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Impossibility of weakly stable and strategy-proof mechanism[☆]

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A B S T R A C T

Many real matching markets are subject to distributional constraints. When the set of feasible matchings is restricted by some distributional constraints, a stable matching may not exist. In contrast, a weakly stable matching is guaranteed to exist under a very general class of constraints that satisfies a condition called *heredity*. However, it has been an open question whether a weakly stable matching can be obtained by a strategy-proof mechanism. We negatively answer this open question; no weakly stable and strategy-proof mechanism exists under any heredity feasibility constraint in general.

1. Introduction

The theory of two-sided matching has been extensively developed and has been applied to many real-life application domains.[1] As the theory has been applied to increasingly diverse types of environments, researchers and practitioners have encountered various forms of distributional constraints. Two streams of works exist on matching with distributional constraints.[2] One stream scrutinizes constraints that arise from real-life applications, such as regional maximum quotas (Kamada and Kojima, 2015), individual/regional minimum quotas (Fragiadakis et al., 2015; Goto et al., 2017), affirmative actions (Ehlers et al., 2014; Kurata et al., 2017), etc. The other stream mathematically studies an abstract and general class of constraints, such as those that can be represented by a substitute choice function (Hatfield and Milgrom, 2005), matroidal constraints (Kojima et al., 2018), and heredity constraints (Kamada and Kojima, 2017; Goto et al., 2017; Aziz et al., 2021). This paper deals with heredity feasibility constraints, which require that if a matching between doctors and hospitals is feasible, then any matching that places weakly fewer doctors at each hospital is also feasible (Kamada and Kojima, 2017; Goto et al., 2017; Aziz et al., 2021).[3]

One important desideratum for a matching mechanism is strategy-proofness (for doctors), which requires that no doctor has an incentive to misreport her preference over hospitals. Another central desideratum for a matching mechanism is stability (Gale and Shapley, 1962), which requires that there exists no pair of agents who prefer matching with each other to accepting the current matching. When distributional constraints are imposed, a stable matching may not exist. The existence is guaranteed *if and only if* all constraints are trivial (Kamada and Kojima, 2017). Given the incompatibility of stability under distributional constraints, mechanism designers have considered weaker stability notions to guarantee existence in various settings (Fragiadakis et al., 2015; Fragiadakis and Troyan, 2017; Goto et al., 2016, 2017; Kamada and Kojima, 2017; Kurata et al., 2017; Yahiro et al., 2020). Arguably, the most acknowledged stability concept that works with any heredity constraint is *weak stability* (Kamada and Kojima, 2017) since the existence of a weakly stable matching is guaranteed under any heredity constraint.

However, whether a weakly stable matching can always be obtained by using a strategy-proof mechanism has been an open question (Kamada and Kojima, 2017). In this paper, we negatively

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1 See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

2 See Aziz et al. (2022) for a comprehensive survey on various distributional constraints.

3 Although our paper is described in the context of a doctor–hospital matching problem, the obtained result is applicable to matching problems in general.

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answer this open question, that is, we prove that no weakly stable and strategy-proof (for doctors) mechanism exists under any heredity feasibility constraint in general.[4]

2. Model

We apply the model presented in Kamada and Kojima (2017).[5] Let there be a finite set of doctors D and a finite set of hospitals H. Each doctor d has a strict preference relation $\succ_d$ over the set of hospitals and the outside option $\varnothing$ (which means that d is unmatched). For any $h, h' \in H \cup \{\varnothing\}$, we write $h \succeq_d h'$ if and only if $h \succ_d h'$ or $h = h'$. Each hospital h has a strict preference relation $\succ_h$ over $D \cup \{\varnothing\}$. Let $\succ_D := (\succ_d)_{d \in D}$ denote the preference profile of all doctors, and $\succ_H := (\succ_h)_{h \in H}$ denote the preference profile of all hospitals. Furthermore, let $\succ := (\succ_D, \succ_H)$ denote the preference profile of all doctors and hospitals. Doctor d is said to be **acceptable** to hospital h if $d \succ_h \varnothing$. Similarly, hospital h is **acceptable** to doctor d if $h \succ_d \varnothing$. Since only rankings of acceptable partners matter for our analysis, we write only acceptable partners to denote preferences. For example,

$\succ_d$: hh'

means that hospital h is the most preferred, h' is the second most preferred, and h and h' are the only acceptable hospitals under preference $\succ_d$ of doctor d.

