Ordinal concavity and representation theorems for path-independent choice rules

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March 2023

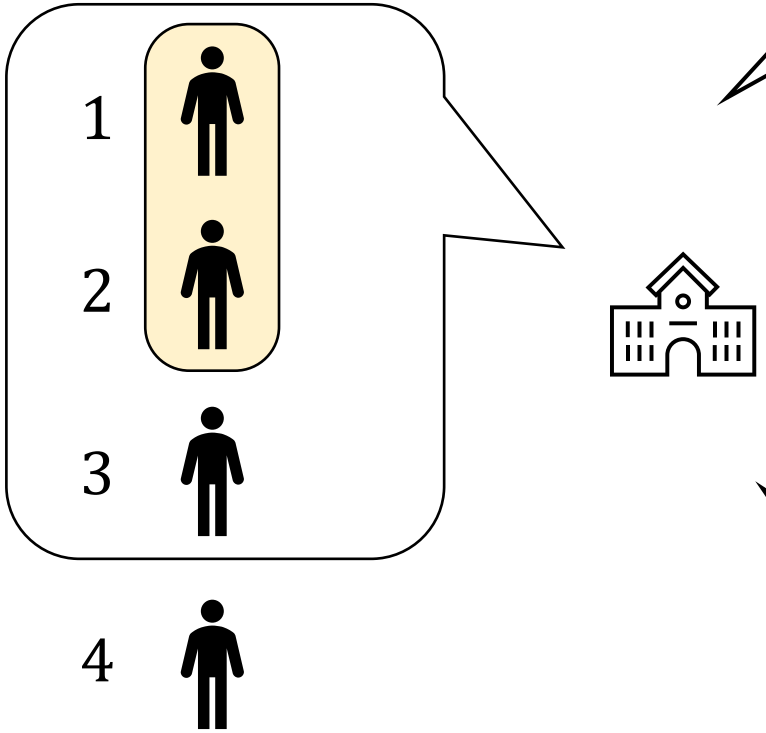
Motivation

- Market Design deals with markets with **indivisibilities**.
- Essential assumption: utility functions satisfy **M$^\natural$-concavity**.
  - Guarantee existence and efficient computation of equilibrium outcomes.
- It is a cardinal property, not an ordinal one.
  - Not preserved under *monotonic transformation*.

$u(\cdot)$ is M$^\natural$-concave
$\not\Rightarrow g(u(\cdot))$ is M$^\natural$-concave.

- In economics, utilities/preferences are ordinal concepts.
- Ordinal versions were introduced, but not yet applied to economics.
  - cf. Murota and Shioura (2003), Chen and Li (2021).
- **This study**: apply *ordinal concavity* to the analysis of matching markets.

What We Do



Choice rule C

Utility function u

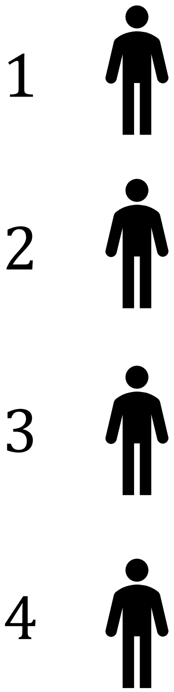
Path independence
($\Leftrightarrow$ Substitutability$^+$)

"Rationalize" if C chooses a utility-maximizing set.

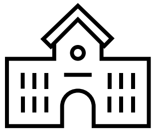
?

- **Representation theorem**:
  C is path-independent
  $\Leftrightarrow$ C is rationalizable by an ordinally concave utility function u.

What We Do



Choice rule C



Two goals:

- Accept students with high **merits**.
- Promote **diversity** (cf. affirmative action).

C maximizes diversity and merit.

- If preferences for diversity are represented by an ordinally concave function, C is **path-independent** and **computed in polynomial time**.

Structure of the Talk

1. Introduction
2. Representation theorem
3. Choice rule with diversity goals
4. Conclusion

Go to 2

Choice rule

Choose a *unique* subset.

