The Super-Stable Common Independent Set Problem

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Aim of This Talk

Matroid

Fundamental object in Discrete Math. and Optimization

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[This talk] Example of nice interaction

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Stable matching

Fundamental model in Economics and Game Theory

Matroid

- **Matroid** is a pair $\mathbf{M} = (U, \mathcal{I})$ consisting of
  - a finite set U
  - a family $\mathcal{I}$ of subsets of U (i.e., $\mathcal{I} \subseteq 2^U$)

  satisfying **the following three conditions**

Definition of Matroid

(I0) $\emptyset \in \mathcal{I}$

(I1) $\forall I, J \subseteq U \colon I \in \mathcal{I}, J \subseteq I \implies J \in \mathcal{I}$

(I2) $\forall I, J \in \mathcal{I} \colon |I| < |J| \implies [\exists u \in J \setminus I \colon I + u \in \mathcal{I}]$

Example of Matroid: Partition Matroid

- For a partition $U_1, U_2, \dots, U_k$ of U,

$$\mathcal{I} := \{I \subseteq U \mid \forall i \in \{1, 2, \dots, k\} \colon |I \cap U_i| \le 1\}$$

Proof:

(I0) $|\emptyset \cap U_i| = 0 \le 1$

(I1) $J \subseteq I \in \mathcal{I} \implies |J \cap U_i| \le |I \cap U_i| \le 1$

(I2) $\forall I, J \in \mathcal{I},\ |I| < |J|$

$$\implies \exists i \in \{1, 2, \dots, k\} \colon |I \cap U_i| = 0, |J \cap U_i| = 1$$

$\implies$ we can add the element in $J \cap U_i$ to I

Remarks on Matroids

- Matroid is a generalization of the family of forests in a graph
- When we consider a computational problem on matroids...
  - we are given the ground set U
  - we are not explicitly given the family $\mathcal{I}$
  - we assume an **oracle** that can determine $I \in \mathcal{I}$
  - we care about the number of calls of the oracle
- For two matroids $\mathbf{M}_1 = (U, \mathcal{I}_1)$, $\mathbf{M}_2 = (U, \mathcal{I}_2)$
  - $I \in \mathcal{I}_1 \cap \mathcal{I}_2$ is a **common independent set** of $\mathbf{M}_1, \mathbf{M}_2$

Matroid Intersection

Matroid optimization

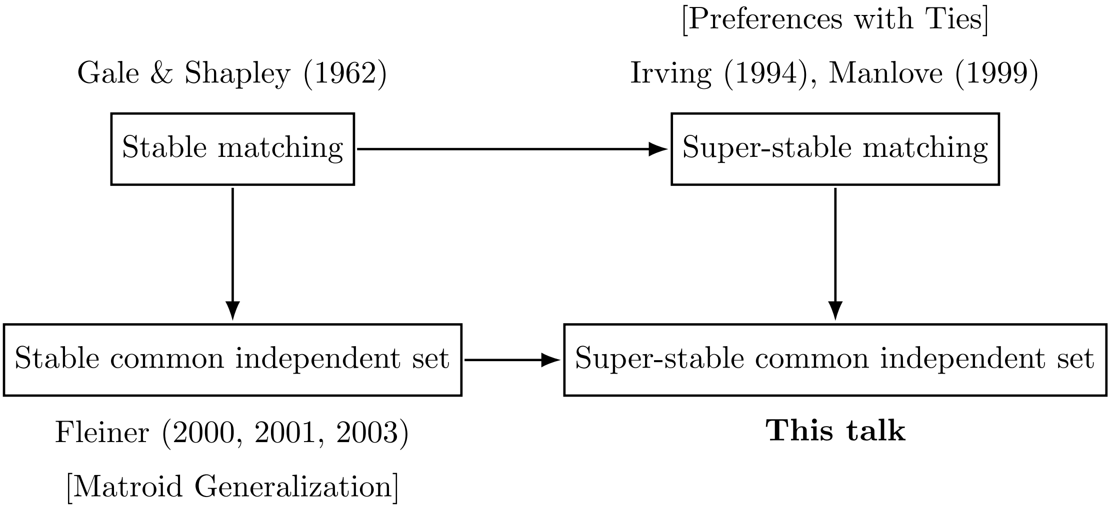
Common independent sets with a **weight function**

[This talk] Modification

Matching under preferences

Common independent sets with **orders**

Outline of This Talk



Outline of This Talk

1. Stable matching
2. Super-stable matching
3. Stable common independent set
4. Super-stable common independent set (Main Result)

Section 4 is based on

Naoyuki Kamiyama, A Matroid Generalization of the Super-Stable Matching Problem, SIAM Journal on Discrete Mathematics, 36(2):1467-1482, 2022.