A **matching** μ is a mapping that satisfies (i) $\mu_d \in H \cup \{\varnothing\}$ for all $d \in D$, (ii) $\mu_h \subseteq D$ for all $h \in H$, and (iii) for any $d \in D$ and $h \in H$, $\mu_d = h$ if and only if $d \in \mu_h$.

A **feasibility constraint** is a map $f : \mathbb{Z}_+^{|H|} \to \{0, 1\}$. Let w denote an $|H|$-element vector, where each coordinate in w corresponds to a hospital, and the number in that coordinate represents the number of doctors matched to the hospital. $f(w) = 1$ means that w is **feasible** and $f(w) = 0$ means it is not. We say that matching μ is **feasible** if and only if $f(w(\mu)) = 1$, where $w(\mu) := (|\mu_h|)_{h \in H}$ is a vector of non-negative integers indexed by hospitals whose coordinate corresponding to hospital h is $|\mu_h|$.

In this paper, we restrict our attention to feasibility constraints that satisfy the property called *heredity*. For two $|H|$-element vectors $w, w' \in \mathbb{Z}_+^{|H|}$, we say $w \leq w'$ if for all $h \in H$, $w_h \leq w'_h$ holds.

Definition 2.1. We say f satisfies **heredity** if $f(0) = 1$ and $f(w) \geq f(w')$ whenever $w \leq w'$, where argument 0 is the zero vector.

A matching μ is **individually rational** if (i) for each $d \in D$, $\mu_d \succeq_d \varnothing$, and (ii) for each $h \in H$, $d \succeq_h \varnothing$ for all $d \in \mu_h$. That is, no agent is matched with an unacceptable partner.

A matching μ is **fair** (or satisfies the **no-justified-envy** property) if there exists no pair of doctors $d, d' \in D$ such that (i) $\mu_{d'} \succ_d \mu_d$ and (ii) $d \succ_{\mu_{d'}} d'$ or $\mu_{d'} = \varnothing$. In the definition, (i) says that d envies d', and (ii) says that the envy is justified.

For each $h \in H$, let e_h be an $|H|$-element vector whose coordinate corresponding to h is one and other coordinates are zero. A matching μ is **non-wasteful** if there is no doctor–hospital pair (d, h) such that (i) $h \succ_d \mu_d$ and $d \succ_h \varnothing$, and (ii) $f(w(\mu) + e_h) = 1$.[6]

Now we are ready to define *weak stability*.

Definition 2.2. A matching μ is **weakly stable** if it is feasible, individually rational, fair, and non-wasteful.

A mechanism φ is a function that maps preference profiles to matchings. The matching under φ at preference profile $\succ$ is denoted $\varphi(\succ)$ and agent i's match is denoted by $\varphi_i(\succ)$ for each $i \in D \cup H$.

Definition 2.3. A mechanism φ is

1. **weakly stable** if it always returns a weakly stable matching.
2. **strategy-proof** (for doctors) if there exists no preference profile $\succ$, doctor $d \in D$, and preference $\succ'_d$ of doctor d such that

$$\varphi_d(\succ'_d, \succ_{-d}) \succ_d \varphi_d(\succ).$$

3. Impossibility theorem

Theorem 3.1. *No weakly stable and strategy-proof mechanism exists in general.*

The following example is enough to prove this impossibility result.

Example 3.1. Consider a matching market with three doctors $D = \{d_1, d_2, d_3\}$ and three hospitals $H = \{h_1, h_2, h_3\}$. The hospitals' preferences are as follows.

$\succ_{h_1}$: $d_1 d_2 d_3$

$\succ_{h_2}$: $d_2 d_3 d_1$

$\succ_{h_3}$: $d_3 d_1 d_2$

Furthermore, suppose $f(w) = 1$ if and only if $w \leq w'$ for some $w' \in \{(2, 0, 0), (0, 2, 0), (0, 0, 2)\}$.

For the above-mentioned market, we deductively show that no weakly stable mechanism is strategy-proof. Assume for contradiction that there exists a weakly stable and strategy-proof mechanism φ. Consider the preference profile $\succ_D^1 = (h_2 h_3 h_1, h_3 h_1 h_2, h_1 h_2 h_3)$.[7] By weak stability of φ,[8]

$$\varphi(\succ_D^1, \succ_H) \in \Big\{[h_1 : \{d_1, d_2\}], [h_2 : \{d_2, d_3\}], [h_3 : \{d_1, d_3\}]\Big\}.$$

We distinguish the following three cases.