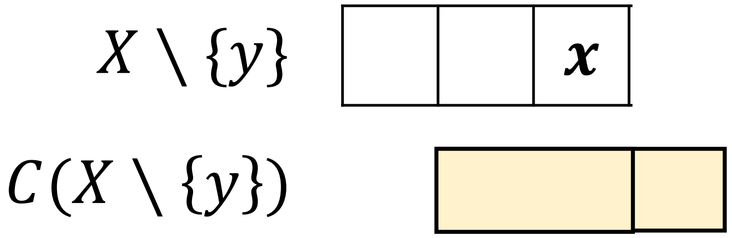
- $\mathcal{X}$: set of **contracts**.
- A **choice rule** is a function $C: 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ s.t. for any $X \subseteq 2^{\mathcal{X}}$, $C(X) \subseteq X$.
- C satisfies **path independence** if, for any $X, X' \subseteq \mathcal{X}$,

$$C(X \cup X') = C(C(X) \cup X').$$

Substitutability

- C satisfies the **substitutes condition** if, for any $X \subseteq \mathcal{X}$ and any distinct $x, y \in X$,

$$x \in C(X) \Longrightarrow x \in C(X \setminus \{y\}).$$

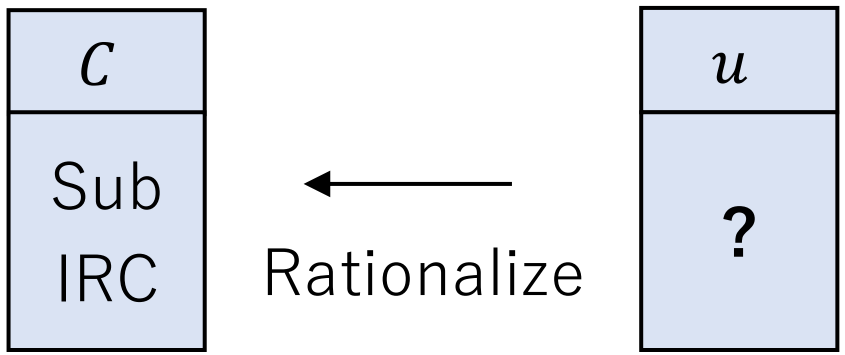


- C satisfies the **irrelevance of rejected contracts (IRC)** if, for any $X \subseteq \mathcal{X}$ and $x \in X$,

$$x \notin C(X) \Longrightarrow C(X \setminus \{x\}) = C(X).$$

- Path independence $\Leftrightarrow$ Substitutes condition + IRC.
  - Aizerman and Malishevski (1981).
- Guarantee existence of a stable matching.

Utility function



- A **utility function** is $u: 2^{\mathcal{X}} \to \mathbb{R}$.
- C is **rationalizable** by u if, for any $X \subseteq \mathcal{X}$,

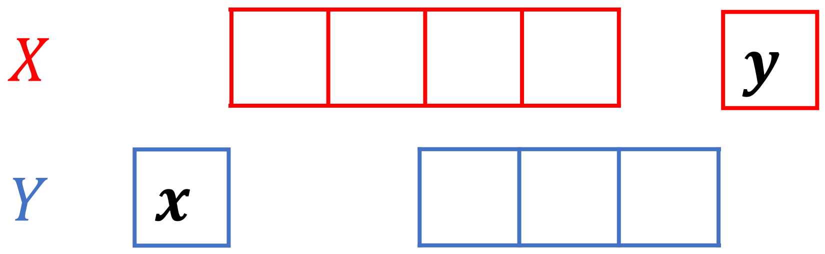
$$u(C(X)) > u(Y) \quad \forall Y \subseteq X \text{ with } Y \neq C(X).$$

- C is rationalizable by some $u \Longrightarrow C$ satisfies IRC.
- Eguchi, Fujishige and Tamura (2003):
  C is rationalizable by u satisfying $M^{\natural}$*-concavity*
  $\Longrightarrow C$ satisfies the substitutes condition.

$\mathrm{M}^{\natural}$-concavity

- Notation:
  - For $X \subseteq \mathcal{X}$ and $x \in \mathcal{X}$, let $X + x = X \cup \{x\}$, $X - x = X \setminus \{x\}$.
  - $X + \phi = X$, $X - \phi = X$.

- $u: 2^{\mathcal{X}} \to \mathbb{R}$ is **$\mathrm{M}^{\natural}$-concave** if, for any $X, Y \subseteq \mathcal{X}$ and $x \in X \setminus Y$, there exists $y \in (Y \setminus X) \cup \{\phi\}$ such that $\underline{u(X)} + \underline{u(Y)} \leq \underline{u(X - x + y)} + \underline{u(Y + x - y)}$.



- Introduce an **ordinal** version.

  cf. Murota and Shioura (2003), Chen and Li (2021).