Outline of This Talk

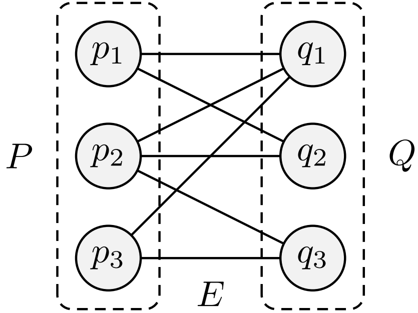
1. Stable matching
2. Super-stable matching
3. Stable common independent set
4. Super-stable common independent set

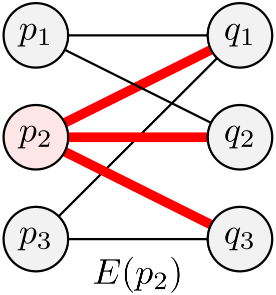
Setting

- We are given a simple bipartite graph $G = (P \cup Q; E)$
  - $P, Q =$ disjoint vertex sets (Define $V := P \cup Q$)
  - $E =$ a set of edges between P and Q
- For $v \in V$ and $F \subseteq E$,

  $F(v) :=$ the set of edges $e \in F$ incident to v
- Each $v \in V$ has a strict preference $\succ_v$ on $E(v)$

Setting





$$p_2 \colon \quad (p_2, q_1) \succ_{p_2} (p_2, q_3) \succ_{p_2} (p_2, q_2)$$

Setting

- We are given a simple bipartite graph $G = (P \cup Q; E)$
  - $P, Q =$ disjoint vertex sets (Define $V := P \cup Q$)
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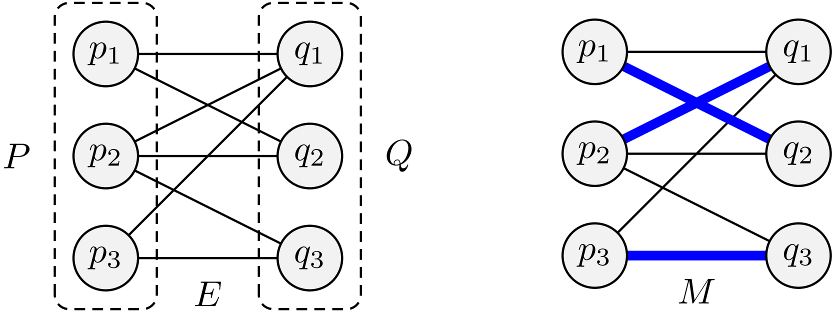
- For $v \in V$ and $F \subseteq E$,

  $F(v) :=$ the set of edges $e \in F$ incident to v
- Each $v \in V$ has a strict preference $\succ_v$ on $E(v)$
- $M \subseteq E$ is a **matching** $\overset{\text{def}}{\Longleftrightarrow}$

  For every $v \in V$, we have $|M(v)| \leq 1$

Setting



- For each matching M and $v \in V$ such that $M(v) \neq \emptyset$, we do not distinguish between $M(v)$ and the edge in $M(v)$

Stable Matching

- Let M be a matching
- $e = (p, q) \in E \setminus M$ **blocks** M $\overset{\text{def}}{\iff}$

  For every $v \in \{p, q\}$, $M(v) = \emptyset$ or $e \succ_v M(v)$
- M is said to be **stable** $\overset{\text{def}}{\iff}$

  Any edge in $E \setminus M$ does not block M

Theorem (Gale & Shapley 1962)

