion can only be based on the reported preferences. However, this argument also has its limitation, since opting for strong incentive properties would also imply automatically the favoring of stability over efficiency in the case of PRP rules. In light of the substantial trade-offs, in order to understand the cost of improved efficiency the extent and specifics of manipulability are of considerable interest to the market designer. Our main result on incentives provides guidance on this, as it sheds light on how students can manipulate PRP rules, and contributes to a more sophisticated understanding of PRP rules than afforded by the rough comparison of the Boston and DA rules only. Several different ways of compromising may be considered when searching for an appropriate matching rule between these two extreme mechanisms, as evidenced by the fact that the family of PRP rules encompasses multiple subfamilies that bridge these two matching rules, such as the First Preference First rules, the Deferred

[9] Taiwan Deduction rules generalize the Application-Rejection rules of Chen and Kesten (2017), but they are not a subset of the class of PRP rules, since there are score deductions that cannot be captured by partitioning the priority and preference ranks.

Boston rules, the Application-Rejection rules or the Favored Students rules. Since quite a few members of the class of PRP rules are already used in real-life school choice and student placement systems, and potentially more may be considered for use if theoretically well understood, our theoretical analysis provides practically relevant insight, and thus our findings should be useful for the design of matching mechanisms that rely on the preference rankings directly when choosing among competing applicants.

Declaration of competing interest

None

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Appendix A. Standard representation of PRP rules

Constructing the standard representation (v^*, x^*)

We use the following convention to specify a unique representation of a PRP rule in terms of its priority and preference rank classes: choose the finest priority rank partitions possible and the coarsest preference rank partitions possible. We refer to this as the **standard representation** of a PRP rule.

Formally, the unique standard representation of a PRP rule f can be found as follows. We say that (t, u) is a **priority reversal** for two priority ranks t, $u \leq n$, where $t < u$, for school $c \in C$ (i.e., for ranks t and u, respectively, in the priority ordering of school c) if there exists a profile $(\succ, P) \in \Pi \times \mathbb{P}$ at which the following holds. Let $s, \hat{s} \in S$ such that the rank of s is t and the rank of $\hat{s}$ is u in $\succ_c$. Then $c\ P_s\ f_s(\succ, P)$ and $f_{\hat{s}}(\succ, P) = c$. Since f is a PRP rule, it follows that if (t, u) is a priority reversal for some school c then ranks t and u are in the same priority rank class for c.

We now prove that for any PRP rule f if (t, u) is a priority reversal for school c and u' satisfies $t < u' < u$ then (t, u') and (u', u) are also priority reversals for c. Fix school $c \in C$ and let (t, u) be a priority reversal for c. Fix u' such that $t < u' < u$. Let the rank of student s be t, the rank of student $\hat{s}$ be u, and the rank of student s' be u' in $\succ_c$. Let $(\succ, P)$ be such that $c\ P_s\ f_s(\succ, P)$ and $f_{\hat{s}}(\succ, P) = c$.

We show first that (t, u') is a priority reversal for c. Let $\tilde{\succ}_c$ be the same as $\succ_c$, except for the position of $\hat{s}$: $\hat{s}$ is moved up in the priority ordering of school c and is ranked directly above s' in $\tilde{\succ}_c$, while all other orderings remain the same. Since ranks t and u are in the same priority rank class of c, where $t < u$, and $\hat{s}$ and s' have ranks u' and $u' + 1$ respectively in $\tilde{\succ}_c$, where $t < u' < u$, $\hat{s}$ and s' are still in the same priority rank class in $\tilde{\succ}_c$, and therefore each student is in the same priority rank class of c in $\tilde{\succ}_c$ as in $\succ_c$. Moreover, the relative priority ranks only differ for a student at most when compared to $\hat{s}$, and otherwise they are the same.

Let $\tilde{\succ} \equiv (\tilde{\succ}_c, \succ_{-c})$. Given that all students who are in the same priority rank class of c as $\hat{s}$ in $\succ_c$ are also in the same priority rank class of c as $\hat{s}$ in $\tilde{\succ}_c$, and since the preference profile is unchanged between $(\succ, P)$ and $(\tilde{\succ}, P)$, the only possibility for a different matching at $(\tilde{\succ}, P)$ compared to the matching at $(\succ, P)$ is if $\hat{s}$ is accepted by school c instead of some other student due to the tie-breaking according to the strict priorities of c, since only the relative position of $\hat{s}$ has changed. However, this is not the case, since $\hat{s}$ was already accepted by c at $(\succ, P)$, given that $f_{\hat{s}}(\succ, P) = c$. Therefore, the matching at $(\tilde{\succ}, P)$ is determined by PRP rule f exactly the same way as at $(\succ, P)$ and $f(\tilde{\succ}, P) = f(\succ, P)$, which implies that (t, u') is a priority reversal for c.

We can show that (u', u) is a priority reversal for c in a similar fashion, by letting $\bar{\succ}_c$ be the same as $\succ_c$, except for the position of s: s is moved down in the priority ordering of school c and is ranked directly below s' in $\bar{\succ}_c$. Then, using the fact that $c\ P_s\ f_s(\succ, P)$, we can show that moving s down in $\succ_c$ will not change the outcome, and therefore $f(\bar{\succ}, P) = f(\succ, P)$, where $\bar{\succ} \equiv (\bar{\succ}_c, \succ_{-c})$.

Now fix t', t'' such that $t \leq t' < t'' \leq u$. Then, by the above arguments, if (t, u) is a priority reversal for some school $c \in C$ then (t', u) is also a priority reversal for c, and thus, applying the above arguments again, (t', t'') is also a priority reversal for c. Therefore, we can define the priority rank classes for each school $c \in C$ as follows: for all $t, u \leq n$ such that $t < u$, let rank t and rank u be in the same priority rank class in v_c if and only if (t, u) is a priority reversal for school c. This unambiguously determines a unique priority rank partition profile v, as we have just shown. Moreover, since, for any priority reversal (t, u) for a school c, ranks t and u must be in the same priority rank class of a PRP rule, and given that according to the above definition for each school two priority ranks belong to different priority rank classes unless a priority reversal requires otherwise, this definition satisfies the convention that v_c is as fine as possible for each school $c \in C$.

We identify next the unique preference rank partition profile x that is as coarse as possible for each student. If the PRP rule f is the DA then there is no priority reversal for any school, and hence for each school $c \in C$ the constructed priority rank partition is $v_c = (1, \ldots, 1)$. Thus, we can let x_s be the coarsest for all students $s \in S$, since the preference rank partition does not affect f in this case. Otherwise, if f is not the DA, let $c \in C$ such that there is at least one priority rank class in v_c that contains at least two ranks, as constructed above. For PRP rule f with such a priority rank partition v_c for $c \in C$, fix $\succ \in \Pi$ and let two students $i, j \in S$ with $i \succ_c j$ be in the same priority rank class of school c according to $\succ_c$. We construct x in the standard representation of f as follows.

Find the lowest rank $t_i^1 > 1$ for which there exists a preference profile $P \in \mathbb{P}$ such that $f_j(\succ, P) = c$, $c\ P_i\ f_i(\succ, P)$, and c has rank t_i^1 in P_i and rank 1 in P_j. That is, for any $t < t_i^1$ if c has rank t in P_i while P_j ranks c first then, regardless of the rest of the preference profile, there is no priority reversal for c involving i and j. Then $1, \ldots, t_i^1 - 1$ are the ranks in student i's first preference rank class in x_i (i.e., $x_i^1 = t_i^1 - 1$). Now reverse the roles of i and j to find the first preference rank class for j similarly, say ranks $1, \ldots, t_j^1 - 1$ (i.e., $x_j^1 = t_j^1 - 1$). Next, find the lowest rank $t_i^2 > t_i^1$ (if it exists) for which there exists a preference profile $P \in \mathbb{P}$ such that $f_j(\succ, P) = c$, $c\ P_i\ f_i(\succ, P)$, and c has rank t_i^2 in P_i and rank t_j^1 in P_j. Since i and j are in the same priority rank class for c according to v_c with $i \succ_c j$, this determines student i's second preference rank class comprised of ranks $t_i^1, \ldots, t_i^2 - 1$ (i.e., $x_i^2 = t_i^2 - t_i^1$). Now reverse the roles of i and j to find the second preference rank class for j similarly, say ranks $t_j^1, \ldots, t_j^2 - 1$ (i.e., $x_j^2 = t_j^2 - t_j^1$).

And so on, by iterating the same procedure we can determine all preference rank classes in x_i and x_j, and we can find all preference rank classes for all students similarly, leading to the preference rank partition profile x in the standard representation of f. Note that since f is a PRP rule, we get consistent results for the preference rank classes of all students using this method. Moreover, the unique preference rank partition profile x which is found this way satisfies the convention that it is as coarse as possible, since for each student two preference ranks belong to the same preference rank class unless a priority reversal requires otherwise.

In sum, we have identified the unique standard representation of each PRP rule f which satisfies the convention that v is as fine as possible and x is as coarse as possible. We denote the standard representation of a PRP rule by (v^*, x^*).

