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[Appendix]

Unprecedented

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Online Supplementary Appendix

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B Supplementary Results

B.1 Supplementary Results for Section 4.2

Consider the synchronous model and suppose that a public randomization device is not available. We call such a model the *synchronous model without a public randomization device*. Note that the strategy $\sigma_i^{(T,1)}$ is well defined in such a model as well.

First, we show that a public randomization device is not necessary to obtain multiple equilibria when $x_i \leq 1$ for each $i = 1, 2$.

Proposition 5. *Suppose $x_i \leq 1$ for each $i = 1, 2$. For any $p_1, p_2 \in (0, 1]$, there exist $\delta' \in (0, 1)$ and $T' < \infty$ such that if $\delta_i \in (\delta', 1)$ for each $i = 1, 2$ and $T > T'$, then $\sigma^{(T,1)}$ is a PBE in the synchronous model without a public randomization device.*

The proof of this result as well as the next are relegated to Appendix B.4.1.

Second, we consider the case when $x_i > 1$ for some $i = 1, 2$ and demonstrate that there is a region of parameter values such that there are multiple equilibria in the synchronous model without a public randomization device while there is a unique PBE in our main model.

Proposition 6. *Suppose $x_i > 1$. For any $p_{-i} \in (0, 1]$, there exists $\delta' \in (0, 1)$ such that, for any $\delta_i \in (\delta', 1)$, the strategy $\sigma_i^{(T,1)}$ is a best response against $\sigma_{-i}^{(T,1)}$ if and only*

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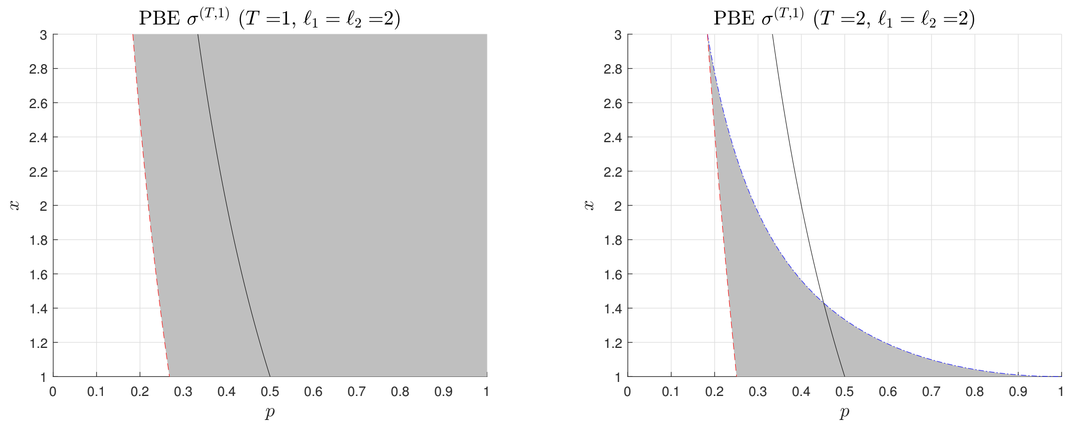


Figure 4: Illustration of Proposition 6 for different values of (p, x). The shaded region depicts the pairs of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors in the synchronous model without a public randomization device. *Left:* $T = 1$. *Right:* $T = 2$. In both panels, we set $\ell_1 = \ell_2 = 2$. The dashed and dashed-dotted curves illustrate the constraints given by (A.1) and (A.2), respectively. The solid curve illustrates the threshold below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves.

if the following two conditions hold:

$$(1-p_{-i})^T x_i \geq (1-(1-p_{-i})^T)(-\ell_i) + (1-p_{-i})^T \left(T + (1-p_{-i})^T x_i\right); \qquad \text{(A.1)}$$

$$\frac{x_i - 1}{x_i} < (1-p_{-i})^T. \qquad \text{(A.2)}$$

Letting $p = p_1 = p_2$ and $x = x_1 = x_2 \geq 1$, Figure 4 uses Proposition 6 to depict the set of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors for a fixed T. Figure 5 then depicts the set of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors for some T. Both figures also depict the threshold probability $p = \frac{2}{1+x+\ell}$ below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves.

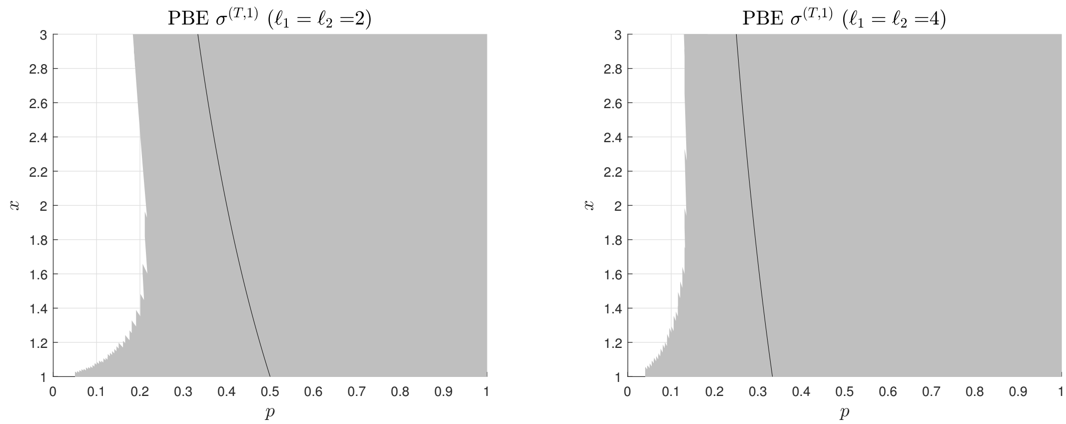


Figure 5: Illustration of Proposition 6 for different values of (p, x). The shaded region depicts the pairs of (p, x) such that $\sigma^{(T,1)}$ is a PBE for sufficiently high discount factors for some T in the synchronous model without a public randomization device. *Left:* $\ell_1 = \ell_2 = 2$. *Right:* $\ell_1 = \ell_2 = 4$. In both panels, the solid curve illustrates the threshold below which σ^{G} is a unique PBE for sufficiently high discount factors in our main model with asynchronous moves. By Proposition 5, the shaded area would occupy the entire region (except at $p = 0$) when $x \leq 1$.

B.2 Supplementary Discussions for Section 4.4

We provide two examples in which the assumptions made in Section 4.4 are violated and the conclusion of Theorem 2 does not hold. In the first example, the assumption on payoffs is violated. The second example pertains to the assumption on the evolution of action sets.

