Δ-substitutes and Indivisible Goods

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What

Competitive equilibria (CE) with indivisible goods.

1. New sufficient condition for existence of CE (quasi and non-quasi linear) that supersedes all prior ones (with one exception).
2. Identify prices at which the excess demand for each good is bounded by a preference parameter *independent* of the size of the economy (social approximate equilibrium).

Why

CE outcomes are a benchmark for the design of markets for allocating goods and services.

When they exist they are in the core and can be pareto optimal.

Under certain conditions they satisfy fairness properties like equal treatment of equals and envy-freeness.

Prior Work

Restrict preferences

Kelso & Crawford (1982), Gul & Stachetti (1999), Danilov, Koshevoy & Murota (2001), Sun & Yang (2006), Baldwin & Co. (2020)

Drown indivisibilities in ‘large’ markets.

Azevedo, Weyl & White (2013)

Prior Work

Eliminate the indivisibilities via lotteries.

Hylland & Zeckhauser (1979), Budish et. al. (2013), Gul, Pesendorfer & Zhang (2020)

Approximate CE outcomes based on cardinal notions of welfare; approximations scale slowly with size of economy.

Dobzinski et al (2014), Feldman et al (2014)

Social Approximate Equilibrium; mismatch between supply and demand grows with size of economy.

Broome (1971), Dierker (1970), Starr (1969)

Notation

M = set of indivisible goods.

Bundle of goods is denoted by vector $\mathbf{x} \in \{\mathbf{0}, \mathbf{1}\}^{\mathbf{m}}$.

X a finite set of bundles one can choose from.

Utility for a bundle x and transfer t when endowed with b units of money is denoted $U(x, b - t)$.

Conditions

$U(x, b-t)$ is continuous and **strictly decreasing** in t, $U(\vec{0}, 0) = 0$.

If $x \notin X$, $U(x, b-t) = -\infty$ for all t.

There exists B such that $U(x, b-B) = -\infty$ for all $x \in X$.

Quasi-linearity means $U(x, b-t) = v(x) + b - t$.

Notation

$p \in \mathbb{R}^m$ is a price vector.

Choice correspondence, denoted $Ch(p)$:

$$Ch(p) = \arg\max\{U(x, b - p \cdot x) : x \in X\}.$$

$(x - y)^+$ is vector whose i^{th} component is $\max\{x_i - y_i, 0\}$.

$$||x - y||_1 = \vec{1} \cdot (x - y)^+ + \vec{1} \cdot (y - x)^+.$$

Equilibrium

N set of agents, $s_i \in \mathbb{Z}_+$ the supply of good $i \in M$.

$\vec{s} \in \mathbb{Z}^m_+$ is the supply vector.

$\vec{b}$ the vector of cash endowments.

An economy is the collection $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$.

Equilibrium

A CE for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ is a price vector p and demands $x^j \in Ch_j(p)$ for all $j \in N$ such that $\sum_{j\in N} x^j \leq s$ with equality for each $i \in M$ for which $p_i \neq 0$.

A <u>α-approximate CE</u> for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s}\}$ is a competitive equilibrium for the economy $\{\{U_j\}_{j\in N}, \vec{b}, \vec{s'}\}$, where $|s_i - s_i'| \leq \alpha$ for every good $i \in M$.

Δ-substitutes

For all b and p, if $|Ch_j(p)| \geq 2$, then, for each $x \in Ch_j(p)$ there exists a $y \in Ch_j(p)$ such that $||x - y||_1 \leq \Delta$.

THEOREM: If all agent's preferences satisfy Δ-substitutes, then for every supply vector s the economy $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$ has a $\Delta - 1$-approximate CE.

Δ-Improvement

$U(x, b - p \cdot x)$ satisfies Δ-improvement property if for any price vector p, any two bundles $x, y \in Ch(p)$ and any price change $\delta p \in \mathbb{R}^m$ satisfying $\delta p \cdot x > \delta p \cdot y$, there exist $a \leq (x - y)^+$ and $d \leq (y - x)^+$ such that

1. $\vec{1} \cdot a + \vec{1} \cdot d \leq \Delta$,
2. $\delta p \cdot a > \delta p \cdot d$, and
3. $x - a + d \in Ch(p)$.

Δ-substitutes = Δ-improvement property.

Geometric Substitutes

For all b and p, if $|Ch_j(p)| \geq 2$, then, for each $x \in Ch_j(p)$ there exists a $y \in Ch_j(p)$ such that $x - y$ is a $\{0, \pm 1\}$ vector with at most two non-zero entries and these being of opposite sign.

THEOREM: An economy $\{\{U_j\}_{j \in N}, \vec{b}, \vec{s}\}$ in which each U_j satisfies geometric substitutes has a competitive equilibrium.