Homogeneous preference rank partition profile x^*

Now we show that if a PRP rule with standard representation (v^*, x^*) can be specified with a homogeneous preference rank partition profile then x^* is homogeneous.[10] Suppose by way of contradiction that there exists a PRP rule for which $f^{v^*,x^*} = f^{\hat{v},\hat{x}}$, where (v^*, x^*) is the standard representation of f, x^* is not homogeneous, and $\hat{x}$ is homogeneous. Since x^* is not homogeneous, there exist $s, s' \in S$ and $t \geq 1$ such that ranks t and $t+1$ belong to different preference rank classes in x_s^* and the same preference rank class in $x_{s'}^*$. Since x^* consists of the unique coarsest possible specification of the preference rank classes for each student, ranks t and $t+1$ belong to different preference rank classes in $\hat{x}_s$ and thus, since $\hat{x}$ is homogeneous, ranks t and $t+1$ belong to different preference rank classes in $\hat{x}_{s'}$. Given that $x_{s'}^*$ and $\hat{x}_{s'}$ can both be used in the specification of the same PRP rule, there is no priority reversal due to $t+1$ being in a lower preference rank class than t in the preference rank partition of s'. However, this can only depend on t and it must be the same for all students. Thus, there is no priority reversal due to $t+1$ being in a lower preference rank class than t for s either. Therefore, letting ranks t and $t+1$ be in the same preference rank class for s also works for the specification of the PRP rule. This implies, however, that x_s^* is not the coarsest preference rank partition, which contradicts the construction of x^*. Therefore, if there is any representation $(\hat{v}, \hat{x})$ of a PRP rule such that $\hat{x}$ is homogeneous, then in the standard representation (v^*, x^*) of this PRP rule the preference rank partition profile x^* is homogeneous.

Appendix B. Proofs

Proof of Proposition 1.

One can readily verify from the definition of modified priority profiles that at each profile $(\succ, P) \in \Pi \times \mathbb{P}$ the PRP rule $f^{v,x}$ selects the DA matching at $(\bar{\succ}((\succ, P), (v, x)), P)$. This also implies that for all $\succ, \succ' \in \Pi$ and $P \in \mathbb{P}$, if $\bar{\succ}((\succ, P), (v, x)) = \bar{\succ}((\succ', P), (v, x))$ then $f^{v,x}(\succ, P) = f^{v,x}(\succ', P)$. Thus, by the stability of the DA matching (Gale and Shapley, 1962), PRP rules are rank-partition stable. Moreover, each PRP rule $f^{v,x}$ assigns the unique (v, x)-optimal rank-partition stable matching at $(\succ, P)$ to each profile $(\succ, P) \in \Pi \times \mathbb{P}$ due to the classic optimality result of Gale and Shapley (1962). □

Proof of Theorem 1.

An equitable-rank-partition stable matching rule is PP-stable.

Let φ be an equitable-rank-partition stable matching rule with respect to (v, x). Then x is homogeneous. Suppose by way of contradiction that there exists a profile $(\succ, P) \in \Pi \times \mathbb{P}$ such that $\varphi(\succ, P)$ is not PP-stable at $(\succ, P)$. Then, since all rank-partition stable matching rules are individually rational and non-wasteful, there is a student $s \in S$ who has PP-justified envy in $\varphi(\succ, P)$ at profile $(\succ, P)$. That is, there exist school $c \in C$ and student $\hat{s} \in S$ such that $c\ P_s\ \varphi_s(\succ, P)$, $s \succ_c \hat{s}$, $r_s(c) \leq r_{\hat{s}}(c)$ at P, and $\varphi_{\hat{s}}(\succ, P) = c$.

[10] We cannot show the same for homogeneous priority rank partition profiles, but an analogous argument to the one presented here for preference rank partition profiles implies such a result for the weaker notion of homogeneity for priority rank partition profiles defined in footnote 5.

Let $k, k' \geq 1$ such that $s \in V^k(\succ_c)$ and $\hat{s} \in V^{k'}(\succ_c)$. Then $s \succ_c \hat{s}$ implies that $k \leq k'$. Let $t, t' \geq 1$ such that $c \in X^t(P_s)$ and $c \in X^{t'}(P_{\hat{s}})$. Then, given that $r_s(c) \leq r_{\hat{s}}(c)$ at P and x is homogeneous, $t \leq t'$. Let $\bar{\succ} \equiv \bar{\succ}((\succ, P), (v, x))$ denote the modified priority profile. Note that $k < k'$ would imply that $s \,\bar{\succ}_c\, \hat{s}$. Thus, given that $c\ P_s\ \varphi_s(\succ, P)$ and $\varphi_{\hat{s}}(\succ, P) = c$, $k < k'$ would imply that s has justified envy in $\varphi(\bar{\succ}, P)$ at $(\succ, P)$, which is a contradiction since φ is a rank-partition stable matching rule with respect to (v, x). Therefore, $k \leq k'$ implies that $k = k'$. Then $t \leq t'$ implies similarly that $t = t'$. Hence, the strict priority ordering $\succ_c$ determines the relevant ordering of s and $\hat{s}$ in $\bar{\succ}_c$ at school c, and $s \succ_c \hat{s}$ implies that $s \,\bar{\succ}_c\, \hat{s}$. However, this means that s has justified envy in $\varphi(\bar{\succ}, P)$ at $(\succ, P)$, which is a contradiction. Therefore, $\varphi(\succ, P)$ is PP-stable at $(\succ, P)$, and since this holds for all $(\succ, P) \in \Pi \times \mathbb{P}$, φ is PP-stable.

If a rank-partition stable matching rule is PP-stable then it is equitable-rank-partition stable.

Let φ be a rank-partition stable matching rule which is PP-stable. Let φ be rank-partition stable with respect to (v, x). Suppose by way of contradiction that φ is not equitable-rank-partition stable. Then x is not homogeneous and thus there exist $(\succ, P) \in \Pi \times \mathbb{P}$, $c \in C$ and $s, \hat{s} \in S$ such that $\hat{s} \,\bar{\succ}_c\, s$, where $\bar{\succ} \equiv \bar{\succ}((\succ, P), (v, x))$ denotes the modified priority profile at profile $(\succ, P)$, and s envies $\hat{s}$ for being assigned c at $(\succ, P)$, which is due to the preference rank classes of s and $\hat{s}$ that contain c, even though s ranks c at least as high as $\hat{s}$ at P. That is,

1. $\varphi_{\hat{s}}(\succ, P) = c$ and $c\ P_s\ \varphi_s(\succ, P)$;
2. s and $\hat{s}$ are in the same priority rank class of c, given $\succ_c$ and v_c: $s, \hat{s} \in V^k(\succ_c)$ for some $k \geq 1$;
3. $c \in X^t(P_s)$ and $c \in X^{\hat{t}}(P_{\hat{s}})$, where $t > \hat{t} \geq 1$;
4. $r_s(c) \leq r_{\hat{s}}(c)$ at P.

Since φ is PP-stable, 1. and 4. imply that $\hat{s} \succ_c s$. Let $T = \{s' \in S : s' \in V^k(\succ_c) \text{ and } c \in X^t(P_{s'})\}$ be the set of students who are in the same priority rank class as both s and $\hat{s}$ and rank c in preference rank class t at P, given x. Let $\succ'_c$ be the same as $\succ_c$ except that all members of T are moved directly above $\hat{s}$ in $\succ'_c$, while preserving the relative priority ordering within T and also for any pair of students who are not in T. Note that $s \in T$ and thus $s \succ'_c \hat{s}$.

Let $\succ' \equiv (\succ'_c, \succ_{-c})$, and let $\bar{\succ}' \equiv \bar{\succ}((\succ', P), (v, x))$ denote the modified priority profile at $(\succ', P)$. We will show that $\bar{\succ} = \bar{\succ}'$. Given that $\succ$ and $\succ'$ only differ in the priority ordering of c, $\bar{\succ}_{-c} = \bar{\succ}'_{-c}$. Thus, we only need to show that $\bar{\succ}_c = \bar{\succ}'_c$. Note that since $\hat{s} \in V^k(\succ_c)$ and $T \subset V^k(\succ_c)$, the set of students in this priority rank class of c has not changed, and thus $\hat{s}$ and the members of T are still in the same priority rank class at $\succ'_c$. Thus, $V^k(\succ_c) = V^k(\succ'_c)$, and for all $k' \geq 1$, $V^{k'}(\succ_c) = V^{k'}(\succ'_c)$. Then, since the preference profile P is unchanged and hence the preference rank classes remain the same, the only possible reason for $\bar{\succ}_c$ to differ from $\bar{\succ}'_c$ is the strict priorities $\succ_c$ and $\succ'_c$, respectively. However, the strict priorities only determine the priority ordering within priority rank classes and within the sets of students who rank c in the same preference rank class at P, given x, and the ordering of students within T, as well as within all other relevant sets of students, have been preserved. Therefore, $\bar{\succ}_c = \bar{\succ}'_c$ and thus $\bar{\succ} = \bar{\succ}'$.

Hence, given that φ is rank-partition stable, $\varphi(\succ, P) = \varphi(\succ', P)$. Thus, $\varphi_{\hat{s}}(\succ', P) = \varphi_{\hat{s}}(\succ, P) = c$ and $c\ P_s\ \varphi_s(\succ', P)$. However, since $s \succ'_c \hat{s}$ and $r_s(c) \leq r_{\hat{s}}(c)$ at P, s has PP-justified envy in $\varphi(\succ', P)$ at $(\succ', P)$. This is a contradiction, since φ is PP-stable. Therefore, φ is equitable-rank-partition stable. □

Proof of Proposition 2.

An Equitable PRP rule is PP-stable.

This follows from the first part of the proof of Theorem 1, since an Equitable PRP rule is equitable-rank-partition stable.

A PP-stable PRP rule is an Equitable PRP rule.