Ordinal concavity

- $u: 2^{\mathcal{X}} \to \mathbb{R}$ is **ordinally concave** if, for any $X, Y \subseteq \mathcal{X}$ and $x \in X \setminus Y$, there exists $y \in (Y \setminus X) \cup \{\phi\}$ such that
    i. $u(X) < u(X - x + y)$, or
    ii. $u(Y) < u(Y + x - y)$, or
    iii. $u(X) = u(X - x + y)$ and $u(Y) = u(Y + x - y)$.

Representation theorem

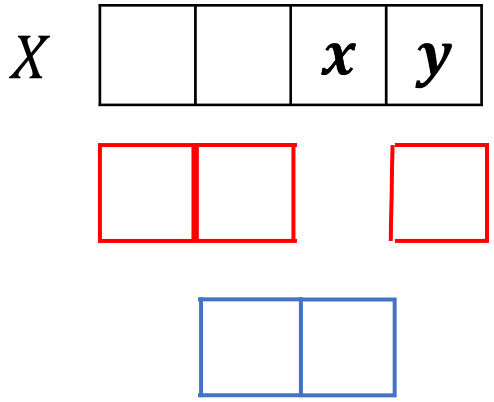
Theorem 1

The following are equivalent:

i. C satisfies the **substitutes condition** and **IRC**.

ii. C is rationalizable by an **ordinally concave** utility function.

- Proof sketch of **ii** $\Longrightarrow$ **i**.
- Main result is **i** $\Longrightarrow$ **ii**:
  - When we analyze substitutable C, it is w.l.o.g. to assume ordinally concave u.
  - Proof is constructive.



Remarks

- Another key property of C is the **law of aggregate demand**.
  - Guarantee *strategy-proofness* of the deferred acceptance algorithm. cf. Hatfield and Milgrom 2005.
- We can provide a representation theorem for C satisfying Sub and LAD.
  - $\mathrm{M}^\natural$-concavity satisfies "exchange property under cardinality constraint"
    $\rightarrow$ C rationalizable by $\mathrm{M}^\natural$-concave u satisfies LAD (Murota and Yokoi 2015).
    $\rightarrow$ Ordinal version.
- Comparison with Fujishige and Yang (2003).
  - u satisfies *gross* substitutes $\Leftrightarrow$ u is $\mathrm{M}^\natural$-concave.
- Gross Sub $\Leftrightarrow$ $\mathrm{M}^\natural$-concave, Sub $\Leftrightarrow$ ordinal concavity (under IRC).

  FY This study

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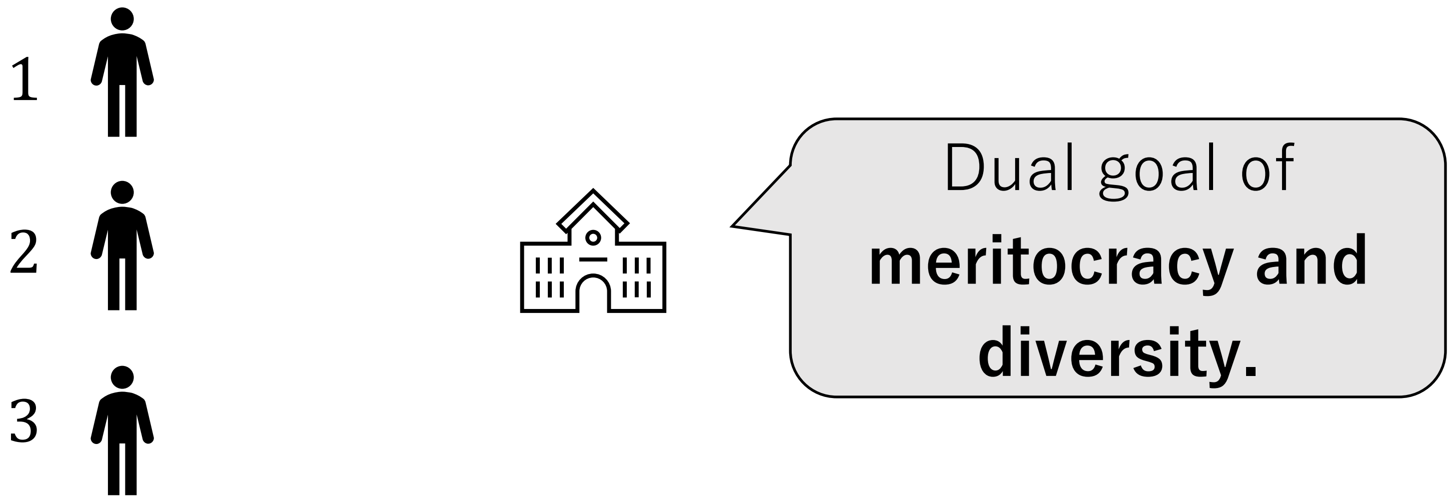
Choice rule with diversity goals



- $\mathcal{X}$: set of **contracts**.
- $\mathcal{T}$: set of **types**.
  - e.g. $\mathcal{T} = \{Female, Male\}$.
- Each contract $x \in \mathcal{X}$ is associated with one type $t \in \mathcal{T}$.

