School Choice and the Housing Market

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Abstract

I develop a unified framework with schools and residential choices to study the welfare and distributional consequences of public schools' switching from the traditional neighborhood assignment to the Deferred Acceptance mechanism. I show that when families receive higher priorities at neighborhood schools, the Deferred Acceptance mechanism improves aggregate or average welfare compared to neighborhood assignment. Additionally, under general conditions, the Deferred Acceptance mechanism improves the welfare of lowest-income families, both with and without neighborhood priorities. My work lays theoretical foundations for analyzing assignment games with externalities.

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1 Introduction

According to the Brookings Institution's Center on Children and Families, the proportion of large school districts in the US that allow parental choice over public schools has doubled from 2000 through 2016 (Whitehurst, 2017). Many school districts have replaced the traditional neighborhood assignment with choice-based assignment mechanisms which reflect recent advances in matching theory and market design. A prominent example is the widespread application of the celebrated Deferred Acceptance mechanism of Gale and Shapley (1962). Following a scholarly article by Abdulkadiroğlu and Sönmez (2003), Deferred Acceptance has been adopted for student assignment by school districts in New York City, Boston, Chicago, Denver, Washington DC and Newark, among many others.

Under neighborhood assignment families enroll their children to the designated neighborhood schools. In contrast, the Deferred Acceptance mechanism assigns children to schools based on families' reported preferences and their priorities at schools. The Deferred Acceptance mechanism and its welfare properties are extensively studied in the matching theory literature. However, previous papers predominantly assume that the preferences and priorities are exogeneously given, while in reality, they depend on families' endogeneous neighborhood choices. It has been empirically documented that families strategically choose where to live, and they do so by taking into account the schooling options (Chung, 2015; Kane, Riegg, and Staiger, 2006; Reback, 2005). Those strategic neighborhood choices affect families' probabilities of being assigned to different schools through their effects on preferences and priorities.[1]

In this paper I develop a unified framework with school choice and a housing market,

[1]Neighborhood choices affect preferences since families prefer schools closer to their homes (e.g., Glazerman (1998), Burgess, Greaves, Vignoles, and Wilson (2011), Abdulkadiroğlu, Agarwal, and Pathak (2017a), Abdulkadiroğlu, Pathak, Schellenberg, and Walters (2020b)), and they affect priorities since schools typically grant higher priorities to neighborhood applicants.

where families make residential choices prior to school admission, and I evaluate the welfare and distributional consequences of the widespread education reform of switching from neighborhood assignment to Deferred Acceptance. The timing of the model is as follows. First, the city or the school district announces the school assignment mechanism. Then, families make neighborhood choices optimally, given the school assignment mechanism, other families' neighborhood choices and the market-clearing neighborhood prices. Lastly, children are assigned to schools through the announced assignment mechanism. There are two major findings. First, I show that when there are neighborhood priorities (i.e., when families receive higher priorities at neighborhood schools), the Deferred Acceptance mechanism improves aggregate (or average) welfare compared to neighborhood assignment. Second, I show that under general conditions the lowest-income families prefer the Deferred Acceptance mechanism, both with and without neighborhood priorities, to neighborhood assignment. To the best of my knowledge, mine is the first theoretical evaluation of welfare and distributional effects of school choice in a general model with unrestricted preference domain and endogenous neighborhood choices. I now elaborate on the results and their policy implications.

When there are no neighborhood priorities, the welfare comparison between the Deferred Acceptance and neighborhood assignment is ambiguous. On one hand, the Deferred Acceptance mechanism may generate higher welfare as it gives families more flexibility to reside in their preferred neighborhoods and enroll their children to their preferred schools, potentially outside of the neighborhoods. On the other hand, neighborhood assignment may generate higher welfare as families with highest valuations for certain schools can guarantee enrollment there by purchasing a house in the corresponding neighborhoods. Despite this welfare trade-off, I show that with neighborhood priorities, the Deferred Acceptance mechanism always generates higher aggregate welfare than neighborhood assignment. I also show that, under some conditions such as identical ordinal preference rankings over neighborhoods and schools, Deferred Acceptance generates higher aggregate welfare with neighborhood priorities than without those.

Neighborhood priorities may be thought of as a compromise between neighborhood assignment and open enrollment without neighborhood priorities (i.e., where all families receive a fair shot at each school). In Boston Public School (BPS) there have been constant debates about using neighborhood assignment or allowing choice (Daley, 1999; Dur, Kominers, Pathak, and Sönmez, 2018; Menino, 2012). Those debates have resulted in redesigning the school assignment system by granting higher priorities to families (at a fraction of seats[2]) at their neighborhood schools (Dur et al., 2018). Neighborhood priorities are applied not only in BPS, but in predominant majority of the US school districts allowing parental choice. My theoretical findings that Deferred Acceptance with neighborhood priorities improves aggregate welfare compared to neighborhood assignment and, under some conditions, compared to Deferred Acceptance without neighborhood priorities, potentially provide a rationale for the widespread application of neighborhood priorities for school assignment. To the best of my knowledge, this is the first theoretical justification of using neighborhood priorities in school assignment for welfare considerations.

Although the aggregate welfare is always larger under Deferred Acceptance with neighborhood priorities compared to neighborhood assignment, some families may be better-off under the latter mechanism. Since the welfare of low-income and disadvantaged communities is a major consideration for education policy (Fuller, 1996; Orfield and Frankenberg, 2013), the question that I ask next is how the mechanisms compare in terms of lowest-income families' welfare. In Section 5 I extend the model so that families are differentiated by incomes or budgets. A budget denotes the maximum amount a family can pay for a house. Proponents argue that school choice weakens the links between schools and the housing market, and potentially leads to more equitable

[2] In 1999, BPS adopted what is known as the '50-50 seat split', where families are granted higher priorities at only half of the seats at their neighborhood schools.

outcomes by allowing families in less affluent neighborhoods to apply to higher quality schools outside of their neighborhoods (Bedrick and Burke, 2015; Coons and Sugarman, 1978). Although the argument is intuitive, it has limited theoretical foundation. Papers on the topic typically analyze stylized models where families have identical preferences over neighborhoods and schools. Such an assumption is unrealistic for the school choice setting.[3] To the best of my knowledge, mine is the first work to compare distributional effects of the Deferred Acceptance mechanism in a general matching model with rich preferences and residential choices. I show that under general conditions lowest-income families prefer both versions of the Deferred Acceptance mechanism to neighborhood assignment. The sufficiency conditions have two parts: (1) underdemanded neighborhoods remain underdemanded when the school assignment mechanisms is switched from neighborhood assignment to Deferred Acceptance, (2) underdemanded (or to put it simply, cheapest) neighborhoods have underdemanded (least selective) schools.[4] The former condition is intuitive: the poorest neighborhoods are unlikely to significantly gain in value to become overdemanded when the school assignment mechanism is switched from neighborhood assignment to Deferred Acceptance. The latter condition is consistent with the empirical evidence: Owens and Candipan (2019) document that in large metropolitan areas in the US the poorest neighborhoods typically have underperforming schools. I show that the conditions are satisfied for natural special cases. Thus, my findings provide a theoretical justification for a major argument in favor of school choice, namely, that lowest-income families benefit from choice.

Finally, my work builds theoretical foundations for studying assignment games with externalities. In my model a family's valuation for a neighborhood depends on other families' neighborhood choices through the latter's effects on the family's school as-

[3]For example, it has been shown that families prefer schools that are closer to their homes (Abdulkadiroğlu et al., 2017a,2; Burgess et al., 2011; Glazerman, 1998).

[4]I show that (1) and (2) are also 'necessary' conditions for lowest-income families to prefer Deferred Acceptance over neighborhood assignment if one requires 'robustness'. The result is formally stated in Theorem 6.

signment probabilities. These externalities may preclude the existence of a competitive equilibrium in discrete economies. However, I show that a competitive equilibrium always exists in a large economy with a continuum of families. The result builds on the fact that in a large economy a family's school assignment probabilities are continuous in other families' neighborhood choices. This allows me to use a novel application of Schauder-Tychonoff fixed point theorem to establish equilibrium existence. Not only does the continuum model circumvent the equilibrium non-existence issue, but it also buys us tractability. In the continuum model I derive closed-form expressions for school assignment probabilities which are used for proving many of the results. In Appendix B I show that the continuum model is an arbitrarily close approximation of sufficiently large discrete ones. This implies that all results, such as the existence of a competitive equilibrium and welfare comparisons across the mechanisms, hold in an approximate sense for every sufficiently large discrete economy. Equilibrium existence and large market approximation results extend to general assignment games with externalities, such as complementarities or peer preferences.

The remainder of the paper is organized as follows. Section 2 reviews related literature. Section 3 describes the model and establishes the existence of competitive equilibria. Section 4 compares aggregate welfare across the school assignment mechanisms. Section 5 studies a model with budget constraints and studies the implications of school choice for lowest-income families. Section 6 discusses simulation results. Section 7 concludes. All omitted proofs are in the main Appendix. The main Appendix also discusses the existence of approximate equilibria in discrete economies. Alternative school assignment mechanisms (such as Top Trading Cycles and Immediate Acceptance) and further extensions are studied in the Supplementary Appendix.

2 Related Literature

Welfare and distributional consequences of school choice have been theoretically analyzed by several earlier works (Avery and Pathak, 2020; Barseghyan, Clark, and Coate, 2013; Calsamiglia, Martínez-Mora, and Miralles, 2015; Epple and Romano, 2003; Lee, 1997; Xu, 2019). These papers feature stylized models, where: (1) types are described by a single parameter which reflects ability or income, (2) families' have no preferences over neighborhoods, (3) schools are ranked by quality and all families prefer the higher quality schools, (4) valuations for schools are supermodular in income and school quality. Some of these assumptions may be highly unrealistic in the context of school choice. For example, assumption (3) implies that families have identical ordinal preference rankings over schools. My work, on the other hand, features a general preference domain with arbitrary valuations over neighborhoods and schools. Such unrestricted heterogeneity is an important aspect in Gale and Shapley (1962) and the vast literature on the two-sided matching literature that followed this seminal work.

The generality of my model allows me to reveal novel insights on welfare and distributional consequences of school assignment mechanisms which are missing from prior papers on the topic. First, unlike most works above that are mainly interested in distributional outcomes of school choice, my paper also compares school assignment mechanisms in terms of aggregate or average welfare. Such an analysis becomes trivial when families have no intrinsic preferences for neighborhoods, or when families have identical ordinal preference rankings over neighborhoods and schools, and supermodular valuations. In those settings, if the number of seats at schools equals the neighborhood's housing supply, then neighborhood assignment maximizes aggregate welfare. I show that this is not true in the general model: Deferred Acceptance may create strictly higher aggregate welfare than neighborhood assignment, and it always creates weakly higher aggregate welfare when there are neighborhood priorities. Second, some of the conclusions on lowest-income families' welfare in previous theoretical papers depend on the preference restrictions those works impose. For example, in Calsamiglia et al. (2015) and Xu (2019) lowest-income families always prefer Deferred Acceptance to neighborhood assignment. In contrast, in my model lowest-income families may not necessarily benefit from the Deferred Acceptance mechanism.[5] I provide general sufficient conditions under which the lowest-income families' prefer both versions of the Deferred Acceptance mechanism to neighborhood assignment, and I show that the conditions are satisfied for some natural special cases.

My setup corresponds to a two-sided matching problem with endogenous preferences and/or priorities. Papers on the topic, such as Peters and Siow (2002) and Bodoh-Creed and Hickman (2018), typically assume unidimensional family types, supermodular valuations and identical preference rankings for tractability. Bodoh-Creed and Hickman (2018) write that *"the richness of the preferences admitted by most models building on Gale and Shapley ... makes it very difficult to include an element of endogenous student quality".* I allow general preferences and I apply a continuum framework to gain tractability. Despite the generality of my model, I obtain strong results on welfare comparisons across the mechanisms.

The second part of my work is related to papers that study assignment problems with budget-constrained agents. In particular, Che, Gale, and Kim (2013a) and Che, Gale, and Kim (2013b) show that in that environment a random assignment with resale improves aggregate welfare compared to the market equilibrium. In my model, there is no resale option for school assignment and therefore aggregate welfare comparisons are ambiguous. However, I show that under fairly general conditions random assignment improves the welfare for agents with the smallest budgets.

My work contributes to the relatively new strand of matching theory literature on 'pri-

[5] Avery and Pathak (2020) also observe that lowest-income families may prefer neighborhood assignment to open enrolment in a model with endogenously priced outside options or multiple school districts. In my model, the result is driven by the richness of families' preferences over neighborhoods and schools.

ority design' (Celebi and Flynn, 2021; Shi, 2021). These papers study optimal priority structures for general assignment mechanisms. In contrast, I am interested in the role of neighborhood priorities and its welfare implications for a particular assignment mechanism, namely the Deferred Acceptance. My results suggest that using priorities may potentially improve aggregate welfare. First, I show that with neighborhood priorities the Deferred Acceptance mechanism always generates higher aggregate welfare than neighborhood assignment. This is not necessarily true without neighborhood priorities. Second, I show that, under some conditions, such as when families have identical ordinal preferences over neighborhoods and schools, the Deferred Acceptance mechanism generates higher aggregate welfare with neighborhood priorities than without those. The last finding is in the spirit papers that show that incorporating 'signaling devices' into matching problems without money may be welfare improving (Abdulkadiroğlu, Che, and Yasuda, 2015; Coles, Cawley, Levine, Niederle, Roth, and Siegfried, 2010; Hylland and Zeckhauser, 1979; Lee and Niederle, 2015). When there are neighborhood priorities, families are allowed to signal their high valuations for schools by choosing the corresponding neighborhoods. Thus, neighborhood choices act as signaling devices in my model, potentially improving aggregate welfare.

Lastly, my work contributes to the literature on large matching markets (Abdulkadiroğlu et al., 2015; Azevedo and Leshno, 2016; Gretsky, Ostroy, and Zame, 1992,9; Kamecke, 1992; Leshno and Lo, 2017) and assignment externalities (Pycia and Yenmez, 2019; Sasaki and Toda, 1996). Unlike the continuum assignment game of Gretsky et al. (1992) and Gretsky et al. (1999), in my model there are assignment externalities: a family cares not only about her own neighborhood choice, but also those of other families since those affect the family's school assignment probabilities. Although externalities preclude the existence of competitive equilibrium in finite discrete markets, I show that a competitive equilibrium always exists in large markets with a continuum of families. Analogous results have been established in alternative matching environments with complementarities and externalities, e.g., Azevedo, Weyl, and White (2013)

Azevedo and Hatfield (2018), Che, Kim, and Kojima (2019) and Greinecker and Kah (2021). My model is potentially closest to the last paper. The authors assume that agents have continuous preferences over a superset of assignments to prove the existence of a competitive equilibrium. The assumption is abstract, and the paper does not clarify whether it is satisfied for specific matching problems. I do not impose such an assumption, but instead I prove that in my model families' expected utilities are equicontinuous in neighborhood choices,[6] which is sufficient to guarantee the existence of a competitive equilibrium. My existence proof technique can be applied more broadly for general assignment games with externalities, such as peer preferences or complementarities.

3 Preliminaries

3.1 The Continuum Model

There is a unit mass of families with a single child and a finite and equal number of neighborhoods H and schools S. There is a unique school in each neighborhood $h \in H$. Let us denote this school by $s_h \in S$. The capacity $q_h \in \mathbb{N}$ of neighborhood h is the mass of families that can reside in the neighborhood. Similarly, the capacity $q_s \in \mathbb{N}$ of school s is the mass of families that can enroll (their children) in school s. I assume that $q_h \leq q_{s_h}$ for all $h \in H$. The assumption is necessary for defining neighborhood assignment, i.e., schools need to have enough seats for all neighborhood families.

Each family has a type $v \in [0,1]^{|H|\times|S|} := V$, where $v(h,s) \in [0,1]$ denotes the type's valuation for living in neighborhood h and enrolling in school s. The economy is described by a (Borel) probability measure η over the type space V.

[6]The last result uses that school assignment probabilities under the Deferred Acceptance mechanism change continuously with families' neighborhood choices.

My model abstracts away from direct externalities or peer preferences. Namely, families' valuations over neighborhood-school pairs are exogenously given and do not depend on other families' neighborhood choices.[7] In Supplementary Appendix B, I establish the robustness of some of my main results for an extension where valuations depend on the housing values. This is motivated by that property taxes are a major source of school expenditure funds (Chetty and Friedman, 2011).

Valuations induce preference rankings, which are complete, reflexive and anti-symmetric relations on S. Let P be the space of preference rankings. Conditional on living in neighborhood h, the preference ranking $\succ_{vh} \in P$ of type v satisfies

$$v(h,s) > v(h,s') \Rightarrow s \succ_{vh} s'. \tag{1}$$

When $v(h,s) = v(h,s')$, ties are broken arbitrarily. For example, we may assume that a fixed ordering over schools is used to break ties.

Let $\bar{H} := H \cup \{0\}$, where 0 denotes the outside option of not buying a house in the school district. Neighborhood choices τ is a probability measure on $V \times \bar{H}$, with the property that

$$\tau\Big((v,h) \in V \times \bar{H} : v \in U, h \in H\Big) = \eta(U),$$

for any measurable $U \subseteq V$. The interpretation of neighborhood choices τ is that for each measurable $U \subseteq V$ and $H' \subseteq \bar{H}$, $\tau\Big((v,h) \in V \times \bar{H} : v \in U, h \in H'\Big)$ denotes the mass of families whose types are in U and who choose some neighborhood in H'. I denote the space of neighborhood choices by $\mathcal{T}$.

In general, school assignment probabilities depend on the reported preference rankings of families. I consider strategyproof school assignment mechanisms, where each family has a dominant strategy to report preferences truthfully. Thus, assuming truthful preference reports, valuations and neighborhood choices uniquely pin down the preference reports of families through equation 1 (and the tie-breaker). Let $\lambda^{\phi}_{vs}(h,\tau) \in [0,1]$

[7] In my model there are 'indirect' externalities as a family's expected utilities depend on other families' neighborhood choices through their effects on the first family's school assignment probabilities.

denote the probability that type v is assigned to school s. This probability depends on the valuation, chosen neighborhood h, the population's neighborhood choices τ and the school assignment mechanism ϕ.

Given school assignment probabilities and neighborhood price vector $p \in [0,1]^{|H|}$, the expected utility of type v choosing neighborhood $h \in H$ is equal to

$$u_v^\phi(h,\tau) - p_h.$$

where $u_v^\phi(h,\tau) := \sum_{s\in S} \lambda_{vs}^\phi(h,\tau)\ v(h,s)$. Also, let $u_v^\phi(0,\tau) := 0$ for all $v \in V$ and $\tau \in \mathcal{T}$. I now define the solution concept.

Definition 1. *For neighborhood choices $\tau \in \mathcal{T}$ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (τ, p) is a **competitive equilibrium (CE)** of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v,h) \in V \times \bar{H} : h = \arg\max_{h'\in\bar{H}}\ u_v^\phi(h',\tau) - p_{h'}\Big) = 1,$ *where* $p_0 := 0$,
2. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) \leq q_{h'}, \forall h' \in H$,
3. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0.$

The definition is standard. The first two conditions in Definition 1 are the optimality and feasibility of neighborhood choices, respectively. The third condition says that neighborhoods with excess capacity are priced at zero. This would guarantee that the sellers of vacant houses in the neighborhood have no incentives to undercut the prices.

3.2 School Assignment Mechanisms

I now describe the school assignment mechanisms and derive school assignment probabilities under each of them.

Neighborhood Assignment.

Under neighborhood assignment (NA), families are assigned to their neighborhood schools. Then, for all $s \in S, h \in H$ and $\tau \in \mathcal{T}$,

$$\lambda_{vs}^{NA}(h,\tau) = \begin{cases} 1 & \text{if } s = s_h, \\ 0 & \text{otherwise.} \end{cases}$$

Deferred Acceptance.

Deferred Acceptance for the continuum model is defined as in Azevedo and Leshno (2016) and Abdulkadiroğlu et al. (2017a). I consider two versions of the Deferred Acceptance mechanism: in the first version families do not receive higher priorities at neighborhood schools, and in the second version they do.

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences, lottery numbers and market clearing admission cutoffs, or simply cutoffs. Preferences are decided by neighborhood choices through equation 1. Lottery numbers are drawn uniformly and independently from the unit interval. Formally, neighborhood choices τ result in a probability measure G_τ over $P \times [0,1]$, given by

$$G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$ and $(r_0, r_1) \subseteq [0,1]$. Here, $G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$ is the mass of types with preferences in P' and lottery numbers in the interval (r_0, r_1).[8]

Cutoffs are derived through an iterative procedure that I describe below. For a vector

[8] The versions of Deferred Acceptance described in this section apply a single tie-breaking rule, i.e., a family has a single lottery number which is commonly used for tie-breaking at all schools. My main results (with slightly modified proofs) hold for the case of multiple tie-breaking, i.e., when different schools use different lottery numbers for a given family. The extension is discussed in Appendix D.