Example 1. Consider the symmetric normal-form game as depicted in Table 2. Suppose $\delta_1 = \delta_2 =: \delta$.

	A	B	C
A	$2, 2$	$-2, 3$	$-2, 3$
B	$3, -2$	$-1, -1$	$0, 1$
C	$3, -2$	$1, 0$	$-2, -2$

Table 2: The payoff matrix of the stage game in Example 1

Note that this game does not satisfy our conditions on the payoffs: If it were to satisfy the conditions, then, since $u_i(A, A) > u_i(B, B) > u_i(C, C)$ for each i, we must have $(a^1, a^2, a^3) = (A, B, C)$ by condition (A). However, we have $u_i(C, B) \geq u_i(C, C)$, violating condition (B).

Suppose, in contrast, that the evolution of action sets satisfies the assumption specified in this subsection, where the actions are ordered as $(a^1, a^2, a^3) = (A, B, C)$. We assume there is $p \in (0, 1)$ such that $p^i_{k,k'} = p$ for each i, k, and k'.

The following strategy profile, in which the players play A forever on the path of play, is a PBE when $\delta \geq \frac{1}{2}$: (i) player i plays A as long as only A has been taken; (ii) i plays C if i has privately learned C and $-i$'s current action is B; (iii) otherwise, i plays B. See Appendix B.4.2 for the proof of this claim.

The following strategy profile σ^* does not play A forever and constitutes a PBE when δ is large and p is small.

- At a history at which player i has taken C, i plays C (this is her only choice).
- At a history at which player i has not taken C:
  - if the opponent has taken C, then player i takes B.
  - if the opponent has not taken C either, then:
    - if action C is available to player i, then i takes C.
    - if action C is not available to player i, then:
      - if the opponent has played B, then player i takes B.
      - if the opponent has not played B, then player i takes A.

Under this strategy profile, the players play C as soon as it becomes available. The reason why this constitutes an equilibrium is that in the small game that excludes action A, there are two Pareto-unranked Nash equilibria: (B, C) and (C, B), where each player prefers the one in which she plays C. Given that the opponent $-i$ will play C as soon as possible, it is i's best response to play C as soon as possible. This sort of construction depends on the multiple Pareto-unranked equilibria in the small game, and our assumption on the payoff functions excludes such a possibility. Appendix B.4.2 formally shows that the above strategy profile is a PBE. □

Example 2. Consider the symmetric normal-form game as depicted in Table 3. Suppose $\delta_1 = \delta_2 =: \delta$.

Note that this game satisfies the conditions on the payoffs, where the actions are ordered as $(a^1, a^2, a^3) = (A, B, C)$.

Suppose, however, that the evolution of action sets does not satisfy our condition but is given as follows.

	A	B	C
A	$2, 2$	$-2, 3$	$-3, 4$
B	$3, -2$	$1, 1$	$-1, 2$
C	$4, -3$	$2, -1$	$0, 0$

Table 3: The payoff matrix of the stage game in Example 2

1. Each player's action set is $\{A\}$ at the beginning.

2. At each period, with probability $p \in (0, 1)$, each player's action set $\{A\}$ becomes $\{A, B, C\}$.

3. Once i plays B or C, her action set becomes $\{B, C\}$ forever.

4. If i has been always playing A and $-i$ has played B or C in the past, then i's action set is $\{A, B, C\}$.

The following strategy profile, in which the players play A forever on the path of play, is a PBE when $\delta \geq \frac{1}{2}$: Play A as long as only A has been taken; otherwise, play C. See Appendix B.4.2 for the proof of this claim.

We argue that the following strategy profile σ^* does not play A forever and is a PBE when the players are sufficiently patient. For each i, construct a PBE strategy profile $\sigma^{(i)}$ and denote by $u_j^{(i)}$ the payoff this strategy profile yields player j. The standard argument shows that, if the players are sufficiently patient, we can construct $(\sigma^{(i)})_{i=1,2}$ satisfying $u_1^{(1)} > u_1^{(2)}$ and $u_2^{(2)} > u_2^{(1)}$. Now we define σ^* as follows:

- Player i plays A if it is the only available action.

- Player i plays C if C is available to i and no player has taken C in the past.

- Once some player has taken C, the players play $\sigma^{(j)}$ forever, where j is the first player who has taken C.

We show in Appendix B.4.2 that σ^* is a PBE when the players are sufficiently patient. The reason why this constitutes an equilibrium is analogous to the analysis of the reversible model in Section 4.1 because the "subgame" after each player chooses an action other than A is that of a reversible model. Recall that the reversible model has multiple equilibria due to the flexibility of action switches. Our assumption on the evolution of action sets in this section shuts down such flexibility of action switches. $\square$

B.3 Supplementary Results for Section 4.6

We provide formal statements of the alternative characterization of PBE discussed in Section 4.6 and comparative-statics results based on the characterization. In our comparative statics, we show that the set of the profiles of discount factors (δ_1, δ_2) under which σ^{G} is a unique PBE is weakly decreasing in p_i, x_i and ℓ_i (in the set-inclusion sense). To that end, we denote by

$$S = S(p_1, p_2, x_1, x_2, \ell_1, \ell_2)$$

the set of profiles of discount factors $(\delta_1, \delta_2) \in (0,1)^2$ such that σ^{G} is a unique PBE for given parameters $(p_1, p_2, x_1, x_2, \ell_1, \ell_2)$.

The results are stated for each of the following three cases that are exhaustive besides nongeneric cases: (i) $x_i > 1$ for each $i = 1, 2$; (ii) $x_i < 1$ for each $i = 1, 2$; and (iii) $x_i > 1 > x_{-i}$ for some $i = 1, 2$. The proofs are relegated to Appendix B.4.3.

First, we start with the case in which $x_i > 1$ for each $i = 1, 2$.