Case 1: Suppose $\varphi(\succ_D^1, \succ_H) = [h_1 : \{d_1, d_2\}]$.

Consider the preference profiles (together with all weakly stable matchings) in Table 1. Given the fact $\varphi(\succ_D^1, \succ_H) = [h_1 : \{d_1, d_2\}]$ (assumption of Case 1), only a subset of these matchings are consistent with strategy-proofness, which we have highlighted in blue. Detailed arguments for why only the highlighted matchings are possible for φ are given below.

Since $\varphi(\succ_D^1, \succ_H) = [h_1 : \{d_1, d_2\}]$, by strategy-proofness of φ, we have $\varphi_{d_2}(\succ_D^2, \succ_H) = h_1$. This, together with weak stability of φ (see Table 1), implies

$$\varphi(\succ_D^2, \succ_H) = [h_1 : \{d_1, d_2\}]. \quad (1)$$

Continuing in this manner, by moving from $\succ_D^t$ to $\succ_D^{t+1}$ for all $t \in \{2, \dots, 5\}$ one by one, and by using similar logic each time, we obtain

$$\varphi(\succ_D^3, \succ_H) = [h_1 : \{d_1, d_2\}], \ \varphi(\succ_D^4, \succ_H) = [h_1 : \{d_1, d_2\}],$$
$$\varphi(\succ_D^5, \succ_H) = [h_1 : \{d_1\}], \text{ and}$$

[4] Apart from weak stability, *cutoff stability* (Aziz et al., 2021) is another stability concept that is compatible with any heredity constraint. Cutoff stability implies weak stability, and consequently, our impossibility result carries over to cutoff stability.

[5] One minor difference is that the maximum quota of each hospital is embedded into a feasibility constraint in our model.

[6] If we replace condition (ii) to $f(w(\mu) + e_h - e_{\mu_d}) = 1$, we obtain a stronger notion than non-wastefulness. Then, by replacing non-wastefulness with this stronger notion in the requirement of weak stability, we obtain the standard stability.

[7] By $(h_2 h_3 h_1, h_3 h_1 h_2, h_1 h_2 h_3)$, we denote a preference profile where doctors d_1, d_2, and d_3 have preferences $h_2 h_3 h_1$, $h_3 h_1 h_2$, and $h_1 h_2 h_3$, respectively.

[8] By $[h_1 : \{d_1, d_2\}]$, we denote the matching $\big[(\{d_1, d_2\}, h_1), (d_3, \varnothing), (\varnothing, h_2), (\varnothing, h_3)\big]$.

Table 1
Weakly stable matchings for preference profiles $(\succ_D^2, \succ_H)$ to $(\succ_D^6, \succ_H)$.

Preference profiles	Doctor d_1	Doctor d_2	Doctor d_3	Weakly stable matchings
$\succ_D^2$	$h_2h_3h_1$	h_3h_1	$h_1h_2h_3$	$[h_1:\{d_1,d_2\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^3$	h_1	h_3h_1	$h_1h_2h_3$	$[h_1:\{d_1,d_2\}]$, $[h_2:\{d_3\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^4$	h_1	h_3h_1	h_3	$[h_1:\{d_1,d_2\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^5$	h_1	h_3	h_3	$[h_1:\{d_1\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^6$	h_2h_1	h_3	h_3	$[h_1:\{d_1\}]$, $[h_2:\{d_1\}]$, $[h_3:\{d_2,d_3\}]$

Table 2
Weakly stable matchings for preference profiles $(\succ_D^7, \succ_H)$ to $(\succ_D^{20}, \succ_H)$.

