Recurring Themes in Auction Theory and Mechanism Design

Part I: Virtual Value and Revenue

Dan Quint
U of Wisconsin
Visitor, UTokyo and UTMD

February 13, 2023
University of Tokyo

Overview

- Goal: highlight a few ideas that help us understand lots of results in auction theory/mechanism design
- Today:

> The expected revenue from any sales mechanism is the expected value of the **virtual value** of the buyer who receives the good.
>
> (one interpretation of Myerson (1981), "Optimal Auction Design")

- What does this mean? Why is it true?
- Why does it "make sense"?
- Several other results that follow from it

What's the best way to sell a thing?

Model

- Single object to sell
- Fixed set of n risk-neutral buyers, with valuations v_i
- v_i are independent random variables $v_i \sim F_i$ on $[a_i, b_i]$
- Seller can specify (and commit to) any game for buyers to play, subject to two constraints:
  - Participation is voluntary
  - Players will understand the game and play equilibrium
- What outcomes can seller achieve?
  What game maximizes expected revenue?

R. Myerson (1981), Optimal Auction Design, *Mathematics of Operations Research* 6(1)
J. Riley and W. Samuelson (1981), Optimal Auctions, *American Economic Review* 71(3)

max E(revenue) s.t. participation, equilibrium

literally all possible games

- For any game the seller could design, let
  - $p_i(v_1, v_2, \ldots, v_n)$ be equilibrium probability buyer i gets object
  - $x_i(v_1, v_2, \ldots, v_n)$ be expected payment buyer i makes to seller
- What if instead of playing the game, seller…
  - asked buyers their valuations
  - committed to implementing allocation $p(\ )$ and payments $x(\ )$
- "Revelation principle"
  - any outcome implemented by any game can be implemented by "direct revelation mechanism"
  - WLOG, maximize revenue only over DRMs

$\max_{\text{direct revelation mechanisms}}$ E(revenue) s.t. participation, equilibrium truth-telling

- For any game the seller could design, let
  - $p_i(v_1, v_2, \ldots, v_n)$ be equilibrium probability buyer i gets object
  - $x_i(v_1, v_2, \ldots, v_n)$ be expected payment buyer i makes to seller

- What if instead of playing the game, seller…
  - asked buyers their valuations
  - committed to implementing allocation $p(\ )$ and payments $x(\ )$

- "Revelation principle"
  - any outcome implemented by any game can be implemented by "direct revelation mechanism"
  - WLOG, maximize revenue only over DRMs
  - Constraints: voluntary participation and truth-telling

max E(revenue)
direct revelation mechanisms s.t. participation, truth-telling

- Truth-telling requires two things:
  - Allocation rule p must be "monotone" $(E_{v_{-i}} p_i(v_i, v_{-i})) \uparrow v_i)$
  - Each bidder's equilibrium expected payoff $U_i(v_i)$ must satisfy

$$U_i(v_i) = U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i}) ds$$

- Voluntary participation requires $U_i(v_i) \geq 0$
  - $U_i(\cdot)$ is increasing, so $U_i(a_i) \geq 0$ suffices

- Now…
  - If p determines each buyer's chance of winning given v_i…
  - …and p and $U_i(a_i)$ together determine i's expected payoff…
  - …then p and $\{U_i(a_i)\}_i$ determine seller's expected revenue!

max E(revenue)

p monotone, $\{U_i(a_i)\} \geq 0$

s.t. participation, truth-telling

- Truth-telling requires two things:
  - Allocation rule *p* must be "monotone" $(E_{v_{-i}} p_i(v_i, v_{-i})) \uparrow v_i)$
  - Each bidder's equilibrium expected payoff $U_i(v_i)$ must satisfy

$$U_i(v_i) = U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i}) ds$$

- Voluntary participation requires $U_i(v_i) \geq 0$
  - $U_i(\cdot)$ is increasing, so $U_i(a_i) \geq 0$ suffices
- Now…
  - If *p* determines each buyer's chance of winning given v_i…
  - …and *p* and $U_i(a_i)$ together determine *i*'s expected payoff…
  - …then *p* and $\{U_i(a_i)\}_i$ determine seller's expected revenue!