Proposition 7. *Fix $p_1, p_2 \in [0,1]$. Let $\delta_1, \delta_2 \in (0,1)$. Suppose $x_i > 1$ for each $i = 1, 2$. Then, $0 < \hat{\delta}_i < \overline{\delta}_i(p_{-i})$ for each $i = 1, 2$. Moreover, the following hold.*

1. *If there is $i = 1, 2$ such that $\delta_i > \hat{\delta}_i$ and $\delta_{-i} > \overline{\delta}_{-i}(p_i)$, then σ^{G} is a unique PBE.*
2. *If $\delta_i \in (\hat{\delta}_i, \overline{\delta}_i(p_{-i}))$ for each $i = 1, 2$, then both σ^{G} and σ^N are PBE.*
3. *If there is $i = 1, 2$ such that $\delta_i \in (0, \hat{\delta}_i)$ and $\delta_{-i} \in (\hat{\delta}_{-i}, \overline{\delta}_{-i}(p_i))$, then σ^N is a unique PBE.*
4. *If there is $i = 1, 2$ such that $\delta_i \in (0, \hat{\delta}_i)$ and $\delta_{-i} \in (\overline{\delta}_{-i}, 1)$, then $(\sigma_i^N, \sigma_{-i}^{\mathrm{G}})$ is a unique PBE.*

This result is illustrated in the left panel of Figure 2 in Section 4.6 of the main text. Figure 6 illustrates the characterization of PBE for different values of (p_{-i}, δ_i). For the symmetric cases, the left panel of Figure 3 in Section 4.6 of the main text illustrates the characterization. The specific shapes and cutoff values in these graphs can be obtained by algebra (the same comment applies to the other figures to be introduced in this subsection).

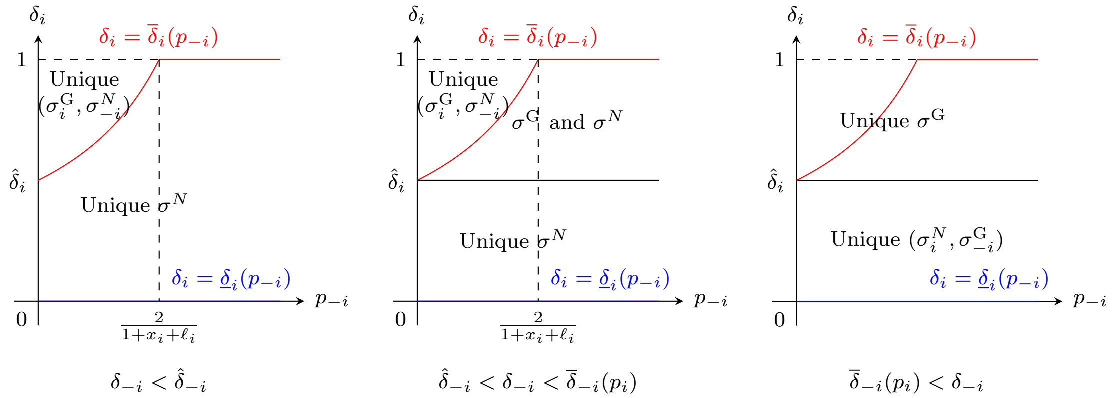


Figure 6: Illustration of Proposition 7 for different values of (p_{-i}, δ_i). For example, "Unique σ^{G}" means that σ^{G} is a unique PBE in the interior of the corresponding region. Also, "σ^{G} and σ^N" means that both σ^{G} and σ^N are a PBE when the parameters satisfy the coditions in the corresponding region. The same comment applies to the subsequence figures. *Left*: The case with $\delta_{-i} < \hat{\delta}_{-i}$. *Center*: The case with $\hat{\delta}_{-i} < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

Although the proposition above does not cover the cases of some knife-edge parameter values, the equilibrium characterization in such cases can be easily obtained by using the upper-hemicontinuity of the set of PBE.[1]

Proposition 7 implies that, when $x_i > 1$ for each $i = 1, 2$, we have

$$S = \{(\delta_1, \delta_2) \in (0,1)^2 \mid \delta_i > \hat{\delta}_i \text{ and } \delta_{-i} > \overline{\delta}_{-i}(p_i) \text{ for some } i = 1, 2\}. \tag{A.3}$$

Now, we provide the comparative-statics result.

Remark 1. Suppose $x_j > 1$ for each $j = 1, 2$. Fix $i = 1, 2$.

1. If $p_i \geq p_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i', p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.
2. If $x_i \geq x_i' > 1$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i', x_{-i}, \ell_i, \ell_{-i})$.
3. If $\ell_i \geq \ell_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell_i', \ell_{-i})$.

That is, the uniqueness of PBE holds for a wider range of discount factors when private learning is less likely, the instantaneous gain from deviation is smaller, or the loss from the opponent deviating is smaller. These conditions are intuitive.

[1] For example, if $\delta_1 = \hat{\delta}_1$ and $\delta_2 > \overline{\delta}_2$, parts 1 and 4 of Proposition 7 imply that both σ^{G} and $(\sigma_1^N, \sigma_2^{\mathrm{G}})$ are PBE.

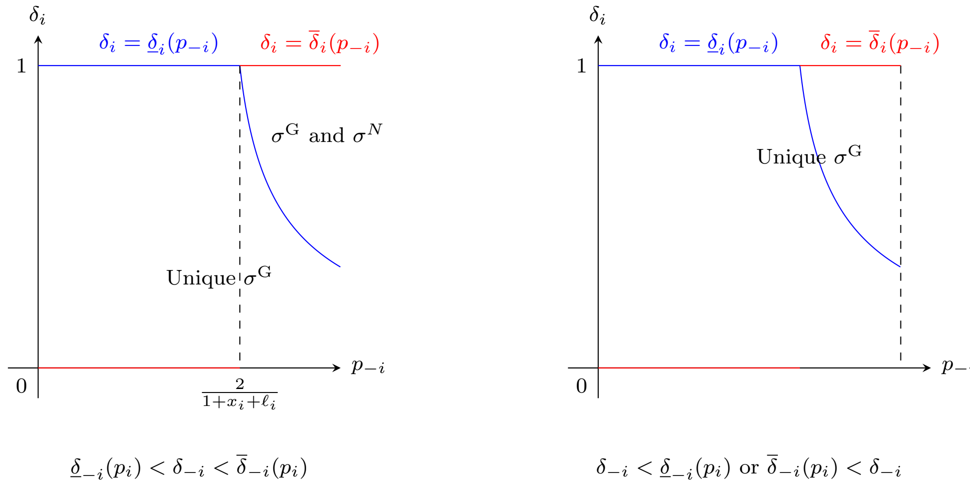


Figure 7: Illustration of Proposition 8 for different values of (p_{-i}, δ_i). *Left*: The case with $\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

Second, we consider the case in which $x_i < 1$ for each $i = 1, 2$.

