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Strategy-proofness and competitive equilibrium with transferable utility: Gross substitutes revisited

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Strategy-proofness and competitive equilibrium with transferable utility: Gross substitutes revisited[*]

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Abstract

We study the strategy-proofness (SP) of the competitive equilibrium selection (CE) mechanism in many-to-one matching with continuous transfers and quasilinear utility. The gross substitutes condition is known to guarantee the existence of CE and SP mechanisms. We show the converse: If a CE and SP mechanism exists, then the valuation of each firm must satisfy the gross substitutes condition. Various conditions for the existence of competitive equilibria have been proposed in the literature. Our results suggest that only the gross substitutes condition guarantees the existence of CE and SP mechanisms. We provide additional implications of our results for investment incentives and policy design. We also examine the two other models—the one with non-quasilinear utility and the one with discrete transfers. In contrast, the gross substitutes condition is not necessary in either model.

1 Introduction

The existence of competitive equilibria has been extensively studied in many economic models. In the literature of matching theory, attention has been paid to many-to-one matching models. These models consider matching between two types of agents, such as firms and workers, sellers and buyers, hospitals and doctors, or schools and

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students. In a many-to-one matching model with continuous transfers and quasilinear utility, Kelso and Crawford (1982) showed the gross substitutes condition (GS), which excludes the complementarity among workers in a firm's valuation, guarantees the existence of competitive equilibria. While GS is known to be a necessary condition in the *maximal domain* sense (Gul and Stacchetti, 1999), competitive equilibria can exist even if GS is violated. Recent developments in this field have revealed that various conditions other than GS can guarantee the existence of competitive equilibria. For example, Sun and Yang (2006) proposed the gross substitutes and complements condition (GSC) and showed that, unlike GS, competitive equilibria exist even when certain types of complement are allowed under GSC. Baldwin and Klemperer (2019) presented general conditions that include both GS and GSC as special cases.

GS offers an additional advantage: GS guarantees the existence of mechanisms that select a competitive equilibrium (CE) and are strategy-proof for workers (SP) (Hatfield et al., 2018). SP is a central notion of incentive compatibility in the literature on matching theory. SP mechanisms are strategically simple because workers do not need to consider how other workers will report, since truthful reporting is a weakly dominant strategy. Moreover, a CE and SP mechanism provides incentives for workers to invest efficiently (Hatfield et al., 2018). While the literature has focused on the existence of competitive equilibria, there has been less focus on SP mechanisms. This naturally leads to the question: Are there conditions, aside from GS, that can guarantee the existence of CE and SP mechanisms?

We show that GS is necessary for the existence of CE and SP mechanisms (Theorem 1). Therefore, various conditions guarantee the existence of CE mechanisms, but GS is the only one if we also require SP. Furthermore, our result differs from a maximal domain result. Specifically, our necessity states that if at least one firm's valuation violates GS, then a CE and SP mechanism does not exist, regardless of other firms' valuations. This necessity is useful for checking the existence of desired mechanisms for a given firm valuation profile. Section 4.1 discusses this point in more detail.

Our result has additional implications—investment incentives and policy design. First, our results show that GS is crucial for the existence of mechanisms that induce efficient investment by workers. Agents typically make certain choices before participating in mechanisms. For example, workers invest considerable time and effort in acquiring skills prior to job matching. This investment can have a significant impact on individual and social welfare. Hatfield et al. (2018) showed the strong link between SP and efficient investment. Together with their result and ours, we show that GS is necessary and sufficient for the existence of CE mechanisms that induce efficient investment by workers. Second, it helps in the design of institutions or policies. A government often imposes a tax or subsidy on a firm (Kojima et al., 2024).[1] In this case, if the policy intervention leads to a violation of GS, our results immediately imply that SP cannot be satisfied (even if the existence of competitive equilibria can be guaranteed). Therefore, if the CE and SP mechanism is desirable for policymakers, a policy must be designed so that a valuation of each firm satisfies GS.

The unique properties of the matching model with continuous transfers and quasi-linear utility are the key to our result. The proof consists of two steps. First, we show that the existence of worker-optimal competitive equilibria is necessary for the existence of CE and SP mechanisms (Proposition 1). The *rural hospital theorem*, which holds without any assumptions in our model (Jagadeesan et al., 2020), plays an important role in proving this. We then use the characterization of GS by local properties (Reijnierse et al., 2002), which is based on $M^\natural$-*concavity*, a concept from discrete convex analysis (Murota and Shioura, 1999).

We examine the other two models—the one with non-quasilinear utility and the one with discrete transfers. We illustrate that GS is not necessary in either model. In a model with continuous transfers and non-quasilinear utility, the rural hospital theorem may not hold. In a model with discrete transfers, the equivalence between the existence of worker-optimal competitive equilibria and that of the CE and SP mechanisms may not hold. Thus, we cannot extend our proof to these models. We illustrate that a CE and SP mechanism exists even if GS is violated. These findings highlight how continuous transfers and quasilinear utility produce clear results.

1.1 Related literature

Our research is most closely related to the works of Hatfield et al. (2018) and Jagadeesan et al. (2018). Hatfield et al. (2018) showed that in our setting, GS is a sufficient condition for the existence CE and SP mechanisms. This generalizes a result in the one-to-one matching model provided by Demange (1982) and Leonard (1983). Jagadeesan et al. (2018) provided an alternative proof using the rural hospital theo-

[1] Kojima et al. (2024) provide a necessary and sufficient condition for a policy to preserve GS.

rem, which also plays an important role in our results. These studies demonstrated the sufficiency of GS. We show the necessity: If a CE and SP mechanism exists, then each firm's valuation must satisfy GS.

There has been considerable research on the sufficient conditions for the existence of competitive equilibria. The classical condition was GS introduced by Kelso and Crawford (1982), and later generalized to GSC by Sun and Yang (2006). Teytelboym (2014) further generalized GSC. Hatfield et al. (2013) generalized GSC in the trading networks model, including our model. Recently, Baldwin and Klemperer (2019) substantially generalized these conditions by introducing the concept of *unimodular demand types*.[2] Furthermore, Shinozaki and Serizawa (2024) introduced a condition that is not a special case of the one Baldwin and Klemperer (2019) proposed. This series of studies has revealed that various conditions other than GS can ensure the existence of CE mechanisms. Our contribution is to show that if SP is also required, GS is the only condition.

CE and SP mechanisms have also been investigated in the model with continuous transfers and non-quasilinear utility. Demange and Gale (1985) proved the existence of CE and SP mechanisms in the one-to-one matching model.[3] Fleiner et al. (2019) studied the trading networks model with frictions and showed the existence of competitive equilibria under *full substitutability*.[4] Schlegel (2022) studied a CE and SP mechanism in this model. His result implies that GS and the additional conditions together guarantee the existence of CE and SP mechanisms in the many-to-one matching model. However, in contrast to the result in the model with the quasilinear utility, GS is not necessary in this model. We discuss this point in more detail in Section 5.

The other issues were also studied in our model. The first is workers' investments. Hatfield et al. (2018) showed the close connection between SP and socially efficient investment. In our model, their result implies that for any CE mechanism, it induces socially efficient investment by workers if and only if it is SP. Thus, we find through our results that GS is the necessary and sufficient condition for the existence of CE mechanisms to induce efficient investments by workers (Corollary 3). The second

[2] Danilov et al. (2001) provided a mathematically similar sufficient condition.

[3] Worker-optimal competitive equilibrium mechanisms can be characterized by desirable properties, including SP (see, e.g., (Miyake, 1998; Morimoto and Serizawa, 2015; Zhou and Serizawa, 2018; Kazumura et al., 2020)).

[4] Baldwin et al. (2023) and Nguyen and Vohra (2022) provided general conditions in a model of exchange economies with indivisible goods.

is policy intervention. Kojima et al. (2024) characterized policy interventions that preserve GS. While they focused on the existence of competitive equilibria, our results can add insights into CE and SP mechanisms (see Section 4.1).