A stable matching always exists, and we can find it in poly-time

Outline of This Talk

1. Stable matching
2. Super-stable matching
3. Stable common independent set
4. Super-stable common independent set

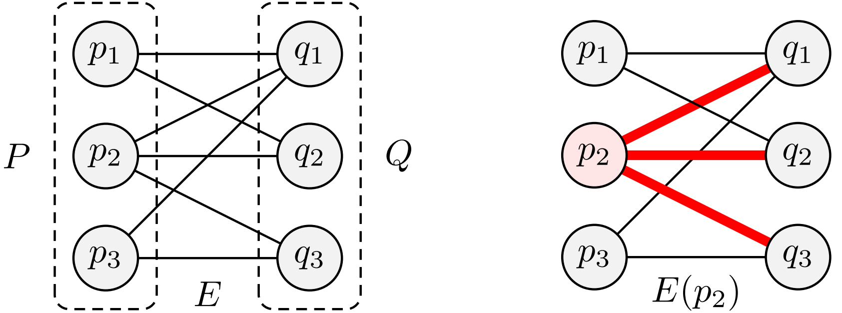
Setting

- We are given a simple bipartite graph $G = (P \cup Q; E)$
  - $P, Q =$ disjoint vertex sets (Define $V := P \cup Q$)
  - $E =$ a set of edges between P and Q
- For $v \in V$ and $F \subseteq E$,

  $F(v) :=$ the set of edges $e \in F$ incident to v
- Each $v \in V$ has a preference $\succsim_v$ on $E(v)$ with ties
- $M \subseteq E$ is a matching $\overset{\text{def}}{\Longleftrightarrow}$

  For every $v \in V$, we have $|M(v)| \leq 1$

Setting



$$p_2 : \quad (p_2, q_1) \sim_{p_2} (p_2, q_3) \succ_{p_2} (p_2, q_2)$$

Weakly Stable Matching

- Let M be a matching
- $e = (p, q) \in E \setminus M$ **strongly blocks** $M \overset{\text{def}}{\iff}$

  For every $v \in \{p, q\}$, $M(v) = \emptyset$ or $e \succ_v M(v)$
- M is said to be **weakly stable** $\overset{\text{def}}{\iff}$

  Any edge in $E \setminus M$ does not strongly block M

Theorem (Gale & Shapley 1962, Irving 1994)

A stable matching always exists, and we can find it in poly-time

Super-Stable Matching

- Let M be a matching
- $e = (p, q) \in E \setminus M$ **weakly blocks** $M \overset{\text{def}}{\iff}$

  For every $v \in \{p, q\}$, $M(v) = \emptyset$ or $e \succsim_v M(v)$
- M is said to be **super-stable** $\overset{\text{def}}{\iff}$

  Any edge in $E \setminus M$ does not weakly block M

Theorem (Irving 1994, Manlove 1999)

We can check the existence of a super-stable matching in poly-time, and find it if exists

Outline of This Talk

1. Stable matching
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3. Stable common independent set
4. Super-stable common independent set

Matroid

- A pair $\mathbf{M} = (U, \mathcal{I})$ of a finite set U and $\mathcal{I} \subseteq 2^U$ is a **matroid**

  $\overset{\text{def}}{\iff}$ (I0), (I1), and (I2) are satisfied

  (I0) $\emptyset \in \mathcal{I}$

  (I1) $\forall I, J \subseteq U \colon I \in \mathcal{I}, J \subseteq I \implies J \in \mathcal{I}$

  (I2) $\forall I, J \in \mathcal{I} \colon |I| < |J| \implies [\exists u \in J \setminus I \colon I + u \in \mathcal{I}]$

- Example of a matroid (**partition matroid**):

  For a partition $U_1, U_2, \ldots, U_k$ of U,

  $$\mathcal{I} = \{I \subseteq U \mid \forall i \in \{1, 2, \ldots, k\} \colon |I \cap U_i| \leq 1\}$$

Matroid

- Let I be a member of $\mathcal{I}$

  (I is called an **independent set**)
- Let u be an element in $U \setminus I$ such that $I + u \notin \mathcal{I}$
- The **fundamental circuit** $\mathsf{C}_{\mathbf{M}}(u, I)$ is defined by

  $$\mathsf{C}_{\mathbf{M}}(u, I) := \{w \in I + u \mid I + u - w \in \mathcal{I}\}$$
- For a partition matroid defined above, $\mathsf{C}_{\mathbf{M}}(u, I)$ is