Other Sufficient Conditions

Gross Substitutes $\subset$ Geometric Substitutes.

Single improvement $\subset$ Geometric Substitutes.

No complementarities $\subset$ Geometric Substitutes.

Net substitutes $\subset$ Geometric substitutes.

Pseudo-equilibrium

A price vector p and $x^j \in conv(Ch_j(p))$ for all $j \in N$ is called a pseudo-equilibrium if $\sum_{j \in N} x^j \leq s$ with equality for every good $i \in M$ with $p_i \neq 0$.

Rounding Lemma

Polytope P **binary** if all of its extreme points are 0-1 vectors and denote its set of extreme points by $ext(P)$.

Binary polytope P is Δ-**uniform** if the ℓ_1 norm of each of its edges is at most Δ.

Rounding Lemma

Lemma

Let $P_1, \ldots, P_n$ *be a collection of binary polytopes in* $\mathbb{R}^m$ *each of which is* Δ*-uniform.*

Let integral $s \in \sum_{i=1}^n P_i$.

There exist vectors $x^i \in ext(P_i)$ *for all* i *such that* $||s - \sum_1^n x^i||_\infty \leq \Delta - 1$.

Shapley-Folkman-Starr

Let $P_1, \ldots, P_n$ be a collection of binary polytopes in $\mathbb{R}^m$ each with diameter at most d and $n > m$.

Let $s \in \sum_{i=1}^n P_i$.

Version 1: There exist vectors $x^i \in P_i$ for all i of which $n - m$ of them are in $ext(P_i)$ such that $||s - \sum_{i=1}^n x^i||_\infty \leq m$.

Version 2: There exist vectors $x^i \in P_i$ for all i of which $n - m$ of them are in $ext(P_i)$ such that $||s - \sum_{i=1}^n x^i||_2 \leq 0.5d\sqrt{m}$.

Course Allocation

Budish (2011): Social approximate equilibria with approximately equal incomes (A-CEEI).

Endowments of artificial currency are allocated at random.

In expectation, each agent receives the same endowment of artificial currency.

Compute a social approximate CE.

Euclidean distance between supply and demand vector is at most $\frac{\sqrt{\min\{2\Delta, m\}m}}{2}$.

Course Allocation

Ordinal preference of student j is $\succeq_j$.

$v_j(x) \geq 0$ utility function that represents these preferences.

Given a budget of \$1, and price p, the **auxiliary** utility of agent j for bundle x is

$$U_j^{\epsilon}(x, p) := v_j(x) + \min\{0, \log(\frac{1 - p \cdot x}{\epsilon})\}$$

Course Allocation

$Ch_j^{\epsilon}(p)$ denote the choice correspondence of auxiliary utility function.

Let $x \in Ch_j^{\epsilon}(p)$, then, there exists a budget b such that $1 \leq b < 1 + \epsilon$ and

$$x = max_{(\succeq_j)}\{x' \in X_j \text{ and } p \cdot x' \leq b\}.$$

Demand Type

Baldwin & Klemperer: characterize preferences over bundles of indivisible goods in terms of how demand changes in response to a small non-generic price change.

Danilov & Koshevoy (2004), tangent cone

Set of vectors that summarize the possible demand changes is called the **demand type**.

In quasi-linear setting, multiple equivalent definitions.

Discrete analog to the rows of a Slutsky matrix.

Demand Type (Baldwin & Klemperer)

The edges of $conv(Ch(p))$ are its 1-dimensional faces and are vectors of the form $v - u$ for some pair $v, u \in Ch(p)$.

If entries of $v - u$ are scaled so that the greatest common divisor of their entries is 1, we call it a **primitive edge direction**.

Demand Type (Baldwin & Klemperer)

A set $\mathcal{D} \subseteq \mathbb{Z}^m$ is the demand type of an agent if it contains the primitive edge directions of $conv(Ch(p))$ for all price vectors p such that $|Ch(p)| > 1$.

In binary case, Δ-substitutes corresponds to the vectors in the demand type having ℓ_1 norm of at most Δ.

Unimodular Demand Type

Matrix is **unimodular** if determinant of every full rank submatrix has value $0, \pm 1$.

A demand type $\mathcal{D}$ is called unimodular if the matrix of its vectors is unimodular.

Network matrix is a $0, \pm 1$ matrix with at most two non-zero entries in each column and these being of opposite sign.

Gross substitutes/ M$^{\#}$-concave corresponds to demand type being a network matrix.

Unimodular Theorem

Theorem

Suppose each agent has quasi-linear preferences and each $v_i(x)$ is concave in x. If all agent's demand types are unimodular, there exists a price vector p and demands $x^j \in Ch_j(p)$ for all $j \in N$ such that $\sum_{j\in N} x^j \leq s$ with equality for each $i \in M$ for which $p_i \neq 0$.