Preference profiles	Doctor d_1	Doctor d_2	Doctor d_3	Weakly stable matchings
$\succ_D^7$	h_3h_2	h_3h_1	$h_1h_2h_3$	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^8$	h_3h_2	h_3h_1	h_1	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_1\}]$, $[h_3:\{d_1,d_2\}]$
$\succ_D^9$	h_3h_1	h_3h_1	h_1	$[h_1:\{d_1,d_2\}]$, $[h_3:\{d_1,d_2\}]$
$\succ_D^{10}$	h_3h_1	h_3	h_1	$[h_1:\{d_1,d_3\}]$, $[h_3:\{d_1,d_2\}]$
$\succ_D^{11}$	h_3h_1	h_3	$h_1h_3h_2$	$[h_1:\{d_1,d_3\}]$, $[h_2:\{d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{12}$	h_3	h_3	h_3	$[h_3:\{d_1,d_3\}]$
$\succ_D^{13}$	h_3h_2	h_3	h_3	$[h_2:\{d_1\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{14}$	h_3	h_3	$h_3h_1h_2$	$[h_1:\{d_3\}]$, $[h_2:\{d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{15}$	h_3h_2	h_3	$h_1h_3h_2$	$[h_1:\{d_3\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{16}$	$h_2h_3h_1$	h_3	$h_1h_3h_2$	$[h_1:\{d_1,d_3\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{17}$	$h_2h_3h_1$	h_3	$h_3h_1h_2$	$[h_1:\{d_1,d_3\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{18}$	$h_2h_3h_1$	h_3	h_3	$[h_1:\{d_1\}]$, $[h_2:\{d_1\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{19}$	h_2	h_3	h_3	$[h_2:\{d_1\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{20}$	h_2	$h_3h_2h_1$	h_3	$[h_1:\{d_2\}]$, $[h_2:\{d_1,d_2\}]$, $[h_3:\{d_2,d_3\}]$

$$\varphi_{d_2}(\succ_D^6, \succ_H) = \varnothing. \quad (2)$$

Next, consider the preference profiles (together with all weakly stable matchings) in Table 2. Similarly as before, we have highlighted the matchings in blue which are possible for φ under the assumption of Case 1 (arguments behind these facts are presented below).

By strategy-proofness and weak stability, (1) implies

$$\varphi(\succ_D^7, \succ_H) = [h_1 : \{d_2, d_3\}].$$

By moving from $\succ_D^t$ to $\succ_D^{t+1}$ for all $t \in \{7, \dots, 10\}$ one by one, and by applying strategy-proofness and weak stability each time, we obtain

$$\varphi(\succ_D^8, \succ_H) = [h_1 : \{d_2, d_3\}],\ \varphi(\succ_D^9, \succ_H) = [h_1 : \{d_1, d_2\}],$$
$$\varphi(\succ_D^{10}, \succ_H) = [h_1 : \{d_1, d_3\}], \text{ and}$$

$$\varphi(\succ_D^{11}, \succ_H) = [h_1 : \{d_1, d_3\}]. \quad (3)$$

From Table 2, we have $\varphi(\succ_D^{12}, \succ_H) = [h_3 : \{d_1, d_3\}]$. This, together with strategy-proofness and weak stability, implies

$$\varphi(\succ_D^{13}, \succ_H) = [h_3 : \{d_1, d_3\}], \text{ and} \quad (4a)$$

$$\varphi(\succ_D^{14}, \succ_H) = [h_3 : \{d_1, d_3\}]. \quad (4b)$$

By strategy-proofness, (3) implies $\varphi_{d_1}(\succ_D^{15}, \succ_H) \neq h_3$. Moreover, (4a) together with strategy-proofness, implies $\varphi_{d_3}(\succ_D^{15}, \succ_H) \neq h_2$. Since $\varphi(\succ_D^{15}, \succ_H) \in \{[h_1 : \{d_3\}], [h_2 : \{d_1, d_3\}], [h_3 : \{d_1, d_3\}]\}$ (see Table 2), the facts $\varphi_{d_1}(\succ_D^{15}, \succ_H) \neq h_3$ and $\varphi_{d_3}(\succ_D^{15}, \succ_H) \neq h_2$ together imply

$$\varphi(\succ_D^{15}, \succ_H) = [h_1 : \{d_3\}]. \quad (5)$$

By strategy-proofness and weak stability, (5) implies

$$\varphi(\succ_D^{16}, \succ_H) = [h_1 : \{d_1, d_3\}]. \quad (6)$$

By strategy-proofness, (4b) implies $\varphi_{d_1}(\succ_D^{17}, \succ_H) \neq h_1$. Moreover, by strategy-proofness, (6) implies $\varphi_{d_3}(\succ_D^{17}, \succ_H) \neq h_2$. Since $\varphi(\succ_D^{17}, \succ_H) \in \{[h_1 : \{d_1, d_3\}], [h_2 : \{d_1, d_3\}], [h_3 : \{d_1, d_3\}]\}$ (see Table 2), the facts $\varphi_{d_1}(\succ_D^{17}, \succ_H) \neq h_1$ and $\varphi_{d_3}(\succ_D^{17}, \succ_H) \neq h_2$ together imply