$\max_{\substack{p \text{ monotone}, \\ \{U_i(a_i)\} \geq 0}}$ E(revenue(p, $\{U_i(a_i)\}$)) s.t. participation, truth-telling

- Truth-telling requires two things:
  - Allocation rule p must be "monotone" $(E_{v_{-i}} p_i(v_i, v_{-i})) \uparrow v_i)$
  - Each bidder's equilibrium expected payoff $U_i(v_i)$ must satisfy

$$U_i(v_i) = U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i}) ds$$

- Voluntary participation requires $U_i(v_i) \geq 0$
  - $U_i(\cdot)$ is increasing, so $U_i(a_i) \geq 0$ suffices

- Now…
  - If p determines each buyer's chance of winning given v_i…
  - …and p and $U_i(a_i)$ together determine i's expected payoff…
  - …then p and $\{U_i(a_i)\}_i$ determine seller's expected revenue!

$$\max_{\substack{p \text{ monotone}, \\ \{U_i(a_i)\} \geq 0}} E(\text{revenue}(p, \{U_i(a_i)\}))$$

Expected Revenue = **Gross Surplus from Allocating Object** – **Expected Surplus Earned by Bidders**

$$= E_v \sum_i v_i p_i(v) - \sum_i E_{v_i}\left(U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i})\, ds\right)$$

- After a bunch of algebra, this is

Expected Revenue $= E_v \sum_i p_i(v)\left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}\right] - \cancel{\sum_i U_i(a_i)}$

Call this buyer i's ***virtual value***

Optimizing seller will set to 0

- Expected revenue from any sales mechanism is **EV of winning bidder's virtual value**

$$\max_{\substack{p \text{ monotone}, \\ \{U_i(a_i)\} \geq 0}} E_v \sum_i p_i(v) \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right]$$

Expected Revenue **=** **Gross Surplus from Allocating Object** **–** **Expected Surplus Earned by Bidders**

$$= E_v \sum_i v_i p_i(v) - \sum_i E_{v_i} \left(U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i}) ds \right)$$

- After a bunch of algebra, this is

Expected Revenue $= E_v \sum_i p_i(v) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] - \cancel{\sum_i U_i(a_i)}$

Call this buyer i's ***virtual value***

Optimizing seller will set to 0

- Expected revenue from any sales mechanism is **EV of winning bidder's virtual value**

$$\max_{p \text{ monotone}} E_v \sum_i p_i(v) \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right]$$

Expected Revenue = **Gross Surplus from Allocating Object** − **Expected Surplus Earned by Bidders**

$$= E_v \sum_i v_i p_i(v) - \sum_i E_{v_i} \left(U_i(a_i) + \int_{a_i}^{v_i} E_{v_{-i}} p_i(s, v_{-i}) ds \right)$$

- After a bunch of algebra, this is

Expected Revenue $= E_v \sum_i p_i(v) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] - \cancel{\sum_i U_i(a_i)}$

Call this buyer i's ***virtual value***

Optimizing seller will set to 0

- Expected revenue from any sales mechanism is **EV of winning bidder's virtual value**

$$\max_{p \text{ monotone}} E_v \sum_i p_i(v) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right]$$

- Recapping the main takeaways:

  - Any *allocation* can be implemented as equilibrium of some sales game as long as it's monotone

  - An allocation is implemented by essentially unique payment rule, so equilibrium allocation uniquely* determines expected revenue

  - **The expected revenue from any sales mechanism is the expected value of virtual value of buyer who gets the object**

- Rest of today: why does this make sense? and what does this buy us?

Is there any nice intuition for "virtual value"?

Expected revenue is $E_v \sum_i p_i(v) \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right]$

- Consider a monopolist facing measure *1* of consumers with willingness-to-pay distributed *F*
  - At price *p*, demand is *q = 1 – F*(*p*), revenue is *p* (*1 – F*(*p*))
  - To sell to "one more buyer," seller must cut price by

$$\frac{1}{-dq/dp} = \frac{1}{f(p)}$$

  - Effect on revenue is

$$-\frac{1}{f(p)} \frac{d}{dp} [p(1-F(p))] = -\frac{1}{f(p)} [1 - F(p) - pf(p)] = p - \frac{1-F(p)}{f(p)}$$

- So $p - \frac{1-F(p)}{f(p)}$ is **marginal revenue** from selling to one more buyer, when that buyer has valuation *p*!