  $\{w \in I + u \mid w = u$ or $w \in I \cap U_i\}$, where $u \in U_i$
- Define $\mathsf{D}_{\mathbf{M}}(u, I) := \mathsf{C}_{\mathbf{M}}(u, I) - u$

Stable Common Independent Set

- Let $\mathbf{M} = (U, \mathcal{I})$ be a matroid
- Let $\succ$ be a strict preference U
- $\mathbf{M} = (U, \mathcal{I}, \succ)$ is called an **ordered matroid**

  (An ordered matroid is also considered as a matroid)
- For each $I \in \mathcal{I}$,

  $\mathrm{dom}_{\mathbf{M}}(I) :=$ the set of elements $u \in U \setminus I$ such that

  - $I + u \notin \mathcal{I}$
  - $w \succ u$ for every $w \in \mathsf{D}_{\mathbf{M}}(u, I)$

Stable Common Independent Set

- We are given ordered matroids

$$\mathbf{M}_P = (U, \mathcal{I}_P, \succ_P) \text{ and } \mathbf{M}_Q = (U, \mathcal{I}_Q, \succ_Q)$$

- $I \in \mathcal{I}_P \cap \mathcal{I}_Q$ is a **common independendet set** of $\mathbf{M}_P, \mathbf{M}_Q$
- $I \in \mathcal{I}_P \cap \mathcal{I}_Q$ is **stable** $\overset{\text{def}}{\iff}$

$$U \setminus I = \mathrm{dom}_{\mathbf{M}_P}(I) \cup \mathrm{dom}_{\mathbf{M}_Q}(I)$$

Theorem (Fleiner 2000, 2001, 2003)

A stable common independent set always exists, and we can find it in poly-time

Stable Matching

- Define $U := E$
- For each $X \in \{P, Q\}$,

  $\mathcal{I}_X := \{M \subseteq E \mid \forall v \in X \colon |M(v)| \leq 1\}$
- For each $X \in \{P, Q\}$, $\succ_X$ is defined as follows
  - Assume that $X = \{v_1, v_2, \ldots, v_k\}$
  - $e \in v_i,\ f \in v_j,\ i < j \iff e \succ_X f$
  - $e, f \in v_i \implies [e \succ_{v_i} f \implies e \succ_X f]$
- $M \subseteq E$ is a stable matching in $G \iff$

  M is a stable common independent set of $\mathbf{M}_P, \mathbf{M}_Q$

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This section is based on

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Super-Stable Common Independent Set

- Let $\mathbf{M} = (E, \mathcal{I})$ be a matroid
- Let $\succsim$ be a preference on E with ties
- $\mathbf{M} = (E, \mathcal{I}, \succ)$ is called an ordered matroid

  (An ordered matroid is also considered as a matroid)
- For each $M \in \mathcal{I}$,

  $\mathsf{dom}_{\mathbf{M}}(M) :=$ the set of elements $e \in E \setminus M$ such that

  - $M + e \notin \mathcal{I}$
  - $f \succ e$ for every $f \in \mathsf{D}_{\mathbf{M}}(e, M)$

Super-Stable Common Independent Set

- We are given ordered matroids

  $$\mathbf{M}_P = (E, \mathcal{I}_P, \succsim_P) \text{ and } \mathbf{M}_Q = (E, \mathcal{I}_Q, \succsim_Q)$$

- $M \in \mathcal{I}_P \cap \mathcal{I}_Q$ is a **common independet set** of $\mathbf{M}_P, \mathbf{M}_Q$
- $M \in \mathcal{I}_P \cap \mathcal{I}_Q$ is **super-stable** $\overset{\text{def}}{\iff}$

  $$E \setminus M = \mathrm{dom}_{\mathbf{M}_P}(M) \cup \mathrm{dom}_{\mathbf{M}_Q}(M)$$

Main Result (Kamiyama 2022)

We can check the existence of a super-stable common independent set in poly-time, and find it if exists

The Student-Project Allocation Problem with Ties

Super-stable matching — Irving (1994), Manlove (1999)

↓

Super-stable student-project allocation — Olaosebikan & Manlove (2020)