Proposition 8. *Fix $p_1, p_2 \in [0, 1]$. Let $\delta_1, \delta_2 \in (0, 1)$. Suppose $x_i < 1$ for each $i = 1, 2$.*

1. *If $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$ for each $i = 1, 2$, then both σ^{G} and σ^N are PBE.*
2. *If $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$ for some $i = 1, 2$, then σ^{G} is a unique PBE.*

The central panel of Figure 2 in Section 4.6 of the main text depicts this result when p_i satisfies $\underline{\delta}_{-i}(p_i) \leq \overline{\delta}_{-i}(p_i)$ for each $i = 1, 2$.[2] Figure 7 illustrates Proposition 8 for different values of (p_{-i}, δ_i). For the symmetric cases, the right panel of Figure 3 in Section 4.6 of the main text depicts the characterization for different values of (p, δ).

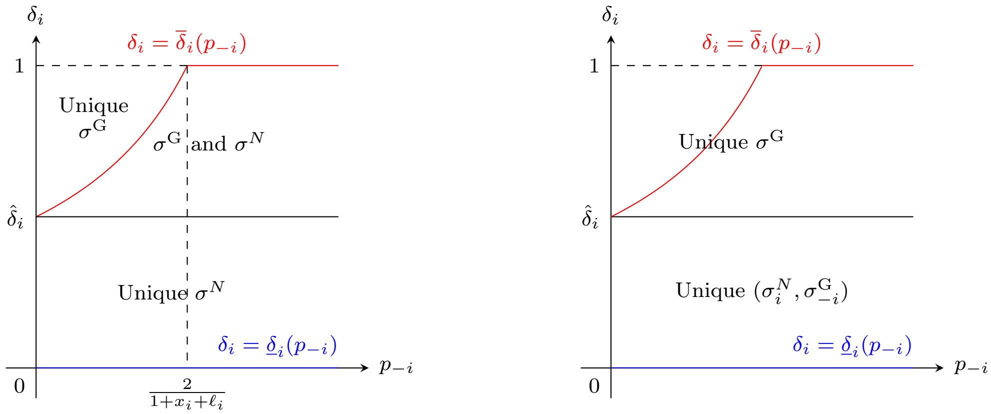
Proposition 8 implies that, when $x_i < 1$ for each $i = 1, 2$, we have

$$S = \{(\delta_1, \delta_2) \in (0, 1)^2 \mid \delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})] \text{ for some } i = 1, 2\}. \tag{A.4}$$

We obtain the same comparative statics as before, as follows:

Remark 2. Suppose $x_j < 1$ for each $j = 1, 2$. Fix $i = 1, 2$.

[2] If $\overline{\delta}_{-i}(p_i) < \underline{\delta}_{-i}(p_i)$, then $\overline{\delta}_{-i}(p_i) = 0$ and $\underline{\delta}_{-i}(p_i) = 1$. Thus, σ^{G} is a unique PBE for any (δ_1, δ_2).



$\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$ $\quad\quad$ $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$

Figure 8: Illustration of Proposition 9 for different values of (p_{-i}, δ_i). *Left*: The case with $\underline{\delta}_{-i}(p_i) < \delta_{-i} < \overline{\delta}_{-i}(p_i)$. *Right*: The case with $\delta_{-i} < \underline{\delta}_{-i}(p_i)$ or $\overline{\delta}_{-i}(p_i) < \delta_{-i}$.

1. If $p_i \geq p_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i', p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.

2. If $x_i \geq x_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i', x_{-i}, \ell_i, \ell_{-i})$.

3. If $\ell_i \geq \ell_i'$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell_i', \ell_{-i})$.

Finally, we consider the case in which $x_i > 1 > x_{-i}$ for some $i = 1, 2$.

Proposition 9. *Fix $p_1, p_2 \in [0,1]$. Let $\delta_1, \delta_2 \in (0,1)$. Suppose $x_i > 1 > x_{-i}$ for some $i = 1, 2$.*

1. *Suppose $\delta_i > \hat{\delta}_i$.*

   (a) *If $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$ and $\delta_i < \overline{\delta}_i(p_{-i})$, then both σ^{G} and σ^N are PBE.*

   (b) *If $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$ or $\delta_i > \overline{\delta}_i(p_{-i})$, then σ^{G} is a unique PBE.*

2. *Suppose $\delta_i < \hat{\delta}_i$.*

   (a) *If $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$, then σ^N is a unique PBE.*

   (b) *If $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$, then $(\sigma_i^N, \sigma_{-i}^{\mathrm{G}})$ is a unique PBE.*

The right panel of Figure 2 in Section 4.6 of the main text illustrates this proposition. The figure depicts the case in which $\underline{\delta}_{-i}(p_i) \leq \overline{\delta}_{-i}(p_i)$. Figure 8 illustrates PBE for different values of (p_{-i}, δ_i).

Proposition 9 implies that, fixing $i = 1, 2$ with $x_i > 1 > x_{-i}$, we have

$$S = \{(\delta_1, \delta_2) \in (0,1)^2 \mid \delta_i > \hat{\delta}_i \text{ and } (\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)] \text{ or } \delta_i > \overline{\delta}_i(p_{-i}))\}. \quad \text{(A.5)}$$

Again, we obtain the following comparative statics:

Remark 3. Suppose there exists $j = 1, 2$ such that $x_j > 1 > x_{-j}$. For each $i = 1, 2$, the following hold.

1. If $p_i \geq p'_i$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p'_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i})$.
2. If $x_i \geq x'_i > 1$ or $1 > x_i \geq x'_i$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x'_i, x_{-i}, \ell_i, \ell_{-i})$.
3. If $\ell_i \geq \ell'_i$, then $S(p_i, p_{-i}, x_i, x_{-i}, \ell_i, \ell_{-i}) \subseteq S(p_i, p_{-i}, x_i, x_{-i}, \ell'_i, \ell_{-i})$.

B.4 Proofs for Supplementary Results

B.4.1 Proofs for Appendix B.1

Proof of Proposition 5. Suppose $\overline{q} = 1$. The proof of Proposition 2 goes through up to (6) under $\overline{q} = 1$. Now, (6) reduces to

$$x_i \leq \frac{1 - \delta_i^s}{1 - \delta_i} + \delta_i^s \min\{x_i, 0\},$$

which is equivalent to

$$\max\{(1 - \delta_i^s)x_i, x_i\} \leq \frac{1 - \delta_i^s}{1 - \delta_i}. \quad \text{(A.6)}$$

Now, suppose that $x_i \leq 1$ for each $i = 1, 2$. Then, the left-hand side of (A.6) is no greater than 1. Also, the right-hand side of (A.6) is no less than $\frac{1-\delta_i^1}{1-\delta_i} = 1$ because it is increasing in s. Hence, (A.6) holds. This implies that the continuation payoff from playing O is no less than the one from playing N. $\square$

Proof of Proposition 6. We consider two cases.