$c \in [0,1]^{|S|}$, the demand function $D : [0,1]^{|S|} \to [0,1]^{|S|}$ is given by

$$D_s(c) = G_\tau\Big((\succ, r) \in P \times [0,1] : r \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r \geq c_{s'}\Big).$$

In words, $D_s(c)$ is the mass of families whose lottery numbers exceed c_s, and who prefer s to any other school s' where their lottery numbers exceed $c_{s'}$. For $c \in [0,1]^{|S|}$ and $x \in [0,1]$ let $c(s,x) \in [0,1]^{|S|}$ denote the vector that differs from c only by that $c_s(s,x) = x$.

Define a sequence of vectors $(c^t)_{t=1}^{\infty}$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

As shown by Abdulkadiroğlu, Angrist, Narita, and Pathak (2017b), $(c^t)_{t\in\mathbb{N}}$ is convergent. Let $c^{DA} := \lim_{t\to\infty} c^t$ denote the **DA cutoffs**. This cutoffs depend on neighborhood choices τ, but I omit this dependence to keep notation simple. The DA cutoffs determine school assignment as follows. A family is assigned to school s if her lottery number exceeds c_s^{DA}, and she prefers s to any school where her lottery number exceeds the corresponding DA cutoff. The probability of this event is

$$\begin{aligned} \lambda_{vs}^{DA}(h,\tau) &= \min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\} \times \max\left\{\frac{\min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\} - c_s^{DA}}{\min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\}},\ 0\right\} \qquad (2) \\ &= \max\Big\{\min\left\{c_{s'}^{DA} : s' \succ_{vh} s\right\} - c_s^{DA},\ 0\Big\}. \end{aligned}$$

The first term in the middle part of equation 2 denotes the probability that v's lottery number does not exceed the cutoff at any school that she prefers more than s. The second term is the probability that her lottery number exceeds the cutoff at s, conditional on it not exceeding those in more preferred schools.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, lottery numbers, priorities and cutoffs. Again, preferences are decided by neighborhood choices through equation 1 and lottery numbers are drawn uniformly and independently from the unit interval. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Formally, neighborhood choices τ result in a probability measure G_τ on $P \times S \times [0,1]$ satisfying

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$. Thus, $G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$ equals the mass of families with preferences in P', who reside in the neighborhood of some school in $S' \subseteq S$ and whose lottery numbers are in the interval (r_0, r_1). For a vector $c \in [0,1]^{|S|}$ the demand function $D : [0,1]^{|S|} \to [0,1]^{|S|}$ is given by

$$D_s(c) = G_\tau\Big((\succ, s', r) \in P \times S \times [0,1] : r + \mathbb{1}[s' = s] \geq c_s \text{ and}$$
$$s \succ s'' \text{ for all } s'' \text{ with } r + \mathbb{1}[s' = s''] \geq c_{s''}\Big).$$

For $c \in [0,1]^{|S|}$ and $x \in [0,1]$ we let $c(s,x) \in [0,1]^{|S|}$ denote the vector that differ from c by that $c_s(s,x) = x$. Consider the sequence of vectors recursively defined by

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise} \end{cases}$$

Again, as shown by Abdulkadiroğlu et al. (2017b), the sequence is convergent. Let $c^{DN} := \lim_{t\to\infty} c^t$ denote the **DN cutoffs**. A family is assigned to school s if her priority at s plus her lottery number exceeds c_s^{DN}, and she prefers s to any school where her priority plus the lottery number exceeds the corresponding DN cutoff. From the description of DN, it can be verified that the probability of this event for a school s is equal to

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} 0 & s_h \succ_{vs} s, \\ \min\left\{c_{s'}^{DN} : s' \succ_{vs} s_h\right\} & s_h = s \\ \max\left\{ \min\left\{c_{s'}^{DN} : s' \succ_{vs} s_h\right\} - c_s^{DN},\ 0\right\} & \text{otherwise.} \end{cases} \tag{3}$$

3.3 Existence of CE

In this subsection I establish the existence of CE under the school assignment mechanisms.

I first discuss existence of a CE of NA. Under NA, families' expected utilities of choosing different neighborhoods do not depend on other families' neighborhood choices. Hence, my problem is equivalent to a continuum assignment game without externalities (Gretsky et al., 1992,9). The existence of a (unique) CE of NA is therefore guaranteed by an analogous result for continuum assignment games.

Theorem 1. *When η is non-atomic and has full support, there is a unique CE of NA.*

Proof. For any $\tau \in \mathcal{T}$, $u_v^{NA}(h,\tau) = v(h, s_h)$. Let $\tilde{\eta}$ be a probability measure on $\tilde{V} := [0,1]^{|H|}$ given by

$$\tilde{\eta}\big(\tilde{U}\big) = \eta\Big(v \in V : (v(h,s_h))_{h\in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since η has full support, so does $\tilde{\eta}$. A CE of NA corresponds to Walrasian equilibrium of the non-atomic assignment model of Gretsky et al. (1999). Therefore, the existence of a unique CE of NA follows from their Proposition 6. □

My next result establishes the existence of CE of DN and DA.

Theorem 2. *When η is absolutely continuous and has full support, there is a CE of DN and DA.*

Proof. Appendix A. □

A crucial step for the proof is establishing that school assignment probabilities change continuously with families' neighborhood choices. I use the Schauder-Tychonoff fixed point theorem to establish CE existence.

In discrete economies assignment externalities may preclude existence of CE of DN and DA. I illustrate this through an example in Appendix B. However, I show that for any level of approximation, approximate equilibria exist in any sufficiently large discrete economies. Moreover, all the welfare comparisons for the continuum case also hold (approximately) for the discrete one as as the market gets large.

4 Aggregate Welfare

In this section I compare the school assignment mechanisms in terms of aggregate (or utilitarian) welfare.[9]

Definition 2. *For two mechanisms ϕ and ψ we say that ϕ* ***creates higher aggregate welfare*** *than ψ if for arbitrary CE neighborhood choices τ^{ϕ} of ϕ and τ^{ψ} of ψ,*

$$\int u_v^{\phi}(h, \tau^{\phi}) d\tau^{\phi} \geq \int u_v^{\psi}(h, \tau^{\psi}) d\tau^{\psi}.$$[10]

My definition of aggregate welfare does not account for neighborhood prices. Therefore, it should not be interpreted as the welfare of the families only, but that of the entire economy. That is, the aggregate welfare in my model is the 'sum' of utilities of all families and house sellers, who may be thought of as passive agents in the model.

[9]In Section 5, I also compare school assignment mechanisms in terms of welfare of lowest-income families, which is a realistic comparison notion to understand the distributional consequences of school choice.

[10]The way I define aggregate welfare comparisons across the mechanisms is robust to the CE selection rule when there is a multiplicity.

4.1 DN versus NA

A major result of this paper is that DN creates unambiguously higher aggregate welfare than NA.

Theorem 3. *DN creates higher aggregate welfare than NA.*

Before proving the result I first provide some intuition behind it. Like NA, DN allows families with high valuations to enroll their children to their preferred schools by choosing the corresponding neighborhood. In addition, it provides more flexibility for families to enroll their children to schools outside of their neighborhoods when those have empty seats (i.e., 'unclaimed' by neighborhood families). In the special case where CE prices are equal under both mechanisms, it is immediate that all families prefer DN to NA, as families can choose the same neighborhood and have superior schooling options. However, this observation does not extend to the general case: when CE prices are not the same under DN and NA, some families may be worse off under DN because of price increases in their preferred neighborhoods. Such an example is provided in Appendix C. Although some families' may prefer NA to DN, Theorem 3 says that the aggregate welfare is always larger under the former mechanism. The proof uses the result that Walrasian equilibria of continuum assignment games maximize aggregate welfare (Gretsky et al., 1992). I outline the proof below.

When fixing school assignment probabilities, my model may be thought of as a continuum assignment game where families' valuations for neighborhoods are their expected utilities from choosing those. Consider an arbitrary CE of NA and DN. If families choose neighborhoods according to the CE of NA, but their expected utilities from choosing neighborhoods are calculated as if the other families choose neighborhoods according to DN and the school assignment mechanism is DN, then the corresponding aggregate welfare (in fact, the welfare of each family) would be larger than that under the CE of NA. This is true since under DN families can be assigned to their neighborhood school due to their higher priorities. Moreover, assuming that the expected utilities are as described above, families' choosing neighborhoods according to DN instead of NA would further improve aggregate welfare. This is true since DN neighborhood choices constitute a Walrasian equilibrium of the corresponding continuum assignment game, and therefore maximize aggregate welfare. For the sake of completeness, I give the formal proof below.

Proof. Let $(\tau^{DN}, p^{DN)}$ be a CE of DN. Also, let $\tilde{V} := [0,1]^{|H|}$ and $\tilde{\eta}^{DN}$ be a measure on $\tilde{V}$ given by

$$\tilde{\eta}^{DN}(\tilde{U}) = \eta\Big(v \in V : (u_v^{DN}(h, \tau^{DN}))_{h\in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since η has full support, so does $\tilde{\eta}^{DN}$. Define a measure $\tilde{\tau}^{DN}$ on $\tilde{V} \times \bar{H}$ by

$$\tilde{\tau}^{DN}\Big((\tilde{u}, h) \in \tilde{V}\times\bar{H} : \tilde{u} \in \tilde{U}, h \in H'\Big) = \tau^{DN}\Big((v, h) \in V\times\bar{H} : \big(u_v^{\phi}(h, \tau^{DN})\big)_{h\in H} \in \tilde{U}, h \in H'\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$ and $H' \subseteq H$. Then, $(\tilde{\tau}^{DN}, p^{DN})$ is a Walrasian equilibrium of the non-atomic assignment game $\tilde{\eta}^{DN}$. Hence, by Theorem 4 of Gretsky et al. (1992),

$$\tau^{DN} = \underset{\tau\in\mathcal{T}}{\arg\max} \int u_v^{DN}(h, \tau^{DN})d\tau, \tag{4}$$

$$\text{s.t. } \tau^{DN}\Big((v, h) \in V \times H : h = h'\Big) \leq q_{h'}, \text{ for all } h' \in H.$$

Hence,

$$\begin{aligned}\int u_v^{DN}(h, \tau^{DN})d\tau^{DN} &\geq \int u_v^{DN}(h, \tau^{DN})d\tau^{NA} \\ &\geq \int v(h, s_h)d\tau^{NA} = \int u_v^{NA}(v, \tau^{NA})d\tau^{NA},\end{aligned}$$

where the first inequality above follows from equation 4, and the second inequality follows from that each type is assigned to a school she weakly prefers to the neighborhood school under DN. □

4.2 DN versus DA

In this subsection I compare aggregate welfare between the two versions of the Deferred Acceptance mechanism.

The welfare comparison across DN and DA is less straightforward. Generally, each mechanism can result in a higher aggregate welfare than the other one. I show that DN outperforms DA in two special case of my model.

Assumption 1. *Suppose* $V = \{v_\alpha\}_{\alpha\in[0,1]}$, $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$ *and for almost all* $\alpha \in [0, 1]$,

- $v_\alpha(h_i, s_m) \geq v_\alpha(h_j, s_n)$ *for all* $h_i, h_j \in H, i \geq j$ *and* $s_m, s_n \in S, m \geq n$,
- $v_\alpha(h_1, s_1) = 0$ *and* $v_\alpha(h_i, s_m) - v_\alpha(h_j, s_n)$ *is increasing in* α *for all* $h_i, h_j \in H, i \geq j$ *and* $s_m, s_n \in S, m \geq n$.

In other words, Assumption 1 says that families, neighborhoods and schools are indexed, all families have a higher valuation for higher indexed neighborhoods and schools and these valuations have increasing differences in $(\alpha; i, j)$. The index of the family may reflect the child's ability, parent's education level, family income or some combination of those. The index of a neighborhood or a school reflects its quality.

The assumptions of same ordinal preference rankings and increasing differences of valuations are also made by papers like Calsamiglia et al. (2015), Xu (2019) and Avery and Pathak (2020). However, those papers also assume that families only care about schools, while I allow valuations for neighborhoods too. None of those works provides welfare comparisons between the two version of the Deferred Acceptance mechanism.

My next assumption relaxes increasing differences, but imposes common and additively separable valuations for neighborhoods.

Assumption 2. *Suppose $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$, and there are constants $(e_i)_{i=1}^N$ such that for almost all $v \in V$,*

- $v(h, s_m) \geq v(h, s_n)$ *for all* $h \in H$ *and* $s_m, s_n \in S, m \geq n$,
- $v(h_i, s) - v(h_j, s) = e_i - e_j \geq 0$ *for all* $h_i, h_j \in H, i \geq j$ *and* $s \in S$.

Theorem 4. *Suppose either Assumption 1 or 2 is satisfied. Then, DN creates higher aggregate welfare than DA.*

Proof. The proof of the first part (for Assumption 1) is in Appendix A.2. I now prove the second part (for Assumption 2.

Let (τ^{DN}, p^{DN}) and (τ^{DA}, p^{DA}) be arbitrary CE of DN and DA, respectively. Suppose $\sum_{h \in H} q_h = 1$. This is without loss of generality since we can add a neighborhood and a school with large enough capacities that are undesirable for all families, or we can add families that are indifferent across all neighborhoods and schools

First, I compute admission cutoffs and school assignment probabilities under DA. Since families have identical ordinal rankings over the schools, all families will demand s_N. Since the school's capacity is q_{s_N}, its cutoff shall be equal to $c_{s_N}^{DA} = \max\{1 - q_{s_N}, 0\}$. Then, all families with lottery numbers smaller than the cutoff $c_{s_N}^{DA}$ will demand school s_{N-1}. Hence, the school's cutoff shall equal to $c_{s_{N-1}}^{DA} = \max\big\{1 - q_{s_N} - q_{s_{N-1}}, 0\big\}$. By an induction argument, we can show that the cutoff at s_k, for any $k \in \{2, ..., N\}$, is equal to

$$c_{s_k}^{DA} = \max\Big\{1 - \sum_{j=k}^{N} q_{s_j}, 0\Big\}.$$

Hence, the probability that a family assigned to s_k or a better school is equal to $\min\big\{\sum_{j=k}^N q_{s_j}, 1\big\}$.

Now consider DN. All families will demand school s_N. Applicants residing in h_N will be guaranteed admission at the school, and the remaining $(q_{s_N} - q_{h_N})$ seats will be be assigned among $(1-q_{h_N})$ applicants not residing in h_N. Hence, the cutoff of the school shall equal to $c_{s_N}^{DN} = \max\left\{1 - \frac{q_{s_N} - q_{h_N}}{1-q_{h_N}}, 0\right\} = \frac{1-q_{s_N}}{1-q_{h_N}}$. All applicants who reside outside of school s_N and whose lottery numbers are smaller than the cutoff $c_{s_N}^{DN}$ will demand school s_{N-1}. Among those applicants, the ones residing in h_{N-1} will be guaranteed admission at the school. Hence, the school will only have $q_{s_{N-1}} - q_{h_{N-1}} c_{s_N}^{DN}$ remaining seats to the remaining ones. Assuming the cutoff c_{N-1}^{DN} is non-zero, it shall satisfy

$$c_{s_N}^{DN} - c_{s_{N-1}}^{DN} = \frac{q_{s_{N-1}} - q_{h_{N-1}} c_{s_N}^{DN}}{1 - q_{h_N} - q_{h_{N-1}}}.$$

Therefore,

$$\begin{aligned}
c_{s_{N-1}}^{DN} &= c_{s_N}^{DN} - \frac{q_{s_{N-1}} - q_{h_{N-1}} c_{s_N}^{DN}}{1 - q_{h_N} - q_{h_{N-1}}} \\
&= c_{s_N}^{DN}\left(1 + \frac{q_{h_{N-1}}}{1 - q_{h_N} - q_{h_{N-1}}}\right) - \frac{q_{s_{N-1}}}{1 - q_{h_N} - q_{h_{N-1}}} \\
&= c_{s_N}^{DN}\left(\frac{1 - q_{h_N}}{1 - q_{h_N} - q_{h_{N-1}}}\right) - \frac{q_{s_{N-1}}}{1 - q_{h_N} - q_{h_{N-1}}} \\
&= \left(\frac{1 - q_{s_N}}{1 - q_{h_N}}\right)\left(\frac{1 - q_{h_N}}{1 - q_{h_N} - q_{h_{N-1}}}\right) - \frac{q_{s_{N-1}}}{1 - q_{h_N} - q_{h_{N-1}}} = \frac{1 - s_N - s_{N-1}}{1 - q_{h_N} - q_{h_{N-1}}}.
\end{aligned}$$

Hence,

$$c_{s_{N-1}}^{DN} = \max\left\{\frac{1 - s_N s_{N-1}}{1 - (q_{h_N} - q_{h_{N-1}})},\ 0\right\}.$$

By an induction argument, we can show that for any $k \in \{2, ..., N\}$,

$$c_{s_k}^{DN} = \max\left\{\frac{1 - \sum_{j=k}^{N} q_{s_j}}{1 - \sum_{j=k}^{N} q_{h_j}},\ 0\right\}.$$

The probability that a family residing in neighborhood h_i is assigned to s_k or a better school is equal to one if $i \geq k$ (since the neighborhood school can be guaranteed) and is equal to

$$\min\left\{1 - \frac{1 - \sum_{j=k}^{N} q_{s_j}}{1 - \sum_{j=k}^{N} q_{h_j}}, 1\right\} = \min\left\{\frac{\sum_{j=k}^{N}\left(q_{s_j} - q_{h_j}\right)}{1 - \sum_{j=k}^{N} q_{h_j}}, 1\right\},$$

if $i < k$.

Consider the 'strategy' of a family where she chooses neighborhood h_j with probability q_{h_j} for all $j \in \{1, 2, ..., N\}$. Let k be such that $\sum_{j=k}^{N} q_{s_j} \leq 1$. Then, the probability that a family employing this strategy is assigned to s_k or a better is equal to

$$\sum_{j=k}^{N} q_{h_j} + \left(1 - \sum_{j=k}^{N} q_{h_j}\right) \left(\frac{\sum_{j=k}^{N} \left(q_{s_j} - q_{h_j}\right)}{1 - \sum_{j=k}^{N} q_{h_j}}\right) = \sum_{j=k}^{N} q_{s_j}$$

The right hand side of the equation is the probability of being assigned to s_k or a better school under DA. Hence, any family can replicate the DA school assignment probabilities by employing the strategy mention above. Formally, neighborhood choices τ corresponding to almost all types playing this strategy is given by

$$\tau\big(U \times \{h_j\}\big) = \eta(U) \times q_{h_j},$$

for all measurable $U \subseteq V$ and $h_j \in H$. Finally, note that τ is a 'feasible' strategy, or

$$\tau\Big((v, h) \in V \times H : h = h'\Big) \leq (=) q_{h'}, \text{ for all } h' \in H.$$

Therefore,

$$\int u_v^{DN}(h, \tau^{DN}) d\tau^{DN} \geq \int u_v^{DN}(h, \tau^{DN}) d\tau = \int u_v^{DA}(h, \tau^{DA}) d\tau^{DA},$$

where the first inequality follows from equation 4 (and 'feasibility' of τ), and the second inequality follows from strategy τ replicates DA assignment probabilities.[11] This complete the proof of the second part of Theorem 4. □

Assumptions 1 and 2 are restrictive as they imply that families have common ordinal preferences over neighborhoods and schools. Such a preference structure has been commonly imposed by the previous works on the topic to gain tractability (Avery and Pathak, 2020; Calsamiglia et al., 2015; Xu, 2019). My work too uses the assumption

[11] Neighborhood choices are not necessarily the same under τ and τ^{DA}, however since the families' valuations over neighborhoods are identical, only school assignment is relevant for the aggregate welfare.

for tractability, and I do not provide more general conditions to give stronger welfare comparisons across DN and DA. Therefore, the superior welfare performance of DN compared to DA in Theorem 4 should be interpreted with care. In Appendix C I provide two counterexamples, where Assumption 1 and 2 fail, and DA outperforms DN in terms of aggregate welfare.

Despite these limitations, my results that DN creates higher aggregate welfare than DA in certain special cases may serve as a potential justification of the fact that school district typically grant higher priorities to neighborhood students.