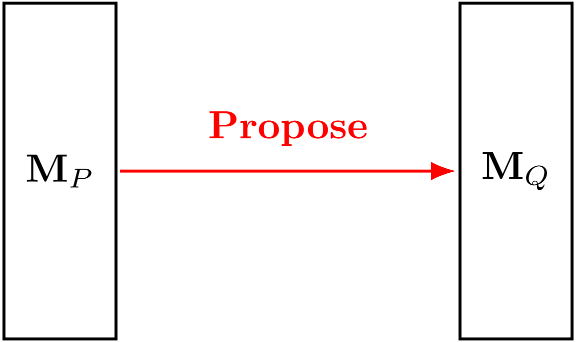
↓

Super-stable common independent set — **This talk**

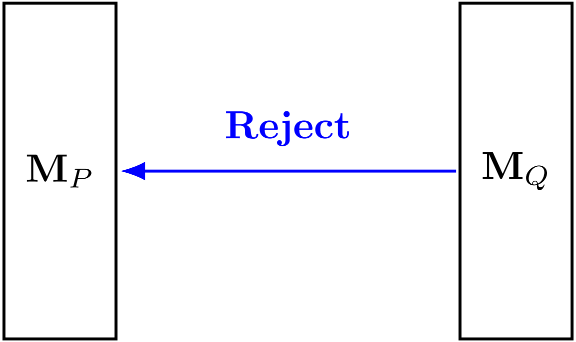
The Student-Project Allocation Problem with Ties

- A set S of students, a set P of projects, a set L of lecturers
- P is partitioned into $\{P_\ell \mid \ell \in L\}$
- The members of S and L have preferences with ties
- The members of P and L have capacities
- $M \subseteq S \times P$ is a matching $\stackrel{\text{def}}{\iff}$
  - At most one pair in M is incident to a student
  - $\#$ of pairs in M incident to a project is at most its capacity
  - $\#$ of pairs in M incident to a lecturer is at most its capacity

Gale-Shapley (Stable Matching with Strict Preferences)



Gale-Shapley (Stable Matching with Strict Preferences)



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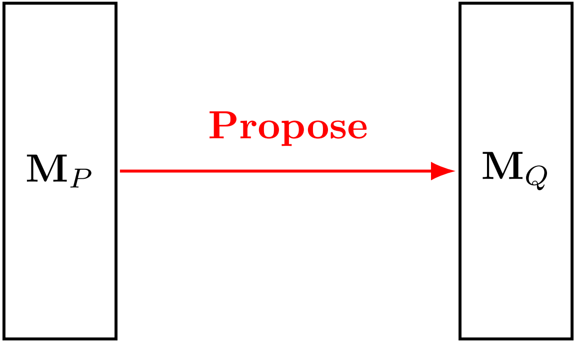

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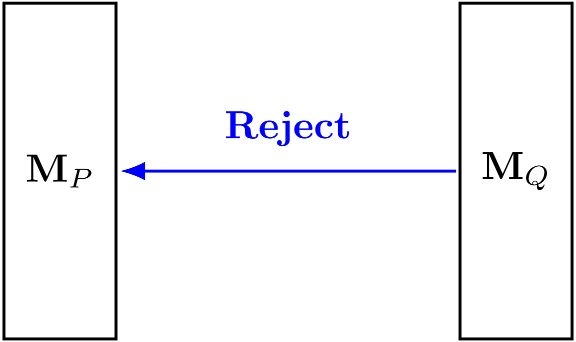


Gale-Shapley (Stable Matching with Strict Preferences)

$\mathbf{M}_P$

$$I \subseteq U$$

$\mathbf{M}_Q$

A **fixed point** I is obtained

Gale-Shapley (Stable Matching with Strict Preferences)

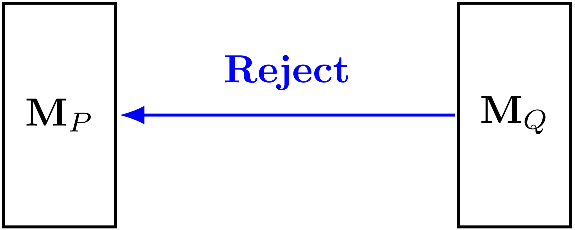
$\mathbf{M}_P$ $\quad I \subseteq U \quad$ $\mathbf{M}_Q$

A **fixed point** I is obtained

$\implies$ I is a **stable matching**

Irving (Stable Matching with Ties)

Irving (Stable Matching with Ties)



Irving (Stable Matching with Ties)