Merit

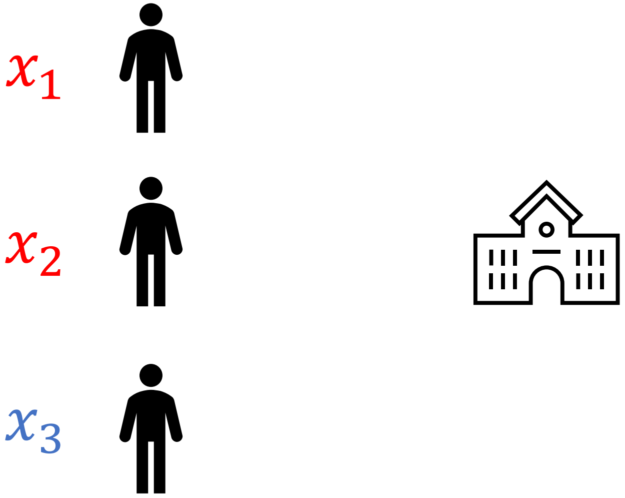
- A **merit ranking** is a linear order $\succ$ over $\mathcal{X}$.
  - $x \succ y$: x has a higher merit than y.
  - $x \succeq y$: $x \succ y$ or $x = y$.
- Consider two sets $X, Y \subseteq \mathcal{X}$.

X	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\cdots$	$\cdots$	$\boldsymbol{x}_{\lvert X\rvert}$	$x_1 \succ x_2 \succ \cdots \succ x_{\lvert X\rvert}$
	$\succeq$	$\succeq$				
Y	$\boldsymbol{y}_1$	$\boldsymbol{y}_2$	$\cdots$	$\boldsymbol{y}_{\lvert Y\rvert}$		$y_1 \succ y_2 \succ \cdots \succ y_{\lvert Y\rvert}$

- X **merit dominates** Y if $|X| \geq |Y|$ and $x_i \succeq y_i$ for all $i \in \{1, \ldots, |Y|\}$.

Distribution

- A **distribution** is a vector ξ in $\mathbb{Z}_+^{\mathcal{T}}$.
- For $X \subseteq \mathcal{X}$, $\xi(X) \in \mathbb{Z}_+^{\mathcal{T}}$ denotes the **distribution induced from $\boldsymbol{X}$**.
  - $\xi_t(X)$ is the number of type t-contracts in X.
- e.g.



- $X = \{x_1, x_2, x_3\}$.
- $T = \{t, \ t'\}$.
- $\xi(X) = (2, 1)$.

Diversity index

- $\Xi \subseteq \mathbb{Z}_+^{\mathcal{T}}$: set of **feasible distributions** (assume $\mathbf{0} \in \Xi$).
  - e.g. $\Xi = \{\xi \in \mathbb{Z}_+^{\mathcal{T}} \mid \sum_{t \in \mathcal{T}} \xi_t \leq q\}$ for some $q \in \mathbb{Z}_+$.
- The **diversity index** $f: \Xi \to \mathbb{R}_+$ measures desirability of $\xi \in \Xi$.
- e.g. **Saturated diversity**.
  - For each $t \in \mathcal{T}$, there is a **reserve** $r_t \in \mathbb{Z}_+$.

$$f(\xi) = \sum_{t \in \mathcal{T}} \min\{\xi_t, r_t\} \text{ for all } \xi \in \Xi.$$

  - cf. Hafalir, Yenmez, Yildirim (2013).
  - Used in real-life school choice programs.
    e.g. Chile (Dogan, Imamura, Yenmez 2022), India (Sönmez and Yenmez 2022).

Objective

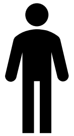
- **Lexicographic maximization** of diversity and merit.
- Given a set of contracts, choose a subset that

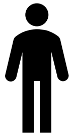
  (i) maximizes the diversity index among feasible distributions, and

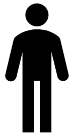
  (ii) merit dominates other subsets that attain the highest diversity.