5 Budget Constraints and Lowest-Income Family Welfare

In this section a family's type is her valuations $v \in [0,1]^{|H|\times|S|} := V$ for neighborhoods and schools and her budget (or income) $b \in [0,1]$, which denotes the maximum amount she can pay for a neighborhood. The economy is described by a probability measure η on $V \times [0,1]$. In this section I study the welfare of lowest-income families, i.e., those whose budget is zero (or sufficiently close to zero).[12]

Neighborhood choices τ is a probability measure on $V \times [0,1] \times \bar{H}$ satisfying $\tau\big(U \times I \times \bar{H}\big) = \eta\big(U \times I\big)$ for all measurable $U \times I \subseteq V \times [0,1]$.

Definition 3. *For neighborhood choices τ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair*

[12] Instead of modelling budget constraints, an alternative way of incorporating income levels would be through assuming that families are differentiated by an income parameter, and those with a smaller income parameter have a 'higher valuation for money' (Avery and Pathak, 2020; Epple and Romano, 1998; Xu, 2019). My results would extend to that environment if lowest-income families would have sufficiently high valuation for money so that they would choose the cheapest neighborhood in equilibrium. I find it natural to model income levels through budget, and therefore, all the analysis is described for this setup.

*(τ, p) is a **competitive equilibrium (CE)** of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = \arg\max_{h' \in \bar{H}_b} \; u^{\phi}_{vb}(h', \tau) - p_{h'}\Big) = 1,$
   where $p_0 := 0$ and $\bar{H}_b := \{h \in \bar{H} : p_h \leq b\}$,
2. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = h'\Big) \leq q_{h'}, \forall h' \in H.$
3. $\tau\Big((v,b,h) \in V \times [0,1] \times \bar{H} : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0.$

In general, existence of CE is not guaranteed under any of the studied mechanisms. This is an immediate consequence of an analogous result for Walrasian equilibria of assignment games with budget constraints (e.g., see van der Laan, Talman, and Yang (2018)). I restrict attention to economies that admit a CE.

Definition 4. *A mechanism ϕ **creates higher welfare for lowest-income families** than mechanism ψ if for arbitrary CE (τ^{ϕ}, p^{ϕ}) of ϕ and (τ^{ψ}, p^{ϕ}) of ψ, there is a number $\bar{b} > 0$, such that for any measurable $U \times I \subseteq V \times [0, \bar{b}]$,*

$$\int_{U \times I} \big[u^{\phi}_{vb}(h, \tau^{\phi}) - p^{\phi}_h\big] d\tau^{\phi} \geq \int_{U \times I} \big[u^{\psi}_{vb}(h, \tau^{\psi}) - p^{\psi}_h\big] d\tau^{\psi}.$$

Throughout this section I assume that $\sum_{h \in H} q_h \geq 1$. This is without loss of generality, since otherwise there will be no zero-priced neighborhood in equilibrium, making the analysis for lowest-income family welfare trivial as they all will choose the outside option 0.

Definition 5. *For a CE (τ^{ϕ}, p^{ϕ}) of ϕ, we say neighborhood h is **underdemanded** if $p^{\phi}_h = 0$. Similarly, for $\phi \in \{DN, DA\}$, we say school s is **underdemanded** if $c^{\phi}_s = 0$.*

My next result gives a sufficient condition under which lowest-income families prefer DN and DA to NA.

Theorem 5. *The following is true:*

1. *If underdemanded neighborhoods under NA are also underdemanded under DN, then DN creates higher welfare for lowest-income families than NA.*

2. *If underdemanded neighborhoods under NA are also underdemanded under DA, and moreover, these neighborhoods have underdemanded schools, then DN creates higher welfare for lowest-income families than NA.*

For CE (τ^ϕ, p^ϕ) of a mechanism ϕ, let H^ϕ_- and S^ϕ_- denote the set of underdemanded neighborhoods and schools at (τ^ϕ, p^ϕ), respectively. I now prove Theorem 5.

Proof. First, I prove point 1. Let $\bar{b} := \min_{h \in H \setminus H^{NA}_-} p^{NA}_h/2$. Then, by Definition 3,

$$\tau^{NA}\Big((v, b, h) \in V \times [0,1] \times \bar{H} : b \in [0, \bar{b}], h \in H^{NA}_-\Big)$$
$$= \eta\Big((v, b) \in V \times [0,1] : b \in [0, \bar{b}]\Big). \tag{5}$$

In words, equations 5 says that almost all families with budgets in $[0, \bar{b}]$ choose a neighborhood in H^{NA}_- under τ^{NA}. Consider an arbitrary measurable $U \times I \subseteq V \times [0, \bar{b}]$. Then,

$$\int_{U \times I} \big[u^{DN}_{vb}(h, \tau^{DN}) - p^{DN}_h\big] d\tau^{DN} \geq \int_{U \times I} \big[u^{DN}_{vb}(h, \tau^{DN}) - p^{DN}_h\big] d\tau^{NA}$$
$$= \int_{U \times I} u^{DN}_{vb}(h, \tau^{DN}) d\tau^{NA} \geq \int_{U \times I} v(h, s_h) d\tau^{NA}$$
$$= \int_{U \times I} u^{NA}_{vb}(h, \tau^{NA}) d\tau^{NA} = \int_{U \times I} \big[u^{NA}_{vb}(h, \tau^{NA}) - p^{NA}_h\big] d\tau^{NA}. \tag{6}$$

The first inequality in equation 6 follows from equation 5 and the optimality of neighborhood choices. The first equality follows from that $H^{NA}_- \subseteq H^{DN}_-$. The second inequality follows from that under DN each family is guaranteed a weakly better school than the neighborhood school. The last equality again follows from equation 5.

I now prove point 2. Let $\bar{b}$ be as before and consider an arbitrary measurable $U \times I \subseteq V \times [0, \bar{b}]$. Then,

$$\int_{U \times I} \big[u^{DA}_{vb}(h, \tau^{DA}) - p^{DA}_h\big] d\tau^{DA} \geq \int_{U \times I} \big[u^{DA}_{vb}(h, \tau^{DA}) - p^{DA}_h\big] d\tau^{NA}$$

$$= \int_{U\times I} u_{vb}^{DA}(h,\tau^{DA})d\tau^{NA} \geq \int_{U\times I} v(h,s_h)d\tau^{DA}$$

$$= \int_{U\times I} u_{vb}^{NA}(h,\tau^{NA})d\tau^{NA} = \int_{U\times I} \left[u_{vb}^{NA}(h,\tau^{NA}) - p_h^{NA}\right]d\tau^{NA}. \tag{7}$$

The second inequality in equation 7 follows from that $\left\{s_h \in S : h \in H_{-}^{NA}\right\} \subseteq \bar{S}^{DA}$. The condition along with equation 5 implies that schools in neighborhoods chosen by types in $U \times I$ are underdemanded. Thus, under DA each of these families is guaranteed a weakly better school than the neighborhood school. The arguments for other steps in equation 7 are as in point 1. □

In other words, the conditions in in Theorem 5 say the following. First, they say that underdemanded neighborhoods remain underdemanded once we switch from a CE of NA to a CE of DN or DA. Second, they say that schools in underdemanded neighborhoods are underdemanded. The conditions are intuitive, and later in this section I show that they are satisfied for some natural special cases (Corollaries 1 and 2).

Conditions in Theorem 5 are sufficient, but not necessary. That is, there are economies that do not satisfy the conditions, but where all lowest-income families prefer DN or DA to NA. However, for any such economy, or for any economy in general, one can find another economy that is arbitrarily close to the original one, such that either the conditions in Theorem 5 hold, or a positive measure of zero-income families prefer NA to DN or DA. Thus, in a sense, the conditions in Theorem 5 are necessary if we also require 'robustness' of lowest-income family welfare comparisons to small perturbations.

Theorem 6. *Consider an arbitrary economy η' and $\epsilon > 0$.*

1. *There is an economy η satisfying,*

$$\|\eta - \eta'\|_2 < \epsilon$$

(where $\|\cdot\|_2$ denotes the L^2 norm), such that $H_-^{NA} \subseteq H_-^{DN}$, or DN does not create higher welfare for lowest-income families than NA.

2. *There is an economy η satisfying*

$$\|\eta - \eta'\|_2 < \epsilon,$$

such that $H_-^{NA} \subseteq H_-^{DA}$ and $\left\{s_h \in S : h \in H_-^{NA}\right\} \subseteq S_-^{DA}$, or DA does not create higher welfare for lowest-income families than NA.

Proof. Appendix A.3. □

I finish this section by showing that the conditions is Theorem 5 are satisfied for natural special cases. The first case assumes common ordinal preference rankings over neighborhoods and schools.

Assumption 3. *Suppose $H = \{h_i\}_{i=1}^N, S = \{s_i\}_{i=1}^N$, and for almost all $v \in V$,*

- *$v(h, s_m) \geq v(h, s_n)$ for all $h \in H$ and $s_m, s_n \in S, m \geq n$,*
- *$v(h_i, s) \geq v(h_j, s)$ for all $h_i, h_j \in H, i \geq j$ and $s \in S$.*

Note that Assumption 3 is weaker than Assumptions 1 and 2.

Corollary 1. *Suppose Assumption 3 is satisfied. Then, DN and DA create higher welfare for lowest-income families than NA.*

Proof. Appendix A.4. □

The result is intuitive: with a common ordinal preferences rankings over neighborhoods and schools, the least preferred neighborhoods and schools are the underdemanded ones for all school assignment mechanisms. Thus, the conditions Theorem 5 are satisfied.

Although widely applied for tractability (e.g., Abdulkadiroğlu, Che, and Yasuda (2011); Avery and Pathak (2020); Calsamiglia et al. (2015); Neilson, Akbarpour, Kapor, van Dijk, and Zimmerman (2020); Xu (2019)), the assumption of common ordinal preference rankings is restrictive. My next result (Corollary 2) provides another condition that guarantees that families prefer DA over NA. We say an economy is uniform if each valuation profile is equally likely. Formally, η is a **uniform economy** if for each measurable $U \times I \subseteq V \times [0,1]$ and $U' \times I \subseteq V \times [0,1]$, U and U' have the same Lebesgue measure only if $\eta\big(U \times I\big) = \eta\big(U' \times I\big)$.

Suppose $\sum_{h \in H} q_h = 1$ and

$$\bar{h} \in \arg\max_{h \in H} q_h \Rightarrow q_{s_{\bar{h}}} \in \arg\max_{s \in S} q_s.$$

In words, the first condition says that the total capacity at neighborhoods equals to the total mass of families, and the second condition says that largest neighborhoods have the largest schools. I show that the uniform economy satisfies the second part of conditions in Theorem 5 for this special case, and therefore DA creates higher welfare for lowest-income families than NA.

Corollary 2. *Let η be the uniform economy. Then, DA creates higher welfare for lowest-income families than NA.*

Proof. Appendix A.5. $\square$

The proof uses the fact that in the uniform economy the underdemanded neighborhoods and schools are the ones with the largest capacities.

The uniform economy framework is commonly applied in matching theory literature to obtain analytical results without restricting the preference domain (Abdulkadiroğlu, Che, Pathak, Roth, and Tercieux, 2020a; Che and Tercieux, 2017; Grigoryan, 2020). The uniform economy can be thought of as an 'average' economy. Hence, the result may be interpreted as that ‘on average’ lowest-income income families prefer DA over NA.

6 Simulations

I compare school assignment mechanisms in a simulated environment 1000 students, 10 neighborhoods and 10 schools. The valuation of family f for the joint assignment to neighborhood h and school s is equal to

$$v_f(h, s) = \alpha U_h + (1-\alpha)U_s + \beta \mathbb{1}[s = s_h] + \epsilon_{fhs},$$

where

- U_h is the common valuation for neighborhood h,
- U_s is the common valuation for school s,
- ϵ_{fhs} is the idiosyncratic valuation of family f for the joint assignment to h and s,
- α and β are parameters.

Values of U_h, U_s and ϵ_{fhs} are iid uniform draws from the unit interval. The capacity of school s is equal to $100 + \kappa_s$, where κ_s is a random draw from the set $\{1, 2, ..., 100/\gamma\}$. Thus, a larger value of γ means a smaller variance in schools’ capacities. I report simulations results for the following parameters: $\alpha \in \{0, 0.5, 1\}$, $\beta \in \{0.1, 0.2\}$ and $\gamma \in \{2, 4\}$.

Table 4 reports the percentage gains or losses in aggregate welfare under DN and DA compared to NA. As the theory predicts, DN always generate larger aggregate welfare than NA. The average aggregate welfare gains are 2.40%. Those gains are larger when neighborhood schools are less desirable (i.e., β is smaller) and when schools have more

α	β	γ	DN	DA
0	0	2	4.96	−3.75
		4	3.17	−7.93
	0.1	2	2.73	−9.88
		4	1.54	−14.53
	0.2	2	2.01	−13.30
		4	0.69	−18.26
0.5	0	2	4.32	1.50
		4	3.19	−0.43
	0.1	2	2.10	−3.87
		4	1.13	−5.52
	0.2	2	0.98	−5.79
		4	0.49	−7.82
1	0	2	5.57	5.65
		4	5.55	5.58
	0.1	2	1.82	1.61
		4	1.55	1.54
	0.2	2	0.71	0.55
		4	0.70	0.55
Average			2.40	−4.12

Table 1: Aggregate welfare, % gains/losses compared to NA

seats (i.e., γ is larger). The table also illustrates that NA and DA are not comparable in terms of aggregate welfare: the former mechanism performs better for smaller values of α, while the latter mechanism performs better for larger ones. Here is the intuition behind the comparative statics. When $\alpha = 0$, families' preferences for schools are aligned. In that case, NA (and also DN) allows families' with highest cardinal valuations for schools to guarantee admission there by choosing the corresponding neighborhood. Hence, NA generates higher aggregate welfare than DA. In contrast, when $\alpha = 1$, families have no common valuations for schools, and preferences for schools are not aligned. Hence, DA (and also DN) manages to assign almost all families to their most preferred schools, and generates higher aggregate welfare than NA.

Finally, Table 5 illustrates how DN and DA compare to NA in terms of welfare of lowest-income family.

The welfare of lowest-income families is computed by assuming that 10 out of 1000 individuals have budgets of 0.05 and the remaining ones have infinite budgets.[13] As the table illustrates, DN and DA create larger welfare for lowest-income families compared to NA. The average gains are 26.51% and 38.25%, respectively. Thus, simulations show that the superior performance of the Deferred Acceptance mechanism in terms of lowest-income families' welfare extends beyond the special cases in Corollaries 1 and 2.

[13] I restrict attention to this simple case for tractability: in general, as discussed in Section 5, CE may not even exist.

α	β	γ	DN	DA
0	0	2	47.80	68.40
		4	37.54	62.58
	0.1	2	29.24	47.16
		4	21.68	41.23
	0.2	2	24.43	41.62
		4	15.75	35.64
0.5	0	2	35.78	46.67
		4	29.80	43.11
	0.1	2	24.22	31.55
		4	13.64	31.14
	0.2	2	13.36	19.85
		4	8.94	18.71
1	0	2	47.05	50.64
		4	46.77	50.34
	0.1	2	28.93	31.72
		4	28.45	32.42
	0.2	2	12.10	17.49
		4	11.62	18.19
Average			26.51	38.25

Table 2: Lowest-income welfare, % gains compared to NA

7 Discussion

My findings suggest that the Deferred Acceptance mechanism has superior welfare and distributional properties compared to neighborhood assignment. The results potentially justify the mechanisms' widespread application for school assignment.

School choice programs take diverse forms, including open enrollment, expansion of magnet or charter schools, private schools and voucher programs. Hence, my findings should not be interpreted as arguments for school choice programs in general, but arguments for open enrollment, potentially through the Deferred Acceptance mechanism. Arguments against (and for) other school choice programs are numerous. For example, Epple and Romano (2003) show that voucher programs may lead to higher ability stratification (at schools) compared to neighborhood assignment, which, in turn, leads to more stratification compared to open enrollment. This, as the authors describe it as another example of the truism that *"all choice programs are not alike"*. A different argument has been made against charter schools by Zheng (2019). She shows that opening a charter school may hurt low-income families through its effect on neighborhood prices. Other papers provide arguments against open enrollment through some alternative school assignment mechanism. For example, the well-studied Immediate Acceptance mechanism, also known as the 'Boston' mechanism, has been criticized on the grounds that it is not strategyproof: families have incentives to 'game the system' by misreporting preferences to obtain better choices (Abdulkadiroğlu and Sönmez, 2003). Moreover, the Immediate Acceptance mechanism may exacerbate inequalities as low-income families might be disproportionately hurt if they are worse at 'gaming the system' (Pathak and Sönmez, 2008) or if they have worse outside options (Calsamiglia et al., 2015; Neilson et al., 2020). In the Supplementary Appendix A.3 of this paper I show that, when there are neighborhood priorities, the lowest-income families may prefer Deferred Acceptance to Immediate Acceptance. Although I do not directly model heterogeneous outside options, neighborhood schools act as outside options in my model. This is because those schools are guaranteed for neighborhood applicants due to neighborhood priorities. Therefore, my result is analogous to those in Calsamiglia et al. (2015) and Neilson et al. (2020).

In my model families' valuation for neighborhood-schools pairs are exogenously given and do not depend on other families' neighborhood choices. However, in reality there are multiple sources of endogenous valuations or externalities. For example, school quality oftentimes depends on the level of local public expenditure, which is typically financed through property taxes (Chetty and Friedman, 2011). Families may also have direct preferences for higher-income neighbors, better performing peers (Bachas, Fonseca, and Pakzad-Hurson, 2021), or higher representation of their own race or ethnic group (Bridge and Blackman, 1978; Glazerman, 1998). Matching models with general externalities and rich preference domains are intractable due to the non-existence of desirable solution concepts (Sasaki and Toda, 1996) or the computational complexity issues (Ronn, 1990). Tractable analyses are possible when one puts restriction on the nature of externalities. For example, in the context of education economics, papers allowing peer effects typically study a stylized setup with ordered family types and simple peer preferences where all families' prefer the higher-type peers (e.g., Epple and Romano (2003), Calsamiglia et al. (2015), Barseghyan et al. (2013) and Avery and Pathak (2020)).

To understand the possible implications of endogenous valuations, in the Supplementary Appendix B, I extend the model to allow for local public financing, so that valuations for schools depend on the housing values in the corresponding neighborhood. I find that my findings on the aggregate welfare comparisons across the Deferred Acceptance mechanism and neighborhood assignment may not extend to this environment. Under Deferred Acceptance (with or without neighborhood priorities), schools at higher priced neighborhoods will attract applicants from other neighborhoods (because of the school spending financed by neighborhood families). This may diminish social welfare if those schools are a bad match for the applicants absent the school spending.[14] However, I show that most of my other main results, including the sufficiency conditions for comparing lowest-income family welfare (i.e., Theorem 5), extend to the environment with local public financing.

My work restricts attention to the setting with fixed outside options. In contrast, Avery and Pathak (2020) study a setting with endogenously priced outside options, which for example can be motivated by having multiple school districts. In their setting, moving from neighborhood assignment to open enrollment may increase the price of the cheapest neighborhood as open enrollment makes that neighborhood more attractive. This may hurt lowest-income family welfare as they will be forced to leave the school district. In my model with fixed outside options, if there is enough housing for all families, cheapest neighborhoods are always priced at zero. Hence, my model does not account for the possibility of hurting lowest-income families through the channel described by Avery and Pathak (2020). However, my results would extend to the setting with endogenously priced outside options if for example lowest-income families are guaranteed public housing.

There are other arguments (both in favor or against) on school choice that my work does not address. For example, I do not consider schools' incentives to improve education quality, whereas proponents consider it as a major argument in favor of school choice. They argue that parental choice enhances school quality through competitive pressures (Chubb and Moe, 1990; Friedman, 1962; Hoxby, 2003). My work also ignores the possibility of sorting on dimensions other than income (O'Neil, 1996; Smith, 1995). For example, when families have same-race preferences, parental choice may exacerbate racial segregation, which is another major concern in public policy of school choice.

[14]This finding is analogous to that of Barseghyan et al. (2013), who show that in a model with peer preferences and endogenous school quality open enrollment may reduce aggregate welfare compared to neighborhood assignment.

Despite the potential limitations above, my results provide a unique theoretical justification for using the Deferred Acceptance mechanism as an alternative to neighborhood assignment by establishing its superior welfare and distributional properties under general conditions. Additionally, I develop a theoretical framework that can be used for future research and potential extensions which would address the limitations above.

References

Abdulkadiroğlu, A., N. Agarwal, and P. A. Pathak (2017a): "The Welfare Effects of Coordinated Assignment: Evidence from the NYC HS Match," *American Economic Review*, 107(12).

Abdulkadiroğlu, A., J. D. Angrist, Y. Narita, and P. A. Pathak (2017b): "Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation," *Econometrica*, 85, 1373–1432.