J. Bulow and J. Roberts (1989), The Simple Economics of Optimal Auctions, *Journal of Political Economy* 97(5)

Expected revenue is $E_v \sum_i p_i(v) \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right]$

- Think of virtual value as the incremental revenue from selling to buyer *i* when he has valuation v_i

$$v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

Revenue we can earn from selling to buyer with valuation v_i

Selling to *i* at this valuation lowers what we can get from him when his valuation is higher

Ratio of buyer *i* with "higher valuations" to "this valuation"

What do we get from this formulation of expected revenue?

Expected revenue is $E_v \sum_i p_i(v)\left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right]$

- Corollary 1: revenue equivalence
  - If p and $\{U_i(a_i)\}$ determine expected revenue…
  - …then any two games with same p and $\{U_i(a_i)\}$ have same expected revenue

- Usually stated as:
  - "Suppose bidders have symmetric, independent private values.
  - Define a *standard auction* as any auction where

    (i) the bidder with the highest valuation wins in equilibrium, and

    (ii) a bidder with their lowest possible valuation gets 0 payoff
  - All standard auctions have the same expected revenue."

$$\max_{p \text{ monotone}} E_v \sum_i p_i(v) \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right]$$

- Corollary 2: solution to seller’s problem
  - Without “monotone *p*” constraint, maximize pointwise: for each *v*, give object to whoever has the highest virtual value (if ≥ 0)
  - If $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ is increasing in v_i (F_i is “regular”) for each *i*, this allocation rule is monotone → feasible → optimal
  - In symmetric case, “highest value wins” – optimal mechanism is a second price auction with reserve price!
  - Reserve price is set such that $0 = r - \frac{1-F(r)}{f(r)}$
  - (Prevents sale when winner’s *virtual value* is negative)
  - (Note that optimal reserve does not depend on *n*!)

  - If distributions not regular, optimal mechanism more complicated

What else is this formulation good for?

Revenue rankings of asymmetric auction formats

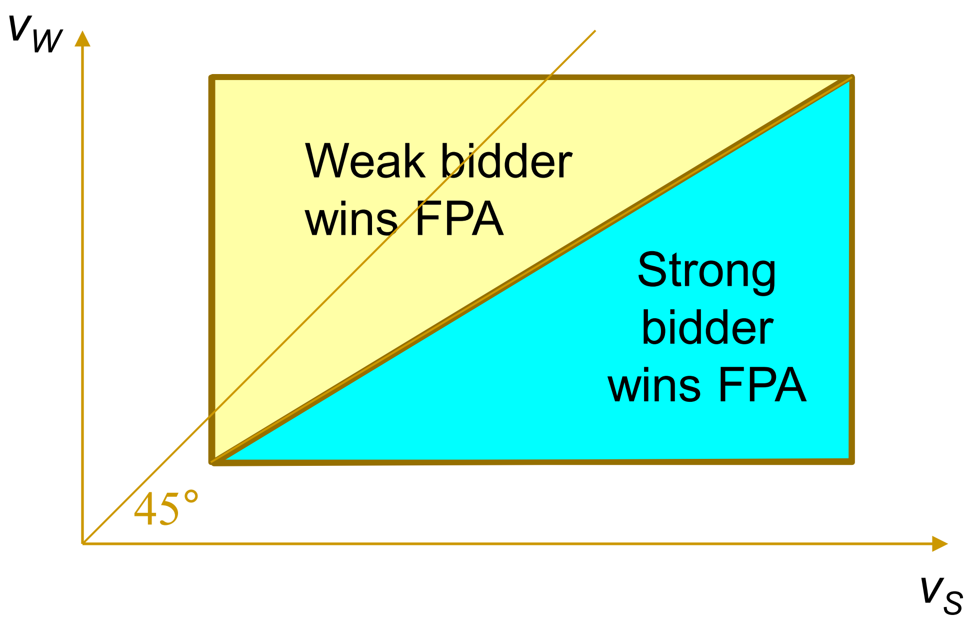
First Price vs Second Price Auctions

- If $F_i = F_j = F$ for all bidders, revenue equivalence
  - FP and SP auctions "equally good"
- If F_i vary across bidders, no longer true
  - Neither is optimal (but optimal auction never used)
  - Which is better?
- Suppose two bidders, $F_S \geq_{FOSD} F_W$
  - SP auction: still a dominant strategy to bid valuation
  - FP auction: equilibrium bids solve pair of diff eq's
  - Strong bidder prefers SP auction, weak bidder prefers FP
  - General revenue ranking is not available