Case 1. Consider any history $h_{i,t} \in \overline{H}_{i,t}$ such that $t = n \cdot T$ for some $n = 1, 2, \ldots$. The continuation payoff from N is $(1 - p_{-i})^T (1 - \delta_i) x_i$. The continuation payoff from O is

$$(1 - (1 - p_{-i})^T)(1 - \delta_i)(-\ell_i) + (1 - p_{-i})^T((1 - \delta_i^T) \cdot 1 + \delta_i^T (1 - \delta_i)(1 - p_{-i})^T x_i).$$

Thus, the continuation payoff from N is no less than the one from O if and only if

$$(1-p_{-i})^T(1-\delta_i)x_i \geq (1-(1-p_{-i})^T)(1-\delta_i)(-\ell_i)+(1-p_{-i})^T((1-\delta_i^T)\cdot 1+\delta_i^T(1-\delta_i)(1-p_{-i})^T x_i),$$

or

$$(1-p_{-i})^T x_i \geq (1-(1-p_{-i})^T)(-\ell_i) + (1-p_{-i})^T \left(\frac{1-\delta_i^T}{1-\delta_i} + \delta_i^T(1-p_{-i})^T x_i\right).$$

There exists $\delta'' < 1$ such that this holds for all $\delta_i > \delta''$ if and only if:

$$(1-p_{-i})^T x_i \geq (1-(1-p_{-i})^T)(-\ell_i) + (1-p_{-i})^T \left(T + (1-p_{-i})^T x_i\right),$$

which we obtain by letting $\delta_i \to 1$. This is condition (A.1).

Case 2. Consider any history $h_{i,t} \in \overline{H}_{i,t}$ such that $t \neq n \cdot T$ for any $n = 1, 2, \ldots$. Let $s \in \{1, \ldots, T-1\}$ be such that the current period is $t = nT - s$ for some $n = 1, 2, \ldots$. If i plays N at $h_{i,t}$, then her continuation payoff is $(1-\delta_i)x_i$. If she plays O instead, then her continuation payoff is

$$(1-\delta_i^s)\cdot 1 + \delta_i^s(1-p_{-i})^T(1-\delta_i)x_i. \tag{A.7}$$

Notice that this is a convex combination of 1 and $(1-p_{-i})^T(1-\delta_i)x_i$.

Case 2-1. If $1 \leq (1-p_{-i})^T(1-\delta_i)x_i$, then (A.7) is no greater than $(1-p_{-i})^T(1-\delta_i)x_i$. Thus, the continuation payoff from O is no less than the one from N only if

$$(1-p_{-i})^T(1-\delta_i)x_i \geq (1-\delta_i)x_i,$$

which is equivalent to $(1-p_{-i})^T \geq 1$, a contradiction. Hence, we need $1 > (1-p_{-i})^T(1-\delta_i)x_i$ for $\sigma^{(T,1)}$ to be a PBE.

Case 2-2. If $1 > (1-p_{-i})^T(1-\delta_i)x_i$, then (A.7) is minimized at $s = 1$, and the minimized value is

$$(1-\delta_i)\cdot 1 + \delta_i(1-p_{-i})^T(1-\delta_i)x_i.$$

Thus, at any history $h_{i,t}$ that we consider in Case 2, the continuation payoff from O is no less than the one from N if and only if

$$(1-\delta_i)x_i \leq (1-\delta_i)\cdot 1 + \delta_i(1-p_{-i})^T(1-\delta_i)x_i,$$

which is equivalent to

$$x_i \leq 1 + \delta_i(1-p_{-i})^T x_i,$$

or

$$(1-p_{-i})^T \geq \frac{x_i - 1}{\delta_i x_i}.$$

Combining with $1 > (1-p_{-i})^T(1-\delta_i)x_i$, we need

$$\frac{x_i - 1}{\delta_i x_i} \leq (1-p_{-i})^T < \frac{1}{(1-\delta_i)x_i}.$$

There exists $\delta'' < 1$ such that this holds for all $\delta_i > \delta''$ if and only if:

$$\frac{x_i - 1}{x_i} < (1-p_{-i})^T,$$

which we obtain by letting $\delta_i \to 1$. This is condition (A.2).

□

B.4.2 Proofs for Appendix B.2

Proof for Example 1. For the first strategy profile, we show that this constitutes a PBE if $\delta \geq \frac{1}{2}$. We consider player i's incentives. If $-i$ has been taking A and A is available to i, then i's following the given strategy yields a payoff of 2. If i deviates, her payoff is at most $(1-\delta)\cdot 3 + \delta \cdot 1$. This is no greater than 2 if $\delta \geq \frac{1}{2}$. After any history in which some player has taken B or C, following the given strategy is weakly better than following any other strategy at any period for any realization of private learning against the opponent's given strategy.

Now we show that the strategy profile σ^* is a PBE. First, at a history at which player i has taken C, the only available action to her is C. Thus, below we consider a history at which player i has not taken C.

Second, suppose that player $-i$ has taken C. Since player $-i$ will keep taking

C in the future, it is player i's best response to take B. For this reason, below we assume that no player has taken C.

Third, suppose that action C is available to player i. If her opponent $-i$ has taken B (note that this implies that player $-i$ has taken B in the last period), then it is player i's best response to follow the prescribed strategy to obtain the maximum continuation payoff (after the opponent takes action B) of 1. Thus, assume that player $-i$ has taken only A. Now, if player i takes C, then her continuation payoff, which we denote by V_C, is $V_C = (1-\delta)\cdot 3 + \delta \cdot 1$. If player i takes A (this means that both players have been taking A), then her continuation payoff, which we denote by V_A, is

$$V_A = (1-\delta)\cdot 2 + \delta\left(\tilde{p}\left((1-\delta)(-2) + \delta\cdot 0\right) + (1-\tilde{p})\left((1-\delta)\cdot 2 + \delta \cdot V_C\right)\right),$$

where $\tilde{p}$ is the probability that player $-i$ takes C in the next period, which is a convex combination of $p^{-i}_{A,C}$ and $p^{-i}_{B,C}$ and thus is p. Hence, in the limit as $\delta \to 1$, the right-hand side becomes $(1-p)V_C$. Thus, there is $\delta' \in (0,1)$ such that if $\delta > \delta'$ then $V_C \geq V_A$ and thus the deviation is not profitable.