Abdulkadiroğlu, A., Y.-K. Che, P. Pathak, A. Roth, and O. Tercieux (2020a): "Efficiency, Justified Envy, and Incentives in Priority-based Matching," *American Economic Review: Insights*.

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2011): "Resolving conflicting preferences in school choice: The "Boston" mechanism reconsidered," *American Economic Review*, 101(1), 399–410.

Abdulkadiroğlu, A. and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Abdulkadiroğlu, A. A., P. A. Pathak, J. Schellenberg, and C. R. Walters (2020b): "Do Parents Value School Effectiveness?" *American Economic Review*.

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2015): "Expanding "Choice" in School Choice," *American Economic Journal: Microeconomics*, 7, 1–42.

Avery, C. and P. A. Pathak (2020): "The distributional consequences of public school choice," American Economic Review, forthcoming.

Azevedo and Hatfield (2018): "Existence of Equilibrium in Large Matching Markets with Complementarities," Working Paper.

Azevedo, E. M. and J. D. Leshno (2016): "A Supply and Demand Framework for Two-sided Matching Markets," *Journal of Political Economy*, 124, 1235–1268.

Azevedo, E. M., G. E. Weyl, and A. White (2013): "Walrasian Equilibrium in Large, Quasilinear Markets," *Theoretical Economics*, 108, 3154–3169.

Bachas, N. M., R. Fonseca, and B. Pakzad-Hurson (2021): "Do Peer Preferences Matter in School Choice Market Design? Theory and Evidence," Working Paper.

Barseghyan, L., D. Clark, and S. Coate (2013): "Peer Preferences, School Competition, and the Effects of Public School Choice," *American Economic Journal: Economic Policy*, 11, 124–158.

Bedrick, J. and L. Burke (2015): "Breaking the link between home prices and school quality; A decent education shouldn't require living in a ritzy neighborhood," https://www.politico.com/magazine/story/2015/09/the-poor-deserve-good-schools-too-213141.

Billingsley, P. (2013): *Convergence of Probability Measures*, John Wiley Sons.

Bodoh-Creed, A. L. and B. R. Hickman (2018): "College Assignment as a Large Contest," *Journal of Economic Theory*, 175, 88–126.

Bridge, G. R. and J. Blackman (1978): "Family Choice in Schooling," A Study of Alternatives in American Education, Vol. IV.

Burgess, S., E. Greaves, A. Vignoles, and D. Wilson (2011): "Parental Choice of Primary School in England: What Types of School do Different Types of Family Really Have Available to Them?" *Policy Studies*, 32, 531–547.

Calsamiglia, C., F. Martínez-Mora, and A. Miralles (2015): "School Choice Mechanisms, Peer Effects and Sorting," Leicester, Department of Economics.

Celebi, O. and J. P. Flynn (2021): "Priority Design in Centralized Matching Markets," Working Paper.

Che, Y., J. Kim, and F. Kojima (2019): “Stable Matching in Large Economies,” *Econometrica*, 87, 65–110.

Che, Y.-K., I. Gale, and J. Kim (2013a): “Assigning Resources to Budget-constrained Agents,” *Review of Economic Studies*, 88, 73–107.

——— (2013b): “Efficient Assignment Mechanisms for Liquidity-constrained Agents,” *International Journal of Industrial Organization*, 31, 659–665.

Che, Y.-K. and O. Tercieux (2017): “Top Trading Cycles in Prioritized Matching: An Irrelevance of Priorities in Large Markets,” Manuscript, Columbia Univ. and Paris School Econ.

Chetty, R. and J. N. Friedman (2011): “Does Local Tax Financing of Public Schools Perpetuate Inequality?” *In National Tax Association Proceedings*, 103, 112–118.

Chubb, J. E. and T. M. Moe (1990): “Politics, Markets, and America’s Schools,” Brookings Institution Press.

Chung, I. H. (2015): “School Choice, Housing Prices, and Residential Sorting: Empirical Evidence from Inter-and intra-district Choice,” *Regional Science and Urban Economics*, 52, 39–49.

Coles, P., J. Cawley, P. B. Levine, M. Niederle, A. E. Roth, and J. J. Siegfried (2010): “The Job Market for New Economists: A Market Design Perspective,” *Journal of Economic Perspectives*, 24, 187–206.

Coons, J. E. and S. D. Sugarman (1978): “Education by choice: The case for family control,” University of California Press.

Daley, B. (1999): “Plan drops race role in enrollment, compromise misses point, critics say,” Boston Globe B1.

Dur, U., S. D. Kominers, P. A. Pathak, and T. Sönmez (2018): “Reserve Design: Unintended Consequences and the Demise of Boston’s Walk Zones,” *Journal of Political Economy*, 6, 2457–2479.

Epple, D. and R. Romano (1998): “Competition Between Private and Public Schools, Vouchers, and Peer Group Effects,” *American Economic Review*, 88(1), 33–62.

Epple, D. and R. E. Romano (2003): "Neighborhood Schools, Choice, and the Distribution of Educational Benefits," *In The economics of school choice, University of Chicago Press*, 227–286.

Friedman, M. (1962): "Capitalism and Freedom: With the Assistance of Rose D. Friedman," *University of Chicago Press.*

Fuller, B. (1996): "School Choice: Who Gains, Who Loses?" *Issues in Science and Technology*, 12, 61–67.

Gale, D. and L. S. Shapley (1962): "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 69, 9–15.

Glazerman, S. M. (1998): "School Quality and Social Stratification: The Determinants and Consequences of Parental School Choice," Dissertation, University of Chicago.

Greinecker, M. and C. Kah (2021): "Pairwise Stable Matching in Large Economies," *Working Papers in Economics and Statistics.*

Gretsky, N. E., J. M. Ostroy, and W. R. Zame (1992): "The Monatomic Assignment Model," *Economic Theory*, 2, 103–127.

——— (1999): "Perfect Competition in the Continuous Assignment Model," *Journal of Economic Theory*, 88, 60–118.

Grigoryan, A. (2020): "Top Trading Cycles with Reordering: Improving Priorities in School Choice," Working paper.

——— (2022): "On the Convergence of Deferred Acceptance in Large Matching Markets," Working Paper.

Hoxby, C. (2003): "School Choice and School Productivity (Or, Could School Choice be a Rising Tide that Lifts All Boats)," in *The Economics of School Choice*, ed. by C. Hoxby, Chicago: University of Chicago Press.

Hylland, A. and R. J. Zeckhauser (1979): "The Efficient Allocation of Individuals to Positions," *Journal of Political Economy*, 87(2), 293–314.

Kamecke, U. (1992): "On the Uniqueness of the Solution to a Large Linear Assignment Problem," *Journal of Mathematical Economics*, 21, 509–521.

Kane, T. J., S. K. Riegg, and D. O. Staiger (2006): "School Quality, Neighborhoods, and Housing prices," *American Law and Economics Review*, 8, 183–212.

Lee, K. (1997): "An Economic Analysis of Public School Choice Plans," *Journal of Urban Economics*, 41, 1–22.

Lee, S. and M. Niederle (2015): "Propose with a Rose? Signaling in Internet Dating Markets," *Experimental Economics*, 18, 731–755.

Leshno, J. and I. Lo (2017): "The Simple Structure of Top Trading Cycles in School Choice," Working Paper.

Menino, T. (2012): "Recommendation to Implement a New BPS Assignment Algorithm," State of the City, Januar 17, http://www.cityofboston.gov/.

Neilson, C., M. Akbarpour, A. Kapor, W. van Dijk, and S. Zimmerman (2020): "Centralized School Choice with Unequal Outside Options," Working Paper.

O'Neil, J. (1996): "New Options, Old Concerns," *Educational Leadership*, 54, 6–8.

Orfield, G. and E. Frankenberg (2013): *Educational Delusions?: Why Choice Can Deepen Inequality and How to Make Schools Fair*, Univ of California Press.

Owens, A. and J. Candipan (2019): "Social and Spatial Inequalities of Educational Opportunity: A Portrait of Schools Serving High-and Low-income Neighbourhoods in US Metropolitan Areas," *Urban Studies*, 3178–3197.

Pathak, P. A. and T. Sönmez (2008): "Leveling the Playing Field: Sincere and Sophisticated Players in the Boston Mechanism," *American Economic Review*, 98(4), 1636–1652.

Peters, M. and A. Siow (2002): "Competing Premarital Investments," *Journal of Political Economy*, 110, 592–608.

Pycia, M. and B. M. Yenmez (2019): "Matching with Externalities," Working Paper.

Reback, R. (2005): "House Prices and the Provision of Local Public Services: Capitalization under School Choice Programs," *Journal of Urban Economics*, 57, 275–301.

Ronn, E. (1990): "NP-Complete Stable Matching Problems," *Journal of Algorithms*, 11, 285–304.

Sasaki, H. and M. Toda (1996): "Two-sided Matching Problems with Externalities," *Journal of Economic Theory*, 70, 93–108.

Shapley, L. S. and M. Shubik (1971): "The Assignment Game I: The Core," *International Journal of game theory*, 1, 111–130.

Shi, P. (2021): "Optimal Priority-based Allocation Mechanisms," *Management Science*.

Smith, A. G. (1995): "Public School Choice and Open Enrollment: Implications for Education, Desegregation, and Equity," *Nebraska Law Review*, 74, 255–303.

van der Laan, G., D. Talman, and Z. Yang (2018): "Equilibrium in the Assignment Market under Budget Constraints," Working Paper.

Whitehurst, G. J. R. (2017): "Education choice and competition index 2016: Summary and commentary," Washington: Brookings Institution Center on Children and Families, https://www.brookings.edu/wp-content/uploads/2017/03/ccf$_2$0170329$_e cci_f ull_r eport.pdf$.

Xu, J. (2019): "Housing choices, sorting, and the distribution of educational benefits under deferred acceptance," *Journal of Public Economic Theory*, 21, 558–595.

Zheng, A. (2019): "Residential Sorting, School Choice, and Inequality," Working Paper.

A Omitted Proofs

A.1 Proof of Theorem 2

The proof below is for DN. The result for DA is proved analogously.

In what follows, whenever discussing continuity and convergence on measure spaces, the topology under consideration is the topology of weak convergence of measures. Let us $\tau_n \rightharpoonup \tau$ to denote that the sequence $(\tau_n)_{n\in\mathbb{N}}$ converges to τ in that topology.

As mentioned in Section 3, each $\tau \in \mathcal{T}$ results in a measure G_τ on $P \times S \times [0,1]$ given by

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P, S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For two measures G and G' on $P \times S \times [0,1]$, let us define a distance between them by

$$d(G, G') := \sup_{P' \subseteq P, S' \subseteq S, r_0, r_1 \in [0,1]} \Big| G\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$- G'\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)\Big|.$$

Lemma 1. *Let (τ, p) be an arbitrary competitive equilibrium of DN and let c denote the corresponding cutoffs vector. Consider a sequence of economies $(\tau_n)_{n\in\mathbb{N}}$ converging to τ and the corresponding sequence of DN cutoffs $(c_n)_{n\in\mathbb{N}}$. Then, $c_n \to c$.*

Proof. The proof has two part.

Part 1. First, I show that $G_{\tau_n} \xrightarrow{d} G_\tau$. By definitions of the distance function d and measures G_{τ_n} and G_τ,

$$d(G_{\tau_n}, G_\tau) =$$
$$\sup_{P' \subseteq P, S' \subseteq S, r_0, r_1 \in [0,1]} \Big(r_1 - r_0\Big) \times \Big| \tau_n\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P', s_h \in S'\Big)$$

$$
\begin{aligned}
&-\tau\Big((v,h)\in V\times\bar{H}:\succ_{vh}\in P', s_h\in S'\Big)\Big| \\
&\leq \max_{P'\subseteq P, S'\subseteq S}\Big|\tau_n\Big((v,h)\in V\times\bar{H}:\succ_{vh}\in P', s_h\in S'\Big) \\
&\quad -\tau\Big((v,h)\in V\times\bar{H}:\succ_{vh}\in P', s_h\in S'\Big)\Big|. \qquad (8)
\end{aligned}
$$

Since $\tau_n \rightharpoonup \tau$, by Portmanteau theorem (e.g., Billingsley (2013)) the last term in equation 8 converges to zero. This establishes Part 1.

Part 2. We say G_τ has rich preferences if a positive measure of each preference type resides in each neighborhood. Formally,

Definition 6. *G_τ has **rich preferences** if for all $\succ\in P$ and $s\in S$,*

$$
\tau\Big((v,h)\in V\times\bar{H}:\succ_{vh}=\succ, s_h=s\Big)>0.
$$

The condition of rich preferences is stronger than that in Grigoryan (2022). Hence, by their Theorem 3, rich preferences are sufficient to to have $c_n\to c$. I now show that the condition is satisfied.

First, I will show that $p_h<1$ for all $h\in H$. Suppose, for the sake of contradiction, that $p_h\geq 1$ for some $h\in H$. Then,

$$
0<q_h=\eta\Big(v\in V: u_v(h,\tau)-p_h=\arg\max_{h'\in\bar{H}}\ u_v(h',\tau)-p_{h'}\Big)
$$

$$
\leq\eta\Big(v\in V: u_v(h,\tau)-p_h\geq 0\Big)\leq\eta\Big(v\in V: \max_{s\in S} v(h,s)-p_h\geq 0\Big)=0,
$$

a contradiction.

Now, consider an arbitrary $h\in H$ and let $\epsilon:=1-p_h>0$. Define a subset $V_h\subseteq V$ by

$$
V_h:=\Big\{v\in V: v(h,s)>1-\epsilon/2, v(h',s)<\epsilon/2, \forall s\in S, h'\in H\setminus\{h\}\Big\}.
$$

When choosing h, type $v\in V_h$ guarantees a payoff strictly larger than

$$
1-\epsilon/2-p_h=1-\epsilon/2-(1-\epsilon)=\epsilon/2,
$$

and when choosing $h' \in H \setminus \{h\}$, she can obtain at most

$$\epsilon/2 - p_{h'} \leq \epsilon/2.$$

Thus, almost all types in V_h choose h at any CE and

$$\tau\big(v \in V_h : s_h = s\big) = \eta(V_h) > 0.$$

The last inequality follows from that η has full support. Again, by full support of η, for each $\succ \in P$ there is a positive measure of types in V_h whose preferences are $\succ$. Denoting by δ the smallest of these measures, we obtain the desired result. $\square$

As Part 2 of Lemma 1 demonstrates, all types in V_h choose neighborhood h in any CE (τ, p). Consider an arbitrary $\tau \in \mathcal{T}$ satisfying

$$\tau\Big((v,h) \in V \times \bar{H} : v \in V_{h'}, h = h'\Big) = \eta\big(V_{h'}\big) \text{ for all } h' \in H. \tag{9}$$

Let $\tau_n \rightharpoonup \tau$ be an arbitrary sequences of neighborhood choices with cutoffs the corresponding cutoffs sequence c_n. Then, with similar arguments as in Lemma 1 one can establish that $c_n \to c$, where c denotes the cutoff of τ. Thus, the continuity of cutoffs hold for any neighborhood choices τ satisfying equation 9. For the rest of the proof, let us restrict attention to such neighborhood choices. With abuse of notation, this set is denoted by $\mathcal{T}$.

Lemma 2. *The collection of functions $(u_v(h,\tau))_{v \in V, h \in H}$ is equicontinuous in τ.*

Proof. Recall that $u_v(h,\tau) = \sum_{s \in S} \lambda_s(\succ_{v,h}, h, \tau) v(h,s)$. First, I show that $\lambda_s(\succ, h, \tau)$ is continuous in τ. That $\lambda_{vs}(h,\tau)$ is continuous in c is immediate from equation 3. Thus, by Lemma 1, $\lambda_{vs}(h,\tau)$ is continuous in τ.

Since $\big\{\lambda_{vs}(h,\tau)\big\}_{v \in V, h \in H}$ is a finite collection of functions, it is equicontinuous. Since v is bounded, $\big\{u_v(h,\tau)\big\}_{h \in H, v \in V}$ is equicontinuous too. $\square$

Lemma 3. *For any $\tau \in \mathcal{T}$, there is a unique price vector $\mathcal{P}(\tau) \in \mathbb{R}_+^{|H|}$ such that for all $h \in H$,*

$$\eta\Big(v \in V : u_v(h,\tau) - \mathcal{P}_h(\tau) = \underset{h' \in \bar{H}}{\arg\max}\ u_v(h',\tau) - \mathcal{P}_{h'}(\tau)\Big) \leq q_h. \tag{10}$$

and the equality is strict only if $\mathcal{P}_h(\tau) = 0$. Moreover, $\mathcal{P}(\tau)$ is continuous in τ.

Proof. Let $\tilde{V} := [0,1]^{|H|}$ and define a measure $\tilde{\eta}$ over $\tilde{V}$ by

$$\tilde{\eta}\big(\tilde{U}\big) = \eta\Big(v \in V : (u_v(h,\tau))_{h\in H} \in \tilde{U}\Big),$$

for all measurable $\tilde{U} \subseteq \tilde{V}$. Since $\tilde{\eta}$ is absolutely continuous and full support, the existence of the unique vector $\mathcal{P}(\tau) \in \mathbb{R}^N$ satisfying equation 10 follows from Gretsky et al. (1999).

I prove the continuity of $\mathcal{P} : \mathcal{T} \to \mathbb{R}_+^{|H|}$ in two parts.

Part 1. Suppose $\tau_n \rightharpoonup \tau$. Let us show that $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$. Consider an arbitrary $\epsilon > 0$ and $\tilde{U} \subseteq \tilde{V}$ with a measure zero boundary $\partial\tilde{U}$. By Portmanteau theorem, it is sufficient to show that $\tilde{\eta}_n(\tilde{U}) \to \tilde{\eta}(\tilde{U})$.

By absolute continuity of $\tilde{\eta}$, there is an open cover $\{\mathcal{O}\}_{i\in I}$ of $\partial\tilde{U}$ such that $\tilde{\eta}\Big(\cup_{i\in I}\mathcal{O}_i\Big) < \epsilon$. Since $\partial\tilde{U}$ is a compact set, there is $\underline{\delta} > 0$ such that for any $\tilde{u} \in \partial\tilde{U}$, the $\underline{\delta}$-ball around $\tilde{u}$ is contained in some element of the open cover $\{\mathcal{O}_i\}_{i\in I}$. For any $\delta \in [0,\underline{\delta}]$, let $\tilde{E}^\delta$ denote the union of δ-balls around each point in $\partial\tilde{U}$. Then, $\partial\tilde{U} \subseteq \tilde{E}^\delta$ and

$$\tilde{\eta}(\tilde{E}^\delta) \leq \tilde{\eta}\Big(\cup_{i\in I}\mathcal{O}_i\Big) < \epsilon. \tag{11}$$

By Lemma 2, for any sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned}\tilde{\eta}_n(\tilde{E}^{\delta/2}) = \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h\in H} \in \tilde{E}^{\delta/2}\Big) &\leq \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h\in H} \in \tilde{E}^{\delta}\Big) \\ &= \tilde{\eta}(\tilde{E}^\delta) < \epsilon. \end{aligned} \tag{12}$$

By equation 11,

$$\tilde{\eta}(\tilde{U}) \leq \tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^\delta\big) + \tilde{\eta}(\tilde{E}^\delta) < \tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^\delta\big) + \epsilon. \tag{13}$$

By Lemma 2, potentially for a larger $n \in \mathbb{N}$,

$$\tilde{\eta}\big(\tilde{U} \setminus \tilde{E}^{\delta}\big) = \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h \in H} \in \tilde{U} \setminus \tilde{E}^{\delta}\Big)$$

$$\leq \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h \in H} \in \tilde{U}\Big) = \tilde{\eta}_n(\tilde{U}). \tag{14}$$

Combining equations 13 and 14,

$$\tilde{\eta}(\tilde{U}) < \tilde{\eta}_n(\tilde{U}) + \epsilon.$$

Similarly, by equation 12,

$$\tilde{\eta}_n(\tilde{U}) \leq \tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) + \tilde{\eta}_n(\tilde{E}^{\delta/2}) < \tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) + \epsilon, \tag{15}$$

and by Lemma 2,

$$\tilde{\eta}_n\big(\tilde{U} \setminus \tilde{E}^{\delta/2}\big) = \eta\Big(v \in V : \big(u_v(h,\tau_n)\big)_{h \in H} \in \tilde{U} \setminus \tilde{E}^{\delta/2}\Big)$$

$$\leq \eta\Big(v \in V : \big(u_v(h,\tau)\big)_{h \in H} \in \tilde{U}\Big) = \tilde{\eta}(\tilde{U}). \tag{16}$$

Combining 15 and 16,

$$\tilde{\eta}_n(\tilde{U}) < \tilde{\eta}(\tilde{U}) + \epsilon.$$

Part 2. Now, let us show that $\mathcal{P}$ is continuous in $\tilde{\eta}$. Let $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$ and $(\mathcal{P}^n)_{n \in \mathbb{N}}$ be the corresponding sequence of prices. Note that $\mathcal{P}^n_h < 1$ for all $h \in H$. Suppose, for the sake of contradiction, that $\mathcal{P}^n_h \geq 1$ for some $h \in H$ and $n \in \mathbb{N}$. Then,

$$0 < q_h = \eta\Big(v \in V : u_v(h,\tau^n) - \mathcal{P}^n_h = \operatorname*{arg\,max}_{h' \in \bar{H}} u_v(h',\tau^n) - \mathcal{P}^n_{h'}\Big)$$

$$\leq \eta\Big(v \in V : u_v(h,\tau^n) - p_h \geq 0\Big) \leq \eta\Big(v \in V : \max_{s \in S} v(h,s) - \mathcal{P}^n_h \geq 0\Big) = 0,$$

a contradiction. By Bolzano-Weierstrass theorem, $(\mathcal{P}^n)_{n \in \mathbb{N}}$ has a convergent subsequence. Without loss of generality, suppose $\mathcal{P}^n \to \mathcal{P}^*$. It is sufficient to show that $\mathcal{P}^*$ satisfies equation 10. By uniqueness, this would imply $\mathcal{P}^* = \mathcal{P}$ and the desired continuity result. For all $h \in H$ define

$$\tilde{V}_h := \Big\{\tilde{v} \in \tilde{V} : h = \operatorname*{arg\,max}_{h' \in \bar{H}} \ \tilde{v}(h') - \mathcal{P}^*_{h'}\Big\}.$$

Suppose, for the sake of contradiction, that $\tilde{\eta}(\tilde{V}_h) \neq q_h$ for some $h \in H$. Without loss of generality, let $\tilde{\eta}(\tilde{V}_h) > q_h$. Since $\tilde{\eta}_n \rightharpoonup \tilde{\eta}$ there are $\delta > 0$ and $M \in \mathbb{N}$ such that $\tilde{\eta}_n(\tilde{V}_h) > q_h + \delta, \forall n > M$. By selecting a large enough n, one can make $\mathcal{P}^n$ arbitrarily close to $\mathcal{P}^*$ and therefore

$$\tilde{\eta}_n\Big(\tilde{v} \in \tilde{V} : h = \underset{h' \in \bar{H}}{\arg\max}\ \tilde{v}(h') - \mathcal{P}^n_{h'}\Big) > q_h,$$

a contradiction. This completes the proof of Lemma 3. $\square$

Define families' best response mapping $\mathcal{B} : \mathcal{T} \to \mathcal{T}$ by

$$\mathcal{B}_\tau\big(U, h\big) = \eta\Big(v \in U : h = \underset{h' \in \bar{H}}{\arg\max}\ u_v(h', \tau) - \mathcal{P}_{h'}(\tau)\Big),$$

for all $h \in H$ and measurable $U \subseteq V$.