E. Maskin and J. Riley (2000), Asymmetric Auctions, *Review of Economic Studies* 67

Expected revenue is $E_v \sum_i p_i(v)\left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right]$

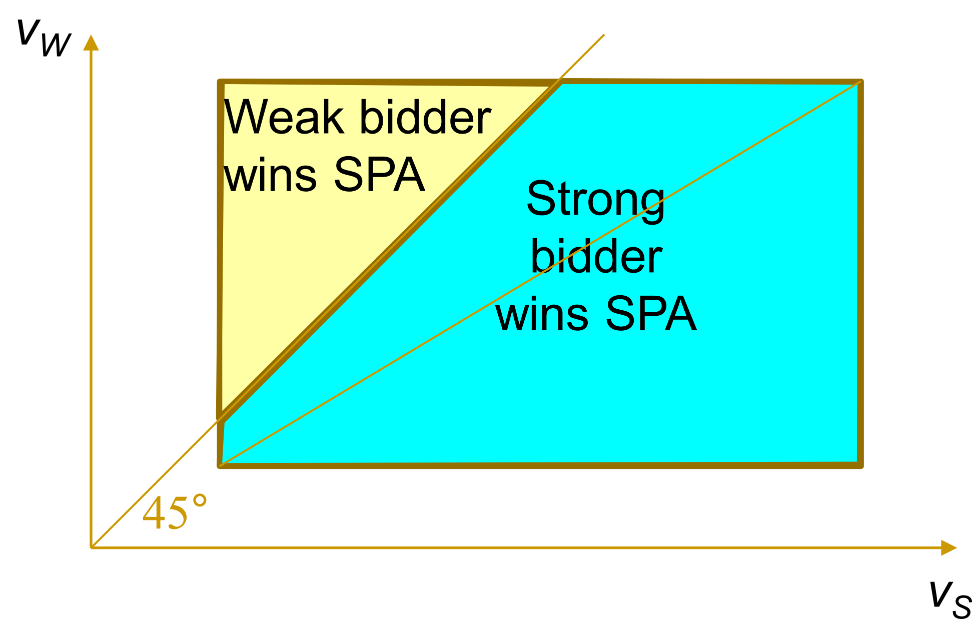
- In FP auction with asymmetric bidders, strong bidder “shades bid more” than weak
- So weak bidder sometimes wins with lower valuation



R. Kirkegaard (2012), A Mechanism Design Approach to Ranking Asymmetric Auctions, *Econometrica* 80(5)

Expected revenue is $E_v \sum_i p_i(v) \left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)} \right]$

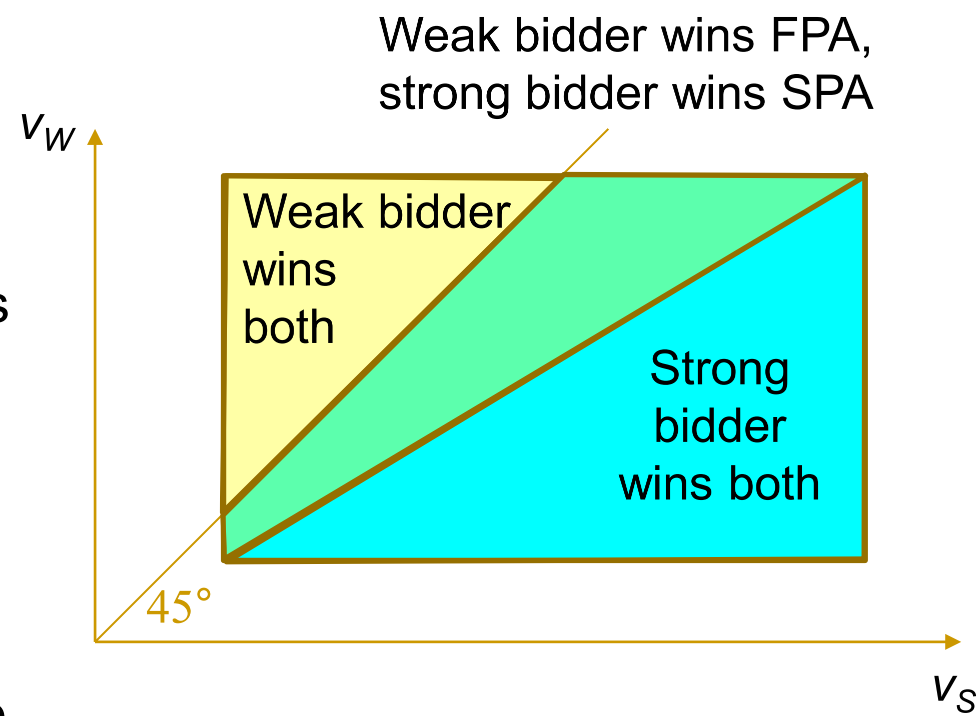
- In SP auction, bidders bid their valuations
- So bidder with higher valuation wins