The model of matching with contracts proposed by Hatfield and Milgrom (2005) is closely related to our model. In this model, *substitutability* and *stable matchings* are the counterparts of GS and competitive equilibria, respectively. They showed that substitutability is not a sufficient condition for the existence of stable and SP mechanisms. Moreover, they showed that an additional condition called the *law of aggregate demand*, together with substitutability, guarantees the existence of stable and SP mechanisms. Furthermore, it is known that each of the two conditions is not a necessary condition (see, e.g., (Hatfield and Kojima, 2010; Kominers and Sönmez, 2016)). Hirata and Kasuya (2017) studied stable and SP mechanisms without assuming any conditions on firms' preferences, such as substitutability. They showed that if a worker-optimal stable matching always exists, then a stable and SP mechanism must select it.[5] However, they also demonstrated that the existence of worker-optimal stable matchings is neither a necessary nor a sufficient condition for the existence of stable and SP mechanisms. In contrast, optimality for workers is both necessary and sufficient in our model. Compared to this series of studies, the model with continuous transfers and quasilinear utility gives surprisingly clear results.

The remainder of the paper is organized as follows. Section 2 introduces the model and concepts. Section 3 presents our main results and proofs. Section 4 presents applications of Theorem 1. Section 5 investigates other models.

2 Preliminaries

2.1 Model

Let F and W be the sets of firms and workers, respectively. Each worker is matched with a firm or remains unmatched, and receives some amount of money. Given $w \in W$, a typical consumption bundle is a pair $(f, s_w) \in F \times \mathbb{R}$. Each firm is matched with a set of workers or remains unmatched, and pays some amount of money. Given $f \in F$,

[5] Hatfield and Kojima (2010) introduced *unilateral substitutability*, which is weaker than substitutability, and showed that it is sufficient to guarantee the existence of worker-optimal stable matchings. Kasuya (2021) showed that unilateral substitutability is necessary in the maximal domain sense.

a typical consumption bundle is a pair $(W', s_f) \in 2^W \times \mathbb{R}$.

Each worker $w \in W$ has a valuation over firms, $v_w : F \to \mathbb{R}$, normalized so that $v_w(\emptyset) = 0$. The set of possible valuations for worker w is $V_w \equiv \mathbb{R}^F$. We let $V \equiv \Pi_{w \in W} V_w$ be the set of possible (worker) valuation profiles. The valuation v_w induces a *quasilinear utility function* $u_w(\cdot|v_w) : F \times \mathbb{R} \to \mathbb{R}$,

$$u_w(f, s|v_w) = v_w(f) + s.$$

The utility function $u_w(\cdot|v_w)$ of worker $w \in W$ induces a *demand correspondence* as follows. For each price vector $p \in \mathbb{R}^{F \times W}$,

$$D_w(p|v_w) = \{f \in F \mid u_w(f, p_{(f,w)}|v_w) \geq u_w(f', p_{(f',w)}|v_w) \text{ for all } f' \in F\}$$

where $p_{(\emptyset,w)} \equiv 0$.

Each firm $f \in F$ has a valuation over sets of workers, $v_f : 2^W \to \mathbb{R}$, normalized so that $v_f(\emptyset) = 0$. The valuation v_f induces a *quasilinear utility function* $u_f : 2^W \times \mathbb{R} \to \mathbb{R}$,

$$u_f(W', s) = v_f(W') - s.$$

We fix the valuation v_f throughout this paper and use $u_f(\cdot)$ instead of $u_f(\cdot|v_f)$. The utility function u_f of firm $f \in F$ induces a *demand correspondence* as follows. For each price vector $p \in \mathbb{R}^{F \times W}$,

$$D_f(p) = \{W' \in 2^W \mid u_f(W', p_{(f,W')}) \geq u_f(W'', p_{(f,W'')}) \text{ for all } W'' \subseteq W\}$$

where $p_{(f,W')} \equiv \sum_{w \in W'} p_{(f,w)}$ for all $W' \neq \emptyset$ and $p_{(f,W')} \equiv 0$ for $W' = \emptyset$.

For any $\mu \subseteq F \times W$, we denote $\mu_w = \{f' \in F \mid (f', w) \in \mu\}$ and $\mu_f = \{w' \in W \mid (f, w') \in \mu\}$ for each $w \in W$ and $f \in F$. We say that $\mu \subseteq F \times W$ is a (many-to-one) *matching* if $|\mu_w| \leq 1$ for all $w \in W$. With abuse of notation, we denote $\mu_w = f$ when $\mu_w = \{f\}$ for some $f \in F$ given a matching μ. Let $\mathcal{M}$ denote the set of all matching.

For each $f \in F$, $w \in W$, and $p_{(f,w)} \in \mathbb{R}$, $(f, w, p_{(f,w)})$ is called a *contract*. A set of contracts A is called an *outcome* when each worker has at most one contract in A. For each outcome A and $w \in W$, let $A_w = \{(f, w, p_{(f,w)}) | (f, w, p_{(f,w)}) \in A\}$ be the set of contracts in A involving w. The utility function of a worker u_w naturally extend to outcome as $u_w(A|v_w) = v_w(f) + p_{(f,w)}$ if $A_w = \{(f, w, p_{(f,w)})\}$ and $u_w(A|v_w) = 0$ if $A_w = \emptyset$. A pair $(\mu, p) \in 2^{F \times W} \times \mathbb{R}^{F \times W}$ is called an *arrangement* when μ is a matching. The utility function of a worker u_w also naturally extend to arrangement as $u_w((\mu, p)|v_w) \equiv v_w(\mu_w) + p_{(\mu_w, w)}$.

An arrangement (μ, p) is a *competitive equilibrium* at $v \in V$ if $\mu_w \in D_w(p|v_w)$ for all $w \in W$ and $\mu_f \in D_f(p)$ for all $f \in F$. A price vector $p \in \mathbb{R}^{F \times W}$ is a *competitive equilibrium price vector* at $v \in V$ if (μ, p) is a competitive equilibrium at $v \in V$ for some matching μ. A matching μ is a *competitive equilibrium matching* at $v \in V$ if (μ, p) is competitive equilibrium at $v \in V$ for some $p \in \mathbb{R}^{F \times W}$. A competitive equilibrium (μ, p) at $v \in V$ is a *worker-optimal competitive equilibrium* at $v \in V$ if $u_w((\mu, p)|v_w) \geq u_w((\mu', p')|v_w)$ for all $w \in W$ and competitive equilibrium (μ', p') at $v \in V$. For a (worker-optimal) competitive equilibrium (μ, p) at $v \in V$, $\{(f, w, p_{(f,w)}) \mid (f, w) \in \mu\}$ is called a *(worker-optimal) competitive equilibrium outcome* at $v \in V$. A matching μ is *efficient* at $v \in V$ if μ maximizes the total surplus: i.e., $\mu \in \arg\max_{\mu' \in \mathcal{M}} \sum_{w \in W} v_w(\mu'_w) + \sum_{f \in F} v_f(\mu'_f)$. It is known that any competitive equilibrium matching is efficient.

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2.2 Mechanism

A *mechanism* is defined as a function φ that specifies an outcome $\varphi(v)$ for each valuation profile $v \in V$. A mechanism φ is a *competitive equilibrium selection* (CE) if for all $v \in V$, $\varphi(v)$ is a competitive equilibrium outcome at v. A CE mechanism φ is *worker-optimal* if for all $v \in V$, $\varphi(v)$ is a worker-optimal competitive equilibrium outcome at v.

Now, we introduce the property of a mechanism that is central to our analysis.

Definition 1. A mechanism φ is *strategy-proof for workers* (SP) if for all $v \in V$, $w \in W$, and $v'_w \in V_w$, we have

$$u_w(\varphi(v)|v_w) \geq u_w(\varphi(v'_w, v_{-w})|v_w),$$

where $v_{-w} = v_{W \setminus \{w\}} \in V_{W \setminus \{w\}}$.

SP states that reporting the true valuation v_w is a weakly dominant strategy for any worker w under φ.

2.3 Gross substitutes

Kelso and Crawford (1982) introduced the gross substitutes condition. Intuitively, this excludes the complementarity among workers in a firm's valuation.

Definition 2. A firm f's valuation v_f satisfies the *gross substitutes* condition (GS) if for any two price vectors p, p' with $p \leq p'$ and $W' \in D_f(p)$, there exists $W'' \in D_f(p')$ such that $\{w \in W' : p_{(f,w)} = p'_{(f,w)}\} \subseteq W''$.

GS states that if a firm f demands a set of workers (contained in a demand set) at a given price vector, then even if the salaries of other workers increase, f will continue to demand that set.

3 Results

3.1 Optimality of CE and SP

To prove our main result, we first show that a worker-optimal competitive equilibrium is a key to the existence of CE and SP mechanisms. Jagadeesan et al. (2018) showed that if a worker-optimal competitive equilibrium exists for any valuation profile, then a CE and SP mechanism exists. We show the converse: The existence of the worker-optimal competitive equilibria is necessary for that of CE and SP mechanisms.