Lemma 4. *$\mathcal{B}$ is continuous.*

Proof. Suppose $\tau_n \rightharpoonup \tau$ and $U \subseteq V$ is a measure zero boundary set.

For any $v \in V$ and $h \in H$, define $\mathcal{F}_\tau(v, h) : \mathcal{T} \to \mathbb{R}$ by

$$\mathcal{F}_\tau(v, h) = u_v(h, \tau) - \mathcal{P}_h(\tau) - \max_{h' \in \bar{H} \setminus \{h\}} \big(u_v(h', \tau) - \mathcal{P}_{h'}(\tau)\big).$$

By Portmanteau theorem, it is sufficient to show

$$\mathcal{B}_{\tau_n}(U, h) = \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v, h) \geq 0\Big) \to \eta\Big(v \in V : \mathcal{F}_\tau(v, h) \geq 0\Big) = \mathcal{B}_\tau(U, h).$$

By Lemmas 2 and 3, the collection of functions $\{\mathcal{F}_\tau(v, h)\}_{v \in V, h \in H}$ is equicontinuous. Fix an arbitrary $\epsilon > 0$. By absolute continuity of η, there is $\delta > 0$ such that

$$\eta\Big(v \in U : \mathcal{F}_\tau(v, h) \in [0, \delta)\Big) < \epsilon. \tag{17}$$

By equicontinuity of $\{\mathcal{F}_\tau(v, h)\}_{v \in V, h \in H}$, for any sufficiently large $n \in \mathbb{N}$,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v, h) \in [0, \delta/2)\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v, h) \in [0, \delta)\Big) < \epsilon. \tag{18}$$

By equation 17,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) = \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) + \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \in [0,\delta)\Big)$$
$$< \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) + \epsilon. \tag{19}$$

By equicontinuity of $\{\mathcal{F}_\tau(v,h)\}_{v\in V, h\in H}$, and potentially larger $n \in \mathbb{N}$,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq \delta\Big) < \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big). \tag{20}$$

Combining 19 and 20,

$$\eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) < \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) + \epsilon.$$

Similarly, by equation 18,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) = \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) + \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \in [0,\delta/2)\Big)$$
$$< \eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) + \epsilon, \tag{21}$$

and by equicontinuity of $\{\mathcal{F}_\tau(v,h)\}_{v\in V, h\in H}$,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq \delta/2\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big). \tag{22}$$

Combining 21 and 22,

$$\eta\Big(v \in U : \mathcal{F}_{\tau_n}(v,h) \geq 0\Big) < \eta\Big(v \in U : \mathcal{F}_\tau(v,h) \geq 0\Big) + \epsilon.$$

This completes the proof of Lemma 4. □

Each fixed point τ^* of $\mathcal{B}$ corresponds to a CE $\big(\tau^*, \mathcal{P}(\tau^*)\big)$ of DN. Since $\mathcal{T}$ is (weakly) compact and $\mathcal{B} : \mathcal{T} \to \mathcal{T}$ is (weakly) continuous, the existence of CE of DN follows from Schauder-Tychonoff fixed point theorem.

A.2 Proof of Theorem 4 (First Part)

The proof for the second part has been given in the main text. I now prove the first part. Suppose the economy satisfies Axiom 1.

Let us assume that $\sum_{i=1}^{N} q_{h_i} = 1$. The assumption is without loss of generality, since when $\sum_{i=1}^{N} q_{h_i} < 1$ one can add a neighborhood that no family likes, and when $\sum_{i=1}^{N} q_{h_i} > 1$ one can add families who are indifferent across all schools and neighborhoods, and who in equilibrium will choose lower indexed neighborhoods and schools.

Define the numbers $0 = a_0 \leq a_1 \leq ... \leq a_N = 1$ by

$$\eta\Big(\{v_\alpha\}_{\alpha\in[a_{k-1},a_k]}\Big) = q_{h_k}, \forall k \in \{1, 2, ..., N\}.$$

Then, it is immediate from the increasing differences property of valuations that a CE exist, and in any CE (τ^ϕ, p^ϕ) of $\phi \in \{DN, DA\}$,

$$\tau^\phi\Big(\{v_\alpha\}_{\alpha\in[a_{k-1},a_k]} \times \{h_k\}\Big) = q_{h_k}, \forall k \in \{1, 2, ..., N\}.$$

Let us compute school assignment probabilities and expected utilities under DA and DN.

Under DA, school assignment is solely determined by lottery numbers. Any type v_α is assigned to a school she prefers weakly more than s_k if and only if her lottery number is in the interval $\big[1 - \sum_{j=1}^{k} q_{s_j}, 1\big]$, the probability of which is $\min\big\{\sum_{j=1}^{k} q_{s_j}, 1\big\}$.

Under DN, school assignment is determined based on both neighborhood choice and lottery numbers. DN assignment can be given by the following procedure.

Round 1: *Let $V_N = V$ and $\bar{V}_N = \big\{v_\alpha \in V_N : \alpha \in [a_{N-1}, a_N]\big\}$. Each family in $\bar{V}_N$ is assigned to s_N with probability one. Remaining seats at s_N are assigned to families $q_{s_N} - \eta(\bar{V}_N) = q_{s_N} - q_{h_N}$ highest lottery numbers among the remaining ones.*

Round $k > 1$: *Let V_{N-k+1} denote the set of families that are unassigned by Round*

k and $\bar{V}_{N-k+1} = \{v_\alpha \in V_{N-k+1} : \alpha \in [a_{N-k}, a_{N-k+1}]\}$. If $\eta(V_{N-k+1}) < q_{s_{N-k+1}}$, all remaining families are assigned to s_{N-k+1}. Otherwise, families in $\bar{V}_{N-k+1}$ is assigned to s_{N-k+1} with probability one and remaining seats at s_{N-k+1} are assigned to families with $q_{s_{N-k+1}} - \eta(\bar{V}_{N-k+1}) = q_{s_{N-k+1}} - q_{s_{N-k+1}}$ highest lottery numbers among the remaining ones.

Consider an alternative school assignment procedure, where one applies only Round 1 of DN, and assigns remaining students to schools uniform randomly. By an induction argument, in order to show that DN creates higher aggregate welfare than DA, it is sufficient to the alternative procedure creates higher aggregate welfare than DA.

The alternative procedure is equivalent to applying DA first, then switching the assignment of types in $\bar{V}_N$ who are not assigned to s_N, with types not in $\bar{V}_N$ who are assigned to s_N. By increasing differences assumption, this reallocation improves aggregate welfare. This completes the proof.

A.3 Proof of Theorem 6

First, I prove point 1. Consider an arbitrary economy η' and $\epsilon > 0$. Suppose $H^{NA}_- \not\subseteq H^{DN}_-$ for all η with $\|\eta - \eta'\|_2 < \epsilon$. Consider the economy

$$\eta = \left(1 - \frac{\epsilon}{|H| + 1}\right) \times \eta + \sum_{h \in H} \frac{\epsilon}{|H| + 1} \times \delta_{(v_h, 0)},$$

where $\delta_{(v_h,0)}$ is the Dirac measure that puts all probability mass on the point $(v_h, 0)$, and for each $h \in H$,

$$v_h(h', s') = \begin{cases} 1 & \text{if } (h', s') = (h, s_h), \\ 0 & \text{otherwise.} \end{cases}$$

Consider a neighborhood $h \in H^{NA}_- \setminus H^{DN}_-$. It is immediate that all families with type $(v_h, 0)$ prefer NA to DN.

Now I prove point 2. Consider an arbitrary economy η' and $\epsilon > 0$. Suppose $H^{NA}_- \not\subseteq$

H_-^{DA} or $\left\{s_h \in S : h \in H_-^{NA}\right\} \not\subseteq S_-^{DA}$ for all η with $\|\eta - \eta'\|_2 < \epsilon$. Consider the economy

$$\eta = \left(1 - \frac{\epsilon}{2|H|+1}\right) \times \eta + \sum_{h \in H} \frac{\epsilon}{2|H|+1} \times \delta_{(v_h,0)} + \sum_{h \in H} \frac{\epsilon}{2|H|+1} \times \delta_{(v_s,0)},$$

where

$$v_h(h', s') = \begin{cases} 1 & \text{if } (h', s') = (h, s_h), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$v_s(h, s') = \begin{cases} 1 & \text{if } s' = s, \\ 0 & \text{otherwise.} \end{cases}$$

If there is a neighborhood $h \in H_-^{NA} \backslash H_-^{DA}$, then all families with type $(v_h, 0)$ prefer NA to DN. Otherwise, consider a school $s \in \left\{s_h \in S : h \in H_-^{NA}\right\} \setminus S_-^{\prime DA}$. It is immediate that all families with type $(v_s, 0)$ prefer NA to DA.

A.4 Proof of Corollary 1

Since neighborhoods and schools are ranked, the set of underdemanded neighborhoods are those with index k satisfying $\sum_{j=k}^{N} q_{h_j} \geq 1$ under all three mechanisms. Moreover, the set of underdemanded schools under DN and DA are those with index k satisfying $\sum_{j=k}^{N} q_{s_j} \geq 1$. The result therefore follows from that $q_{s_j} \geq q_{h_j}$ for all $j \in \{1, 2, ..., N\}$.

A.5 Proof of Corollary 2

The proof has two parts. Part 1 establishes that $H_-^{NA} = H_- := \arg\max_{h \in H} q_h \subseteq H_-^{DA}$, and Part 2 establishes that $\left\{s_h \in S : h \in H_-^{NA}\right\} \subseteq S_-^{DA}$. By Theorem 5, these two conditions are sufficient to prove Corollary 2.

Part 1. Let us first show that $H_-^{NA} \subseteq H_-$. Suppose, for the sake of contradiction, that $p_h^{NA} = 0$ for some $h \in H \setminus H_-$. Consider an arbitrary $\bar{h} \in H_-$. Then,

$$
\begin{aligned}
q_h = \eta\Big(&(v,b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} v(h', s_{h'}) - p_{h'}^{NA}\Big) \\
= \eta\Big(&(v,b) \in V \times B : v(h, s_h) \geq v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \\
&\text{and } v(h, s_h) \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
\geq \eta\Big(&(v,b) \in V \times B : v(h, s_h) \geq v(\bar{h}, s_{\bar{h}}) \text{ and } v(h, s_h) \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
= \eta\Big(&(v,b) \in V \times B : v(\bar{h}, s_{\bar{h}}) \geq v(h, s_h) \text{ and } v(\bar{h}, s_{\bar{h}}) \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
\geq \eta\Big(&(v,b) \in V \times B : v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h, s_h) \\
&\text{and } v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
\geq \eta\Big(&(v,b) \in V \times B : b \geq p_{\bar{h}}^{NA}, v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h, s_h) \\
&\text{and } v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
\eta\Big(&(v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} v(h', s_{h'}) - p_{h'}^{NA}\Big) = q_{\bar{h}} > q_h,
\end{aligned}
$$

a contradiction. The second equality above follows from uniformity (therefore, symmetry) of η. The remaining steps are immediate. It is left to show that $H_- \subseteq H_-^{NA}$, or equivalently, $p_h^{NA} = 0$ for all $h \in H_-$. Let $\bar{h} \in H_-$ be such that $p_{\bar{h}}^{NA} = 0$. Such a neighborhood exists since $H_- \supseteq H_-^{NA} \neq \emptyset$. Suppose, for the sake of contradiction, that $p_h^{NA} > 0$. Then,

$$
\begin{aligned}
q_h = \eta\Big(&(v,b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} v(h', s_{h'}) - p_{h'}^{NA}\Big) \\
= \eta\Big(&(v,b) \in V \times B : v(h, s_h) - p_h^{NA} \geq v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \\
&\text{and } v(h, s_h) - p_h^{NA} \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
< \eta\Big(&(v,b) \in V \times B : v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(\bar{h}, s_h) - p_h^{NA} \\
&\text{and } v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
= \eta\Big(&(v,b) \in V \times B : b \geq p_{\bar{h}}^{NA}, v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h, s_h)
\end{aligned}
$$

$$\text{and } v(\bar{h}, s_{\bar{h}}) - p_{\bar{h}}^{NA} \geq v(h', s_{h'}) - p_{h'}^{NA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big)$$

$$\eta\Big((v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} v(h', s_{h'}) - p_{h'}^{NA}\Big) = q_{\bar{h}},$$

a contradiction. The strict inequality above follows from uniformity of η and from that $p_h^{NA} > 0 = p_{\bar{h}}^{NA}$.

Let us now show that $H_- \subseteq H_-^{DA}$. Suppose, for the sake of contradiction, that $p_h^{DA} > 0$ for some $h \in H_-$. Consider an arbitrary $\bar{h} \in H_-^{DA}$. Then,

$$\begin{aligned}
q_h &= \eta\Big((v,b) \in V \times B : h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}\Big) \\
&= \eta\Big((v,b) \in V \times B : u_v^{DA}(h, \tau^{DA}) - p_h^{DA} \geq u_v^{DA}(\bar{h}, \tau^{DA}) - p_{\bar{h}}^{DA} \\
&\quad \text{and } u_v^{DA}(h, \tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
&= \eta\Big((v,b) \in V \times B : u_v^{DA}(\bar{h}, \tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h, \tau^{DA}) - p_{\bar{h}}^{DA} \\
&\quad \text{and } u_v^{DA}(h, \tau^{DA}) - p_h^{DA} \geq u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
&< \eta\Big((v,b) \in V \times B : u_v^{DA}(\bar{h}, \tau^{DA}) - p_{\bar{h}}^{DA} \geq u_v^{DA}(h, \tau^{DA}) - p_h^{DA} \\
&\quad \text{and } u_v^{DA}(\bar{h}, \tau^{DA}) - p_{\bar{h}}^{DA} \geq u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}, \forall h' \in \bar{H}_b \setminus \{h, \bar{h}\}\Big) \\
&= \eta\Big((v,b) \in V \times B : \bar{h} = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}\Big) = q_{\bar{h}},
\end{aligned}$$

a contradiction. The third equality follows from the uniformity of η and from that school assignment probabilities do not depend on the neighborhood choices. The strict inequality follows $p_h^{DA} > 0 = p_{\bar{h}}^{DA}$.

Part 2. Consider an arbitrary $\bar{h} \in H_-$. I show that $\bar{s} := s_{\bar{h}} \in S_-^{DA}$. Suppose, for the sake of contradiction, that

$$\lambda := \lambda_{\bar{v}\bar{s}}^{\tau^{DA}}(h, \tau^{DA}) < \lambda_{vs}^{\tau^{DA}}(h, \tau^{DA}) = 1,$$

for some $s \in S$ and types $\bar{v}$ and v that rank $\bar{s}$ and s as first choices, respectively. Note that these probabilities are the same for all $h \in H$.

For each $h \in H$, let $U_h \subset V_h$ denote the set of types that rank $\bar{s}$ and s as the first two choices, in arbitrary order. Then,

$$\sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,\bar{s}) > v(h,s) \text{ and } h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}\Big)$$

$$= \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,\bar{s}) > v(h,s)$$

$$\text{and } \lambda v(h,\bar{s}) + (1-\lambda) v(h,s) - p_h^{DA} \geq \lambda v(h',\bar{s}) + (1-\lambda) v(h',s) - p_{h'}^{DA}, \forall h' \in H\Big)$$

$$< \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,s) > v(h,\bar{s})$$

$$\text{and } v(h,s) - p_h^{DA} \geq h = v(h',\bar{s}) - p_{h'}^{DA}, \forall h' \in H\Big)$$

$$= \sum_{h \in H} \eta\Big((v,b) \in U_h \times B : v(h,s) > v(h,\bar{s}) \text{ and } h = \arg\max_{h' \in \bar{H}_b} u_v^{DA}(h', \tau^{DA}) - p_{h'}^{DA}\Big).$$

The strict inequality follows from uniformity of η and from that $\lambda < 1$. Thus, in equilibrium the mass of types that rank $\bar{s}$ as first choice and s as second choice is larger than the mass of types that rank s as first choice and $\bar{s}$ as second choice. With analogous arguments, one can show that this is true for any position of choices for $\bar{s}$ and s. This contradicts that the probability of being assigned to $\bar{s}$ is smaller than the probability of being assigned to s. This completes the proof.

B Continuum Economy as a Limit of Discrete Economies

B.1 The Discrete Model

There is a finite set of families F with a single child and equal number of neighborhoods H and schools S. There is a unique school in neighborhood $h \in H$, which we denote by $s_h \in S$. Each neighborhood h has capacity $q_h \in \mathbb{N}$ which denotes the maximum number of families that can reside in the neighborhood. Similarly, each school s has a capacity $q_s \in \mathbb{N}$, which denotes the maximum number of families that can enrol in the school. Each family $f \in F$ has a valuation $v_f(h, s) \in [0, 1]$ for living in neighborhood h and enrolling (their child) at school s. Valuations of all families are commonly known. Families' valuations induce preference rankings over schools. Conditional on living in neighborhood h, the preference ranking of family f satisfies

$$v_f(h, s) > v_f(h, s') \Rightarrow s \succ_{fh} s'. \tag{23}$$

When $v_f(h, s) = v_f(h, s')$, ties are broken arbitrarily.

Let $\bar{H} := H \cup \{0\}$. Neighborhood choices of families is a mapping $\sigma : F \to \bar{H}$.

Family's expected utilities of choosing a certain neighborhood depend on other families' neighborhood choices and the school assignment mechanism, as they jointly determine the family's school assignment probabilities. For a school assignment mechanism ϕ, let $\lambda^{\phi}_{fs}(h, \sigma) \in [0, 1]$ denote the probability that family f is assigned to school s when she chooses neighborhood h and other families' choose neighborhoods according σ.