R. Kirkegaard (2012), A Mechanism Design Approach to Ranking Asymmetric Auctions, *Econometrica* 80(5)

Expected revenue is $E_v \sum_i p_i(v)\left[v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right]$

- Since revenue is EV of winner's VV…
- …revenue ranking depends on who has higher average virtual value in middle region
- (Lets us better understand known special cases, prove some new ones)



R. Kirkegaard (2012), A Mechanism Design Approach to Ranking Asymmetric Auctions, *Econometrica* 80(5)

Optimal search auctions

"Search auctions"

- Suppose seller needs to search for buyers
  - Instead of "*n* buyers ready to go"…
  - …there are *n potential* buyers
  - Costs seller c_i to find buyer *i* and educate them about object
  - Buyer *i* then learns their v_i and can participate in auction
- Seller knows $\{c_i, F_i\}$ for each of *n* potential buyers
- What should seller do?

J. Crémer, Y. Spiegel, and C.Z. Zheng (2007), Optimal Search Auctions, *Journal of Economic Theory* 134

Without private information, optimal search is a solved problem

- "Pandora's problem" (Weitzman 1979)
  - *n* boxes
  - Box *i* has cost c_i to open, contains prize worth $v_i \sim F_i$
  - You know (c_i, F_i) for each box, can open as many as you want in any order, claim any one opened box's prize at any time
  - Solution: calculate index for each box

$$a_i = E\max\{a_i, v_i\} - c_i$$

$$c_i = \int_{a_i}^{\infty} (v_i - a_i) f_i(v_i)\, dv_i$$

  - Open boxes in decreasing index order
  - Stop when best prize so far > highest remaining index

M.L. Weitzman (1979), Optimal Search for the Best Alternative, *Econometrica* 47
J.C. Gittins and D. Jones (1974), A Dynamic Allocation Index for the Sequential Design of Experiments, in *Progress In Statistics*, Ed. J. Gani, North Holland

But here, buyers have private info

- Once you contact a buyer, *they* know v_i, you don't
  - Still need to design a mechanism for them to play
  - How should seller proceed?
- Hint: revenue of a mechanism is EV of VV of winner
- Treat this as a *full-info* search problem, where "prize" is $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$ rather than v_i
  - Let G_i be distribution of bidder *i*'s *virtual* value $v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$
  - Solve Pandora's problem with (c_i, G_i)
  - Run dynamic direct-revelation mechanism: each time a buyer is contacted, ask them their valuation
  - Allocation rule: winner is highest VV when search ends
  - Payment rule is determined by allocation rule

The importance of *n* relative to auction format

How important is auction choice?

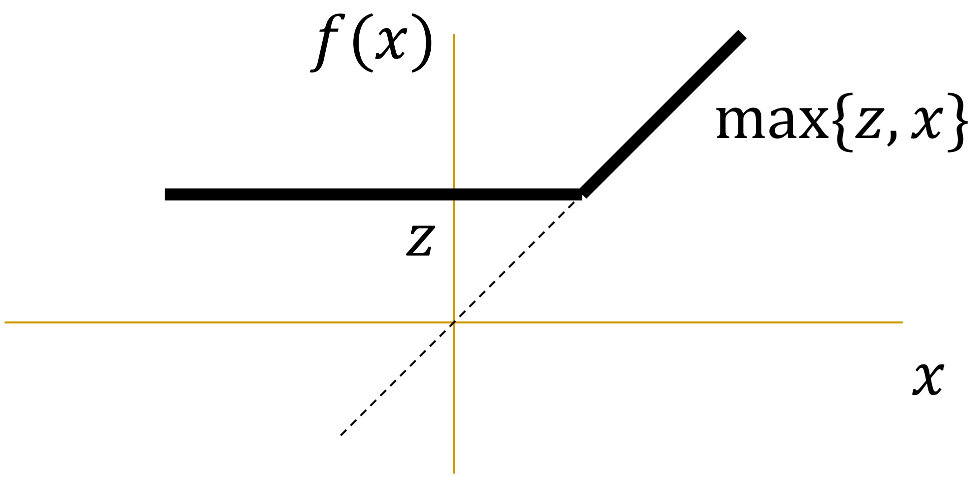
- We've considered revenue maximization with fixed set of bidders and valuation distributions
- In some sense, increasing participation matters more than getting the mechanism right
- Adding *one more bidder* is always more valuable than setting correct reserve price
  - Optimal auction with n symmetric bidders is lower-revenue than "pure auction" (no reserve price) with $n+1$