The *rural hospital theorem* is crucial for our necessity. This theorem states that if a worker receives strictly positive utility in some competitive equilibrium, then she is matched in every competitive equilibrium matching. Equivalently, if a worker is unmatched in some competitive equilibrium matching, then she receives zero utility, which is the same as being unmatched, in every competitive equilibrium.

Lemma 1 ((Jagadeesan et al., 2020))**.** For any valuation profile v, competitive equilibria (μ, p) and (μ', p') at v, and worker w, $u_w((\mu, p)|v_w) > 0$ implies $\mu'_w \neq \emptyset$.

It is worth noting that the rural hospital theorem holds in our model without any assumption about firms' valuations. This contrasts with the other models, such as the one with non-quasilinear utility. In fact, optimality is not necessary in this model. We will discuss this point more in detail in Section 5.

Proposition 1. If a CE and SP mechanism exists, then a worker-optimal competitive equilibrium exists for all valuation profile $v \in V$.

Proof. Suppose not: i.e., a CE and SP mechanism φ exists, and a worker-optimal competitive equilibrium does not exist for some valuation profile v. Let $B = \varphi(v)$. Then, there exist a competitive equilibrium outcome A and a worker $w \in W$ such that $u_w(A|v_w) > u_w(B|v_w)$. Let (μ, p) and (μ', p') be competitive equilibria for A and B, respectively. We let ϵ be such that $u_w(A|v_w) > \epsilon > u_w(B|v_w)$. Consider a valuation $\tilde{v}_w$ with $\tilde{v}_w(f) = v_w(f) - \epsilon$ for all $f \in F$. We show that (μ, p) is a competitive equilibrium at $\tilde{v} = (\tilde{v}_w, v_{-w})$. By the construction of $\tilde{v}$, we have $\tilde{v}_{\hat{w}} = v_{\hat{w}}$ for any $\hat{w} \in W$ with $\hat{w} \neq w$. This implies $\mu_{\hat{w}} \in D_{\hat{w}}(p|\tilde{v}_{\hat{w}})$ for any $\hat{w} \in W$ with $\hat{w} \neq w$. Since v_f is fixed for all $f \in F$, we have $\mu_f \in D_f(p)$. For all $f \in F$, $\mu_w \in D_w(p|v_w)$ implies $u_w(\mu_w, p_{(\mu_w,w)}|v_w) \geq u_w(f, p_{(f,w)}|v_w)$, which implies $u_w(\mu_w, p_{(\mu_w,w)}|v_w) - \epsilon \geq u_w(f, p_{(f,w)}|v_w) - \epsilon$. Thus, we have $u_w(\mu_w, p_{(\mu_w,w)}|\tilde{v}_w) \geq u_w(f, p_{(f,w)}|\tilde{v}_w)$ for all $f \in F$. Note $u_w(A|v_w) - \epsilon > 0$, which implies $u_w(\mu_w, p_{(\mu_w,w)}|\tilde{v}_w) > u_w(\emptyset, p_{(\emptyset,w)}|\tilde{v}_w) = 0$. These facts together imply $\mu_w \in D_w(p|\tilde{v}_w)$.

Note $u_w(A|\tilde{v}_w) = u_w(A|v_w) - \epsilon > 0$ by the construction of ϵ. Since (μ, p) is a competitive equilibrium at $(\tilde{v}_w, v_{-w})$, w is matched with some firm at any competitive equilibrium matching at $(\tilde{v}_w, v_{-w})$ by Lemma 1. Since φ is CE, $\varphi(\tilde{v}_w, v_{-w})$ is a competitive equilibrium outcome for some competitive equilibrium $(\hat{\mu}, \hat{p})$ at $(\tilde{v}_w, v_{-w})$. Then, we have

$$
\begin{aligned}
u_w(\varphi(\tilde{v}_w, v_{-w})|\tilde{v}_w) &= \tilde{v}_w(\hat{\mu}_w) + \hat{p}_{(\hat{\mu}_w,w)} \\
&= v_w(\hat{\mu}_w) - \epsilon + \hat{p}_{(\hat{\mu}_w,w)} \\
&= u_w(\varphi(\tilde{v}_w, v_{-w})|v_w) - \epsilon \\
&\geq 0,
\end{aligned}
$$

where the second equality follows from $\hat{\mu}_w \neq \emptyset$, and the fourth inequality follows from $u_w(\varphi(\tilde{v}_w, v_{-w})|\tilde{v}_w) \geq 0$ by the definition of CE. Thus, we have $u_w(\varphi(\tilde{v}_w, v_{-w})|v_w) \geq \epsilon$. By the construction of ϵ, we have $u_w(\varphi(\tilde{v}_w, v_{-w})|v_w) > u_w(B|v_w) = u_w(\varphi(v)|v_w)$. This contradicts that φ is SP. $\square$

Together with the sufficiency by Jagadeesan et al. (2018), we can characterize the CE and SP mechanisms using the optimality of competitive equilibrium.

Corollary 1. A CE and SP mechanism exists if and only if a worker-optimal competitive equilibrium exists for all valuation profile $v \in V$.

3.2 Necessity for CE and SP: Gross substitutes

Now we provide our main result.

Theorem 1. If a CE and SP mechanism exists, then a valuation v_f satisfies GS for any $f \in F$.

Together with the sufficiency by Hatfield et al. (2018), we find that GS is the unique condition for the existence of CE and SP mechanisms.

Corollary 2. A CE and SP mechanism exists if and only if a valuation v_f satisfies GS for any $f \in F$.

Several points related to Theorem 1 should be mentioned. First, our result is useful for checking the existence of CE and SP mechanisms. Specifically, our necessity states that if at least one firm's valuation violates GS, then a CE and SP mechanism does not exist, regardless of other firms' valuations. Note that our necessity differs from the usual *maximal domain*. A condition is a maximal domain for desired mechanisms if, for any violation of this condition by a firm, there always exist valuations of other firms that satisfy the condition such that there are no desired mechanisms at the profile. GS is known to be a maximal domain for the existence of CE mechanisms (Gul and Stacchetti, 1999). Our necessity is useful for checking the existence of desired mechanisms for a given firm valuation profile. Consider a firm valuation profile where a valuation of some firm violates GS. Since GS is the maximal domain for the existence of competitive equilibria, we cannot determine whether a CE mechanism exists for this profile. However, we can immediately determine from Theorem 1 that a CE and SP mechanism does not exist. We will illustrate this point in Section 4.1.

Second, our result also has implications for efficient investment. Agents typically make certain choices before participating in mechanisms. For example, workers invest considerable time and effort in acquiring skills prior to job matching. This investment can have a significant impact on individual and social welfare. Hatfield et al. (2018) showed the strong link between SP and socially efficient investment. Together with their result and ours, we show that GS is the necessary and sufficient condition for CE mechanisms to induce socially efficient investment by workers. This point is discussed in more detail in Section 4.2.

Third, our result has implications for policy design. A government often imposes a tax or subsidy on a firm (Kojima et al., 2024). In this case, if the policy intervention leads to a violation of GS, Theorem 1 immediately implies that SP cannot be satisfied (even if the existence of competitive equilibria can be guaranteed). Therefore, if the CE and SP mechanism is desirable for policymakers, a policy must be designed so that a valuation of each firm satisfies GS. Furthermore, Kojima et al. (2024) characterized policy interventions that preserve GS. While they focused on the existence of competitive equilibria, our results can add insights into CE and SP mechanisms. Example 3 in Section 4.1 illustrates this point.

Finally, we provide a variant of Theorem 1. The proof of Theorem 1 exploits the fact that a worker can have a positive valuation for firms. Therefore, we allow for the possibility of negative salaries in a competitive equilibrium. This may seem unnatural in the context of worker-firm matching. Theorem 1 still holds even if we restrict the valuation of each worker to be non-positive by assuming that v_f is non-decreasing for each $f \in F$.[6] Furthermore, in this case, salaries will be non-negative at every competitive equilibrium. We provide an example (Example 6) as an application of this result in Section 4.1.

Proposition 2. Suppose that $V_w = \mathbb{R}^F_{\leq 0}$ for any $w \in W$ and v_f is non-decreasing for any $f \in F$.[7] If a CE and SP mechanism exists, then a valuation v_f satisfies GS for any $f \in F$.