A PRP rule $f^{v,x}$ is rank-partition stable by Proposition 1. Thus, if $f^{v,x}$ is PP-stable then the second part of the proof of Theorem 1 applies and hence $f^{v,x}$ is equitable-rank-partition stable. Then x is homogeneous and therefore $f^{v,x}$ is an Equitable PRP rule. □

Proof of Theorem 2.

A Near-Boston rule is rank-partition stable and Pareto-efficient.

Near-Boston rules are PRP rules and thus they are rank-partition stable by Proposition 1. We need to show that Near-Boston rules are Pareto-efficient. Let $f^{v,x}$ be a Near-Boston rule. Fix a profile $(\succ, P) \in \Pi \times \mathbb{P}$ and let $\mu \equiv f^{v,x}(\succ, P)$. We will prove that μ is Pareto-efficient. Since $f^{v,x}$ is a Near-Boston rule, all priority rank partitions are the coarsest in v, and there exists $j_1 \in S$ such that for all $s \in S \setminus \{j_1\}$, s's preference rank partition is the finest in x.

For all $s \in S$ such that $\mu_s \neq 0$, let t_s denote the round in the iterative $f^{v,x}$ procedure at $(\succ, P)$ in which s is accepted by school $\mu_s \in C$ for the first time. Let $z > 0$ be the total number of rounds of $f^{v,x}$ at $(\succ, P)$. For all $s \in S$ such that $\mu_s = 0$, let $t_s = z + 1$. We will say that student s *replaces* student s' at school c in round t if, due to s's application to school c, student s', who was tentatively assigned to school c for at least one round prior to t, is rejected from c in round t. In the following we will call such tentative assignments that don't become final *temporary* assignments. Now we show that at most one student replaces a student in any round for $f^{v,x}$, so this is an unambiguous definition for Near-Boston rules even with arbitrary school capacities. If there is any student who replaces another student in the Near-Boston procedure $f^{v,x}$ at $(\succ, P)$, then the first one in any round must be student j_1, since all students other than j_1 have the finest preference rank partition and thus cannot replace another student, given that the finest preference rank partitions ensure that all acceptances are final (just like in the Boston rule). Furthermore, j_1's acceptance is final in step t_{j_1} by μ_{j_1}, since all other students have the finest preference rank partition and thus any potential applicant $i \in S \setminus \{j_1\}$ for μ_{j_1} in any round after round t_{j_1} would have μ_{j_1} in the preference rank class of student i that corresponds to the number of the rounds in the iterative procedure up to that point, while j_1 has μ_{j_1} in a higher (i.e., lower-indexed) preference rank class, given that j_1 replaced another student in round t_{j_1}, and hence j_1 would be selected by μ_{j_1} in each round over any new applicant.

The second student (if any) who may replace another student is the student who was replaced at μ_{j_1} by j_1 in round t_{j_1}, since this is the only student who may have a school, when the student applies to it, in a lower-indexed preference rank class than other students who are tentatively assigned to it or are applying to it (given that j_1 has a final assignment in a previous round), since this student, say j_2, has the school she is applying to in a preference rank class which is at least one class higher (i.e., lower-indexed), due to being temporarily assigned to μ_{j_1} for at least one round, compared to other students who are applying to this school and have been rejected in each previous round by a school. Furthermore, for the same reasons j_1's acceptance was final in round t_{j_1}, the acceptance of j_2 is final when she replaces another student in round t_{j_2}.

In general, there is at most one student in any round whose application to a school causes the replacement of a student, and this student is either j_1 or a student further along the replacement chain initiated by j_1. We will call this chain consisting of students who replace another student when they get matched the *replacement list* and denote it by $(j_1, \ldots, j_{\bar{k}})$, indicating the order in which they appear in the iterative rounds. The replacement list depends on the profile and may have no members at all, which would imply that no student is replaced during the entire $f^{v,x}$ procedure at $(\succ, P)$ and hence the matching selected by $f^{v,x}$ is the same as the matching selected by the Boston rule at $(\succ, P)$. Once a student who was replaced by the previous student in the replacement list does not replace another student the chain ends, as there won't be any more students who are "out of sync" with the preference rank classes of other students who are not yet accepted by a school, and the rest of the rounds will be just like in the Boston rule. For student $j_{\bar{k}}$, the last student in the replacement list, there exists student $i \in S$ who is replaced by $j_{\bar{k}}$ in round $t_{j_{\bar{k}}}$ such that i was temporarily assigned to $\mu_{j_{\bar{k}}}$ in a previous round, but i does not replace any student in a later round. Although student i is not a member of the replacement list, for notational convenience let $j_{\bar{k}+1}$ denote this student. Note that each student in the replacement list has a final assignment in a round prior to the round when the final assignment was made for subsequent students in the replacement list, that is, $t_{j_1} < \ldots < t_{j_{\bar{k}}}$.

For all $k \in \{1, \ldots, \bar{k}\}$, let $\hat{t}_{j_k}$ denote the round in which j_{k+1} was first assigned to μ_{j_k}. Then, for all $k \in \{1, \ldots, \bar{k}\}$, j_{k+1} applies to μ_{j_k} in round $\hat{t}_{j_k}$, where $\hat{t}_{j_k} < t_{j_k}$, and thus, since j_{k+1} has the finest preference rank partition, $\mu_{j_{k+1}} \in X^t(P_{j_{k+1}})$ such that $t > \hat{t}_{j_k}$. Also, for all $k \in \{1, \ldots, \bar{k}-1\}$, j_{k+2} applies to $\mu_{j_{k+1}}$ in round $\hat{t}_{j_{k+1}}$ and therefore, since j_{k+2} has the finest preference rank partition, $\mu_{j_{k+1}} \in X^{\hat{t}_{j_{k+1}}}(P_{j_{k+2}})$. Thus, since j_{k+1} replaces j_{k+2} at $\mu_{j_{k+1}}$, $t \leq \hat{t}_{j_{k+1}}$. Then $\hat{t}_{j_k} < t \leq \hat{t}_{j_{k+1}}$, and therefore $\hat{t}_{j_1} < \ldots < \hat{t}_{j_{\bar{k}}}$.

It is clear that for any permutation σ of S if students are assigned at $(\succ, P)$ their favorite school with at least one remaining seat according to the order of students specified by σ, that is, if matching μ corresponds to such a *sequential matching* that is found exactly as in a serial dictatorship (Satterthwaite and Sonnenschein, 1981), then μ is Pareto-efficient. We will demonstrate that it is possible to specify a permutation σ depending on the profile $(\succ, P) \in \Pi \times \mathbb{P}$ such that $f^{v,x}(\succ, P)$ corresponds to the sequential matching based on σ.

Fix a permutation σ of the set of students S for $(\succ, P)$ with the following requirements[11]:

a. Let σ follow the order in which the final assignments are made by the iterative procedure of $f^{v,x}$ at $(\succ, P)$. That is, let σ be such that for all $s, s' \in S$, if $\sigma(s) < \sigma(s')$ then $t_s \leq t_{s'}$, where $\sigma(s)$ is the rank number of student s in σ.
b. Make the following exceptions to the requirement in a. above: for all $k \in \{1, \ldots, \bar{k}\}$, treat student j_k in the replacement list at $(\succ, P)$ as if j_k was matched in the round where j_{k+1} was first assigned (temporarily) to μ_{j_k}, namely in round $\hat{t}_{j_k}$. That is, for all $k \in \{1, \ldots, \bar{k}\}$, let $\sigma(j_k) < \sigma(s)$ for all students s who were accepted by school μ_s for the first time after round $\hat{t}_{j_k}$ in the $f^{v,x}$ procedure at $(\succ, P)$. Moreover, for all $k \in \{1, \ldots, \bar{k}\}$, let $\sigma(j_k) > \sigma(s)$ for all students s who were accepted by school μ_s for the first time in round $\hat{t}_{j_k}$ or prior to this round in the $f^{v,x}$ procedure at $(\succ, P)$.

Let $f^{\sigma}(\succ, P)$ denote the sequential matching at $(\succ, P)$ in which first student s_1 with $\sigma(s_1) = 1$ is assigned to her first-ranked school, then student s_2 with $\sigma(s_2) = 2$ is assigned to her first-ranked school among the schools with remaining vacant seats, and so on.

If $f^{v,x}$ is the Boston rule, that is, if j_1 has the finest preference rank partition along with all other students then, for all $(\succ, P) \in \Pi \times \mathbb{P}$, there exists σ, as a function of $(\succ, P)$, such that $f^{\sigma}(\succ, P) = \mu$ and σ satisfies requirement a. above. In this case there are no temporary assignments in the procedure and the replacement list is empty. Thus, in each round each student who is still unmatched applies to her highest-ranked acceptable school that hasn't rejected her yet, and once a

[11] Since permutation σ is a function of the profile $(\succ, P)$, we would need to write $\sigma(\succ, P)$ formally, but we suppress the argument for the sake of notational convenience.

school accepts a student the assignment is final. This means that the exceptions specified by requirement b. are not relevant if $f^{v,x}$ is the Boston rule, and thus μ is a sequential matching and hence Pareto-efficient.

Observe that, given requirements a. and b. for σ at $(\succ, P)$, the only possibility for a Near-Boston rule that may cause $f^{\sigma}(\succ, P) \neq \mu$ is if there exist $c \in C$ and $k \in \{1, \ldots, \bar{k}\}$ such that j_k was rejected by school c in round t, where $\hat{t}_{j_k} < t < t_{j_k}$. In this case, when it is j_k's turn to be assigned to a school in the sequential matching according to σ, if there is a vacant seat at c then j_k would be assigned to c or to a higher-ranked school by j_k in $f^{\sigma}(\succ, P)$, given that $c \, P_{j_k} \, \mu_{j_k}$.