J. Bulow and P. Klemperer (1996), Auctions Versus Negotiations, *Amer. Economic Review* 86(1)

Mathematical preliminaries

- The max operator is convex
  - For fixed z, $f(x) = \max\{z,x\}$ is a convex function of x
  - By Jensen, $E_X \max\{z, X\} \geq \max\{z, E(X)\}$
  - By iterated expectations, if z is a random variable,

$$E_{X,Z} \max\{Z, X\} \geq E_Z \max\{Z, E(X)\}$$



Mathematical preliminaries

- The max operator is convex
  - For fixed *z*, *f*(*x*) = max{*z*,*x*} is a convex function of *x*
  - By Jensen, $E_X \max\{z, X\} \geq \max\{z, E(X)\}$
  - By iterated expectations, if *z* is a random variable,

$$E_{X,Z} \max\{Z, X\} \geq E_Z \max\{Z, E(X)\}$$

- The expected value of bidder *i*'s virtual value is a_i

$$E(VV_i) = \int_{a_i}^{b_i} \left[v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \right] f_i(v_i) dv_i$$

  - (Turns out to be a_i, trust me on this one) ☺

The result

- Assume symmetric IPV with regular distribution
- Auction with no reserve price, *n*+1 bidders gives revenue

$$E\,\max\{VV_1, VV_2, \ldots, VV_n, VV_{n+1}\}$$

- Optimal auction with *n* bidders gives revenue

$$E\,\max\{VV_1, VV_2, \ldots, VV_n, 0\}$$

- Just showed $E\,\max\{Z, X\} \geq E\,\max\{Z, E(X)\}$, so

$$\begin{aligned} E\,\max\{\underbrace{VV_1, VV_2, \ldots, VV_n}_{Z}, \underbrace{VV_{n+1}}_{X}\} &\geq E\,\max\{VV_1, VV_2, \ldots, VV_n, E(VV_{n+1})\} \\ &= E\,\max\{VV_1, VV_2, \ldots, VV_n, a_{n+1}\} \\ &\geq E\,\max\{VV_1, VV_2, \ldots, VV_n, 0\} \end{aligned}$$

- Adding one bidder is more valuable than learning *F* and running the optimal mechanism!

If you face one buyer and don’t know *F*

What if seller doesn't know *F*?

- With enough buyers, doesn't really matter
- But what if $n = 1$?
- Cool result: if you don't know F but know it's regular, you can get at least ½ the expected revenue of the optimal mechanism if you get *one* sample draw from F
- How? Set posted price equal to the sample draw!

T. Roughgarden (2014), Approximately Optimal Mechanism Design: Motivation, Examples, and Lessons Learned, *ACM SIGEcom Exchanges* 10.2

Guaranteeing half the optimal revenue with one sample draw from *F*

- Let v_0 be outcome of sample draw, demand price $p = v_0$
- You sell whenever actual buyer's valuation v_1 is above v_0
- Expected revenue is $E\{VV_1 \times 1_{v_1 > v_0}\}$
- Since *F* regular, this is $E\{VV_1 \times 1_{VV_1 > VV_0}\}$
- By symmetry, $= \frac{1}{2}E\{VV_1 \times 1_{VV_1 > VV_0}\} + \frac{1}{2}E\{VV_0 \times 1_{VV_0 > VV_1}\}$

$$= \frac{1}{2}E \max\{VV_1, VV_0\}$$

- Recall $E \max\{VV_1, VV_0\}$ is revenue of a two-bidder auction with no reserve…
- …which is more than optimal one-bidder auction
- So if *F* regular, revenue is at least half the optimal revenue!