- Develop the **diversity choice rule**.

Diversity choice rule: example

x_1 

x_2 

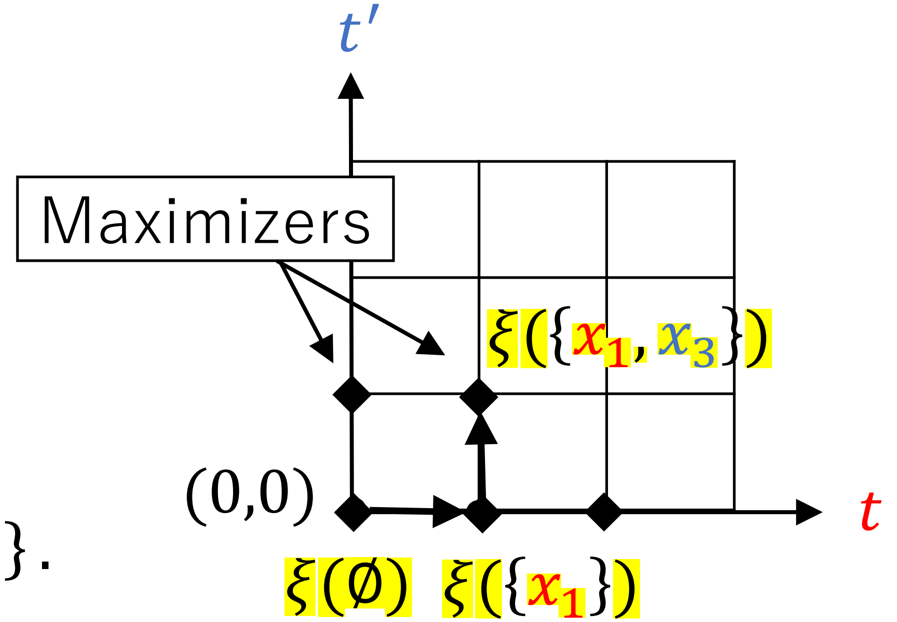
x_3 

$x_1 \succ x_2 \succ x_3$

$\mathcal{T} = \{t, t'\}.$

$\Xi = \{\xi: \xi_t + \xi_{t'} \leq 2\}.$

$f(\xi) = \min\{\xi_t, 0\} + \min\{\xi_{t'}, 1\}.$



Step 1 Find the set of maximizers of $f(\xi)$ s.t. $\xi \in \Xi$ and $\xi \leq \xi(\{x_1, x_2, x_3\})$.

Step 2 Start from ∅.

Add a contract with the highest merit if the resulting distribution becomes closer to some maximizer.

Diversity choice rule: formal definition

Input A set of contracts X.

Step 1 $\max f(\xi)$ subject to $\xi \in \Xi$ and $\xi \leq \xi(X)$. Let $\Xi^*(X)$ be the set of distributions that solve this maximization problem. Set $X_0 = \emptyset$ and $k = 0$.

Step 2 If there exist $x \in X \setminus X_k$ and $\xi \in \Xi^*(X)$ such that $\xi(X_k + x) \leq \xi$, then choose such a contract x_{k+1} of maximum merit, let $X_{k+1} = X_k + x_{k+1}$, and go to Step 3. Otherwise, go to Step 4.

Step 3 Add 1 to k and go to Step 2.

Step 4 Return X_k and stop.

- Denoted $C^{\mathrm{d}}(X)$.
- **Q:** When is this choice rule well behaved?

Ordinal concavity

- For $t \in \mathcal{T}$, let $\mathbf{1}_t \in \{0,1\}^{\mathcal{T}}$ denote the ***t*-th unit vector.**
  - $\mathbf{1}_\phi$: zero vector.
- $f: \Xi \to \mathbb{R}_+$ is **ordinally concave** if,