Suppose by way of contradiction that there exist such c, k and t. Since j_k replaces j_{k+1} in round t_{j_k} at μ_{j_k}, and since j_{k+1} is temporarily assigned to μ_{j_k} in round $\hat{t}_{j_k} < t_{j_k}$, j_k has μ_{j_k} in preference rank class $\hat{t}_{j_k}$ or a lower-indexed preference rank class, that is, $\mu_{j_k} \in X^{t'}(P_{j_k})$ such that $t' \leq \hat{t}_{j_k}$, otherwise j_k would not replace j_{k+1} in round t_{j_k} at μ_{j_k}. Then, since $c \, P_{j_k} \, \mu_{j_k}$, $c \in X^{t''}(P_{j_k})$, where $t'' \leq t' \leq \hat{t}_{j_k}$.

Let $h \in S$ such that h is tentatively assigned to c in round t. We will consider two cases depending on whether h is in the replacement list at $(\succ, P)$. If h is in the replacement list $(j_1, \ldots, j_{\bar{k}})$ at $(\succ, P)$, let $h = j_{k'}$. First note that $k' \neq k$. If $k' = k+1$ then $\hat{t}_{j_k} < t < t_{j_k}$ implies that h is temporarily assigned to μ_{j_k} in round t, which is a contradiction, since $\mu_{j_k} \neq c$. If $k' > k+1$ then $t < t_{j_k} < t_{j_{k'-1}}$ and h is only temporarily assigned to c in round t. Thus, $c \in X^{\hat{t}_{j_{k'-1}}}(P_{j_{k'}})$. Moreover, given that $t'' \leq \hat{t}_{j_k}$, where $c \in X^{t''}(P_{j_k})$, and given $\hat{t}_{j_k} < \hat{t}_{j_{k'-1}}$, we have $t'' < \hat{t}_{j_{k'-1}}$. This means that j_k cannot be rejected by c in round t, since j_k has school c in a lower-indexed preference rank class than $j_{k'}$ does. Thus, $k' < k$. Note that requirement b. implies that $h = j_{k'}$ precedes all students in σ who were assigned after round $\hat{t}_{j_{k'}}$. Therefore, since $k' < k$ implies that $\hat{t}_{j_{k'}} < \hat{t}_{j_k}$, $\sigma(h) = \sigma(j_{k'}) < \sigma(j_k)$, and we have a contradiction.

Thus, for all $h \in S$ such that h is tentatively assigned to c in round t, h is not in the replacement list at $(\succ, P)$, and hence h is permanently assigned to c in round t. Then, without loss of generality, we can let $h \in \mu_c$ such that for all $h' \in \mu_c$, $t_h \geq t_{h'}$. Since j_k is rejected by c in round t and h is not in the replacement list at $(\succ, P)$, c has to be in at least as low-indexed a preference rank class for h as for j_k. Thus, $c \in X^{t_h}(P_h)$, where $t_h \leq t'' \leq \hat{t}_{j_k}$ and $c \in X^{t''}(P_{j_k})$. Therefore, given that for all $h' \in \mu_c$, $t_h \geq t_{h'}$, for all $h' \in \mu_c$, $\sigma(h') < \sigma(j_k)$ due to requirement b.

In sum, given the exceptions made by requirement b. in requirement a. for σ, there are no vacant seats remaining at school c immediately after round $\hat{t}_{j_k}$, when it is j_k's turn to receive an assignment in the sequential matching according to permutation σ. Since a similar argument holds for all $c \in C$ and $k \in \{1, \ldots, \bar{k}\}$ such that j_k was rejected by school c in round t, where $\hat{t}_{j_k} < t < t_{j_k}$, j_k's favorite school with remaining vacant seats is μ_k when j_k receives her assignment, after all the students who were assigned to a school in round $\hat{t}_{j_k}$. Thus, if σ satisfies requirements a. and b. at $(\succ, P)$, $f^{\sigma}(\succ, P) = \mu$, and since for all $s, s' \in S$, $\sigma(s) < \sigma(s')$ implies that either $\mu_s \, P_s \, \mu_{s'}$ or $\mu_s = \mu_{s'}$, this means that the sequential matching μ is Pareto-efficient.

A rank-partition stable and Pareto-efficient matching rule is a Near-Boston rule.

We will show that a Pareto-efficient PRP rule $f^{v,x}$ is a Near-Boston rule. Note that this is sufficient to prove the claim since, by Proposition 1, $f^{v,x}$ assigns the unique (v, x)-optimal rank-partition stable matching at $(\succ, P)$ to each profile, which Pareto-dominates all other matchings that are stable with respect to $(\bar{\succ}((\succ, P), (v, x)), P)$.

Step 1: *Coarsest priority rank partition profile*

We show first that if $f^{v,x}$ is a Pareto-efficient PRP rule then each school has the coarsest priority rank partition. Let $f^{v,x}$ be a Pareto-efficient PRP rule. Suppose that there exists $a \in C$ such that v_a is not the coarsest. Assume without loss of generality that $x_i^1 \leq x_l^1 \leq x_j^1$. For now we assume that $q_a = q_b = q_c = 1$ and generalize the arguments at the end of Step 1.

Specify $(\succ, P) \in \Pi \times \mathbb{P}$ as follows. Let $i, j, l, h \in S$ and assume that

- i has the top priority and j has the lowest priority in $\succ_a$.
- j has the top priority and i has the lowest priority in $\succ_b$.
- h has the top priority in $\succ_c$.

All other priorities in $\succ$ are arbitrary. Let P be given as shown in the table below.

P_i	P_j	P_l	P_h
c	a	c	c
b	b	b	
a		0	

Assume that further preferences for students i, j, l, h are arbitrary at P, and for all students $h' \in S \setminus \{i, j, l, h\}$, there is no acceptable school according to $P_{h'}$. The rounds of the $f^{v,x}$ procedure at $(\succ, P)$ are displayed below, with the selections underlined in each round.

Round	a	b	c
1	$\underline{j}$		$i, l, \underline{h}$
2	$\underline{j}$	$i, \underline{l}$	$\underline{h}$
3	$\underline{i}, j$	$\underline{l}$	$\underline{h}$
4	$\underline{i}$	$\underline{j}, l$	$\underline{h}$

<u>Round 1:</u> Student h is selected over i and l by c, since either h is in a higher priority rank class than i and l for c, or h wins on the tie-breaker, given that $r_i(c) = r_l(c) = r_h(c) = 1$ and $h \succ_c i$, $h \succ_c l$.
<u>Round 2:</u> Student l is selected over i by b, since either l is in a higher priority rank class than i for b, or l is selected based on preference rank classes, given that $x_i^1 \leq x_l^1$ and $r_i(b) = r_l(b) = 2$, or if neither the priority rank classes nor the preference rank classes provide a basis for selecting l over i, then l wins on the tie-breaker due to $l \succ_b i$.
<u>Round 3:</u> Given the assumption that v_a is not the coarsest, $i \in V^1(\succ_a)$ and $j \notin V^1(\succ_a)$. Thus, in Round 3 student i is selected over j by school a based on the priority rank classes.
<u>Round 4:</u> Student j is selected over l by b, since either j is in a higher priority rank class than l for b, or j is selected based on the preference rank classes, given that $x_l^1 \leq x_j^1$ and $r_j(b) = r_l(b) = 2$, and if neither the priority rank classes nor the preference rank classes provide a basis for selecting j over l, then j wins on the tie-breaker due to $j \succ_b l$.

Given that only schools c and b are acceptable according to P_l, student l remains unmatched and the final matching is as indicated for Round 4. Since i is assigned a and j is assigned b, i and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency.

In order to relax the assumption that $q_a = q_b = q_c = 1$ and generalize the previous arguments, since there exist $a, b, c \in C$ such that $q_a + q_b + q_c < n$, we can introduce additional students with the top priorities for the relevant schools such that each student with a top priority for a school (i.e., among the top q_c highest-priority students for school c) ranks the school first. Since it can easily be verified that PRP rules satisfy *mutual best*,[12] that is, students among the q_c highest-priority students for c who rank school c first are matched to c by any PRP rule, the addition of these students would allow for getting the same contradiction as above. In order to include the remaining schools and students, if there are any, we can simply assume that these schools are not acceptable to any student, and these students don't find any school acceptable. Since this is a straightforward extension, we omit the details. Therefore, if $f^{v,x}$ is a Pareto-efficient PRP rule then each school has the coarsest priority rank partition.

Step 2: *Near-Boston preference rank partition profile*

Given Step 1, we will show that if $f^{v,x}$ is a Pareto-efficient PRP rule then it is a Near-Boston rule. Suppose that $f^{v,x}$ is Pareto-efficient and v_c is the coarsest for each school $c \in C$, but it is not a Near-Boston rule. Then there exist $j, l \in S$ with at least one preference rank class each which are minimally size 2. Since the preference rank classes which are larger than size 1 have to be relevant, that is, there must exist a profile where these preference rank classes matter, this means that there are enough students who can have top priorities at relevant schools, and since we can assume that these students rank their top-priority schools first, to reduce the tedious details we will assume that the minimally size 2 preference rank classes are the top preference rank classes for students j and l, and that $q_a = q_b = q_c = 1$ for schools $a, b, c \in C$.