3.3 Proof sketch of Theorem 1

In this section, we provide a proof sketch of Theorem 1. First, we illustrate that two types of violations of GS lead non-existence of CE and SP mechanisms (Examples 1 and 2). Then, we see that any violation of GS can be attributed to these two types of violations. This follows from the characterization of GS by local properties (Reijnierse et al., 2002).

The first example is the violation of *submodularity*, which is implied by GS in our model.[8] The following example illustrates that the violation of submodularity would lead to the non-existence of CE and SP mechanisms.

Example 1. Let $F = \{f\}$ and $W = \{w_1, w_2\}$. The valuation profile is given as

[6]A firm's valuation v_f is non-decreasing if $v_f(W') \geq v_f(W'')$ for any $W', W'' \subseteq W$ with $W'' \subseteq W'$.
[7]Let $\mathbb{R}_{\leq 0}$ denote the set of non-positive real numbers.
[8]A valuation v_f is submodular if for all $X, Y \subseteq W$, we have $v_f(X)+v_f(Y) \geq v_f(X \cup Y)+v_f(X \cap Y)$.

follows:

$$v_f(\emptyset) = v_f(\{w_1\}) = v_f(\{w_2\}) = 0, v_f(\{w_1, w_2\}) = 1,$$
$$v_{w_1}(f) = v_{w_2}(f) = 0.$$

Note that μ with $\mu_f = \{w_1, w_2\}$ is a unique efficient matching, and thus a competitive equilibrium matching has to be μ. There are two competitive equilibria (μ, p) and (μ, q) such that $p = (p_{(f,w_1)}, p_{(f,w_2)}) = (1, 0)$ and $q = (0, 1)$. However, any (μ, r) with $r_{(f,w_1)}, r_{(f,w_1)} \geq 1$ is not a competitive equilibrium since $r_{(f,w_1)} + r_{(f,w_1)} > 1$. This means that a worker-optimal competitive equilibrium does not exist for this valuation profile. Thus, Proposition 1 implies that there is no CE and SP mechanism in this example.

The second example was provided by Kelso and Crawford (1982). While GS implies submodularity in our model, the converse does not hold. Kelso and Crawford (1982) illustrates this fact using the following example. We show that there is no CE and SP mechanism in this example.

Example 2. Let $F = \{f\}$ and $W = \{w_1, w_2, w_3\}$. The valuation profile is given as follows:

$$v_f(\emptyset) = 0, v_f(\{w_1\}) = v_f(\{w_2\}) = 4, v_f(\{w_3\}) = 4.25,$$
$$v_f(\{w_1, w_2\}) = 7.5, v_f(\{w_1, w_3\}) = v_f(\{w_2, w_3\}) = 7,$$
$$v_f(\{w_1, w_2, w_3\}) = 9,$$
$$v_{w_1}(f) = v_{w_2}(f) = 0, v_{w_3}(f) = -3.$$

While v_f is submodular, it violates GS: We have $D_f(p) = \{\{w_1, w_2\}\}$ for $p = (p_{(f,w_1)}, p_{(f,w_2)}, p_{(f,w_3)}) = (3, 3, 3)$ and $D_f(q) = \{\{w_3\}\}$ for $q = (3.5, 3, 3)$. Note that μ with $\mu_f = \{w_1, w_2\}$ is a unique efficient matching, and thus a competitive equilibrium matching has to be μ. There are two competitive equilibria (μ, p) and (μ, q) such that $p = (3.5, 2.75, 3)$ and $q = (2.75, 3.5, 3)$. Consider any (μ, r) with $r_{(f,w_1)}, r_{(f,w_2)} \geq 3.5$. If $\{w_1, w_2\} \in D_f(r)$, we have $r_{(f,w_3)} \geq 3.75$, which imply $D_{w_3}(r|v_{w_3}) = \{f\}$. Thus, (μ, r) is not a competitive equilibrium. This means that a worker-optimal competitive equilibrium does not exist for this valuation profile. Thus, Proposition 1 implies that there is no CE and SP mechanism in this example.

The violation of GS can be attributed to the two types of violations illustrated in Examples 1 and 2. This follows from the characterization of GS by two local properties (Reijnierse et al., 2002).

Proposition 3 ((Reijnierse et al., 2002))**.** A firm's valuation v_f satisfies GS if and only if the following two conditions hold.

Condition 1. For any $X \subseteq W$, $w_1, w_2 \in W \setminus X$ with $w_1 \neq w_2$,

$$v_f(X \cup \{w_1, w_2\}) + v_f(X) \leq v_f(X \cup \{w_1\}) + v_f(X \cup \{w_2\}).$$

Condition 2. For any $X \subseteq W$, $w_1, w_2, w_3 \in W \setminus X$ such that w_1, w_2, and w_3 are distinct,

$$\begin{aligned} & v_f(X \cup \{w_1, w_2\}) + v_f(X \cup \{w_3\}) \\ \leq & \max\{v_f(X \cup \{w_1, w_3\}) + v_f(X \cup \{w_2\}), v_f(X \cup \{w_2, w_3\}) + v_f(X \cup \{w_1\})\}. \end{aligned}$$

The conditions above are "local" in the sense that they focus only on two sets X, Y with $\max(|X \setminus Y|, |Y \setminus X|) \leq 2$.[9] This characterization is related to $M^\natural$-*concavity*, a concept from discrete convex analysis (Murota and Shioura, 1999). In the model with continuous transfers and quasilinear utility, Fujishige and Yang (2003) showed that GS is equivalent to $M^\natural$-concavity of a valuation. Proposition 3 utilizes this equivalence. See Murota (2016) for more details.

Based on Proposition 3, any violations of GS can be classified into that of Conditions 1 or 2. A firm's valuation v_f in Example 1 violates Condition 1. A firm's valuation v_f in Example 2 satisfies Condition 1 but violates Condition 2. The proof of Theorem 1 generalizes the intuition of these examples.

3.4 Proof of Theorem 1

Proof. We begin with a lemma.

[9] In general, we need additional conditions on $\mathrm{dom} f = \{X : f(X) > -\infty\}$ for this result. In our model, we have $\mathrm{dom} f = 2^W$, and thus these conditions are always satisfied. The condition $\mathrm{dom} f = 2^W$ also holds in the models studied by Kelso and Crawford (1982); Gul and Stacchetti (1999); Reijnierse et al. (2002). For general models with $\mathrm{dom} f \neq 2^W$, see (Murota, 2016).

Lemma 2 ((Jagadeesan et al., 2020))**.** Let (μ, p) be any competitive equilibrium. For any competitive equilibrium matching μ', (μ', p) is a competitive equilibrium.

Suppose toward a contradiction that there exists a firm f such that v_f violates GS. Then either Condition 1 or Condition 2 in Proposition 3 is violated. We have two cases to consider. To simplify notation, define $u_f(X, p) \equiv v_f(X) - \sum_{w \in X} p_{(f,w)}$ for each $X \subseteq W$, and $p \in \mathbb{R}^{F \times W}$.

Case 1: v_f violates Condition 1.

There exist $X \subseteq W$ and $w_1, w_2 \in W \setminus X$ with $w_1 \neq w_2$ such that

$$v_f(X \cup \{w_1, w_2\}) + v_f(X) > v_f(X \cup \{w_1\}) + v_f(X \cup \{w_2\}).$$

Consider the following worker valuation profile v;

$$\begin{aligned} v_w(f) &= \alpha \text{ for all } w \in X \cup \{w_1, w_2\}, \\ v_w(f) &< -\max_{Z \in 2^W} 2|v_f(Z)| \text{ for all } w \in W \setminus (X \cup \{w_1, w_2\}), \text{ and} \\ v_w(f') &< -\max_{Z \in 2^W} 2|v_{f'}(Z)| \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}, \end{aligned}$$

where $\alpha \equiv -\min_{Z \subseteq Y \subseteq X \cup \{w_1, w_2\}} v_f(Y) - v_f(Z)$. Note $\alpha \geq 0$.