$$\varphi(\succ_D^{17}, \succ_H) = [h_3 : \{d_1, d_3\}].$$

By moving from $\succ_D^t$ to $\succ_D^{t+1}$ for all $t \in \{17, 18, 19\}$ one by one, and by applying strategy-proofness and weak stability each time, we obtain

$$\varphi(\succ_D^{18}, \succ_H) = [h_3 : \{d_1, d_3\}],\ \varphi(\succ_D^{19}, \succ_H) = [h_3 : \{d_2, d_3\}], \text{ and}$$

$$\varphi(\succ_D^{20}, \succ_H) = [h_3 : \{d_2, d_3\}]. \quad (7)$$

Next, consider the preference profiles (together with all weakly stable matchings) in Table 3. As before, we have highlighted the matchings in blue which are possible for φ under the assumption of Case 1.

Since $\varphi(\succ_D^1, \succ_H) = [h_1 : \{d_1, d_2\}]$, by strategy-proofness and weak stability, we have

$$\varphi(\succ_D^{21}, \succ_H) = [h_1 : \{d_1, d_2\}].$$

By moving from $\succ_D^t$ to $\succ_D^{t+1}$ for all $t \in \{21, \dots, 24\}$ one by one, and by applying strategy-proofness and weak stability each time, we obtain

$$\varphi(\succ_D^{22}, \succ_H) = [h_1 : \{d_1, d_2\}],\ \varphi(\succ_D^{23}, \succ_H) = [h_1 : \{d_1, d_2\}],$$
$$\varphi(\succ_D^{24}, \succ_H) = [h_1 : \{d_2, d_3\}], \text{ and}$$

$$\varphi(\succ_D^{25}, \succ_H) = [h_1 : \{d_2, d_3\}]. \quad (8)$$

From Table 3, we have $\varphi(\succ_D^{26}, \succ_H) = [h_2 : \{d_2, d_3\}]$. This, together with strategy-proofness and weak stability, implies

$$\varphi(\succ_D^{27}, \succ_H) = [h_2 : \{d_2, d_3\}], \text{ and} \quad (9a)$$

Table 3
Weakly stable matchings for preference profiles $(\succ_D^{21}, \succ_H)$ to $(\succ_D^{34}, \succ_H)$.

Preference profiles	Doctor d_1	Doctor d_2	Doctor d_3	Weakly stable matchings
$\succ_D^{21}$	$h_2h_3h_1$	h_1	$h_1h_2h_3$	$[h_1:\{d_1,d_2\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_1,d_3\}]$
$\succ_D^{22}$	h_2h_1	h_1	$h_1h_2h_3$	$[h_1:\{d_1,d_2\}]$, $[h_2:\{d_1,d_3\}]$, $[h_3:\{d_3\}]$
$\succ_D^{23}$	h_2h_1	h_1	h_2h_1	$[h_1:\{d_1,d_2\}]$, $[h_2:\{d_1,d_3\}]$
$\succ_D^{24}$	h_2	h_1	h_2h_1	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_1,d_3\}]$
$\succ_D^{25}$	h_2	$h_1h_2h_3$	h_2h_1	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2\}]$
$\succ_D^{26}$	h_2	h_2	h_2	$[h_2:\{d_2,d_3\}]$
$\succ_D^{27}$	h_2	h_2	$h_2h_3h_1$	$[h_1:\{d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_3\}]$
$\succ_D^{28}$	h_2	$h_2h_1h_3$	h_2	$[h_1:\{d_2\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2\}]$
$\succ_D^{29}$	h_2	$h_1h_2h_3$	$h_2h_3h_1$	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{30}$	h_2	$h_1h_2h_3$	$h_3h_2h_1$	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{31}$	h_2	$h_2h_1h_3$	$h_3h_2h_1$	$[h_1:\{d_2,d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{32}$	h_2	h_2h_3	$h_3h_2h_1$	$[h_1:\{d_3\}]$, $[h_2:\{d_2,d_3\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{33}$	h_2	h_2h_3	h_3	$[h_2:\{d_1,d_2\}]$, $[h_3:\{d_2,d_3\}]$
$\succ_D^{34}$	h_2h_1	h_2h_3	h_3	$[h_1:\{d_1\}]$, $[h_2:\{d_1,d_2\}]$, $[h_3:\{d_2,d_3\}]$