Given the school assignment probabilities and neighborhood price vector $p \in [0, 1]^{|H|}$, the expected utility of family f choosing neighborhood h is equal to

$$u^{\phi}_f(h, \sigma) - p_h.$$

where $u^{\phi}_f(h, \sigma) := \sum_{s \in S} \lambda^{\phi}_{fs}(h, \sigma) v_f(h, s)$. Also, let $u^{\phi}_f(0, \sigma) := 0$. The housing market is competitive: families choose neighborhoods to maximize expected utilities, given other families' neighborhood choices and the market clearing neighborhood prices.

Definition 7. *For a neighborhood choices σ and price vector $p \in \mathbb{R}^{|H|}_+$, we say a pair (σ, p) is a **competitive equilibrium (CE)** of ϕ is it satisfies the following conditions:*

1. $u^{\phi}_f(\sigma(f), \sigma) - p_{\sigma(f)} = \arg\max_{h \in \bar{H}}\ u^{\phi}_f(h, \sigma) - p_h, \forall f \in F$, where $p_0 := 0$,

2. $|\sigma^{-1}(h)| \leq q_h, \forall h \in H$,

3. $|\sigma^{-1}(h)| < q_h \Rightarrow p_h = 0$.

School Assignment Mechanisms

In this section, I describe the school assignment mechanisms for the discrete model.

Neighborhood Assignment.

Under neighborhood assignment (NA), families are assigned to the neighborhood schools. Therefore, school assignment probabilities are trivial:

$$\lambda_{fs}^{NA}(h,\sigma) = \begin{cases} 1 & \text{if } s = s_h, \\ 0 & \text{otherwise} \end{cases}$$

Deferred Acceptance.

As in the continuum model, I study two versions of DA, which differ on how schools' priority ranking is determined.

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences and lottery numbers. Preferences are induced by neighborhood choices through equation 23. A lottery number for each family is uniformly and independently drawn from the unit interval. All schools rank families according to their lottery numbers, i.e., a higher lottery numbers denotes a higher rank. The assignment is determined through the following algorithm by Gale and Shapley (1962): until there are no more rejections,

- each family f with $\sigma(f) \neq 0$ applies to her most preferred school that has not rejected her,
- each school considers all new applicants and previous applicants, tentatively accepts up to q_s of them according to its ranking, and rejects the rest.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, lottery number and priorities. Preferences and lottery numbers are decided as under DA. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Schools rank families according to their lottery numbers plus the priority. Again, DN assignment is determined by the Gale and Shapley (1962) algorithm as described above.

B.2 (Non)Existence of CE in the Discrete Model

Under NA, the existence of CE follows from Shapley and Shubik (1971). In contrast, assignment externalities may preclude the existence of CE under DN and DA. The example below shows the nonexistence result for DN.

Example 1. *Suppose there are two families $F = \{f_1, f_2\}$, two neighborhoods $H = \{h_1, h_2\}$ and two schools $S = \{s_1, s_2\}$. Each neighborhood and school has a unit capacity. Families' valuations are shown in Table 3.*

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
f_1	0	0.3	0	0.1
f_2	0.1	0.2	0	0.1

Table 3: Valuations

Suppose, for the sake of contradiction, that there is a CE (σ, p) of DN. Consider cases:

(i) Suppose $\sigma(f_1) = h_1$ and $\sigma(f_2) = h_2$. Then, f_1's utility is 0, as she is rejected by s_2, where f_2 has a higher priority. If f_1 chooses h_2 instead of h_1, her utility is $\frac{1}{2} \times 0.1 = 0.05$ as she has $\frac{1}{2}$ change of being assigned to s_2. Thus, $\sigma(f_1) = h_1$ implies $p_{h_2} - p_{h_1} \geq 0.05$. Also, f_2's utility is 0.1 as she is guaranteed being assigned to s_2. If f_2 chooses h_1, she has a $\frac{1}{2}$ chance of being assigned to s_1 and $\frac{1}{2}$ chance of being assigned to s_2, thus her

utility is $\frac{1}{2} \times 0.1 + \frac{1}{2} \times 0.2 = 0.15$*. Thus,* $\sigma(f_2) = h_2$ *implies* $p_{h_2} - p_{h_1} \leq -0.05$*, a contradiction.*

(ii) Now suppose $\sigma(f_1) = h_2$ *and* $\sigma(f_2) = h_1$*. Then,* f_1*'s utility is* 0.1*. If* f_1 *chooses* h_1 *instead of* h_2*, her utility is* $\frac{1}{2} \times 0.3 = 0.15$*. Thus,* $\sigma(f_1) = h_2$ *implies* $p_{h_2} - p_{h_1} \leq -0.05$*. Also,* f_2*'s utility is* 0.1 *as she is rejected by* s_2*. If* f_2 *chooses* h_2 *instead of* h_1*, her utility is* $\frac{1}{2} \times 0.1 = 0.05$*. Thus,* $\sigma(f_2) = h_1$ *implies* $p_{h_2} - p_{h_1} \geq 0.05$*, a contradiction.*

The next example shows the nonexistence result for DA.

Example 2. *Consider a discrete economy with two families* $F = \{f_1, f_2\}$*, two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$*. Assume* $q_{h_1} = q_{h_2} = q_{s_1} = 1$ *and* $q_{s_2} = 2$*. Valuations are given in Table 4.*

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
f_1	0.5	0	0	0.4
f_2	0	0.1	0.3	0

Table 4: Valuations

Suppose, for the sake of contradiction, that (σ, p) *is a CE. Consider cases:*

(i) Suppose $\sigma(f_1) = h_1$ *and* $\sigma(f_2) = h_2$*. Then,* f_1*'s utility is* $\frac{1}{2} \times 0.5 = 0.25$ *when choosing* h_1 *and* 0.4 *when choosing* h_2*. Thus,* $\sigma(f_1) = h_1$ *implies* $p_2 - p_1 \geq 0.15$*. Also,* f_2*'s utility is* 0.1 *when choosing* h_1 *and* $\frac{1}{2} \times 0.3 = 0.15$ *when choosing* h_2*. Thus,* $\sigma(f_2) = h_2$ *implies* $p_2 - p_1 \leq 0.05$*, a contradiction.*

(ii) Now suppose $\sigma(f_1) = h_2$ *and* $\sigma(f_2) = h_1$*. Then,* f_1*'s utility is* 0.5 *when choosing* h_1 *and* 0.4 *when choosing* h_2*. Thus,* $\sigma(f_1) = h_2$ *implies* $p_1 - p_2 \geq 0.1$*. Also,* f_2*'s utility is* 0.1 *when choosing* h_1 *and* 0.3 *when choosing* h_2*. Thus,* $\sigma(f_2) = h_1$ *implies* $p_1 - p_2 \leq -0.2$*, a contradiction.*

As the proof demonstrates, the nonexistence results are due to assignment externalities. A family's expected utility from different neighborhood choices depend on other families' neighborhood choices through the latter's effect on school assignment probabilities. In contrast, as shown in Section 3, a CE always exists in the continuum model. In Section B.3, I show that continuum economies are arbitrarily good approximations of finite discrete economies when the number of families is sufficiently large. In particular, this implies that approximate CE exist in sufficiently large discrete economies, and welfare comparisons for the continuum model carry over to the discrete one.

B.3 Existence of Approximate CE in Large Markets

Let η be an absolutely continuous and fully supported probability measure on $V := [0,1]^{|H|\times|S|}$. For a fixed $k \in \mathbb{N}$ let $\{v_f\}_{f\in F}, |F| = k$, be k independent draws from V according to η. Suppose neighborhood $h \in H$ has a capacity $\lfloor q_h k \rfloor$ and each school $s \in S$ has a capacity $\lfloor q_s k \rfloor$.

Definition 8. *For an $\epsilon > 0$, neighborhood choices $\sigma : F \to \bar{H}$ and a price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (σ, p) is an $\epsilon-$**competitive equilibrium (ϵ-CE)** of ϕ if it satisfies the following conditions:*

1. $u_f^\phi(\sigma(f), \sigma) - p_{\sigma(f)} + \epsilon \geq \max\left\{u_f^\phi(h, \sigma) - p_h, 0\right\}, \forall f \in F, h \in H,$
2. $|\sigma^{-1}(h)| \leq (q_h + \epsilon)k, \forall h \in H,$
3. $|\sigma^{-1}(h)| < (q_h - \epsilon)k \Rightarrow p_h = 0.$

Let (τ^ϕ, p^ϕ) denote a CE of a continuum economy η for $\phi \in \{DN, DA, NA\}$. For each size k discrete economy $(v_f)_{f\in F}$, consider neighborhood choices $\sigma_k : F \to \bar{H}$ satisfying

$$\sigma_k(f) = \arg\max_{h\in\bar{H}} \; u_f^\phi(h, \tau^\phi) - p_h^\phi.$$

Theorem 7. *Let (τ^ϕ, p^ϕ) be a competitive equilibrium of the continuum economy η. Then for any $\epsilon > 0$ the probability that (σ_k, p) is an ϵ-competitive equilibrium of the discrete economy converges to one as k goes to infinity.*

In other words, Theorem 7 says that in a sufficiently large market approximate equilibria exist with a probability that is arbitrarily close to one.

Proof. First, consider $\phi = NA$. Let (τ, p) be a CE of NA for each size $k \in \mathbb{N}$ economy $(v_f)_{f \in F_k}$, define $\sigma_k : F_k \to H$ by

$$\sigma_k(f) = \arg\max_{h \in H} \; v_f(h, s_h) - p_h.$$

I show that for a sufficiently large $k \in \mathbb{N}$, the probability that (σ_k, p) satisfies the conditions of Definition 8 of ϵ-CE approaches to one.

1. The first point is immediate from the definition of σ_k.

2. Let $F_{kh} = \Big\{ f \in F : v_f(h, s_h) - p_h = \arg\max_{h' \in H} v_f(h', s'_h) - p_{h'} \Big\}$ denote the set of families in F whose optimal choice is h (ties broken arbitrarily). Then,
$$\begin{aligned} \frac{\left|\sigma_k^{-1}(h)\right|}{k} = \frac{|F_{kh}|}{k} &\xrightarrow{p} \eta\Big(v \in V : h = \arg\max_{h' \in H} v(h', s_{h'}) - p_{h'} \Big) \\ &= \tau\big(V \times \{h\}\big) \leq q_h < q_h + \epsilon, \end{aligned} \tag{24}$$
where the convergence in probability follows from low of large numbers. Multiplying the first and last terms of equation 24 by k, we obtain the desired result.

3. The proof is by contrapositive. Suppose $p_h \neq 0$. Then,
$$\begin{aligned} \frac{\left|\sigma_k^{-1}(h)\right|}{k} = \frac{|F_{k,h}|}{k} &\xrightarrow{p} \eta\Big(v \in V : h = \arg\max_{h' \in H} v(h', s_{h'}) - p_{h'} \Big) \\ &= \tau\big(V \times \{h\}\big) = q_h > q_h - \epsilon, \end{aligned} \tag{25}$$
where the last equality follows from $p_h \neq 0$. Multiplying the first and last terms of equation 25 by k, we obtain that $\left|\sigma_k^{-1}(h)\right| > (q_h - \epsilon)k$ with probability approaching to one.

Now consider $\phi = DN$. The proof for $\phi = DA$ is similar. Let (τ, p) be a CE of NA for each size $k \in \mathbb{N}$ economy $(v_f)_{f \in F_k}$, define $\sigma_k : F_N \to H$ by

$$\sigma_k(f) = \underset{h \in H}{\arg\max}\ u_f(h, \tau) - p_h.$$

I show that for a sufficiently large $k \in \mathbb{N}$, the probability that (σ_k, p) satisfies the conditions of Definition 8 of ϵ-CE approaches to one.

1. Let $F_{kh} = \Big\{ f \in F : h = \arg\max_{h' \in H} u_f(h', \sigma_k) - p_{h'} \Big\}$ denote the set of families in F whose optimal choice is h (ties broken arbitrarily). Then, by law of large number,

   $$\frac{\left|\sigma_k^{-1}(h)\right|}{k} = \frac{|F_{kh}|}{k} \xrightarrow{p} \eta\Big(v \in V : h = \underset{h' \in H}{\arg\max}\, u_f(h', \sigma_k) - p_{h'}\Big) = \tau\big(V \times \{h\}\big).$$

   Thus, the proportion of individuals with given preferences and priorities in the discrete economy converges to its continuum analog. This, by Lemma 3 of Abdulkadiroğlu et al. (2017b), implies that $u_f(h, \sigma_k)$ converges to $u_f(h, \tau)$ in probability for all $h \in H$, establishing the desired result.

2. The proof for this part is similar to that for $\phi = NA$.

3. The proof for this part is similar to that for $\phi = NA$.

$\square$

The proof of Theorem 7 uses the fact that expected utilities in the discrete markets converge to their continuum analogous. This also implies that all welfare comparisons that I establish for the continuum economy, hold 'approximately' for the corresponding sufficiently large discrete ones.

C Examples

My first example shows that when we relax the common valuations of neighborhoods and the increasing differences assumptions, then DA may create higher aggregate welfare than DN.

Example 3. *There are two neighborhoods* $H = \{h_1, h_2\}$ *and two schools* $S = \{s_1, s_2\}$. *Each neighborhood and school has a capacity* 0.5. *Economy* η *is supported at only two points* v_1 *and* v_2, *with*

$$\eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_2\Big) = 0.5.$$

Valuations are given in in Table 3.

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
v_1	0	0.3	0	0.3
v_2	0.1	0	0.5	0.6

Table 5: Valuations

It is easy to verify that prices $p^{\phi}_{h_1} = 0$ *and* $p^{\phi}_{h_2} = 2$ *support CE* (τ^{ϕ}, p^{ϕ}) *of* $\phi \in \{DN, DA\}$, *satisfying*

$$\tau^{\phi}\Big((v,h) \in V \times \bar{H} : v = v_i, h = h_i\Big) = \eta\Big(v \in V : v = v_i\Big) \text{ for all } i \in \{1, 2\}.$$

Under DN, type v_2 *receives a higher priority at* s_2 *and therefore she is assigned there with probability one. Under DA, each type has an equal probability of being assigned to* s_2. *Expected utilities are*

$$\begin{array}{ll} u^{DN}_{v_1}(h_1, \tau^{DN}) = 0 & \quad u^{DA}_{v_1}(h_1, \tau^{DA}) = 0.15 \\ u^{DN}_{v_2}(h_2, \tau^{DN}) = 0.6 & \quad u^{DA}_{v_2}(h_2, \tau^{DA}) = 0.55 \end{array}.$$

Therefore,

$$\int u^{DN}_{v}(h, \tau^{DN}) d\tau^{DN} = \frac{1}{2} \times u^{DN}_{v_1}(h_1, \tau^{DN}) + \frac{1}{2} \times u^{DN}_{v_2}(h_2, \tau^{DN}) = 0.3$$

$$< 0.35 = \frac{1}{2} \times u_{v_1}^{DA}(h_1, \tau^{DA}) + \frac{1}{2} \times u_{v_2}^{DA}(h_2, \tau^{DA}) = \int u_v^{DA}(h, \tau^{DA}) d\tau^{DA}.$$

My second example maintains the assumption of common and additively separable valuation over neighborhoods, but relaxes the assumption of identical ordinal preferences over schools.

Example 4. *There are three neighborhoods $H = \{h_1, h_2, h_3\}$ and three schools $S = \{s_1, s_2, s_3\}$. Capacities are $q_{h_1} = 0.6$, $q_{h_2} = q_{h_3} = 0.2$ and $q_{s_2} = q_{s_3} = 0.3$. Economy η is supported at only three points v_1, v_2 and v_3, with*

$$\eta\Big(v \in V : v = v_1\Big) = 0.6, \eta\Big(v \in V : v = v_2\Big) = \eta\Big(v \in V : v = v_3\Big) = 0.2.$$

Families only care about schools. Formally, $v_i(h_j, s) = v_i(h_k, s)$ for all $i, j, k \in \{1, 2, 3\}$ and $s \in S$. Thus, a type can be described by its valuation for schools. Valuations are given in Table 6.

	s_1	s_2	s_3
v_1	0	1	0.9
v_2	0	0.9	0
v_3	0	0	0.9

Table 6: Valuations

Let us first compute aggregate welfare under DN. I prove that prices $p_{h_1}^{DN} = 0$, $p_{h_2}^{DN} = 0.7$, and $p_{h_3}^{DN} = 0.6$ supports CE neighborhood choices

$$\tau^{DN}\Big((v, h) \in V \times \bar{H} : v = v_i, h = h_i\Big) = \eta\Big(v \in V : v = v_i\Big) \text{ for all } i \in \{1, 2, 3\}.$$

First, let us show the optimality of type v_1 families' neighborhood choices at (τ^{DN}, p^{DN}). Since families receive higher priorities at neighborhood schools, almost all v_2 type families are assigned to s_2 and almost all v_3 type families are assigned to s_3. The remaining 0.2 cumulative capacity at schools s_2 and s_3 are assigned to highest ranked families of *type v_1. Thus, the probability that v_1 is assigned to either s_2 or s_3 is equal to $\frac{1}{3}$. Therefore,*

$$u_{v_1}^{DN}(h_1, \tau^{DN}) - p_{h_1}^{DN} \geq \frac{1}{3} \times 0.9 = u_{v_1}^{DN}(h_2, \tau^{DN}) - p_{h_2}^{DN} = u_{v_1}^{DN}(h_3, \tau^{DN}) - p_{h_3}^{DN}.$$

Now consider a type v_2 family. If a v_2 family chooses neighborhood h_1 or h_3, she is assigned to s_2 only if she has one of the 0.1 highest lottery numbers among 0.6 mass of type v_1 families. The probability of this event is $\frac{1}{6}$. Therefore,

$$u_{v_2}^{DN}(h_2, \tau^{DN}) - p_{h_2}^{DN} = 0.9 - 0.7 > \frac{1}{6} \times 0.9 = u_{v_2}^{DN}(h_1, \tau^{DN}) - p_{h_1}^{DN} > u_{v_2}^{DN}(h_3, \tau^{DN}) - p_{h_3}^{DN}.$$

Finally, consider a type v_3 family. Conditional on being assigned to h_1 or h_2, type v_3 is assigned to s_3 only she has one of the highest 0.1 highest lottery numbers among mass 0.5 type v_1 families who do not have a high enough lottery number to be assigned to s_2. The conditional lottery numbers' distribution of families not assigned to s_2 is uniform in $\left[0, \frac{5}{6}\right]$, and the probability that v_3 is assigned to s_3 is $\frac{1}{6} + \frac{5}{6} \times \frac{1}{5} = \frac{1}{3}$. Therefore,

$$u_{v_3}^{DN}(h_2, \tau^{DN}) - p_{h_3}^{DN} = 0.9 - 0.6 = \frac{1}{3} \times 0.9 = u_{v_3}^{DN}(h_1, \tau^{DN}) - p_{h_1}^{DN} > u_{v_3}^{DN}(h_2, \tau^{DN}) - p_{h_2}^{DN}.$$

Aggregate welfare under DN is

$$\int u_v^{DN}(h, \tau^{DN}) d\tau^{DN} = 0.1 \times 1 + 0.1 \times 0.9 + 0.2 \times 0.9 + 0.2 \times 0.9 = 0.550.$$

Now consider DA. Since families only care about schools, any neighborhood choice $\tau^{DA} \in \mathcal{T}$ is supported as a CE with prices $p_{h_1}^{DA} = p_{h_2}^{DA} = p_{h_3}^{DA}$. Then, a mass 0.15 of type v_1 families who have valuation 1 for s_2 are assigned to the school. The remaining 0.15 capacity at s_2 is filled with families who have valuation 0.9 for s_2. The entire 0.3 capacity of school s_3 is filled with families who have valuation 0.9 for s_3.[15] *Thus, aggregate welfare under DA is*

$$\int u_v^{DA}(h, \tau^{DA}) d\tau^{DA} = 0.15 \times 1 + 0.15 \times 0.9 + 0.3 \times 0.9 = 0.555.$$

[15] I implicitly assume that types v_2 and v_3 apply to s_1 before s_3 and s_2, respectively. This is without loss of generality, as alternatively one could slightly increase valuations at s_1 for all families and adjust prices accordingly.

The following example demonstrates that some families may prefer NA to DN.

Example 5. *There are two neighborhoods $H = \{h_1, h_2\}$ and two schools $S = \{s_1, s_2\}$. Each neighborhood and school has a capacity 0.5. Economy η is supported at only two points v_1 and v_2, with*

$$\eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_2\Big) = 0.5.$$

Valuations are given in in Table 7.