Consider a matching μ with $\mu_f = X \cup \{w_1, w_2\}$ and $\mu_{f'} = \emptyset$ for all $f' \in F \setminus \{f\}$. We show that there exists a competitive equilibrium (μ, p) such that

$$\begin{aligned} p_{(f,w_1)} &= v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\}), \\ p_{(f,w_2)} &= v_f(X \cup \{w_2\}) - v_f(X), \\ p_{(f,w)} &= -\alpha \text{ for all } w \in X, \\ p_{(f,w)} &= -v_w(f) \text{ for all } w \notin X \cup \{w_1, w_2\}, \text{ and} \\ p_{(f',w)} &= -v_w(f') \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}. \end{aligned}$$

By the construction of p, we have $f \in D_w(p|v_w)$ for each $w \in X \cup \{w_1, w_2\}$. Also, we have $\emptyset \in D_w(p|v_w)$ for each $w \notin X \cup \{w_1, w_2\}$. For any $f' \in F \setminus \{f\}$, $w \in W \setminus \{w\}$, and $Y \subseteq W$, we have

$$
\begin{aligned}
&u_{f'}(Y,p) - u_{f'}(Y \cup \{w\}, p)\\
=&p_{(f',w)} - (v_{f'}(Y \cup \{w\}) - v_{f'}(Y))\\
=&\max_{Z \in 2^W} 2|v_{f'}(Z)| - (v_{f'}(Y \cup \{w\}) - v_{f'}(Y))\\
\geq&0.
\end{aligned}
$$

Thus, we have $\emptyset \in D_{f'}(p)$ for each $f' \in F \setminus \{f\}$. We show that $X \cup \{w_1, w_2\} \in D_f(p)$. Note that for all Y with $Y \cap (W \setminus (X \cup \{w_1, w_2\})) \neq \emptyset$, we have $u_f((X \cup \{w_1, w_2\}) \cap Y, p) \geq u_f(Y, p)$. Thus, it suffices to show that for all $Y \subseteq X \cup \{w_1, w_2\}$, we have $u_f(X \cup \{w_1, w_2\}, p) \geq u_f(Y, p)$. For all $Y \subseteq X \cup \{w_1, w_2\}$ and $w \in X \setminus Y$, we have

$$
\begin{aligned}
&u_f(Y \cup \{w\}, p) - u_f(Y, p)\\
=&v_f(Y \cup \{w\}) - v_f(Y) + \alpha\\
\geq&0,
\end{aligned}
$$

where the second inequality follows from $p_{(f,w)} = -\alpha$. This implies $u_f(X,p) \geq u_f(\emptyset, p) = 0$ and $u_f(X \cup Z, p) \geq u_f(Z, p)$ for all $Z \subseteq W$. Thus, it suffices to show $u_f(X \cup \{w_1, w_2\}, p) \geq u_f(Y, p)$ for $Y \in \{X, X \cup \{w_1\}, X \cup \{w_2\}\}$.

For X,

$$
\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, p)\\
=&v_f(X \cup \{w_1, w_2\})\\
&- ((v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\})) + (v_f(X \cup \{w_2\}) - v_f(X)))\\
&- \sum_{w \in X} p_{(f,w)}\\
=&v_f(X) - \sum_{w \in X} p_{(f,w)}\\
=&u_f(X,p).
\end{aligned}
$$

For $X \cup \{w_2\}$,

$$\begin{aligned}
&u_f(X \cup \{w_2\}, p) \\
&= v_f(X \cup \{w_2\}) - ((v_f(X \cup \{w_2\}) - v_f(X)) - \sum_{w \in X} p_{(f,w)}) \\
&= v_f(X) - \sum_{w \in X} p_{(f,w)} \\
&= u_f(X, p) \\
&= u_f(X \cup \{w_1, w_2\}, p).
\end{aligned}$$

For $X \cup \{w_1\}$,

$$\begin{aligned}
&u_f(X \cup \{w_1\}, p) \\
&= v_f(X \cup \{w_1\}) - (v_f(X \cup \{w_1, w_2\} - v_f(X \cup \{w_2\})) - \sum_{w \in X} p_{(f,w)} \\
&< v_f(X) - \sum_{w \in X} p_{(f,w)} \\
&= u_f(X, p) \\
&= u_f(X \cup \{w_1, w_2\}, p),
\end{aligned}$$

where the second inequality follows from the assumption of Case 1.

Similarly, we can show that there exists a competitive equilibrium (μ, q) such that

$$\begin{aligned}
q_{(f,w_1)} &= v_f(X \cup \{w_1\}) - v_f(X), \\
q_{(f,w_2)} &= v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_1\}), \\
q_{(f,w)} &= -\alpha \text{ for all } w \in X, \\
q_{(f,w)} &= -v_w(f) \text{ for all } w \notin X \cup \{w_1, w_2\}, \text{ and} \\
q_{(f',w)} &= -v_w(f') \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}.
\end{aligned}$$

Suppose that there exists a worker-optimal competitive equilibrium (μ', r). By Lemma 2, (μ, r) is also a competitive equilibrium. Note $u_w((\mu', r)|v_w) = u_w((\mu, r)|v_w)$ for any $w \in W$. Since (μ', r) is a worker-optimal competitive equilibrium, we have

$r_{(f,w_1)} \geq p_{(f,w_1)}$ and $r_{(f,w_2)} \geq q_{(f,w_2)}$. However, we have $X \cup \{w_1, w_2\} \notin D_f(r)$ since

$$\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, r) \\
=&v_f(X \cup \{w_1, w_2\}) \\
&- ((v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\})) + (v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_1\})) \\
&- \sum_{w \in X} r_{(f,w)} \\
<&v_f(X) - \sum_{w \in X} r_{(f,w)} \\
=&u_f(X, r),
\end{aligned}$$

where the second inequality follows from the assumption of Case 1. This contradicts that (μ, r) is a competitive equilibrium. Thus, a worker-optimal competitive equilibrium does not exist for this valuation profile v. By Proposition 1, there is no CE and SP mechanism in this case.

Case 2: While v_f satisfies Condition 1, v_f violates Condition 2.

There exist $X \subseteq W$ and $w_1, w_2, w_3 \in W \setminus X$ such that w_1, w_2, and w_3 are distinct, and

$$\begin{aligned}
&v_f(X \cup \{w_1, w_2\}) + v_f(X \cup \{w_3\}) \\
&> \max\{v_f(X \cup \{w_1, w_3\}) + v_f(X \cup \{w_2\}), v_f(X \cup \{w_2, w_3\}) + v_f(X \cup \{w_1\})\}.
\end{aligned}$$

Consider the following worker valuation profile v;

$$\begin{aligned}
&v_w(f) = \beta \text{ for all } w \in X \cup \{w_1, w_2\}, \\
&v_{w_3}(f) \\
&\quad = -\max\{v_f(X \cup \{w_1, w_3\}) - v_f(X \cup \{w_1\}), v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\})\}, \\
&v_w(f) < -\max_{Z \in 2^W} 2|v_f(Z)| \text{ for all } w \in W \setminus (X \cup \{w_1, w_2, w_3\}), \text{ and} \\
&v_w(f') < -\max_{Z \in 2^W} 2|v_{f'}(Z)| \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\},
\end{aligned}$$

where $\beta \equiv -\min_{Z \subseteq Y \subseteq X \cup \{w_1, w_2, w_3\}} v_f(Y) - v_f(Z)$. Note $\beta \geq 0$.

Consider a matching μ with $\mu_f = X \cup \{w_1, w_2\}$ and $\mu_{f'} = \emptyset$ for all $f' \in F \setminus \{f\}$

We show that there exists a competitive equilibrium (μ, p) such that

$$\begin{aligned}
p_{(f,w_1)} &= v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\}),\\
p_{(f,w_2)} &= v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_3\}),\\
p_{(f,w_3)} &= v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\}),\\
p_{(f,w)} &= -\beta \text{ for all } w \in X,\\
p_{(f,w)} &= -v_w(f) \text{ for all } w \notin X \cup \{w_1, w_2, w_3\}, \text{ and}\\
p_{(f',w)} &= -v_w(f') \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}.
\end{aligned}$$

Note $f \in D_w(p|v_w)$ for each $w \in X \cup \{w_1, w_2\}$, $\emptyset \in D_w(p|v_w)$ for each $w \notin X \cup \{w_1, w_2\}$, and $\emptyset \in D_{f'}(p)$ for each $f' \in F \setminus \{f\}$. We show that $X \cup \{w_1, w_2\} \in D_f(p)$. By $p_{(f,w)} = -\beta$ for all $w \in X$, a similar argument in Case 1 implies that it suffices to show that $u_f(X \cup \{w_1, w_2\}, p) \geq u_f(Y, p)$ for

$$Y \in \{X, X \cup \{w_1\}, X \cup \{w_2\}, X \cup \{w_3\}, X \cup \{w_2, w_3\}, X \cup \{w_1, w_3\}, X \cup \{w_1, w_2, w_3\}\}.$$

For $X \cup \{w_1, w_2, w_3\}$,

$$\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_1, w_2, w_3\}, p)\\
&= (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\})) - (v_f(X \cup \{w_1, w_2, w_3\}) - v_f(X \cup \{w_1, w_2\}))\\
&\geq 0,
\end{aligned}$$

where the second inequality follows from that v_f satisfies Condition 1.