We specify $(\succ, P) \in \Pi \times \mathbb{P}$ as follows. Let the preferences for $i, j, l \in S$ be given as shown in the table below. Student j has both a and b in the first preference rank class, and so does student l: $a, b \in X^1(P_j)$ and $a, b \in X^1(P_l)$.

P_i	P_j	P_l
a	a	b
0	b	a

Let $l \succ_a i \succ_a j$ and $j \succ_b l$. The rounds of the $f^{v,x}$ procedure at $(\succ, P)$ are displayed below, with the selections underlined in each round.

Round	a	b
1	$\underline{i}, j$	$\underline{l}$
2	$\underline{i}$	$\underline{j}, l$
3	$i, \underline{l}$	$\underline{j}$

Note that all priority rank partitions are the coarsest, as shown in Step 1. Therefore, the strict priorities determine the selection in each round (i.e., the tie-breaker), as each application is for a school which is in the first preference rank class of the applying student. Given that only a is acceptable according to P_i, student i remains unmatched and the final matching

[12] See Morrill (2013) for the one-to-one ($q_c = 1$) version of mutual best.

is as indicated for Round 3. Since l is assigned a and j is assigned b, l and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency. This implies that a Pareto-efficient PRP rule is a Near-Boston rule. □

Proof of Theorem 3.

We first show in Step 1 that a PRP rule cannot be strategyproof unless it is the DA rule, and subsequently this fact is used in Step 2 to prove the theorem.

Step 1: *The only strategyproof PRP rule is the DA rule.*

It is well known that the DA rule is strategyproof (Dubins and Freedman, 1981; Roth, 1982). Let $f^{v,x}$ be a PRP rule which is not the DA rule. We need to show that $f^{v,x}$ is not strategyproof. Since $f^{v,x}$ is not the DA rule, there exists $c \in C$ such that v_c is not the finest, and there exists $s \in S$ such that x_s is not the coarsest. Specifically, there is a priority reversal (t, u) for school c, where priority ranks t and u satisfy $t < u \leq n$, as defined in Appendix A. Thus, there exist a profile $(\succ, P) \in \Pi \times \mathbb{P}$ and student $\hat{s} \in S$ such that the rank of s is t and the rank of $\hat{s}$ is u in $\succ_c$, $c \; P_s \; f_s(\succ, P)$ and $f_{\hat{s}}(\succ, P) = c$. Note that ranks t and u are in the same priority rank class for v_c, and x_s is not the coarsest since the priority reversal occurs when the choice function of school c favors $\hat{s}$ over s, and s is not selected due to the preference rank classes. This can only happen when s reports c in a preference rank class that is not the top preference rank class for s, that is, $c \in X^t(P_s)$ such that $t \geq 2$. However, it is easy to see that one can construct examples of $\tilde{P}_s$ such that $c \in X^1(\tilde{P}_s)$ and $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s})) = c$ (we omit the laborious details). This means that s can manipulate at P via $\tilde{P}_s$ and implies that $f^{v,x}$ is not strategyproof.

Step 2: *The only strategyproof rank-partition stable matching rule is the DA rule.*

Suppose by way of contradiction that φ is a rank-partition stable matching rule which is strategyproof, but φ is not the DA rule. Let φ be rank-partition stable with respect to (v, x). By Step 1, φ is not the PRP rule $f^{v,x}$. Thus, there exists $(\succ, P) \in \Pi \times \mathbb{P}$ such that $\varphi(\succ, P) \neq f^{v,x}(\succ, P)$. Then Proposition 1 implies that $f^{v,x}(\succ, P)$ Pareto-dominates $\varphi(\succ, P)$, and hence there exists $s \in S$ such that $f_s^{v,x}(\succ, P) \; P_s \; \varphi_s(\succ, P)$. Since φ is rank-partition stable it is also individually rational, and thus it follows that $f_s^{v,x}(\succ, P) \; P_s \; 0$. Let $\tilde{P}_s \in \mathcal{P}$ be the *truncation* of P_s directly below $f_s^{v,x}(\succ, P)$, so that the ordering in $\tilde{P}_s$ is the same as in P_s from the first-ranked school to $f_s^{v,x}(\succ, P)$, and the schools ranked below $f_s^{v,x}(\succ, P)$ in P_s are unacceptable to s at $\tilde{P}_s$. Let $\tilde{P} \equiv (\tilde{P}_s, P_{-s})$. Note that $f^{v,x}(\succ, \tilde{P}) = f^{v,x}(\succ, P)$.[13] Moreover, as φ is strategyproof and individually rational, $\varphi_s(\succ, \tilde{P}) = 0$. Let $\bar{\succ} \equiv \bar{\succ}((\succ, \tilde{P}), (v, x))$. Since φ is rank-partition stable with respect to (v, x) and $f^{v,x}$ is a PRP rule, both $\varphi(\succ, \tilde{P})$ and $f^{v,x}(\succ, \tilde{P})$ are stable at $(\bar{\succ}, \tilde{P})$. Then, since $f^{v,x}(\succ, \tilde{P}) = f^{v,x}(\succ, P)$, $f^{v,x}(\succ, P)$ is also stable at $(\bar{\succ}, \tilde{P})$. However, $\varphi_s(\succ, \tilde{P}) = 0$ and $f_s^{v,x}(\succ, P) \neq 0$. Since both $\varphi(\succ, \tilde{P})$ and $f^{v,x}(\succ, P)$ are stable at $(\bar{\succ}, \tilde{P})$, this contradicts the rural hospital theorem (Roth, 1986). □

Proof of Theorem 4.

Summary of the proof:

Given a PRP rule $f^{v,x}$ and a fixed profile $(\succ, P)$, we need to compare the assignment of a student s in $f^{v,x}(\succ, P)$ and in $f^{v,x}(\succ, (\check{P}_s, P_{-s}))$, where $\check{P}_s$ is an arbitrary alternative preference ordering to P_s for student s. The heart of the proof is the Manipulation Lemma, in which we consider two simple kinds of transformations of some preference ordering P_s into another preference ordering P'_s, which we refer to as T1 and T2 transformations, assuming that according to both P_s and P'_s all schools are acceptable. In a T1 transformation P'_s preserves the partition of schools into preference rank classes and only a reshuffling of schools within the preference rank classes is allowed compared to P_s. In a T2 transformation only the positions of two schools, a and b, satisfying $a \; P_s \; b$, are switched between the two preference orderings such that these two schools are in different preference rank classes in P_s, and thus their respective preference rank classes are also switched in P'_s. T2 transformations are divided into several cases, depending on the position of school d with respect to a and b, where d is the school that is assigned to s when she reports P_s. For each kind of transformation and for each case one or more scenarios are proved about $f_s^{v,x}(\succ, (P'_s, P_{-s}))$. Namely, we show in the Manipulation Lemma that the new assignment for s at $(\succ, (P'_s, P_{-s}))$ is either b (the school that is moved to a higher preference rank class), or d (the school that is assigned to s at $(\succ, P)$), or a less preferred school than d according to P_s. Then we define a specific sequence of T1 and T2 transformation steps that transforms an arbitrary preference ordering P_s into another arbitrary preference ordering $\check{P}_s$ (but still assuming that all schools are acceptable), which we call a *T-procedure.* We provide the proof of the theorem based on a T-procedure by showing that narrowing down the assignment of s to the three types of outcomes that are identified in the Manipulation Lemma in each step of the transformation of P_s into $\check{P}_s$ implies that s cannot be assigned to school c when she reports $\check{P}_s$ instead of P_s, such that c is preferred to $f_s^{v,x}(\succ, P)$ according to P_s, if c is not ranked in a higher preference rank class in $\check{P}_s$ than in P_s. Finally, we relax the assumption that all schools are acceptable, which completes the proof of the theorem for general preference orderings of s.

We begin with a lemma that will be used repeatedly in the following arguments. It establishes an invariance property of PRP rules which is satisfied by most known matching rules. Let f^{DA} denote the DA rule.

[13] For details, see Lemma 1 in the proof of Theorem 4.

Lemma 1. *Let $f^{v,x}$ be a PRP rule and fix a profile $(\succ, P) \in \Pi \times \mathbb{P}$. Let preferences $P'_s \in \mathcal{P}$ for student $s \in S$ be such that there exists $a \in C$ with $a\ P_s\ 0$, and the preference orderings of all the schools from the first-ranked school to a are identical for P_s and P'_s, while the preference orderings of schools ranked below a in P_s and assignment 0 may differ between P_s and P'_s. Let $\tilde{C} \equiv \{c \in C : c\ R_s\ a\}$ denote the set of schools that are weakly preferred to school a according to P_s. Then, for all $c \in \tilde{C}$, $f^{v,x}_s(\succ, P) = c$ if and only if $f^{v,x}_s(\succ, (P'_s, P_{-s})) = c$. Moreover, if $f^{v,x}_s(\succ, P) = c$ for some $c \in \tilde{C}$ then each round of the $f^{v,x}$ procedure at $(\succ, P)$ is the same as at $(\succ, (P'_s, P_{-s}))$, implying that $f^{v,x}(\succ, P) = f^{v,x}(\succ, (P'_s, P_{-s}))$.*

Proof. For all $S' \subseteq S$ and $c \in \tilde{C}$ the choice functions associated with $f^{v,x}$ lead to the same selection at the two relevant profiles: $Ch_c(S', (\succ, P)) = Ch_c(S', (\succ, (P'_s, P_{-s})))$, and the only possible difference is in the selections made by schools in $C \setminus \tilde{C}$ and for the selection of s only. Since in the $f^{v,x}$ procedure students apply to schools in the order of their preferences, if $f^{v,x}_s(\succ, P) = c$ such that $c \in \tilde{C}$ then s does not apply to any school in $C \setminus \tilde{C}$ at $(\succ, P)$ and thus, given the above observations about the selections made by the schools' choice functions, each round of the $f^{v,x}$ procedure at $(\succ, P)$ is the same as at $(\succ, (P'_s, P_{-s}))$, and s does not apply to any school in $C \setminus \tilde{C}$ at $(\succ, (P'_s, P_{-s}))$ either. This implies that for all $c \in \tilde{C}$, $f^{v,x}_s(\succ, P) = c$ if and only if $f^{v,x}_s(\succ, (P'_s, P_{-s})) = c$, and if $f^{v,x}_s(\succ, P) = c$ for some $c \in \tilde{C}$ then $f^{v,x}(\succ, P) = f^{v,x}(\succ, (P'_s, P_{-s}))$. □

The Manipulation Lemma will be established for the special case where student s, whose alternative preferences are compared, finds all schools acceptable. Let $\bar{\mathcal{P}} \subset \mathcal{P}$ denote the set of strict preferences over $C \cup \{0\}$ that rank 0 last.