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
v_1	0	0.1	0.2	0.3
v_2	0.5	0	0.8	0.3

Table 7: Valuations

It is easy to verify that prices $p^{NA}_{h_1} = p^{NA}_{h_2} = 0$ support CE (τ^{NA}, p^{NA}) of NA, satisfying

$$\tau^{NA}\Big((v, h) \in V \times \bar{H} : v = v_1, h = h_2\Big) = \eta\Big(v \in V : v = v_1\Big),$$

and

$$\tau^{NA}\Big((v, h) \in V \times \bar{H} : v = v_2, h = h_1\Big) = \eta\Big(v \in V : v = v_2\Big).$$

Also, prices $p^{DN}_{h_1} = 0, p^{DN}_{h_2} = 0.2$ support (τ^{DN}, p^{DN}) of DN, satisfying

$$\tau^{DN}\Big((v, h) \in V \times \bar{H} : v = v_1, h = h_1\Big) = \eta\Big(v \in V : v = v_1\Big),$$

and

$$\tau^{DN}\Big((v, h) \in V \times \bar{H} : v = v_2, h = h_2\Big) = \eta\Big(v \in V : v = v_2\Big).$$

Thus,

$$u^{NA}_{v_1}(h_2, \tau^{NA}) = 0.3 > 0.1 = u^{DN}_{v_1}(h_1, \tau^{DN}),$$

and type v_1 families prefer NA to DN.

D Multiple Tie-breaking

Consider the model in Section 3. I first define the Deferred Acceptance mechanisms for multiple tie-breaking. With abuse of notations, I use DA and DN to denote the mechanisms.

D.1 School Assignment Mechanisms

Deferred Acceptance without Neighborhood Priority (DA).

School assignment under DA is determined based on families' preferences, school-specific lottery numbers and market clearing cutoffs, or simply cutoffs. Preferences are decided by neighborhood choices through equation 1. School-specific lottery numbers are drawn uniformly and independently from the unit interval. Formally, neighborhood choices τ result in a probability measure G_τ over $P \times [0,1]^{|S|}$, given by

$$G_\tau\Big((\succ, r) \in P \times [0,1]^{|S|} : \succ \in P', r_s \in (r_{s0}, r_{s1}), \forall s \in S\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \prod_{s \in S} \Big(r_{s1} - r_{s0}\Big),$$

for each $P' \in P$ and $(r_{s0}, r_{s1}) \subseteq [0,1]$. Thus, $G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r_s \in (r_{s0}, r_{s1}), \forall s \in S\Big)$ equals the mass of types with preferences in P' and school s lottery numbers in the interval (r_{s0}, r_{s1}) for each $s \in S$.

Cutoffs are derived through an iterative procedure that I describe below. For a vector $c \in [0,1]^{|S|}$, the demand function $D : [0,1]^{|S|} \to [0,1]^{|S|}$ is given by

$$D_s(c) = G_\tau\Big((\succ, r) \in P \times [0,1]^{|S|} : r_s \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r_{s'} \geq c_{s'}\Big).$$

In words, $D_s(c)$ is the mass of families whose school s lottery numbers exceed c_s, and who prefer s to any other school s' where their school s' lottery numbers exceed $c_{s'}$.

For $c \in [0,1]^{|S|}$ and $x \in [0,1]$ let $c(s,x) \in [0,1]^{|S|}$ denote the vector that differs from c only by that $c_s(s,x) = x$.

Let us define a sequence of vectors $(c^t)_{t=1}^{\infty}$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

As shown by Abdulkadiroğlu et al. (2017b), $(c^t)_{t\in\mathbb{N}}$ is convergent. Let $c^{DA} := \lim_{t\to\infty} c^t$ denote the **DA cutoffs**.

The DA cutoffs determine school assignment as follows. A family is assigned to school s if her school s lottery number exceeds c_s^{DA}, and she prefers s to any school where her school-specific lottery number exceeds the corresponding DA cutoff. The probability of this event is

$$\lambda_{vs}^{DA}(h,\tau) = \prod_{s':s'\succ s} c_{s'}^{DA} \times (1 - c_s^{DA}). \tag{26}$$

The first term in the right-hand side of equation 26 is the probability that the family does not clear the cutoffs at choices more preferred to s, and the second term is the probability that the family clears the school s cutoff.

Deferred Acceptance with Neighborhood Priority (DN).

Under DN, school assignment is determined based on families' preferences, school-specific lottery numbers, priorities and cutoffs. Again, preferences are decided by neighborhood choices through equation 1 and school-specific lottery numbers are drawn uniformly and independently from the unit interval. Families receive priority 1 at neighborhood schools and priority 0 at non-neighborhood ones. Formally, neighborhood choices τ result in a probability measure G_τ on $P \times S \times [0,1]^{|S|}$ satisfying

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1]^{|S|} : \succ \in P', s \in S', r_{s'} \in (r_{s'0}, r_{s'0}), \forall s' \in S\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \prod_{s\in S}\Big(r_{s1} - r_{s0}\Big),$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_{s0}, r_{s1}) \subseteq [0,1]$. For a vector $c \in [0,1]^{|S|}$ the demand function $D : [0,1]^{|S|} \to [0,1]^{|S|}$ is given by

$$D_s(c) = G_\tau\Big((\succ, s', r) \in P \times S \times [0,1]^{|S|} : r_s + \mathbb{1}[s' = s] \geq c_s \text{ and}$$
$$s \succ s'' \text{ for all } s'' \text{ with } r_{s''} + \mathbb{1}[s' = s''] \geq c_{s''}\Big).$$

Consider the sequence of vectors recursively defined by

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise} \end{cases}$$

and let $c^{DN} := \lim_{t\to\infty} c^t$ denotes the **DN cutoffs**. A family is assigned to school s if her priority at s plus her school s lottery number exceeds c_s^{DN}, and she prefers s to any school where her priority plus the school-specific lottery number exceeds the corresponding DN cutoff. Hence,

$$\lambda_{vs}^{DN}(h,\tau) = \begin{cases} 0 & s_h \succ_{vs} s, \\ \prod_{s':s'\succ s} c_{s'}^{DN} & s_h = s \\ \Big(\prod_{s':s'\succ s} c_{s'}^{DN}\Big) \times c_s^{DN} & \text{otherwise.} \end{cases}$$

D.2 Results for Multiple-tie Breaking

Like in the single tie-breaking case, school assignment probabilities under multiple tie-breaking are continuous in the cutoffs. Therefore, Theorem 2 - 7 all hold for this setting too, and the proofs are almost identical to the one with a single tie-breaking.

Supplementary Appendix

A Alternative School Assignment Mechanisms

A.1 Overview

This section studies two additional school assignment mechanisms: Top Trading Cycles (TTC) and Immediate Acceptance (IA).

The TTC mechanism has been originally formulated by Shapley and Scarf (1974) and has been introduced for public school assignment by Abdulkadiroğlu and Sönmez (2003). For the continuum economy model, I use the formulation of TTC developed by Leshno and Lo (2021). I show that most of our results on how Deferred Acceptance mechanism compares to neighborhood assignment in terms of welfare also hold for TTC.

The IA mechanism, also known as the 'Boston' mechanism, is widely applied for public school admissions in the US and around the world. The mechanism has often been criticized on the grounds of being manipulable, i.e., families have incentives to misreport their true preferences to improve their school assignments (Abdulkadiroğlu and Sönmez, 2003). Despite this potential shortcoming, it is also known that IA may improve families' welfare as it allows to 'signal' their valuations by preference manipulation. For example, Abdulkadiroğlu, Che, and Yasuda (2011) show that in a setting without neighborhood priorities and where families have common ordinal preference rankings over schools, all families prefer any (symmetric Bayesian) equilibrium outcome of IA to the outcome of DA. This result does not extend to the setting with neighborhood priorities. An important observation in our analysis of IA is that families who reside in the neighborhoods of the least preferred schools may be worse-off under IA with neighborhood priorities compared to DN. The reason is that when the more preferred schools are sufficiently demanded, the families in the neighborhood of the least preferred schools have no 'safe option' other than the least preferred schools. Therefore, under IA with neighborhood priorities they may find it optimal to rank a moderate school as a first choice to avoid the possibility of being rejected by all higher ranked choices and being assigned to their preferred one. As a consequence, those families are worse off. This observation is analogous to the ones in Calsamiglia, Martínez-Mora, and Miralles (2015) and Neilson, Akbarpour, Kapor, van Dijk, and Zimmerman (2020). Those papers demonstrate that families without outside options may prefer DA to IA. I do not explicitly model outside options, but because of neighborhood priorities, in our setting neighborhood schools correspond to outside options.

In what follows I talk about two versions of TTC and IA: one where families do not receive higher priorities at neighborhood schools, and one where they do so. When it is clear from the context, I do not mention which version of the mechanism is studied.

A.2 TTC

A.2.1 Overview

Consider the model in Section 3.

TTC without neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determine a probability measure G_τ over $P \times$

$[0,1]$, satisfying

$$G_\tau\Big((\succ, r) \in P \times [0,1] : \succ \in P', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{vh} \in P'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$ and $(r_0, r_1) \subseteq [0,1]$.

For the resulting measure G_τ the TTC assignment (without neighborhood priorities) is found by the procedure given by Leshno and Lo (2021). I omit the technical details for the sake of brevity. In a nutshell, TTC assignment is determined by cutoffs $c^{TTC} := (c^{TTC}_{ss'})_{s,s' \in S} \in [0,1]^{|S|^2}$, such that a family is assigned to school s if and only if her lottery number is larger than $\min_{s' \in S} c^{TTC}_{s's}$ and she prefers s to any school $s'' \in S \setminus \{s\}$ such that her lottery number is larger than $\min_{s' \in S} c^{TTC}_{s's''}$. Let $N := |S|$ and suppose the schools are indexed as follows,

$$\min_{s \in S} c^{TTC}_{ss_i} > \min_{s \in S} c^{TTC}_{ss_j} \text{ if and only if } i > j$$

. Also, let $c^{TTC}_{ss_0} := 0, \forall s \in S$. Then, it follows from the TTC description by Leshno and Lo (2021) that the cutoffs c^{TTC} should satisfy

$$\sum_{i=1}^{k} G_\tau\Big((\succ, r) \in P \times [0,1] : \; s_k \succ s, \forall s \in S \backslash \{s_i, \ldots, s_N\}, r \in \big[\min_{s \in S} c^{TTC}_{ss_{i-1}}, \min_{s \in S} c^{TTC}_{ss_i}\big)\Big) \leq q_{s_k},$$

and the equation holds with equality whenever $\min_{s \in S} c^{TTC}_{ss_k} > 0$. It then follows from the description of DA cutoffs c^{DA}, that $c^{DA}_s = \min_{s' \in S} c^{TTC}_{s's}$ for all $s \in S$. Thus, we obtain the following equivalence result.

Proposition 1. *For any $v \in V, h \in H$ and $\tau \in \mathcal{T}$,*

$$u^{DA}_v(h, \tau) = u^{TTC}_v(h, \tau).$$

The result extends an earlier finding about the equivalence of DA (random serial dictatorship) and TTC (core from random endowments) by Abdulkadiroğlu and Sönmez (1998) to the continuum one-to-many matching model. To the best of our knowledge, my Proposition 1 is the first documentation of this observation.

TTC with neighborhood priorities.

Neighborhood choices τ uniquely determine a probability measure G_τ on $P \times S \times [0,1]$, satisfying

$$G_\tau\Big((\succ, s, r) \in P \times S \times [0,1] : \succ \in P', s \in S', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h) \in V \times \bar{H} : \succ_{v,h} = \succ, s_h \in S'\Big) \times \Big(r_1 - r_0\Big),$$

for each $P' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For the resulting measure G_τ the TTC assignment (with neighborhood priorities) is given by cutoffs $c^{TTC} := (c^{TTC}_{ss'})_{s,s' \in S} \in [0,2]^{|S|^2}$, such that a family, choosing the neighborhood of school s', is assigned to school s if and only if her lottery number plus $\mathbb{1}[s' = s]$ is larger than $\min_{s'' \in S} c^{TTC}_{s''s}$ and she prefers s to any school $s''' \in S \setminus \{s\}$ such that her lottery number plus $\mathbb{1}[s' = s''']$ is larger than $\min_{s'' \in S} c^{TTC}_{s''s'''}$.

Therefore, it follows from the TTC description by Leshno and Lo (2021) that the cutoffs c^{TTC} should satisfy

$$\sum_{i=1}^{k} G_\tau\Big((\succ, s', r) \in P \times S \times [0,1] : \; s_k \succ s, \forall s \in S \setminus \{s_i, \dots, s_N\},$$
$$r \in \big[\min_{s \in S} c^{TTC}_{ss_{i-1}}, \min_{s \in S} c^{TTC}_{ss_i}\big) \cup \big[\max\{0, c^{TTC}_{s's_{i-1}}\}, \max\{0, c^{TTC}_{s's_i}\}\big)\Big)$$
$$\leq q_{s_k},$$

and the equation holds with equality whenever $\min_{s \in S} c^{TTC}_{ss_k} > 0$.

When there are neighborhood priorities, TTC is no longer equivalent to the Deferred Acceptance mechanism.[1] However, the equivalence holds for a special case of my problem, where families have common ordinal preference rankings over schools. This observation is important for establishing some of the further results.

[1]In fact, Calsamiglia and Miralles (2020) show that with neighborhood priorities TTC may be a better alternative to Deferred Acceptance with regard to providing better access to non-neighborhood schools. Unlike in my work, the authors take neighborhood choices as exogenously given.

Proposition 2. *Let $S = \{s_i\}_{i=1}^{|S|}$ and suppose there is a $V' \subseteq V$ with $\eta(V') = 1$, such that $v(h, s_i) \geq v(h, s_j)$ for all $v \in V, h \in H$ and $s_i, s_j \in S, i \geq j$. Then, for any $v \in V, h \in H$ and $\tau \in \mathcal{T}$,*

$$u_v^{DN}(h, \tau) = u_v^{TTC}(h, \tau).$$

I now discuss which results established for the Deferred Acceptance mechanism extend to TTC.

The proofs of Theorem 3 and Theorem 5 directly apply to TTC. Moreover, Assumptions 1 and 2 imply same ordinal rankings. Therefore, it is immediate from Propositions 1 and 2, that Theorem 4, Corollary 1 and 2 apply to TTC.

A.3 IA

Unlike the Deferred Acceptance and TTC, the IA mechanism is not strategyproof, i.e., truthfully reporting preferences is not a dominant strategy for families. Since preferences are typically unknown to the central planner, it is realistic to extend the model to allow families to choose not only where to reside, but also what preference ranking to report. Therefore I model families choices τ as a (Borel) probability measure over $V \times \bar{H} \times \mathcal{P}$. Let $\mathcal{T}$ be the space of such measures.

For a given mechanism ϕ and choices τ, I denote by $\lambda_{vs}^{\phi}(h, \succ, \tau) \in [0, 1]$ the probability that type v is assigned to school s conditional on choosing neighborhood h and submitting a preference ranking $\succ$. Later in this section, I derive school assignment probabilities for IA with or without neighborhood priorities. Before that, I define competitive equilibrium in this extended model.

Given school assignment probabilities and neighborhood price vector $p \in [0, 1]^{|H|}$, the expected utility of type v choosing neighborhood $h \in H$ and submitting preference ranking $\succ$ is equal to

$$u_v^\phi(h,\succ,\tau) - p_h.$$

where $u_v^\phi(h,\succ,\tau) := \sum_{s\in S} \lambda_{vs}^\phi(h,\succ,\tau)\ v(h,s)$. Also, let $u_v^\phi(0,\succ,\tau) := 0$ for all $v \in V, \succ \in P$ and $\tau \in \mathcal{T}$.

Definition 1. *For neighborhood choices $\tau \in \mathcal{T}$ and price vector $p \in \mathbb{R}_+^{|H|}$, we say a pair (τ, p) is a* ***competitive equilibrium (CE)*** *of mechanism ϕ if it satisfies the following conditions:*

1. $\tau\Big((v,h,\succ) \in V \times \bar{H} \times P : h = \arg\max_{h'\in\bar{H}}\ u_v^\phi(h',\succ,\tau) - p_{h'}\Big) = 1$, *where* $p_0 := 0$,
2. $\tau\Big((v,h,\succ) \in V \times \bar{H} \times P : h = h'\Big) \leq q_{h'}, \forall h' \in H$,
3. $\tau\Big((v,h,\succ) \in V \times \bar{H} \times P : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0$.

I now derive school assignment probabilities two versions of IA mechanism. To the best of our knowledge, this is the first description of IA for the continuum economy model.

IA without neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determines a probability measure G_τ over $P^2 \times [0,1]$, given by

$$G_\tau\Big((\succ,\succ',r) \in P^2 \times [0,1] : \succ \in P', \succ' \in P'', r \in (r_0, r_1)\Big)$$
$$= \tau\Big((v,h,\succ') \in V \times \bar{H} \times P : \succ_{vh} \in P', \succ' \in P''\Big) \times \Big(r_1 - r_0\Big),$$

for each $P', P'' \subseteq P$ and $(r_0, r_1) \subseteq [0,1]$.

For any $\succ \in P$ and $s \in S$, let $rk_\succ(s)$ denote the rank of school s in the preference ranking P in reverse order (i.e., $rk_\succ(s) = |S|$ when s the highest ranked according to $\succ$, and $rk_\succ(s) = 1$ if it is the lowest ranked).

Like with the Deferred Acceptance mechanism, the IA assignment can be given by cutoffs. Cutoffs are derived through an iterative procedure that I describe below. For a vector $c \in [1, |S|+1]^{|S|}$, the demand function $D : [1, |S|+1]^{|S|} \to [0, 1]$ is given by

$$D_s(c) = G_\tau\Big((\succ, \succ', r) \in P^2 \times [0,1] : r + rk_{\succ'}(s) \geq c_s \text{ and } s \succ s' \text{ for all } s' \text{ with } r + rk_{\succ'}(s') \geq c_{s'}\Big).$$

In other words, one may think of families having scores at schools which equals their lottery number plus the ranking of the school in their reported preferences. Thus, in this way families receive higher 'priorities' at IA when they rank it higher. Then, $D_s(c)$ is the mass of families whose scores exceed c_s, and who prefer s to any other school s' where their scores exceed $c_{s'}$. For $c \in [0, 1]^{|S|}$ and $x \in [1, |S|+1]$ let $c(s, x) \in [1, |S|+1]^{|S|}$ denote the vector that differs from c only by that $c_s(s, x) = x$.

Let us define a sequence of vectors $(c^t)_{t=1}^{\infty}$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0, 1] : D_s\big(c^t(s, x)\big) \leq q_s\Big\} & \text{otherwise.} \end{cases}$$

It follows from similar arguments as in Abdulkadiroğlu, Angrist, Narita, and Pathak (2017), that the sequence $(c^t)_{t \in \mathbb{N}}$ is convergent. Let $c^{IA} := \lim_{t \to \infty} c^t$ denote the **IA cutoffs**.

For cutoffs c^{IA} and preference ranking $\succ$, let $\bar{s}$ denote the most preferred school with $rk(\underline{s}) \geq c_{\underline{s}}^{IA}$. Also, let $\bar{S} \subseteq S$ be the largest set such that for each $s \in \bar{S}$, $s \succ \bar{s}$ or $s = \bar{s}$ and $rk(s) \geq c_s^{IA} - 1$. Then, the probability that type v is assigned to school s when choosing neighborhood h and reporting preference ranking $\succ$ is equal to $\lambda_{vs}^{IA}(h, \succ, \tau) = 0$ if $s \notin \bar{S}$, and otherwise,

$$\lambda_{vs}^{IA}(h, \succ, \tau) = \max\Big\{0, \min\Big\{c_{s'}^{IA} : s' \succ s, s' \in \bar{S}\Big\} - c_s^{DA}\Big\}.$$

IA with neighborhood priorities.

Neighborhood choices $\tau \in \mathcal{T}$ uniquely determines a probability measure G_τ over $P^2 \times S \times [0,1]$, given by

$$G_\tau\Big((\succ, \succ', s, r) \in P^2 \times S \times [0,1] : \succ \in P', \succ' \in P'', s \in S', r \in (r_0, r_1)\Big)$$

$$= \tau\Big((v, h, \succ') \in V \times \bar{H} \times P : \succ_{vh} \in P', s_h \in S', \succ' \in P''\Big) \times \Big(r_1 - r_0\Big),$$

for each $P', P'' \subseteq P$, $S' \subseteq S$ and $(r_0, r_1) \subseteq [0,1]$.