For $X \cup \{w_2\}$,

$$u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_2\}, p) = 0$$

by the definition of $p_{(w_1,f)}$.

For $X \cup \{w_1\}$,

$$\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_1\}, p)\\
&= (v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_1\})) - (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_3\}))\\
&> 0,
\end{aligned}$$

where the second inequality follows from the assumption of Case 2.

For $X \cup \{w_3\}$,

$$u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_3\}, p) = 0$$

since $u_f(X \cup \{w_3\}, p) = u_f(X \cup \{w_2\}, p)$.

For $X \cup \{w_2, w_3\}$,

$$u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_2, w_3\}, p) = 0$$

since $u_f(X \cup \{w_2, w_3\}, p) = u_f(X \cup \{w_2\}, p)$.

For $X \cup \{w_1, w_3\}$,

$$\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, p) - u_f(X \cup \{w_1, w_3\}, p) \\
&= v_f(X \cup \{w_1, w_2\}) - (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_3\})) \\
&\quad - (v_f(X \cup \{w_1, w_3\}) - (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\}))) \\
&= v_f(X \cup \{w_1, w_2\}) + v_f(X \cup \{w_3\}) - (v_f(X \cup \{w_1, w_3\}) + v_f(X \cup \{w_2\})) \\
&> 0,
\end{aligned}$$

where the last inequality follows from the assumption of Case 2.

For X,

$$\begin{aligned}
&u_f(X \cup \{w_1, w_2\}, p) - u_f(X, p) \\
&= (v_f(X \cup \{w_1, w_2\}) - v_f(X)) \\
&\quad - (v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\})) - (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_3\})) \\
&= (v_f(X \cup \{w_2\}) - v_f(X)) - (v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_3\})) \\
&\geq 0,
\end{aligned}$$

where the second inequality follows from that v_f satisfies Condition 1.

Similarly, we can show that there exists a competitive equilibrium (μ, q) such that

$$\begin{aligned}
q_{(f,w_1)} &= v_f(X \cup \{w_1, w_3\}) - v_f(X \cup \{w_3\}),\\
q_{(f,w_2)} &= v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_1\}),\\
q_{(f,w_3)} &= v_f(X \cup \{w_1, w_3\}) - v_f(X \cup \{w_1\}),\\
q_{(f,w)} &= -\beta \text{ for all } w \in X,\\
q_{(f,w)} &= -v_w(f) \text{ for all } w \notin X \cup \{w_1, w_2, w_3\}, \text{ and}\\
q_{(f',w)} &= -v_w(f') \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}.
\end{aligned}$$

Note that $p_{(f,w_1)} \geq q_{(f,w_1)}$ and $q_{(f,w_2)} \geq p_{(f,w_2)}$ by the assumption of Case 2. Suppose that there exists a worker-optimal competitive equilibrium (μ', r). By Lemma 2, (μ, r) is also a competitive equilibrium. Note $u_w((\mu', r)|v_w) = u_w((\mu, r)|v_w)$ for any $w \in W$. Since (μ', r) is a worker-optimal competitive equilibrium, we have $r_{(f,w_1)} \geq p_{(f,w_1)}$ and $r_{(f,w_2)} \geq q_{(f,w_2)}$. Note that $\mu_f = X \cup \{w_1, w_2\} \in D_f(r)$ must hold since this is necessary so that (μ, r) is a competitive equilibrium. Then, we have

$$u_f(X \cup \{w_1, w_2\}, r) \geq u_f(X \cup \{w_3\}, r).$$

This implies

$$\begin{aligned}
&v_f(X \cup \{w_1, w_2\}) - ((v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_2\}))\\
&\quad - (v_f(X \cup \{w_1, w_2\}) - v_f(X \cup \{w_1\})) - \sum_{w \in X} r_{(f,w)}\\
&\geq v_f(X \cup \{w_1, w_2\}) - \sum_{w \in X \cup \{w_1, w_2\}} r_{(f,w)}\\
&\geq v_f(X \cup \{w_3\}) - r_{(f,w_3)} - \sum_{w \in X} r_{(f,w)},
\end{aligned}$$

where the first inequality follows from $r_{(f,w_1)} \geq p_{(f,w_1)}$ and $r_{(f,w_2)} \geq q_{(f,w_2)}$. Thus, we have

$$
\begin{aligned}
& r_{(f,w_3)} \\
& \geq (v_f(X \cup \{w_1, w_2\}) + v_f(X \cup \{w_3\})) - (v_f(X \cup \{w_1\}) + v_f(X \cup \{w_2\})) \\
& > \max\{v_f(X \cup \{w_1, w_3\}) + v_f(X \cup \{w_2\}), v_f(X \cup \{w_2, w_3\}) + v_f(X \cup \{w_1\})\} \\
& \quad - (v_f(X \cup \{w_1\}) + v_f(X \cup \{w_2\})) \\
& = \max\{v_f(X \cup \{w_1, w_3\}) - v_f(X \cup \{w_1\}), v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\})\},
\end{aligned}
$$

where the second inequality follows from the assumption of Case 2. Thus, we have $D_{w_3}(r|v_{w_3}) = \{f\}$, which contradicts that (μ, r) is a competitive equilibrium. This means that a worker-optimal competitive equilibrium does not exist for this valuation profile v. By Proposition 1, there is no CE and SP mechanism in this case. □

Proof of Proposition 2

The proof of Theorem 1 can be applied to Proposition 2 with small modifications. First, we assume v_f is non-decreasing in Proposition 2, which implies $\alpha, \beta = 0$ and

$$
-\max\{v_f(X \cup \{w_1, w_3\}) - v_f(X \cup \{w_1\}), v_f(X \cup \{w_2, w_3\}) - v_f(X \cup \{w_2\})\} \leq 0.
$$

Thus, all valuations of workers in the proof become non-positive. Second, it is easy to see that Proposition 1 holds when $V_w = \mathbb{R}^F_{\leq 0}$ for any $w \in W$.

4 Applications

4.1 Checking CE and SP

In this section, we illustrate that our results can determine the existence of the desired mechanisms. In each example, CE mechanisms exist. However, some valuations violate GS. Therefore, our results show that there is no CE and SP mechanism.

Case of single firm

First, we consider the case where there is a single firm in a market. While this case is very simple, it is theoretically valuable because it guarantees the existence of competitive equilibria regardless of the firm's valuation. Moreover, it also has practical relevance. Policies such as subsidies to a particular group in auctions can be discussed.

Proposition 4. Suppose $F = \{f\}$. Then, a competitive equilibrium exists for any $v \in V$ and $v_f \in V_f$.

Proof. For each worker valuation profile $v \in V$, consider

$$W' \in \arg\max_{W' \subseteq W}(v_f(W') + \sum_{w \in W'} v_w(f)).$$

Then, (μ, p) is a competitive equilibrium at ν where $\mu_f = W'$ and $p_{(f,w)} = -v_w(f)$ for each $w \in W$. □

It is worth noting that Proposition 4 does not follow from the existence theorem by Baldwin and Klemperer (2019). While a firm's valuation violates GS in Examples 1 and 2, CE mechanisms exist in the case of a single firm by Proposition 4. However, there is no CE and SP mechanism in these examples by Theorem 1.

The case of a single firm includes some important applications. Consider the setting of an auction where a single seller (firm) has multiple and homogeneous items and sells them to unit-demand buyers (workers). Suppose that the seller only cares about revenue, and her reservation values are zero. Thus, $v_f(W') = 0$ for all $W' \subseteq W$. It is easy to see that v_f satisfies GS, and the CE and SP mechanism exists.

In practice, to address equity issues, governments often provide support for small businesses and certain targeted groups in government procurement and allocation programs (Athey et al., 2013). Here, we consider a policy in which a government offers financial incentives to the seller based on the number of minority buyers who can obtain the items. A policy intervention is represented by a function $t_f : 2^W \to \mathbb{R}$.

Example 3. Let $W_m \subseteq W$ be the set of minority buyers. A policy intervention t_f is defined as:

$$t_f(W') = \begin{cases} \alpha & \text{if } |W' \cap W_m| \geq a, \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha > 0$ and $1 \leq a \leq |W_m|$.