T. Roughgarden (2014), Approximately Optimal Mechanism Design: Motivation, Examples, and Lessons Learned, *ACM SIGEcom Exchanges* 10.2

Generalizing to other environments

Generalizing to richer environments

- "Marginal revenue maximization" is also used in algorithmic mechanism design at the CS/econ junction
  - "The intuition that profit is optimized by maximizing marginal revenue is a guiding principle in microeconomics.
  - In the classical auction theory for agents with quasi-linear utility and single-dimensional preferences, Bulow and Roberts show that the optimal auction of Myerson is in fact optimizing marginal revenue.
  - In particular Myerson's virtual values are exactly the derivative of an appropriate revenue curve."

S. Alaei, H. Fu, N. Haghpanah, and J. Hartline, The Simple Economics of Approximately Optimal Auctions, available at https://arxiv.org/abs/1206.3541

Generalizing to richer environments

- Now consider a richer setting where…
  - Seller can "serve" multiple buyers,
    faces constraint on which sets of buyers can be served
  - Non-quasilinear preferences (e.g., risk aversion or budget constraints), possible flexibility on other details of "service"
- For a single buyer, find mechanism that maximizes expected revenue for a fixed ex ante probability of service
  - This defines "revenue curve" relating revenue to "quantity," which lets you calculate "marginal revenue" for each buyer at each valuation
  - "Marginal revenue mechanism" then serves set of buyers that maximizes marginal revenue of those served
- Paper shows…
  - condition ("revenue linearity") under which this mechanism is optimal
  - that it is approximately optimal when sufficient condition holds approximately

S. Alaei, H. Fu, N. Haghpanah, and J. Hartline, The Simple Economics of Approximately Optimal Auctions, available at https://arxiv.org/abs/1206.3541

Summing up

Overview

- "Best" way to sell a single object?
  - Revenue of any mechanism determined by equilibrium allocation
  - Any allocation can be implemented if it's monotone
  - Expected revenue is EV of winner's virtual value

- Leads to lots of classic results
  - Revenue equivalence and optimal mechanism
  - Importance of participation over mechanism choice
  - Useful tool for comparing non-optimal auction formats,
    solving sequential search for buyers,
    doing "well enough" without knowing distribution of valuations…
  - …and analogous mechanism can be defined in richer environments

Overview

- Today: how does a seller optimize given a fixed set of buyers with fixed information and fixed valuations
- Tomorrow: pre-auction choices
  - Auctions with endogenous participation…
  - …or endogenous buyer information…
  - …or endogenous pre-auction investments
  - Focus on efficiency rather than revenue maximization
  - **Externalities** give useful lens for simplifying intuition

Thank you!

References

R. Myerson (1981), Optimal Auction Design, *Mathematics of Operations Research* 6(1)

J. Riley and W. Samuelson (1981), Optimal Auctions, *American Economic Review* 71(3)

J. Bulow and J. Roberts (1989), The Simple Economics of Optimal Auctions, *Journal of Political Economy* 97(5)

E. Maskin and J. Riley (2000), Asymmetric Auctions, *Review of Economic Studies* 67

R. Kirkegaard (2012), A Mechanism Design Approach to Ranking Asymmetric Auctions, *Econometrica* 80(5)

J. Crémer, Y. Spiegel, and C.Z. Zheng (2007), Optimal Search Auctions, *Journal of Economic Theory* 134

M.L. Weitzman (1979), Optimal Search for the Best Alternative, *Econometrica* 47

J.C. Gittins and D. Jones (1974), A Dynamic Allocation Index for the Sequential Design of Experiments, in *Progress In Statistics*, Ed. J. Gani, North Holland

J. Bulow and P. Klemperer (1996), Auctions Versus Negotiations, *Amer Econ Review* 86(1)

T. Roughgarden (2014), Approximately Optimal Mechanism Design: Motivation, Examples, and Lessons Learned, *ACM SIGEcom Exchanges* 10.2

S. Alaei, H. Fu, N. Haghpanah, and J. Hartline, The Simple Economics of Approximately Optimal Auctions (short version in FOCS 2013, full version at http://arxiv.org/abs/1206.3541)