If player i takes B instead, then her continuation payoff, which we denote by V_B, is

$$V_B = (1-\delta)\cdot 3 + \delta\left(p\cdot 0 + (1-p)((1-\delta)(-1) + \delta\cdot 1)\right).$$

We have $V_B \leq V_C$ because V_B is a convex combination of 3 and a term which is less than 1, V_C is a convex combination of 3 and 1, and the weights on the convex combinations are the same.

Fourth, assume that action C is not available to player i and that the opponent has played B. If player i follows the prescribed strategy, then letting $\tilde{q}$ be the probability that player $-i$ takes C in the next period, her continuation payoff, which we denote by W_B, satisfies:

$$W_B = (1-\delta)(-1) + \delta\left(\tilde{q}\cdot 0 + (1-\tilde{q})(p\cdot 1 + (1-p)W_B)\right).$$

If A is available to player i and she takes A, then her continuation payoff is

$$(1-\delta)(-2) + \delta\left(\tilde{q}((1-\delta)(-2) + \delta\cdot 0) + (1-\tilde{q})W_B\right).$$

To see that such a deviation is not profitable, for any $\tilde{q}$, it is enough to show that

$p \cdot 1 + (1-p)W_B \geq W_B$, that is, $1 \geq W_B$. This inequality follows because player i's maximum possible continuation payoff (after the opponent takes action B) is 1.

Fifth, suppose that action C is not available to player i and player $-i$ has been taking A. If the only available action to player i is A, then she takes A. Suppose action B is available to player i. Note that the probability that player $-i$ takes C in the next period is p. If player i follows the prescribed strategy, then her continuation payoff, which we denote by X_A, satisfies:

$$
\begin{aligned}
X_A &= (1-\delta)2 + \delta\left(p((1-\delta)(-2) + \delta \cdot 0) + (1-p)((1-\delta)2 + \delta\left(p \cdot V_C + (1-p)X_A\right))\right) \\
&= (1-\delta)2 + \delta\left(p(1-\delta)(-2) + (1-p)((1-\delta)2 + \delta\left(p \cdot ((1-\delta)3 + \delta \cdot 1) + (1-p)X_A\right))\right).
\end{aligned}
$$

Hence, the limit of X_A as $\delta \to 1$, which we denote by X_A^*, satisfies

$$X_A^* = (1-p)p + (1-p)^2 X_A^*.$$

Thus,

$$X_A^* = \frac{(1-p)p}{1-(1-p)^2}.$$

If player i instead takes B, then her continuation payoff, which we denote by X_B, is

$$X_B = (1-\delta) \cdot 3 + \delta Y_B,$$

where Y_B is the continuation payoff when the latest action profile is (B, A) or (B, B), it is $-i$'s turn to move, and i has not privately learned C:

$$Y_B = p \cdot 0 + (1-p)\left((1-\delta)(-1) + \delta\left(p \cdot 1 + (1-p)\left((1-\delta)(-1) + \delta Y_B\right)\right)\right).$$

Hence, the limit of Y_B as $\delta \to 1$, which we denote by Y_B^*, satisfies

$$Y_B^* = (1-p)p + (1-p)^2 Y_B^*.$$

Thus,

$$Y_B^* = \frac{(1-p)p}{1-(1-p)^2}.$$

Hence, the limit of X_B as $\delta \to 1$, which we denote by X_B^*, satisfies

$$X_B^* = Y_B^*.$$

Thus, we have $X_A^* = X_B^*$. Now, a tedious calculation shows that

$$\left.\frac{d(X_A - X_B)}{d\delta}\right|_{\delta=1} = 2 + \frac{4}{2-p} - \frac{3}{p}.$$

Note that this is increasing in p and is negative for a sufficiently small p. Hence, there exist $\delta'' \in (0,1)$ and $p' \in (0,1)$ such that $X_A > X_B$ for all $\delta > \delta''$ and $p < p'$.

In sum, there exist $\overline{\delta} \in (0,1)$ and $\overline{p} \in (0,1)$ such that the given strategy profile constitutes a PBE for all $\delta \in (\overline{\delta}, 1)$ and $p_{A,C} = p_{B,C} \in (0, \overline{p})$. □

Proof for Example 2. For the first strategy profile, we show that this constitutes a PBE if $\delta \geq \frac{1}{2}$. We consider player i's incentives. If player i can choose her action from $\{A, B, C\}$ because she has privately learned B and C and the opponent has been taking A, then she receives a payoff of 2 as long as she follows the strategy. If she deviates by taking C, then she receives a payoff of $(1-\delta)4$ (if she deviates by taking B, her payoff is bounded by $(1-\delta)3 < (1-\delta)4$). Thus, she plays A if

$$(1-\delta) \cdot 4 \leq 2, \text{ that is, } \delta \geq \frac{1}{2}.$$

If player i can choose her action from $\{A, B, C\}$ because the opponent has taken B or C in the past, then it is player i's best response to always choose C.

If player i's available action set is $\{B, C\}$, then it is player i's best response to always choose C.

Second, we show that the second strategy profile σ^* is a PBE when the players are sufficiently patient. We consider two cases. Suppose first that both players have been taking only A. Player i plays C when she privately learns it if

$$(1-\delta)4+\delta u_i^{(i)} \geq (1-\delta)2+\delta\left(p\left\{(1-\delta)(-3)+\delta u_i^{(-i)}\right\}+(1-p)\left\{(1-\delta)2+\delta\left((1-\delta)4+\delta u_i^{(i)}\right)\right\}\right),$$

or,

$$(1-\delta^2(1-p))\left((1-\delta)4+\delta u_i^{(i)}\right) \geq (2-p)(1-\delta)2+\delta p\left\{(1-\delta)(-3)+\delta u_i^{(-i)}\right\}.$$

When $\delta = 1$, since $p > 0$, the above inequality holds because it reduces to $u_i^{(i)} > u_i^{(-i)}$. Thus, there exists $\overline{\delta} \in (0, 1)$ such that the above inequality holds when $\delta \in (\overline{\delta}, 1)$.

Next, suppose that some player has taken B in the past and that C has never been taken. Player i plays C when she privately learns it if

$$(1-\delta)2 + \delta u_i^{(i)} \geq (1-\delta)2 + \delta\left((1-\delta)(-1) + \delta u_i^{(-i)}\right),$$

where the left-hand side is a lower bound of the payoff from playing C and the right-hand side is an upper bound of the payoff from playing A or B. This inequality holds for any δ because $u_i^{(i)} > u_i^{(-i)} \geq -1$: The strict inequality follows by assumption, and the weak inequality follows because $u_{-i}^{(-i)}$ must be feasible in the game with actions B and C while -1 is the worst payoff in such a game.