	$\emptyset$	$\{w_d\}$	$\{w_{n_1}\}$	$\{w_{n_2}\}$	$\{w_d, w_{n_1}\}$	$\{w_d, w_{n_2}\}$	$\{w_{n_1}, w_{n_2}\}$	W
v_{f_1}	0	18	3	3	22	22	4	24
v_{f_2}	0	1	11	11	13	13	20	23
v_{f_2}	0	12	6	6	20	20	10	25

Table 1: Firms' valuations in Example 4

It is easy to see that $v_f + t_f$ violates GS. Therefore, t_f leads to the non-existence of CE and SP mechanisms by Theorem 1, even though a competitive equilibrium exists in the case of the single seller. This observation leads to a natural question: What kind of policy intervention t_f can preserve GS? Kojima et al. (2024) showed that if t_f is represented by sums of *additively separable* and *cardinally concave* functions, then GS can be preserved (for all valuations satisfying it).

Case of multiple firms

Second, we consider the case where there are multiple firms in a market. In this case, various conditions other than GS can guarantee the existence of competitive equilibria.

Example 4. Suppose that there are three firms $F = \{f_1, f_2, f_3\}$, one doctor and two nurses $W = \{w_d, w_{n_1}, w_{n_2}\}$. Firms' valuations are given in Table 1. Sun and Yang (2006) provided this example. Intuitively, from the viewpoint of each firm, nurses are substitutes, but a doctor and nurses are complements. They showed that GSC guarantees the existence of competitive equilibria, and every firm's valuation violate GS but satisfies GSC in Example 4.[10] Thus, a CE mechanism exists, but a CE and SP mechanism does not exist by Theorem 1 in this example.

Example 5. Suppose that there are two firms $F = \{f_1, f_2\}$ and three workers $W = \{w_1, w_2, w_3\}$. Each firm has a valuation defined by

$$v_{f_1}(W') = \begin{cases} \alpha \text{ if } \{w_1, w_2\} \subseteq W', \\ 0 \text{ otherwise}, \end{cases} \quad v_{f_2}(W') = \begin{cases} \beta \text{ if } \{w_2, w_3\} \subseteq W', \\ 0 \text{ otherwise}, \end{cases}$$

[10] The model by Sun and Yang (2006) is a little different from ours. In our model, workers are agents, while in their model they are treated as items. Precisely, the existence of a competitive equilibria under GSC in our model follows from the result by Hatfield et al. (2013).

for each $W' \subseteq W$ where $\alpha, \beta > 0$. Note that each firm's valuation satisfies neither GS nor GSC. Nevertheless, it can be confirmed that the demand type induced from valuations of firms and workers is unimodular defined by Baldwin and Klemperer (2019). Thus, their existence theorem implies that a competitive equilibrium exists for any valuation profile of workers. However, Theorem 1 implies that a CE and SP mechanism does not exist in this example.

Example 6. Consider a situation where the set of workers is partitioned, and for each firm, a set of workers is valuable only when the set is in the partition.[11] Formally, let $\mathcal{W} = \{W_1, \dots, W_n\}$ be a partition of W and suppose that each $f \in F$ has a valuation v_f such that for each $W' \in 2^W$,

$$v_f(W') \begin{cases} > 0 & \text{if } W' \in \mathcal{W}, \\ = \max_{W'' \in \mathcal{W} \cup \{\emptyset\}, W'' \subseteq W'} v_f(W'') & \text{otherwise.} \end{cases}$$

Note that for each f, v_f is non-decreasing. Assume that each worker has a non-positive valuation, that is, $V_w = \mathbb{R}^F_{\leq 0}$ for each $w \in W$. Then, there is a competitive equilibrium for each $v \in V$.

The existence of a competitive equilibrium can be shown by constructing a one-to-one matching economy.[12] Let $v \in V$. Let W^* be the set of n workers such that each worker $i \in W^*$ corresponds to a set W_i of workers in $\mathcal{W}$. For each $i \in W^*$, let $v^*_i = \sum_{w \in W_i} v_w$. Each firm $f \in F$ has a valuation v^*_f over W^* such that for each $i \in W^*$, $v^*_f(i) = v_f(W_i)$.

For this one-to-one economy, it is known that there is a competitive equilibrium (μ^*, p^*) at v^*. Note that $p^* \geq 0$ since firms' valuations are non-negative and non-decreasing and workers' valuations are non-positive. Given this equilibrium, let μ be a matching in the original economy such that for each $f \in F$, $\mu_f = W_{\mu^*_f}$. Let p be a

[11]This example is inspired by Shinozaki and Serizawa (2024). They study an auction model with multi-demand agents, and show that if a mechanism satisfies SP, envy-freeness and other conditions, the set of objects is partitioned into multiple bundles and the mechanism selects a competitive equilibrium allocation with respect to the bundles. The main difference between their result and our example is that a partition is endogenously determined in their result while the partition is given exogenously in our example.

[12]The demand type induced from valuations of firms and workers in this example is not unimodular. Thus, the sufficient condition of Baldwin and Klemperer (2019) cannot be applied to this example.

price vector such that for each $i \in W^*$, each $w \in W_i$ and each $f \in F$,

$$p_{(f,w)} = \begin{cases} -v_w(f) + \frac{p^*_{(\mu^*_i,i)} + v^*_i(f)}{|W_i|} & \text{if } f = \mu_w, \\ -v_w(f) + \frac{p^*_{(\mu^*_i,i)} + v^*_i(\mu_i)}{|W_i|} - t(f,w) & \text{otherwise,} \end{cases}$$

where

$$t(f,w) = \begin{cases} \min\{|v_w(f)|, \lambda\} & \text{if } p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i)) \geq 0, \\ -v_w(f) - \frac{p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i))}{|W_i|} & \text{if } p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i)) < 0, \end{cases}$$

where λ solves the equation $\sum_{w' \in W_i} \min\{|v_{w'}(f)|, \lambda\} = v^*_i(f) - (p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i)))$. By the construction of p, for each $w \in W$ and $f \in F$, $u_w(\mu_w, p_{(\mu_w,w)} \mid v_w) \geq 0$ and $u_w(\mu_w, p_{(\mu_w,w)} \mid v_w) \geq u_w(f, p_{(f,w)} \mid v_w)$. Thus, $\mu_f \in D_w(p \mid v_w)$. Further, by the definition of firms' valuations, for each $f \in F$, $\mu_f \in D_f(p)$. Hence, (μ, p) is a competitive equilibrium (in the original economy) at v.

It is also true that $p \geq 0$. To see this, let $i \in W^*$, $w \in W_i$, and $f \in F$. If $f = \mu_w$ or $p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i)) \geq 0$, then by $p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i) \geq 0$, $p_{(f,w)} \geq 0$. Suppose $f \neq \mu_w$ and $p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i)) < 0$. Then,

$$\begin{aligned} p_{(f,w)} &= -v_w(f) + \frac{p^*_{(\mu^*_i,i)} + v^*_i(\mu_i)}{|W_i|} - t(f,w) \\ &= -v_w(f) + \frac{p^*_{(\mu^*_i,i)} + v^*_i(\mu_i)}{|W_i|} + v_w(f) + \frac{p^*_{(f,i)} - (p^*_{(\mu^*_i,i)} + v^*_i(\mu^*_i))}{|W_i|} \\ &= \frac{p^*_{(f,i)}}{|W_i|}. \end{aligned}$$

Since $p^*_{(f,i)} \geq 0$, $p_{(f,w)} \geq 0$. Thus, $p \geq 0$.

If there is $W' \in \mathcal{W}$ such that $|W'| \geq 2$, firms' valuations violate GS. Thus, in such a case, there is no CE and SP mechanism by Proposition 2.

4.2 Investment

Agents typically make certain investments before participating in mechanisms. The design of a mechanism can affect the incentives for workers to invest because it partly determines the return on their investment. Hatfield et al. (2018) characterized the mechanisms that provide incentives for socially efficient *ex ante* investment in the general model, including ours. Since their characterization is based on SP, we obtain a close relationship between GS and investment efficiency through our results.