For a vector $c \in [1, 2(|S|+1)]^{|S|}$ consider the demand function $D : [1, 2(|S|+1)]^{|S|} \to [0,1]$ given by

$$D_s(c) = G_\tau\Big((\succ, \succ', s', r) \in P^2 \times S \times [0,1] : r + 2rk_{\succ'}(s) + \mathbb{1}[s' = s] \geq c_s$$

$$\text{and } s \succ s'' \text{ for all } s'' \text{ with } r + 2rk_{\succ'}(s'') + \mathbb{1}[s' = s''] \geq c_{s'}\Big).$$

Define a sequence of vectors $(c^t)_{t=1}^\infty$ recursively by $c^1 = 0$ and

$$c_s^{t+1} = \begin{cases} 0 & \text{if } D_s(c^t) < q_s, \\ \min\Big\{x \in [0,1] : D_s\big(c^t(s,x)\big) \leq q_s\Big\} & \text{otherwise,} \end{cases}$$

and let $c^{IA} := \lim_{t\to\infty} c^t$ be the IA cutoffs.

Again, for cutoffs c^{IA}, preference ranking $\succ$ and neighborhood choice h, let $\bar{s}$ denote the most preferred school with $2rk(\underline{s}) + \mathbb{1}[s_h = \underline{s}] \geq c_{\underline{s}}^{IA}$. Also, let $\bar{S} \subseteq S$ be the largest set such that for each $s \in \bar{S}$, $s \succ \bar{s}$ or $s = \bar{s}$ and $rk(s) + \mathbb{1}[s_h = s] \geq c_s^{IA} - 1$. Then, the probability that type v is assigned to school s when choosing neighborhood h and reporting preference ranking $\succ$ is equal to $\lambda_{vs}^{IA}(h, \succ, \tau) = 0$ if $s \notin \bar{S}$, and otherwise,

$$\lambda_{vs}^{IA}(h, \succ, \tau) = \max\Big\{0, \min\Big\{c_{s'}^{IA} : s' \succ s, s' \in \bar{S}\Big\} - c_s^{IA}\Big\}.$$

I briefly discuss how some of the results I established for the Deferred Acceptance mechanism extend to the setting with IA. Theorem 3 applies to IA with neighborhood priorities since a family can guarantee a neighborhood school by ranking it as a first choice. Since lowest-income families can guarantee underdemanded neighborhoods and schools for both versions of IA, Theorem 5 applies to IA as well.

I finish this section by discussing how IA compares to DA in terms of families' welfare. When there are no neighborhood priorities and families have identical ordinal preferences over schools, Abdulkadiroğlu et al. (2011) show that all families prefer IA to DA. I illustrate by an example that this is not necessarily the case when there are neighborhood priorities. In what follows I use IA to denote the version of the mechanism with neighborhood priorities.

Example 1. *There are three neighborhoods* $H = \{h_1, h_2, h_3\}$ *and three schools* $S = \{s_1, s_2, s_3\}$*, with* $q_{h_1} = 2$ *and* $q_{h_2} = q_{h_3} = 0.4$*,* $q_{s_1} = 0.4$ *and* $q_{s_2} = q_{s_3} = 0.58$*. The economy* η *is supported at only three points* v_1, v_2 *and* v_3*, with*

$$\eta\Big(v \in V : v = v_1\Big) = 0.2$$

and

$$\eta\Big(v \in V : v = v_2\Big) = \eta\Big(v \in V : v = v_3\Big) = 0.4,$$

where v_1, v_2 *and* v_3 *are shown in Table 1.*

	(h_1, s_1)	(h_1, s_2)	(h_1, s_3)	(h_2, s_1)	(h_2, s_2)	(h_2, s_3)	(h_3, s_1)	(h_3, s_2)	(h_3, s_3)
v_1	0.95	0.9	0.8	0.15	0.1	0	0.15	0.1	0
v_2	0.95	0.9	0.8	0.95	0.9	0.8	0.15	0.1	0
v_3	0.15	0.1	0	0.15	0.1	0	0.15	0.1	0

Table 1: Valuations

There is a CE of DN, where

- $p^{IA}_{h_1} = 0.5$, $p^{IA}_{h_2} = 0.2$ *and* $p^{IA}_{h_3} = 0$,

- *all type v_1 families choose neighborhood h_1 and submit their true preference rankings,*
- *all type v_2 families choose neighborhood h_2 and submit their true preference rankings,*
- *all type v_3 families choose neighborhood h_3 and submit preference ranking $s_2 \succ s_1 \succ s_3$.*

and a CE of IA, where

- $p^{IA}_{h_1} = 0.5$, $p^{IA}_{h_2} = 0.2$ *and* $p^{IA}_{h_3} = 0$,
- *all type v_1 families choose neighborhood h_1 and submit their true preference rankings,*
- *all type v_2 families choose neighborhood h_2 and submit their true preference rankings,*
- *all type v_3 families choose neighborhood h_3 and submit preference ranking $s_2 \succ s_1 \succ s_3$.*

The expected utility of type v_3 under IA is 0.1, *whereas, under DN her expected utility is*

$$\frac{1}{4} \times 0.15 + \frac{3}{4} \times \frac{58}{60} \times 0.1 > 0.1.$$

Thus, when there are neighborhood priorities, families in less preferred neighborhoods may prefer DN to IA.

B The Model with Local Public Financing

B.1 Preliminaries

In this section I assume that families' utilities from neighborhoods and schools depend on the housing values. This is motivated by the observation that local public expenditure funds are generated by property taxes, which are proportion to property values or equilibrium neighborhood prices.

I extend the model in Section 3 to assume that families' utilities depend on neighborhood prices.

More specifically, given neighborhood choice τ, neighborhood prices $p \in \mathbb{R}_+^{|H|}$ and school assignment mechanism ϕ, a type-v families expected utility when choosing neighborhood $h \in H$ is $U_v^\phi(h,\tau,p) - p_h$, where

$$U_v^\phi(h,\tau,p) := \sum_{s\in S} \lambda_{vs}^\phi(h,\tau)\ \mathcal{X}\big(v(h,s),\ p_h,\ p_{h_s}\big),$$

for all $h \in H$, and $U_v(0,\tau,p) \equiv 0$. Here h_s denotes the neighborhood of school s, $\mathcal{X} : \mathbb{R}^3 \to \mathbb{R}_+$ is a function satisfying

- $\mathcal{X}$ is non-decreasing in the first argument,
- $\mathcal{X}$ is non-increasing in the second argument,
- $\mathcal{X}$ is non-decreasing in the third argument,
- $\mathcal{X}\big(v(h,s),\ 0,\ 0\big) \equiv v(h,s)$.

The second point is motivated by that families pay property taxes which is proportional to the housing values. The third point is motivated by that the school spending is largely generated by local public funds, i.e., the property taxes from neighborhood families. The previous two arguments also motivate the fourth point: namely, valuations are unaffected if there is no taxation and no school spending.

Definition 2. *For neighborhood choices* $\tau \in \mathcal{T}$ *and price vector* $p \in \mathbb{R}_+^{|H|}$, *we say a pair* (τ, p) *is a* ***competitive equilibrium (CE)*** *of mechanism* ϕ *if it satisfies the following conditions:*

1. $\tau\Big((v,h) \in V \times \bar{H} : h = \arg\max_{h' \in \bar{H}} \sum_{s \in S} U_v^{\phi}(h', \tau, p) - p_{h'}\Big) = 1,$ *where* $p_0 := 0$,
2. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) \leq q_{h'}, \forall h' \in H$,
3. $\tau\Big((v,h) \in V \times \bar{H} : h = h'\Big) < q_{h'} \Rightarrow p_{h'} = 0$.

When defining the Deferred Acceptance mechanism, families' preferences over school account for the endogeneity of school valuations. Namely, given neighborhood choices τ and price vector p,

$$\mathcal{X}\big(v(h,s),\ p_h,\ p_s\big) > \mathcal{X}\big(v(h,s'),\ p_h,\ p_{s'}\big) \Rightarrow s \succ_{hv} s'.$$

Otherwise, the school assignment mechanisms are as in Section 3.

B.2 Aggregate Welfare

For CE (τ, p) of mechanism ϕ, let $M_h^{\phi}(\tau)$ and $M_s^{\phi}(\tau)$ denote the measure of types that choose neighborhood h and enroll in school s, respectively. Formally,

$$M_h^{\phi}(\tau) := \tau\Big((v,h') \in V \times \bar{H} : h' = h\Big) \text{ and}$$
$$M_s^{\phi}(\tau) := \int \lambda_{vs}^{\phi}(h,\tau) d\tau.$$

In this subsection I assume that public expenditures enter linearly in the families valuations. Namely, there are constants α and β, with $\alpha, \beta \in [0,1]$, such that for neighborhood choices τ and price vector p,

$$\mathcal{X}\big(v(h,s),\ p_h,\ p_s\big) = v(h,s) - \alpha p_h + \beta p_{h_s}. \tag{1}$$

In equation 1, we may think of α as the property tax rate and of β as the per student school spending.

I first compare school assignment mechanisms in terms of utilitarian or aggregate welfare, which may be interpreted as the 'sum' of families' and house sellers' utilities and public savings. For a CE (τ, p) of mechanism ϕ, families' utilities are

$$\int \Big(U_v^\phi(h, \tau, p) - p_h \Big) d\tau = \int U_v^\phi(h, \tau, p) d\tau - \sum_{h \in H} p_h M_h^\phi(\tau)$$

$$= \int \lambda_{vs}^\phi v(h, s) d\tau - \alpha \sum_{h \in H} p_h M_h^\phi(\tau) + \beta \sum_{s \in S} p_{h_s} M_s^\phi(\tau) - \sum_{h \in H} p_h M_h^\phi(\tau)$$

the sellers utilities are $\sum_{h \in H} p_h M_h^\phi(\tau)$, and the public savings are

$$\alpha \sum_{h \in H} p_h M_h^\phi(\tau) - \beta \sum_{s \in S} p_{h_s} M_s^\phi(\tau).$$

Hence, the aggregate welfare is

$$\int \lambda_{vs}^\phi v(h, s) d\tau.$$

Definition 3. *For mechanisms ϕ and ψ we say that ϕ* ***creates higher aggregate welfare*** *than ψ if for arbitrary CE (τ^ϕ, p^ϕ) of ϕ and (τ^ψ, p^ψ) of ψ,*

$$\int \lambda_{vs}^\phi(h, \tau^\phi) v(h, s) d\tau^\phi \geq \int \lambda_{vs}^\psi(h, \tau^\psi) v(h, s) d\tau^\psi.$$

My first observation is that the welfare comparison across DN and NA (Theorem 3) does not extend to the setting with local public financing, even in this special case where public expenditures enter the valuations linearly. I demonstrate this through the following example.

Example 2. *There are two neighborhoods $H = \{h_1, h_2\}$ and two schools $S = \{s_1, s_2\}$, with $q_{h_1} = 1/3$, $q_{h_2} = 2/3$ and $q_{s_1} = q_{s_2} = 1$. The economy η is supported at only three points v_1, v_2, v_3 with*

$$\eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_1\Big) = \eta\Big(v \in V : v = v_1\Big) = 1/3,$$

	(h_1, s_1)	(h_1, s_2)	(h_2, s_1)	(h_2, s_2)
v_1	1	1	0	0
v_2	0.55	0.55	0	0
v_2	0	0.01	0	0.01

Table 2: Valuations

where v_1, v_2 *and* v_3 *are shown in Table 2.*

Suppose $\alpha = 0.1$ *and* $\beta = 0.05$.

First consider DN. There is a CE of DN, where

- $p_{h_1}^{DN} = 0.5$ *and* $p_{h_1}^{DN} = 0$,
- *all type* v_1 *families choose* h_1,
- *all type* v_2 *and* v_3 *families choose* h_2,
- *all families are assigned to* s_1.

The aggregate welfare is $1/3 \times 1 + 1/3 \times 0 + 1/3 \times 0 = 1/3$.

Now consider NA. There is a CE of NA, where

- $p_{h_1}^{NA} = 0.5$ *and* $p_{h_1}^{NA} = 0$,
- *all type* v_1 *families choose* h_1,
- *all type* v_2 *and* v_3 *families choose* h_2,
- *all families are assigned to their neighborhood schools.*

The aggregate welfare is $1/3 \times 1 + 1/3 \times 0 + 1/3 \times 0.01 > 1/3$.

Next, I study how DN compares to DA in the setting with local public financing. My main result is that, like in the model without local public financing, DN creates higher aggregate welfare than DA for the special cases satisfying Assumptions 1 in Section 3.

Proposition 3. *Suppose Assumption 1 is satisfied. Then, DN creates higher aggregate welfare than DA.*

The proof is (almost) identical to that of the first part of Theorem 4 in the main text.

B.3 Lowest-Income Family Welfare

I now study how the mechanisms compare in terms of lowest-income family welfare. The model is as in Section 5 of the main text, with the difference that the utility of type $(v, b) \in V \times [0, 1]$ is

$$U^{\phi}_{vb}(h, \tau, p) := \sum_{s \in S} \lambda^{\phi}_{vs}(h, \tau)\ \mathcal{X}\Big(v(h, s),\ p_h,\ p_{h_s}\Big),$$

where again $\mathcal{X} : V \times \mathbb{R}^2 \times \mathcal{T} \to \mathbb{R}_+$ satisfies the four points mentioned before.

The definition of CE and underdemanded neighborhoods and schools are as in Section 5. Same is true for the notion of comparing mechanisms in terms of lowest-income family welfare. I (re)state this last definition below.

Definition 4. *A mechanism ϕ **creates higher welfare for lowest-income families** than mechanism ψ if for arbitrary CE (τ^{ϕ}, p^{ϕ}) of ϕ and (τ^{ψ}, p^{ϕ}) of ψ, there is a number $\bar{b} > 0$, such that for any measurable $U \times I \subseteq V \times [0, \bar{b}]$,*

$$\int_{U \times I} \big[U^{\phi}_{vb}(h, \tau^{\phi}) - p^{\phi}_h\big] d\tau^{\phi} \geq \int_{U \times I} \big[U^{\psi}_{vb}(h, \tau^{\psi}) - p^{\psi}_h\big] d\tau^{\psi}.$$

A major finding of this section is that Theorem 5 in the main section extends to the environment with local public financing.

Proposition 4. *The following is true:*

1. *If underdemanded neighborhoods under NA are also underdemanded under DN, then DN creates higher welfare for lowest-income families than NA.*

2. *If underdemanded neighborhoods under NA are also underdemanded under DA, and moreover, these neighborhoods have underdemanded schools, then DN creates higher welfare for lowest-income families than NA.*

The proof of Proposition 4 is (almost) identical to that of Theorem 5. The result uses that $\mathcal{X}\big(v(h,s),\ 0,\ 0\big) \equiv v(h,s)$, and hence the valuations for underdemanded neighborhoods and the corresponding schools are unaffected by the public financing.

Finally, we can establish that the conditions in Proposition 4 are satisfied under Assumption 3, i.e., for the special case of ranked neighborhoods and schools.

Corollary 1. *Suppose Assumption 3 is satisfied. Then, DN and DA create higher welfare for lowest-income family than NA.*

C Aggregate Welfare and (In)Efficiency

C.1 Overview

None of the studied assignment mechanisms maximizes aggregate welfare. The goal of this section is to quantify the inefficiencies admitted by DN, DA and NA by comparing them to two benchmarks: (1) welfare maximizing assignment (first best), and (2) welfare maximizing stable assignment. All subsequent analysis builds on the discrete economy model of Section 6.1.

C.2 Results

The first benchmark that I study is the (aggregate) **welfare maximizing assignment**. A joint neighborhood-school assignment of families is given by a mapping $\mu : F \to H \times S$, satisfying

- $\sum_{s \in S} \left|\mu^{-1}(h,s)\right| \leq q_h, \forall h \in H$,
- $\sum_{h \in H} \left|\mu^{-1}(h,s)\right| \leq q_s, \forall s \in S$.

Let $\mathcal{M}$ denote the set of all assignments. We say assignment μ^* is welfare maximizing assignment if it solves,

$$\mu^* = \arg\max_{\mu \in \mathcal{M}} \sum_{f \in F} v_f(\mu(f)).$$

When families valuations for joint neighborhood-school assignment take values of either zero or one, finding welfare maximizing assignment reduces to the NP-complete 3-dimensional matching problem (Karp, 1972). Therefore, finding a welfare maximizing assignment is NP-hard problem. The problem is tractable in the special case where families' valuations for neighborhood schools are additively separable.

The second benchmark that I study is the **welfare maximizing stable assignment**. When the school district applied neighborhood priorities, stability (also known as elimination of justified-envy) requires that a family is assigned to a school that she prefers less than her neighborhood school only if the latter school does not admit any family residing outside of the school's neighborhood. Formally, an assignment μ is stable if there are no families $f, f' \in F$, such that $\mu(f) = (h,s)$, $\mu(f') = (h', s_h)$, $v_f(h, s_h) > v_f(\mu(f))$ and $h' \neq h$. To simplify the analysis, I consider additively separable valuations for neighborhoods and schools. Moreover, instead of maximizing welfare in the entire set of stable assignments, I first fix families' neighborhood choices $\sigma^* : F \to \bar{H}$ to maximize the sum of neighborhood valuations, and then maximize aggregate welfare in the set of stable assignments that 'agree' with σ^*, i.e., for all $f \in F$, there is an $s \in S$, such that $\mu(f) = (\sigma^*(f), s)$.

Even with the simplifications above, finding a welfare maximizing stable assignment is an NP-hard problem. However, I solve this problem in our simulated environment using the methodology developed by Abdulkadiroğlu, Dur, and Grigoryan (2021). The authors provide an algorithm that is polynomial time in the number of students, but potentially exponential time in the number of schools. Since the number of school districts is typically much smaller than the number of students, the algorithm is tractable for real-life problems.

In the remainder of this section, I compare welfare across assignment mechanisms through simulations. There 1000 students, 10 neighborhoods and 10 schools. The valuation of family f for the joint assignment to neighborhood h and school s is equal to

$$v_f(h, s) = \alpha U_h + (1 - \alpha)U_s + 0.5\epsilon_{fh} + 0.5\epsilon_{fs},$$

where

- U_h and U_s are the common valuation for neighborhood h and schools s, respectively,
- ϵ_{fh} and ϵ_{fs} are the idiosyncratic valuations of family f for neighborhood h and schools s, respectively,
- α is a parameter.

Values of U_h, U_s, ϵ_{fh} and ϵ_{fs} are iid uniform draws from the unit interval. The capacity of school s is $100+\kappa_s$, where κ_s is a random draw from the set $\{1, 2, ..., 100/\gamma\}$. I report results for the following values for our parameters: $\alpha \in \{0, 0.5, 1\}$ and $\gamma \in \{2.4\}$.

α	γ	DN	DA	WM	WMS
0	2	5.28	−7.47	14.15	12.04
	4	3.15	−13.79	10.85	8.19
0.5	2	4.73	0.33	10.28	9.45
	4	3.28	−2.56	8.79	7.74
1	2	7.15	7.15	7.17	7.17
	4	7.14	7.14	7.17	7.17
Average		5.12	−1.58	9.74	8.63

Table 3: Aggregate welfare, % gains/losses compared to NA

The last two columns show the percentage gains in aggregate welfare from the welfare maximizing assignment and welfare maximizing stable assignment compared to NA. The numbers are 9.74% and 8.63%, respectively. Those gains are less than twice as large as those under DN. Therefore, DN eliminates more than half of the inefficiency admitted by NA (and DA).

References

Abdulkadiroğlu, A., J. D. Angrist, Y. Narita, and P. A. Pathak (2017): "Research Design Meets Market Design: Using Centralized Assignment for Impact Evaluation," *Econometrica*, 85, 1373–1432.

Abdulkadiroğlu, A., Y.-K. Che, and Y. Yasuda (2011): "Resolving conflicting preferences in school choice: The "Boston" mechanism reconsidered," *American Economic Review*, 101(1), 399–410.

Abdulkadiroğlu, A., U. Dur, and A. Grigoryan (2021): "School Assignment by Match Quality," National Bureau of Economic Research, Working Paper N28512.

Abdulkadiroğlu, A. and T. Sönmez (1998): "Random Serial Dictatorship and the Core from Random Endowments in House Allocation Problems," *Econometrica*, 66(3), 689–701.

——— (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Calsamiglia, C., F. Martínez-Mora, and A. Miralles (2015): "School Choice Mechanisms, Peer Effects and Sorting," Leicester, Department of Economics.

Calsamiglia, C. and A. Miralles (2020): "Catchment Areas and Access to Better Schools," Working Paper.

Karp, R. M. (1972): "Reducibility among Combinatorial Problems," *In Complexity of Computer Computations, Springer, Boston,MA*, 85–103.

Leshno, J. and I. Lo (2021): "The Simple Structure of Top Trading Cycles in School Choice," *Review of Economics Studies, Forthcoming.*

Neilson, C., M. Akbarpour, A. Kapor, W. van Dijk, and S. Zimmerman (2020): "Centralized School Choice with Unequal Outside Options," Working Paper.

Shapley, L. and H. Scarf (1974): "On Cores and Indivisibility," *Journal of Mathematical Economics*, 1, 23–28.