In sum, σ^* is a PBE when $\delta \in (\overline{\delta}, 1)$. □

B.4.3 Proofs for Appendix B.3

Proof of Proposition 7. For each $i = 1, 2$, when $x_i > 1$, it follows from the proof of Lemma 5 that $\hat{\delta}_i = \frac{x_i - 1}{x_i} > 0$ and $\hat{\delta}_i < \overline{\delta}_i(p_{-i})$. Also, for each $i = 1, 2$, since $\pi_i(0) = 1 < x_i$, we have $\underline{\delta}_i(p_{-i}) = 0$. Hence, $x_i < \pi_i(\delta_i)$ if $\delta_i > \overline{\delta}_i(p_{-i})$, and $x_i > \pi_i(\delta_i)$ if $\delta_i < \overline{\delta}_i(p_{-i})$.

Then, the proposition follows from Theorem 3. □

Proof of Proposition 8. Recall that, for each $i = 1, 2$, if $x_i < 1$ then $\hat{\delta}_i = 0$. Also, for each $i = 1, 2$, since $\pi_i(0) = 1 > x_i$, we have $\underline{\delta}_i(p_{-i}) > 0$.

Since $\hat{\delta}_i = 0$ for each $i = 1, 2$, it follows from parts 1 and 3 of Theorem 3 that σ^{G} is a unique PBE if $x_i < \pi_i(\delta_i)$ for some $i = 1, 2$; and that both σ^{G} and σ^N are a PBE if $x_i > \pi_i(\delta_i)$ for all $i = 1, 2$.

Thus, it suffices to show the following two assertions for each $i = 1, 2$. First, if $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$, then $x_i > \pi_i(\delta_i)$. Second, if $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$, then $x_i < \pi_i(\delta_i)$.

Fix $i = 1, 2$. We consider the following three exhaustive cases. As the first case, suppose $\underline{\delta}_i(p_{-i}) = 1$. Then, the first assertion vacuously follows because $(\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})) = \emptyset$. For the second assertion, it follows from the definition of $\underline{\delta}_i(p_{-i})$ that $x_i < \pi_i(\delta_i)$ for all $\delta_i \in (0, 1)$. Then, we have $x_i < \pi_i(\delta_i)$ for all $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$, as desired.

As the second case, suppose that $\underline{\delta}_i(p_{-i}) \neq 1$ and $\overline{\delta}_i(p_{-i}) = 1$. Then, $\underline{\delta}_i(p_{-i}) \in (0, 1)$. Also, $x_i \geq \pi_i(1)$ (otherwise, $\overline{\delta}_i(p_{-i}) < 1$). This means that the quadratic equation $x_i = \pi_i(\delta_i)$ has a unique interior solution in $(0,1)$, which is $\underline{\delta}_i(p_{-i})$. Thus, $x_i < \pi_i(\delta_i)$ if $\delta_i \in (0, \underline{\delta}_i(p_{-i}))$, that is, $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$. Also, $x_i > \pi_i(\delta_i)$ if $\delta_i \in (\underline{\delta}_i(p_{-i}), 1)$, that is, $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$. Hence, the two assertions hold.

As the third case, suppose that $\underline{\delta}_i(p_{-i}) \neq 1$ and $\overline{\delta}_i(p_{-i}) \neq 1$. Then, $\underline{\delta}_i(p_{-i}) \in (0,1)$. Also, $x_i < \pi_i(1)$ (otherwise, $\overline{\delta}_i(p_{-i}) = 1$). This means that the quadratic equation $x_i = \pi_i(\delta_i)$ has either (i) two interior solutions in $(0,1)$, which are $\underline{\delta}_i(p_{-i})$ and $\overline{\delta}_i(p_{-i})$ with $\underline{\delta}_i(p_{-i}) < \overline{\delta}_i(p_{-i})$, or (ii) a unique solution in $(0,1)$, which is $\underline{\delta}_i(p_{-i}) = \overline{\delta}_i(p_{-i})$. Thus, $x_i < \pi_i(\delta_i)$ if $\delta_i \in (0, \underline{\delta}_i(p_{-i})) \cup (\overline{\delta}_i(p_{-i}), 1)$, that is, $\delta_i \notin [\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i})]$. Also, $x_i > \pi_i(\delta_i)$ if $\delta_i \in (\underline{\delta}_i(p_{-i}), \overline{\delta}_i(p_{-i}))$. Hence, the two assertions hold.

In sum, for each of the three cases, the statement of the proposition holds. □

Proof of Proposition 9. Suppose that $x_i > 1 > x_{-i}$ for some $i = 1, 2$. Then, it follows from the proof of Lemma 5 that $\overline{\delta}_i(p_{-i}) > \hat{\delta}_i = \frac{x_i - 1}{x_i} > 0$. Also, $\hat{\delta}_{-i} = 0$. For player i, as in the proof of Proposition 7, $x_i < \pi_i(\delta_i)$ if $\delta_i > \overline{\delta}_i(p_{-i})$, and $x_i > \pi_i(\delta_i)$ if $\delta_i < \overline{\delta}_i(p_{-i})$. For player $-i$, similarly to the proof of Proposition 8, $x_{-i} < \pi_{-i}(\delta_{-i})$ if $\delta_{-i} \notin [\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i)]$, and $x_{-i} > \pi_{-i}(\delta_{-i})$ if $\delta_{-i} \in (\underline{\delta}_{-i}(p_i), \overline{\delta}_{-i}(p_i))$. Given these conclusions, the statement of the proposition follows from Theorem 3. □

To prove Remarks 1 to 3, we provide the following auxiliary result.

Lemma 10. *Fix $i = 1, 2$.*

1. *The threshold discount factor $\overline{\delta}_i$ is non-decreasing in p_{-i}, x_i, and ℓ_i.*

2. *The threshold discount factor $\underline{\delta}_i$ is non-increasing in p_{-i}, x_i, and ℓ_i.*

Proof of Lemma 10. We prove both parts at once. We define

$$D(p_{-i}, x_i, \ell_i) := \{\delta_i \in [0,1] \mid x_i < \pi_i(\delta_i; p_{-i}, x_i, \ell_i)\},$$

where we made explicit the dependence of π_i on the parameters.