Manipulation Lemma. *Let $f^{v,x}$ be a PRP rule and fix a profile $(\succ, P) \in \Pi \times \mathbb{P}$. Let $s \in S$ and $d \in C \cup \{0\}$ such that $f^{v,x}_s(\succ, P) = d$. Assume that $P_s \in \bar{\mathcal{P}}$ and let $P'_s \in \bar{\mathcal{P}}$ such that P'_s differs from P_s in one of the following two ways, which we refer to as T1 and T2 transformations, respectively.*

T1: *P_s and P'_s have the same preference rank partition of schools: for all $t \geq 1$, $X^t(P_s) = X^t(P'_s)$.*
T2: *Only the positions of two schools, $a, b \in C$, are switched between the two preference orderings such that, letting $a\ P_s\ b$, a is the bottom-ranked school in its preference rank class in P_s, and b is the top-ranked school in its preference rank class in P_s. Thus, by switching the positions of a and b the two schools also switch preference rank classes in P'_s compared to P_s, and hence there exist $t, \bar{t} \geq 1$ with $t < \bar{t}$ such that $a \in X^t(P_s)$, $b \in X^{\bar{t}}(P_s)$, $a \in X^{\bar{t}}(P'_s)$, and $b \in X^t(P'_s)$.*

Then one of three different types of outcomes is reached by $f^{v,x}_s(\succ, (P'_s, P_{-s}))$:

O1: $f^{v,x}_s(\succ, (P'_s, P_{-s})) = b$ *(the new assignment is the school which is moved to a higher preference rank class in a T2 transformation step)*
O2: $f^{v,x}_s(\succ, (P'_s, P_{-s})) = d$ *(the assignment remains the same)*
O3: $d\ P_s\ f^{v,x}_s(\succ, (P'_s, P_{-s}))$ *(the new assignment is less preferred according to P_s than the original assignment)*

Proof. Let the relevant modified priority profiles be denoted as $\bar{\succ} \equiv \bar{\succ}((\succ, P), (v, x))$ and $\bar{\succ}' \equiv \bar{\succ}((\succ, (P'_s, P_{-s})), (v, x))$. By Proposition 1, $f^{v,x}(\succ, P) = f^{DA}(\bar{\succ}, P)$ and $f^{v,x}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}', (P'_s, P_{-s}))$.

T1: We will show that one of the following two scenarios holds:
scenario (i): $f^{v,x}_s(\succ, (P'_s, P_{-s})) = d$ (O2)
scenario (ii): $d\ P_s\ f^{v,x}_s(\succ, (P'_s, P_{-s}))$ and $f^{v,x}_s(\succ, (P'_s, P_{-s}))\ P'_s\ d$ (O3)
Only the set of schools in each preference rank class is relevant for constructing a modified priority profile, while the ordering of the schools within a preference rank class is irrelevant. Hence, for T1 transformations, $\bar{\succ} = \bar{\succ}'$ and $f^{v,x}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}, (P'_s, P_{-s}))$. Therefore, the strategyproofness of the DA rule implies that either $f^{v,x}_s(\succ, (P'_s, P_{-s})) = d$, corresponding to scenario (i), or $d\ P_s\ f^{v,x}_s(\succ, (P'_s, P_{-s}))$ and $f^{v,x}_s(\succ, (P'_s, P_{-s}))\ P'_s\ d$, corresponding to scenario (ii).

T2: Apart from the swapped positions of a and b, for a T2 transformation P'_s is the same as P_s. We will denote the set of schools ranked by both P_s and P'_s between a and b by $E \subset C$. Then $e \in E$ if and only if $a\ P_s\ e\ P_s\ b$, as well as $b\ P'_s\ e\ P'_s\ a$. The proof below is presented for the general case where $E \neq \emptyset$, but it is easy to verify that the proof can be carried out in a simplified and very similar manner if $E = \emptyset$. In the following we consider different cases that depend on the position of school d in P_s with respect to schools a and b.

Case T2-A: $d\ P_s\ a$
By Lemma 1, $f^{v,x}_s(\succ, (P'_s, P_{-s})) = f^{v,x}_s(\succ, P) = d$. (O2)

Case T2-B: $d = a$
Lemma 1 implies one of the following scenarios:
scenario (i) : $f^{v,x}_s(\succ, (P'_s, P_{-s})) = b$ (O1 and O3)
scenario (ii): $f^{v,x}_s(\succ, (P'_s, P_{-s})) = d = a$ (O2)
scenario (iii): $d\ P_s\ f^{v,x}_s(\succ, (P'_s, P_{-s}))$ such that $f^{v,x}_s(\succ, (P'_s, P_{-s})) \neq b$ (O3)

Case T2-C: $a\ P_s\ d\ P_s\ b$ (i.e., $d \in E$)

We will show that one of the following scenarios holds:
scenario (i) : $f_s^{v,x}(\succ, (P'_s, P_{-s})) = b$ (O1 and O3)
scenario (ii): $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d$ (O2)
Consider the following modification of the preferences for s: let $\hat{P}_s \in \bar{\mathcal{P}}$ rank a last (directly above 0), while the preference ordering of all other schools remains the same as in P_s. Then, since the DA rule is strategyproof, $f_s^{DA}(\bar{\succ}, P) = d$ implies that $f_s^{DA}(\bar{\succ}, (\hat{P}_s, P_{-s})) = d$. Furthermore, since only the priority of student s may change, and only for schools a and b, when comparing $\bar{\succ}'$ to $\bar{\succ}$, it must be the case that $f_s^{DA}(\bar{\succ}', (\hat{P}_s, P_{-s})) = d$, since student s does not apply to either school a or to school b in the DA procedure at profile $(\bar{\succ}, (\hat{P}_s, P_{-s}))$, and thus each round of the DA procedure is the same at profiles $(\bar{\succ}, (\hat{P}_s, P_{-s}))$ and $(\bar{\succ}', (\hat{P}_s, P_{-s}))$. Then it follows from the strategyproofness of the DA rule that $f_s^{DA}(\bar{\succ}', (P'_s, P_{-s})) \in \{b, d\}$. This means that either $f_s^{v,x}(\succ, (P'_s, P_{-s})) = b$, corresponding to scenario (i), or $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d$, corresponding to scenario (ii).

Case T2-D: $d = b$
We will show that $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d = b$ (O1 and O2). Since the DA rule is strategyproof, $f_s^{DA}(\bar{\succ}, P) = d = b$ implies that $f_s^{DA}(\bar{\succ}, (P'_s, P_{-s})) = b$. Furthermore, since only the priority of student s may change, and only for schools a and b, when comparing $\bar{\succ}'$ to $\bar{\succ}$, the application of s to school b in the DA procedure at profile $(\bar{\succ}, (P'_s, P_{-s}))$ is the first possibility for a difference from the DA procedure at profile $(\bar{\succ}', (P'_s, P_{-s}))$. Given that s has the same or a higher rank in $\bar{\succ}'_b$ compared to $\bar{\succ}_b$, while all relative rankings of the other students are the same in the two priority rankings of school b, since s is accepted at $(\bar{\succ}, (P'_s, P_{-s}))$ by b, the same is true at $(\bar{\succ}', (P'_s, P_{-s}))$ in the same round of the DA procedure. Moreover, since only the relative ranking of s may differ between $\bar{\succ}'$ and $\bar{\succ}$, and given that s is not ranked lower by $\bar{\succ}'_b$ than by $\bar{\succ}_b$, the rounds of the DA procedure are the same at $(\bar{\succ}, (P'_s, P_{-s}))$ and at $(\bar{\succ}', (P'_s, P_{-s}))$, and s is not rejected by b in a later round in the DA procedure at profile $(\bar{\succ}', (P'_s, P_{-s}))$. Hence, the ranking of s in $\bar{\succ}'_a$ versus $\bar{\succ}_a$ is irrelevant, and $f_s^{DA}(\bar{\succ}', (P', P_{-s})) = b$. Therefore, $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d = b$.