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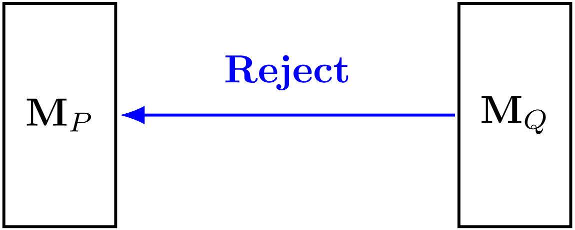

Irving (Stable Matching with Ties)

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A **fixed point** I is obtained

Irving (Stable Matching with Ties)

$\mathbf{M}_P$

$$I \subseteq U$$

$\mathbf{M}_Q$

Check some condition

A **fixed point** I is obtained

Irving (Stable Matching with Ties)

$\mathbf{M}_P$ $\qquad I \subseteq U \qquad$ $\mathbf{M}_Q$

$$\downarrow$$

Check some condition

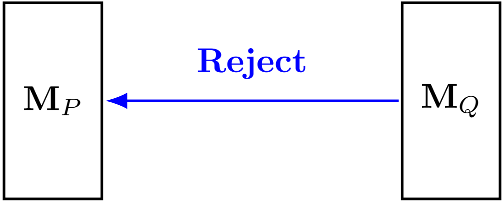
A **fixed point** I is obtained

Yes: I is a **super-stable matching**

No: there does not exist a super-stable matching

Olaosebikan-Manlove and This paper

Olaosebikan-Manlove and This paper



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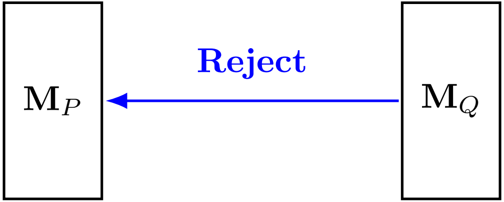


Olaosebikan-Manlove and This paper

$$\boxed{\mathbf{M}_P} \qquad I \subseteq U \qquad \boxed{\mathbf{M}_Q}$$

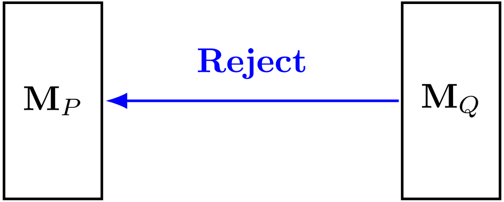
Olaosebikan-Manlove and This paper

Removed elements remove some elements

$\mathbf{M}_P$ $\qquad I \subseteq U \qquad$ $\mathbf{M}_Q$

Olaosebikan-Manlove and This paper

Olaosebikan-Manlove and This paper



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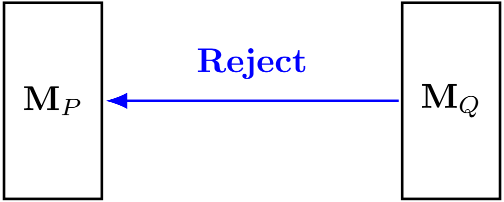

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$\boxed{\mathbf{M}_P}$ $\qquad I \subseteq U \qquad$ $\boxed{\mathbf{M}_Q}$

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Removed elements remove some elements

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Olaosebikan-Manlove and This paper

$\mathbf{M}_P$ $\qquad I \subseteq U \qquad$ $\mathbf{M}_Q$

A **fixed point** I is obtained

Olaosebikan-Manlove and This paper

$\mathbf{M}_P$ $\quad I \subseteq U \quad$ $\mathbf{M}_Q$

Check some condition

A **fixed point** I is obtained

Olaosebikan-Manlove and This paper

$\mathbf{M}_P$ $\qquad I \subseteq U \qquad$ $\mathbf{M}_Q$

$\downarrow$

Check some condition

A **fixed point** I is obtained

Yes: I is a **super-stable matching**
No: there does not exist a super-stable matching

Conclusion

[Preferences with Ties]

Gale & Shapley (1962)

Irving (1994), Manlove (1999)

Stable matching

Super-stable matching

Stable common independent set

Super-stable common independent set

Fleiner (2000, 2001, 2003)

This talk

[Matroid Generalization]

Thank you for your attention!!