(i) <u>Comparative statics with respect to p_{-i}</u>: For any x_i and ℓ_i, we define

$$A(\delta) := 1 + \delta_i(-\ell_i) + \delta_i^2 \cdot 0 \quad \text{and} \quad B(\delta) := 1 + \delta_i \cdot 1 + \delta_i^2 x_i.$$

We consider the two cases.

Case 1. Suppose $A(1) \geq B(1)$ and $x_i < 0$. Since $B(1) = 2 + x_i$, $x_i < \pi_i(1; p_{-i}, x_i, \ell_i)$. Since π_i is concave in δ_i and $\pi_i(1; p_{-i}, x_i, \ell_i) = 0$, it follows that $x_i < \pi_i(\delta_i; p_{-i}, x_i, \ell_i)$ for all $\delta_i \in [0, 1]$. Thus, we have $\overline{\delta}_i(p_{-i}) = 0$ and $\underline{\delta}_i(p_{-i}) = 1$ for all p_{-i}.

Case 2. Suppose $A(1) < B(1)$ or $x_i \geq 0$. We first show $A(\delta_i) \leq B(\delta_i)$ for all $\delta_i \in [0, 1]$. We consider the following two subcases.

Case 2-1. Suppose $x_i \geq 0$. Then, we have $B(\delta) - A(\delta) = x_i \delta_i^2 + (1+\ell_i)\delta_i \geq 0$ for all $\delta_i \in [0, 1]$.

Case 2-2. Suppose $x_i < 0$ and $A(1) < B(1)$. Since B is concave in δ_i and $A(0) = B(0)$, we have $A(\delta_i) \leq B(\delta_i)$ for all $\delta_i \in [0, 1]$.

Since $\pi_i(\delta_i; p_{-i}, x_i, \ell_i) = p_{-i} A(\delta_i) + (1 - p_{-i}) B(\delta_i)$,

$$\pi_i(\delta_i; p''_{-i}, x_i, \ell_i) - \pi_i(\delta_i; p'_{-i}, x_i, \ell_i) = (p''_{-i} - p'_{-i})(B(\delta_i) - A(\delta_i)).$$

If $p'_{-i} < p''_{-i}$, then this is non-negative. Thus, $D(p'_{-i}, x_i, \ell_i) \supseteq D(p''_{-i}, x_i, \ell_i)$. Hence, by the definition of $\overline{\delta}_i$, $\overline{\delta}_i$ is no greater under p'_{-i} than under p''_{-i}. Similarly, by the definition of $\underline{\delta}_i$, $\underline{\delta}_i$ is no smaller under p'_{-i} than under p''_{-i}.

(ii) <u>Comparative statics with respect to x_i:</u> For any δ_i, p_{-i}, and ℓ_i, we have

$$\pi_i(\delta_i; p_{-i}, x''_i, \ell_i) - \pi_i(\delta_i; p_{-i}, x'_i, \ell_i) = (1 - p_{-i})\delta_i^2 (x''_i - x'_i) \leq x''_i - x'_i.$$

Thus,

$$\pi_i(\delta_i; p_{-i}, x'_i, \ell_i) - x'_i \geq \pi_i(\delta_i; p_{-i}, x''_i, \ell_i) - x''_i,$$

which implies $D(p_{-i}, x'_i, \ell_i) \supseteq D(p_{-i}, x''_i, \ell_i)$. Hence, by the definition of $\overline{\delta}_i$, $\overline{\delta}_i$ is no greater under x'_i than under x''_i. Similarly, by the definition of $\underline{\delta}_i$, $\underline{\delta}_i$ is no smaller under x'_i than under x''_i.

(iii) <u>Comparative statics with respect to ℓ_i:</u> For any δ_i, p_{-i}, and x_i, we have $\pi_i(\delta_i; p_{-i}, x_i, \ell'_i) \geq \pi_i(\delta_i; p_{-i}, x_i, \ell''_i)$ if $\ell'_i \leq \ell''_i$. Thus, $D(p_{-i}, x_i, \ell'_i) \supseteq D(p_{-i}, x_i, \ell''_i)$. Hence, by the definition of $\overline{\delta}_i$, $\overline{\delta}_i$ is no greater under ℓ'_i than under ℓ''_i. Similarly, by the definition of $\underline{\delta}_i$, $\underline{\delta}_i$ is no smaller under ℓ'_i than under ℓ''_i.

□

Proof of Remark 1. Recall that the set S is given by equation (A.3). All the statements of this remark follow if, for each $j = 1, 2$, the threshold discount factors $\hat{\delta}_j$ and $\overline{\delta}_j$ are non-decreasing in x_j, p_{-j}, and ℓ_j. Since $x_j > 1$, $\hat{\delta}_j = \frac{x_j - 1}{x_j}$ is indeed non-decreasing in x_j, p_{-j}, and ℓ_j. For $\overline{\delta}_j$, it follows from part 1 of Lemma 10 that $\overline{\delta}_j$ is non-decreasing in x_j, p_{-j}, and ℓ_j. $\square$

Proof of Remark 2. Recall that the set S is given by equation (A.4). All the statements of this remark follow if, for each $i = 1, 2$, (i) the threshold discount factor $\overline{\delta}_i$ is non-decreasing in x_i, p_{-i}, and ℓ_i; and (ii) the threshold discount factor $\underline{\delta}_i$ is non-increasing in x_i, p_{-i}, and ℓ_i. Statement (i) follows from part 1 of Lemma 10, and statement (ii) follows from part 2 of Lemma 10. $\square$

Proof of Remark 3. Take $i = 1, 2$ with $x_i > 1 > x_{-i}$. Recall that the set S is given by equation (A.5). All the statements of this remark follow if the following hold: (i) $\hat{\delta}_i$ is non-decreasing in x_i, p_{-i}, and ℓ_i; (ii) $\overline{\delta}_i$ is non-decreasing in x_i, p_{-i}, and ℓ_i; (iii) $\overline{\delta}_{-i}$ is non-decreasing in x_{-i}, p_i, and ℓ_{-i}; and (iv) $\underline{\delta}_{-i}$ is non-increasing in x_{-i}, p_i, and ℓ_{-i}. Statement (i) follows because, as shown in Remark 1, $\hat{\delta}_i = \frac{x_i - 1}{x_i}$ is non-decreasing in x_i, p_{-i}, and ℓ_i. Statements (ii) and (iii) follow from part 1 of Lemma 10. Statement (iv) follows from part 2 of Lemma 10. $\square$