Case T2-E: $b \; P_s \; d$.
We will show that one of the following scenarios holds:
scenario (i): $f_s^{v,x}(\succ, (P'_s, P_{-s})) = b$ (O1)
scenario (ii): $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d$ (O2)
We show first that $f_s^{DA}(\bar{\succ}', P) \neq a$. Given $a \; P_s \; d$, s is rejected by school a in the DA procedure at profile $(\bar{\succ}, P)$. Since only the priority of student s may change, and only for schools a and b, when comparing $\bar{\succ}'$ to $\bar{\succ}$, the application of s to school a in the DA procedure at profile $(\bar{\succ}, P)$ is the first possibility for a difference from the DA procedure at profile $(\bar{\succ}', P)$. Given that s is ranked the same or lower by $\bar{\succ}'_a$ than by $\bar{\succ}_a$, while all relative rankings of the other students are the same in the two priority rankings of school a, since s is rejected at $(\bar{\succ}, P)$ by a, s is also rejected at $(\bar{\succ}', P)$ by a. Student s may be temporarily accepted at $(\bar{\succ}, P)$ by school a before being rejected, and s may also be temporarily accepted at $(\bar{\succ}', P)$ by school a before being rejected, or possibly rejected immediately or in fewer rounds, but in the end s is rejected by school a at both profiles, since any possible difference in the rounds of the DA procedure between the two profiles that affects the assignment of s to a can only occur if s is rejected by school a sooner at $(\bar{\succ}', P)$ than at $(\bar{\succ}, P)$, whereas if there is no difference then a would reject s anyway at $(\bar{\succ}', P)$. Therefore, $f_s^{DA}(\bar{\succ}', P) \neq a$.

Consider the following modification of the preferences for s: let $\tilde{P}_s \in \bar{\mathcal{P}}$ rank schools a and b last (directly above 0), while the relative preference ordering of all other schools remains the same as in P_s and P'_s. Since $f_s^{DA}(\bar{\succ}, P) = d$, the strategyproofness of the DA rule implies that $f_s^{DA}(\bar{\succ}, (\tilde{P}_s, P_{-s})) = d$. Then, since only the priority of student s may change when comparing $\bar{\succ}$ and $\bar{\succ}'$, and only for schools a and b, while all other students have the same relative ranking in the two priority profiles, it must be the case that $f_s^{DA}(\bar{\succ}', (\tilde{P}_s, P_{-s})) = d$, since the DA procedure has the same rounds at profiles $(\bar{\succ}, (\tilde{P}_s, P_{-s}))$ and $(\bar{\succ}', (\tilde{P}_s, P_{-s}))$, and student s does not apply to either school a or b at $(\bar{\succ}', (\tilde{P}_s, P_{-s}))$. Then, by the strategyproofness of the DA rule, $f_s^{DA}(\bar{\succ}', P) \in \{a, b, d\}$. As shown above, $f_s^{DA}(\bar{\succ}', P) \neq a$, and therefore $f_s^{DA}(\bar{\succ}', P) \in \{b, d\}$. Since the DA rule is strategyproof, if $f_s^{DA}(\bar{\succ}', P) = b$ then $f_s^{DA}(\bar{\succ}', (P'_s, P_{-s})) = f_s^{v,x}(\succ, (P'_s, P_{-s})) = b$, corresponding to scenario (i), and if $f_s^{DA}(\bar{\succ}', P) = d$ then $f_s^{DA}(\bar{\succ}', (P'_s, P_{-s})) = f_s^{v,x}(\succ, (P'_s, P_{-s})) = d$, corresponding to scenario (ii).

Conclusion: We have analyzed all possible cases depending on the position of school d in P_s with respect to schools a and b, and can conclude that one of the following three outcomes is reached in each case:

O1: $f_s^{v,x}(\succ, (P'_s, P_{-s})) = b$ in Case T2-B scenario (i), Case T2-C scenario (i), Case T2-D, and Case T2-E scenario (i).
O2: $f_s^{v,x}(\succ, (P'_s, P_{-s})) = d$ in T1 scenario (i), Case T2-A, Case T2-B scenario (ii), Case T2-C scenario (ii), Case T2-D, and Case T2-E scenario (ii).
O3: $d \; P_s \; f_s^{v,x}(\succ, (P'_s, P_{-s}))$ in T1 scenario (ii), Case T2-B scenarios (i) and (iii), and Case T2-C scenario (i). □

The proof of Theorem 4 below relies not only on the statement of the Manipulation Lemma but also on some details of the proof of the lemma; we presented the Manipulation Lemma separately for the sake of clarity. Observe that outcomes O2 and O3 in the statement of the lemma are the only possibilities for a strategyproof rule, whereas O1 is a feature of PRP rules. As it is possible that $b \; P_s \; f_s^{v,x}(\succ, P)$, O1 is indicative of how PRP rules may be manipulated. Specifically, Case T2-E scenario (i) is the only case where manipulation occurs.

Completion of the proof of Theorem 4.

Let $f^{v,x}$ be a PRP rule and fix a profile $(\succ, P) \in \Pi \times \mathbb{P}$. Let $s \in S$ and $\bar{d} \in C \cup \{0\}$ such that $f_s^{v,x}(\succ, P) = \bar{d}$. For now we assume that $P_s \in \bar{\mathcal{P}}$ and let $\check{P}_s \in \bar{\mathcal{P}}$; we will relax these assumptions at the end of the proof. Let $c \in C$ such that $c\ P_s\ \bar{d}$ and c is in the same or a lower preference rank class in $\check{P}_s$ than in P_s, given x.

Based on the preference rank class that a school is ranked in by P_s compared to $\check{P}_s$, we classify each school in C as an "up," "down" or "stay" school. A school is an *up school* if it is in a higher preference rank class in $\check{P}_s$ than in P_s, and it is a *down school* if it is in a lower preference rank class in $\check{P}_s$ than in P_s, given x. The rest of the schools are *stay schools*. Note that school c is either a down school or a stay school, by assumption.

By repeatedly applying the two kinds of transformations, T1 and T2, that are analyzed in the Manipulation Lemma, we can transform P_s into $\check{P}_s$ using a finite number of transformation steps (i.e., intermediate preference orderings). Let $\tilde{P}_s \in \mathcal{P}$ denote an intermediate preference ordering, including $\tilde{P}_s = P_s$, and let $\tilde{P}_s'$ denote the preference ordering immediately following $\tilde{P}_s$ that results from the T1 or T2 transformation of $\tilde{P}_s$, including $\tilde{P}_s' = \check{P}_s$. We call a school *up-bound in* an intermediate preference ordering $\tilde{P}_s$ if it is in a higher preference rank class in $\check{P}_s$ than in $\tilde{P}_s$, and we call it *down-bound in* $\tilde{P}_s$ if it is in a lower preference rank class in $\check{P}_s$ than in $\tilde{P}_s$, given x. The remaining schools, which are in the same preference rank class in $\tilde{P}_s$ and $\check{P}_s$, are *stay-bound in* $\tilde{P}_s$. Note that if there is any up-bound school in an intermediate preference ordering $\tilde{P}_s$ then there is also a down-bound school in $\tilde{P}_s$, and vice versa. If there is no up-bound (or down-bound) school in $\tilde{P}_s$, then all schools are stay-bound and thus each school is in the same preference rank class in $\tilde{P}_s$ as in $\check{P}_s$.

Let this transformation of P_s into $\check{P}_s$ proceed according to the following specifications, which we refer to as a *T-procedure.*

T-procedure

Given P_s, start with a T1 step and alternate T1 and T2 steps until $\check{P}_s$ is reached. If there is any up-bound (or down-bound) school in P_s, start with a (non-final) T1 step. Otherwise, carry out the final T1 step.

T1 step (non-final): In each T1 step arrange the schools within each preference rank class of $\tilde{P}_s$ such that all up-bound schools in $\tilde{P}_s$ are at the top of the preference rank class and all down-bound schools in $\tilde{P}_s$ are at the bottom of the preference rank class, leaving all other relative orderings of schools within each preference rank class unchanged. That is, $\tilde{P}_s'$ satisfies the following *within each preference rank class* of $\tilde{P}_s$ (and $\tilde{P}_s'$):

(i) all up-bound schools in $\tilde{P}_s$ are ranked above all stay-bound schools in $\tilde{P}_s$;
(ii) all stay-bound schools in $\tilde{P}_s$ are ranked above all down-bound schools in $\tilde{P}_s$;
(iii) all up-bound schools in $\tilde{P}_s$ are in the same relative order in $\tilde{P}_s'$ as in $\tilde{P}_s$;
(iv) all stay-bound schools in $\tilde{P}_s$ are in the same relative order in $\tilde{P}_s'$ as in $\tilde{P}_s$;
(v) all down-bound schools in $\tilde{P}_s$ are in the same relative order in $\tilde{P}_s'$ as in $\tilde{P}_s$.

Carry out a T2 step next.

T2 step: In each T2 step choose one pair of schools (a, b) such that $a\ \tilde{P}_s\ b$, a is down-bound in $\tilde{P}_s$, b is up-bound in $\tilde{P}_s$, and a is positioned in the lowest rank of its preference rank class, while b is positioned in the highest rank of its preference rank class such that all schools ranked between a and b, if there are any, are stay-bound in $\tilde{P}_s$. Switch the positions of schools a and b in this T2 transformation step, so that $b\ \tilde{P}_s'\ a$, and let all other schools have the same rank in $\tilde{P}_s'$ as in $\tilde{P}_s$.
If $\tilde{P}_s' = \check{P}_s$ then the procedure is over.
If $\tilde{P}_s' \neq \check{P}_s$ but each school is in the same preference rank class in $\tilde{P}_s'$ as in $\check{P}_s$, that is, if all schools are stay-bound in $\tilde{P}_s'$, then carry out the final T1 step.
If there is at least one up-bound (or down-bound) school in $\tilde{P}_s'$, carry out a (non-final) T1 step next.