$$\varphi(\succ_D^{28}, \succ_H) = [h_2 : \{d_2, d_3\}]. \tag{9b}$$

By strategy-proofness, (8) implies $\varphi_{d_3}(\succ_D^{29}, \succ_H) \neq h_2$. Moreover, (9a) together with strategy-proofness, implies $\varphi_{d_2}(\succ_D^{29}, \succ_H) \neq h_3$. Since $\varphi(\succ_D^{29}, \succ_H) \in \{[h_1 : \{d_2, d_3\}], [h_2 : \{d_2, d_3\}], [h_3 : \{d_2, d_3\}]\}$ (see Table 3), the facts $\varphi_{d_3}(\succ_D^{29}, \succ_H) \neq h_2$ and $\varphi_{d_2}(\succ_D^{29}, \succ_H) \neq h_3$ together imply

$$\varphi(\succ_D^{29}, \succ_H) = [h_1 : \{d_2, d_3\}]. \tag{10}$$

By strategy-proofness and weak stability, (10) implies

$$\varphi(\succ_D^{30}, \succ_H) = [h_1 : \{d_2, d_3\}]. \tag{11}$$

By strategy-proofness, (9b) implies $\varphi_{d_3}(\succ_D^{31}, \succ_H) \neq h_1$. Moreover, by strategy-proofness, (11) implies $\varphi_{d_2}(\succ_D^{31}, \succ_H) \neq h_3$. Since $\varphi(\succ_D^{31}, \succ_H) \in \{[h_1 : \{d_2, d_3\}], [h_2 : \{d_2, d_3\}], [h_3 : \{d_2, d_3\}]\}$ (see Table 3), the facts $\varphi_{d_3}(\succ_D^{31}, \succ_H) \neq h_1$ and $\varphi_{d_2}(\succ_D^{31}, \succ_H) \neq h_3$ together imply

$$\varphi(\succ_D^{31}, \succ_H) = [h_2 : \{d_2, d_3\}].$$

By moving from $\succ_D^t$ to $\succ_D^{t+1}$ for all $t \in \{31, 32, 33\}$ one by one, and by applying strategy-proofness and weak stability each time, we obtain

$$\varphi(\succ_D^{32}, \succ_H) = [h_2 : \{d_2, d_3\}], \ \varphi(\succ_D^{33}, \succ_H) = [h_2 : \{d_1, d_2\}], \text{ and}$$

$$\varphi(\succ_D^{34}, \succ_H) = [h_2 : \{d_1, d_2\}]. \tag{12}$$

Finally, consider the preference profile $\succ_D^{35} = (h_2h_1, h_3h_2h_1, h_3)$. By weak stability, we have $\varphi(\succ_D^{35}, \succ_H) \in \{[h_1 : \{d_1, d_2\}], [h_2 : \{d_1, d_2\}], [h_3 : \{d_2, d_3\}]\}$.

(i) Suppose $\varphi(\succ_D^{35}, \succ_H) = [h_1 : \{d_1, d_2\}]$. This, together with $\varphi(\succ_D^{34}, \succ_H) = [h_2 : \{d_1, d_2\}]$ (see (12) for details), contradicts strategy-proofness.
(ii) Suppose $\varphi(\succ_D^{35}, \succ_H) = [h_2 : \{d_1, d_2\}]$. This, together with $\varphi(\succ_D^{20}, \succ_H) = [h_3 : \{d_2, d_3\}]$ (see (7) for details), contradicts strategy-proofness.
(iii) Suppose $\varphi(\succ_D^{35}, \succ_H) = [h_3 : \{d_2, d_3\}]$. This, together with $\varphi_{d_2}(\succ_D^{6}, \succ_H) = \varnothing$ (see (2) for details), contradicts strategy-proofness.

Case 2: Suppose $\varphi(\succ_D^1, \succ_H) = [h_2 : \{d_2, d_3\}]$.

By renaming doctors d_1, d_2, d_3 as d_3', d_1', d_2', respectively, and renaming hospitals h_1, h_2, h_3 as h_3', h_1', h_2', respectively, we obtain an identical situation to Case 1.

Case 3: Suppose $\varphi(\succ_D^1, \succ_H) = [h_3 : \{d_1, d_3\}]$.

By renaming doctors d_1, d_2, d_3 as d_2'', d_3'', d_1'', respectively, and renaming hospitals h_1, h_2, h_3 as h_2'', h_3'', h_1'', respectively, we obtain an identical situation to Case 1.

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