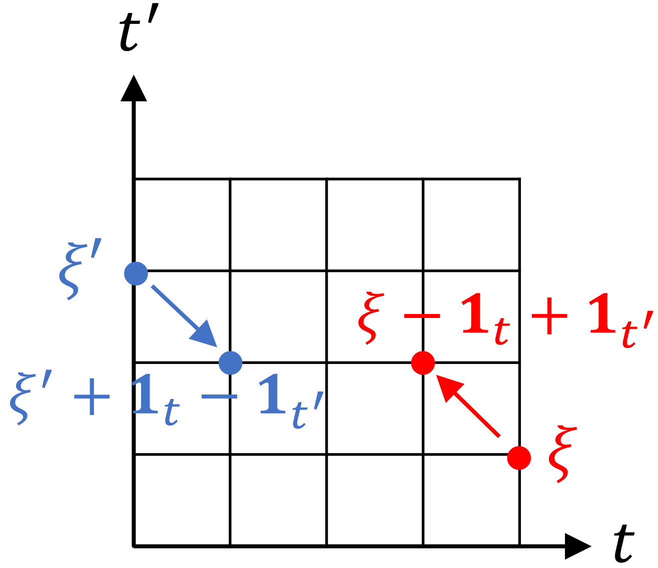
  for any $\xi, \xi' \in \mathbb{Z}_+^{\mathcal{T}}$ and $t \in \mathcal{T}$ with $\xi_t > \xi'_t$,

  there exists $t' \in \mathcal{T} \cup \{\phi\}$ $(\xi'_{t'} > \xi_{t'}$ if $t' \neq \phi)$ s.t.

  i. $f(\xi) < f(\xi - \mathbf{1}_t + \mathbf{1}_{t'})$, or

  ii. $f(\xi') < f(\xi' + \mathbf{1}_t - \mathbf{1}_{t'})$, or

  iii. $f(\xi) = f(\xi - \mathbf{1}_t + \mathbf{1}_{t'})$ and $f(\xi') = f(\xi' + \mathbf{1}_t - \mathbf{1}_{t'})$.



  - Saturated diversity satisfies ordinal concavity under capacity constraint.

Properties of C^{d}

$$f\left(\xi\left(C^{\mathrm{d}}(X)\right)\right) \geq f(\xi(X')) \;\; \forall X' \subseteq X \text{ with } \xi(X') \in \Xi.$$

Theorem 2

Suppose f is ordinally concave. Then, fo … t of contracts $X \subseteq \mathcal{X}$,

i. $C^{\mathrm{d}}(X)$ maximizes f among subsets of X.

ii. $C^{\mathrm{d}}(X)$ merit dominates any subset of X that maximizes f.

iii. $C^{\mathrm{d}}(X)$ can be calculated in $O(|T| \times |X|^2)$,

(assuming $f(\xi)$ can be calculated in constant time for any $\xi \in \Xi$).

- Proof idea:

ii. f is ordinally concave $\Longrightarrow \Xi^*(X)$ has a matroidal structure (M$^\natural$-convex).

$\Longrightarrow C^{\mathrm{d}}(X)$ is a greedy algorithm on a matroid (cf. Gale 1968).

iii. Generalize the **domain reduction algorithm** for M$^\natural$-concave function.

cf. Ch.10 in Murota (2003).

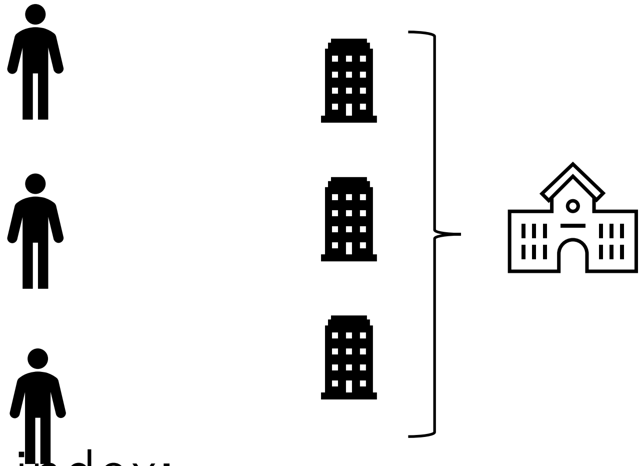
Properties of $C^{\mathbf{d}}$

Theorem 3

Suppose f is ordinally concave. Then, $C^{\mathbf{d}}$ satisfies the **substitutes condition** and **IRC**.

Extensions:

- School has **multiple departments**.



- **Constrained maximization** of the diversity index: Find ξ that achieves $f(\xi) \geq \lambda$ for $\lambda \in \mathbb{R}_+$.

Go to 4

Conclusion

- $\mathrm{M}^\natural$-concavity → Ordinal concavity.
- A choice rule is **substitutable**$^+$ ⇔ rationalized by an **ordinally concave** u.
- The key for designing a stable matching algorithm:
  each agent maximizes an ordinally concave function.
  - Application to choice rules with diversity goals.
- Concavity is crucial in markets with indivisibilities and without transfers.

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