Final T1 step: In the final T1 step, once each school is in the same preference rank class as in $\check{P}_s$, arrange the schools within each preference rank class in the same order as they are in $\check{P}_s$. Then $\tilde{P}_s' = \check{P}_s$ and the procedure is over.

A T-procedure reaches $\check{P}_s$ from P_s in a finite number of steps, alternating T1 and T2 transformation steps, regardless of the order in which (a, b) pairs are picked in T2 steps. Notice that an up school is up-bound in each transformation step until it reaches the preference rank class that it belongs to in $\check{P}_s$, and then it is stay-bound in each subsequent transformation step in a T-procedure. Similarly, a down school is down-bound in each transformation step until it reaches the preference rank class that it belongs to in $\check{P}_s$, and then it is stay-bound in each subsequent transformation step. Moreover, a stay school is stay-bound in each transformation step. Since c is either a down school or a stay school, this also implies that c is either down-bound or stay-bound in each transformation step. Observe, furthermore, that if the positions of two schools are switched in any step, they are not switched back to their previous relative positions again in any later step of a T-procedure.

Now we are ready to show that there is no transformation step (i.e., an intermediate preference ordering $\tilde{P}_s$) in a T-procedure that transforms P_s into $\check{P}_s$ in which student s is assigned to c. Due to the Manipulation Lemma, we only need to consider three different types of outcomes for each step of a T-procedure that transforms P_s into $\check{P}_s$: O1, O2, and O3. Let

$\tilde{P}_s$ be an intermediate preference ordering followed immediately by $\tilde{P}'_s$ in a T-procedure, including $\tilde{P}_s = P_s$ and $\tilde{P}'_s = \check{P}_s$. Assume that $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s})) \neq c$. We will show that then $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) \neq c$. Since $f_s^{v,x}(\succ, P) \neq c$, it will follow by induction that $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$.

O1: $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) = b$, *where* b *is in a higher preference rank class in* $\tilde{P}'_s$ *than in* $\tilde{P}_s$.
Note that b is an up-bound school in $\tilde{P}_s$, and since c is either down-bound or stay-bound in $\tilde{P}_s$, it follows that $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) \neq c$ in a transformation step with O1.

O2: $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) = f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s}))$
A transformation step with O2 cannot yield c, since if $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s})) \neq c$ then $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) \neq c$.

O3: $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s}))\ \tilde{P}_s\ f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s}))$
The Manipulation Lemma shows that a transformation step with O3 can only occur in a T1 step scenario (*ii*), or in a T2 step under Case T2-B scenarios (*i*) and (*iii*), or Case T2-C scenario (*i*). Case T2-B scenario (*i*) and Case T2-C scenario (*i*) have already been analyzed under O1, since these cases fall under both the O1 and O3 outcome types, which leaves T1 scenario (*ii*) and Case T2-B scenario (*iii*). We will look at these two kinds of transformation steps in turn. Let $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s})) = d$ such that $d \neq c$. Suppose by way of contradiction that $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) = c$. Then $d\ \tilde{P}_s\ c$.

T1 step: If this is a T1 step scenario (*ii*) then $c\ \tilde{P}'_s\ d$ holds, as shown in the proof of the Manipulation Lemma, where this T1 step transforms $\tilde{P}_s$ into $\tilde{P}'_s$ and d and c are in the same preference rank class in $\tilde{P}_s$. Given that if the positions of two schools are switched in any step they are not switched back to their original relative positions again in any later step of a T-procedure, $d\ \tilde{P}_s\ c$ and $c\ \tilde{P}'_s\ d$ imply that $d\ P_s\ c$. Then, by transitivity, $c\ P_s\ \bar{d}$ implies that $d\ P_s\ \bar{d}$ and hence, since $f_s^{v,x}(\succ, (\tilde{P}_s, P_{-s})) = d$, according to the Manipulation Lemma there is a previous T2 transformation step corresponding to Case T2-E scenario (*i*) with outcome O1 in which s is assigned to school d. This means that d is an up school. Now note that $d\ P_s\ c$ implies that d is in the same or a higher preference rank class than c in P_s, and since d is an up school while c is either a down school or a stay school, and d and c are in the same preference rank class in $\tilde{P}_s$, it follows that both d and c are in the same preference rank class in P_s as in $\tilde{P}_s$, implying that d is not in a higher preference rank class in $\tilde{P}_s$ compared to P_s. Therefore, since d is an up school, d is up-bound in $\tilde{P}_s$, which implies that the T1 step that transforms $\tilde{P}_s$ into $\tilde{P}'_s$ is non-final. Then, however, since c is either down-bound or stay-bound in $\tilde{P}_s$ and d is up-bound in $\tilde{P}_s$, $d\ \tilde{P}'_s\ c$ holds, which is a contradiction.

T2 step: If this is a T2 step, specifically Case T2-B scenario (*iii*), then d is moved from its preference rank class in $\tilde{P}_s$ to a lower preference rank class in $\tilde{P}'_s$ and thus d is a down school and it is down-bound in $\tilde{P}_s$. We show next for all possible cases of the relationship of d to $\bar{d}$ in P_s that this leads to a contradiction.

(a) If $d\ P_s\ \bar{d}$, then the Manipulation Lemma implies that one of the previous steps correspond to Case T2-E scenario (*i*) with outcome O1 and hence d is an up school. This is a contradiction, since d is a down school, as noted above.
(b) If $d = \bar{d}$ or $\bar{d}\ P_s\ d$, then $c\ P_s\ \bar{d}$ implies that $c\ P_s\ d$. Then, given that d is down-bound in $\tilde{P}_s$, $c\ P_s\ d$ implies that $c\ \tilde{P}'_s\ d$ according to the specifications of a T-procedure. However, given that if the positions of two schools are switched in any step they are not switched back to their original relative positions again in any later step of a T-procedure, $c\ P_s\ d$ and $d\ \tilde{P}_s\ c$ imply that $d\ \tilde{P}'_s\ c$, which is a contradiction.

We reached a contradiction for both T1 and T2 transformation steps with O3, and hence $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) \neq c$ in any transformation step with O3.

We have checked all three different types of outcomes for the steps in a T-procedure, and thus it follows from the Manipulation Lemma that there is no step in a T-procedure that transforms P_s into $\check{P}_s$ in which any intermediate preference ordering $\tilde{P}'_s$ yields $f_s^{v,x}(\succ, (\tilde{P}'_s, P_{-s})) = c$. Therefore, we can conclude by induction that $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$. In sum, this proves that for any P_s and $\check{P}_s$ for which all schools are acceptable to s, if c is preferred to $f_s^{v,x}(\succ, P)$ by s and c is in the same or a lower preference rank class in $\check{P}_s$ than in P_s, then $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$. Therefore, the theorem is proved for the special case where $P_s, \check{P}_s \in \bar{\mathcal{P}}$.

Finally, we need to show that the above arguments generalize to arbitrary P_s and $\check{P}_s$, without assuming that all schools are acceptable to s. Fix arbitrary preference orderings $P_s, \check{P}_s \in \mathcal{P}$. Let $P''_s \in \bar{\mathcal{P}}$ be the same as P_s, except for adding all the schools that are unacceptable to s in P_s to the bottom of the preference ordering of P''_s as acceptable schools. Similarly, let $\check{P}''_s \in \bar{\mathcal{P}}$ be the same as $\check{P}_s$, except for adding all the schools that are unacceptable to s in $\check{P}_s$ to the bottom of the preference ordering of $\check{P}''_s$ as acceptable schools.

Let $c \in C$ such that $c\ P_s\ f_s^{v,x}(\succ, P)$ and c is in the same or a lower preference rank class in $\check{P}_s$ than in P_s, given x. We will show that $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$. Let $d \in C \cup \{0\}$ such that $f_s^{v,x}(\succ, P) = d$. By individual rationality, $d = 0$ or $d\ P_s\ 0$. If $d = 0$ then either $f_s^{v,x}(\succ, (P''_s, P_{-s})) = 0$ or $0\ P_s\ f_s^{v,x}(\succ, (P''_s, P_{-s}))$, by Lemma 1, and if $d\ P_s\ 0$ then Lemma 1 implies that $f_s^{v,x}(\succ, P) = f_s^{v,x}(\succ, (P''_s, P_{-s})) = d$. Thus, it follows from $c\ P_s\ f_s^{v,x}(\succ, P)$ that $c\ P''_s\ f_s^{v,x}(\succ, (P''_s, P_{-s}))$. If $0\ \check{P}_s\ c$ then, since $f^{v,x}$ is individually rational, $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$. Then, since $c \neq 0$, we can assume that $c\ \check{P}_s\ 0$. Also, by individual rationality, $c\ P_s\ 0$. Then c is in the same or a lower preference rank class in $\check{P}''_s$ than in P''_s. As $P''_s, \check{P}''_s \in \bar{\mathcal{P}}$, we can apply the above proof of the theorem to P''_s and $\check{P}''_s$. Therefore, since $c\ P''_s\ f_s^{v,x}(\succ, (P''_s, P_{-s})$ and c is in the same or a lower preference rank class in $\check{P}''_s$ than in P''_s, it follows that $f_s^{v,x}(\succ, (\check{P}''_s, P_{-s})) \neq c$. Then, since $c\ \check{P}_s\ 0$, Lemma 1 implies that $f_s^{v,x}(\succ, (\check{P}_s, P_{-s})) \neq c$. This proves the theorem for the general case where $P_s, \check{P}_s \in \mathcal{P}$. $\square$

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