We follow the model of Hatfield et al. (2018). Before participating in the mechanism, each worker makes an investment decision that shapes her valuation across firms. This investment decision is modeled as a choice of the valuation v_w. The cost associated with this investment is defined by a *cost function* $c_w : V_w \to \mathbb{R}$. We slightly abuse notation and write $\mu(A)$ to mean that $\mu(A)$ is a matching associated with an outcome A. A mechanism φ is *efficient* if $\mu(\varphi(v))$ is an efficient matching at v for any $v \in V$. The *ex ante utility of worker* w given an outcome–investment pair (A, v) is $u_w(A|v_w) - c_w(v_w)$ where $v \in V$. The *social welfare* (or total surplus) of an outcome A at a valuation profile $v \in V$ is denoted by $\mathbf{V}(A|v) \equiv \sum_{w \in W} v_w(\mu(A)_w) + \sum_{f \in F} v_f(\mu(A)_f)$. We define the *ex ante social welfare* of an outcome–investment pair (A, v) as $\mathbf{V}(A|v) - \sum_w c_w(v_w)$.

Definition 3. A mechanism φ induces efficient investment by workers if for all $w \in W$ and $v_{-w} \in V_{W \setminus \{w\}}$,

$$\arg\max_{v'_w \in V_w}\{u_w(\varphi(v'_w, v_{-w}) - c_w(v'_w)|v'_w)\} = \arg\max_{v'_w \in V_w}\{\mathbf{V}(\varphi(v'_w, v_{-w})|(v'_w, v_{-w})) - c_w(v'_w)\}.$$

A mechanism induces efficient investment for workers if, assuming that workers report truthfully, for every valuation profile v_{-w} of other workers, and for every cost function c_w, the valuations that maximize the utility of w are exactly those that maximize social welfare.

Hatfield et al. (2018) clarified the relationship between SP and inducing efficient investment.

Proposition 5 ((Hatfield et al., 2018))**.** Consider an efficient mechanism φ. The following statements are equivalent:

1. The mechanism φ induces efficient investment by workers.

2. The mechanism φ is SP.

By noting that CE mechanisms are efficient, together with our results, we can characterize the valuations of a firm to guarantee the existence of mechanisms that are CE and induce efficient investment by workers.

Corollary 3. A mechanism that is CE and induces efficient investment by workers exists if and only if a valuation v_f satisfies GS for any $f \in F$.

5 Other models

5.1 Model with non-quasilinear utility

The following example shows that Theorem 1 does not hold in the model with continuous transfers and non-quasilinear utility.

Example 7. Let $F = \{f\}$ and $W = \{w_1, w_2\}$. Firm f has a utility function u_f over $2^W \times \mathbb{R}$ defined by

$$u_f(\{w_1, w_2\}, p) = \begin{cases} 2 - p & \text{if } p < 1 - \epsilon \\ -\frac{(1+\epsilon)}{\epsilon}(p - 1) & \text{if } 1 - \epsilon \leq p \leq 1 \\ -p + 1 & \text{if } p > 1, \end{cases}$$

and for $i = 1, 2$,

$$u_f(\{w_i\}, p) = 1 - p$$

where $\epsilon > 0$ is sufficiently small so that $\frac{1}{2+\epsilon} < 1 - \epsilon$, equivalently, $\epsilon^2 + \epsilon < 1$. The utility function induces a demand correspondence D_f as in Section 2.1. Then, the gross substitutes condition can be defined for D_f in the same way as Definition 2. Figure 1 illustrates the demand of f. Note that D_f violates the gross substitutes condition since $D_f(1 - \epsilon, 0) = \{\{w_1, w_2\}\}$ and $D_f(1 - \epsilon, \frac{1}{2+\epsilon}) = \{\{w_2\}\}$.

We assume that each worker has a quasilinear utility. Since there is a single firm, we use v_w to represent $v_w(f)$.

We define a mechanism as follows.

$$\varphi(v_{w_1}, v_{w_2}) = \begin{cases} \emptyset & \text{if } -v_{w_1}, -v_{w_2} > 1, \\ \{(f, w_1, 1)\} & \text{if } -v_{w_1} \leq 1, -v_{w_2} > 1, \\ \{(f, w_1, -v_{w_2})\} & \text{if } -v_{w_2} > -v_{w_1}, \frac{1}{2+\epsilon} < -v_{w_2} \leq 1, \\ \{(f, w_1, \frac{1}{2+\epsilon}), (f, w_2, \frac{1}{2+\epsilon})\} & \text{if } -v_{w_1}, -v_{w_2} \leq \frac{1}{2+\epsilon}, \\ \{(f, w_2, -v_{w_1})\} & \text{if } -v_{w_1} \geq -v_{w_2}, \frac{1}{2+\epsilon} < -v_{w_1} \leq 1, \\ \{(f, w_2, 1)\} & \text{if } -v_{w_2} \geq 1, -v_{w_1} > 1. \end{cases}$$

Note that $-v_{w_i}$ is the minimum salary at which w_i ($i = 1, 2$) is willing to match f, i.e., $\{f\} \in D_{w_i}(p)$ if and only if $p_{w_i} \geq -v_{w_i}$. For $(v_{w_1}, v_{w_2}) \in \mathbb{R}^2$, if $-v_{w_1} \leq 1$ and $-v_{w_2} > 1$, then $(\{(f, w_1)\}, (1, -v_{f_2}))$ is a competitive equilibrium since $\{w_1\} \in D_f(1, -v_{f_2})$

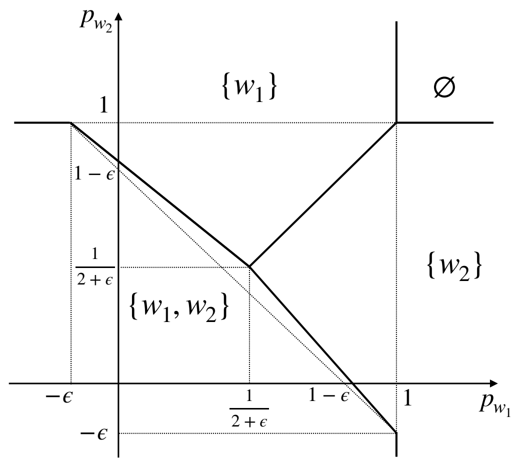

Figure 1: A representation of D_f in price space. Each labeled region represents the set of price vectors in which f demands the set of workers corresponding to the label. Black lines represent the set of price vectors in which f is indifferent among more than one set of workers. For example, $D_f(p) = \{\{w_1\}, \{w_2\}\}$ for all $p \in \{(p_{w_1}, p_{w_2}) \mid \frac{1}{2+\epsilon} < p_{w_1} = p_{w_2} < 1\}$.

by Figure 1, and if $-v_{w_2} > -v_{w_1}$ and $\frac{1}{2+\epsilon} < -v_{w_2} \leq 1$, then $(\{(f, w_1)\}, (-v_{w_2}, -v_{w_2}))$ is a competitive equilibrium since $\{w_1\} \in D_f(-v_{w_2}, -v_{w_2})$ by Figure 1. We can check that φ selects a competitive equilibrium outcome for the other cases. Thus, φ is a CE mechanism.

We show that φ is SP. For each $(v_{w_1}, v_{w_2}) \in \mathbb{R}^2$, let $\varphi_{w_i}(v_{w_1}, v_{w_2}) = \{(f, w, p_w) \in \varphi(v_{w_1}, v_{w_2}) \mid w = w_i\}$ be the allocation to w_i under φ $(i = 1, 2)$. Fix any $v_{w_2} \in \mathbb{R}$. Consider the following three cases.

(1) Suppose that $-v_{w_2} \leq \frac{1}{2+\epsilon}$. Then,

$$\varphi_{w_1}(v'_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, \frac{1}{2+\epsilon})\} & \text{if } -v'_{w_1} \leq \frac{1}{2+\epsilon} \\ \emptyset & \text{if } -v'_{w_1} > \frac{1}{2+\epsilon}. \end{cases}$$

This implies that $u_{w_1}(\varphi(v_{w_1}, v_{w_2})|v_{w_1}) \geq u_{w_1}(\varphi(v'_{w_1}, v_{w_2})|v_{w_1})$ for all $v_{w_1}, v'_{w_1} \in \mathbb{R}$.

(2) Suppose that $\frac{1}{2+\epsilon} < -v_{w_2} \leq 1$. Then,

$$\varphi_{w_1}(v'_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, -v_{w_2})\} & \text{if } -v'_{w_1} < -v_{w_2} \\ \emptyset & \text{if } -v'_{w_1} \geq -v_{w_2}. \end{cases}$$

This implies that $u_{w_1}(\varphi(v_{w_1}, v_{w_2})|v_{w_1}) \geq u_{w_1}(\varphi(v'_{w_1}, v_{w_2})|v_{w_1})$ for all $v_{w_1}, v'_{w_1} \in \mathbb{R}$.

(3) Suppose that $-v_{w_2} > 1$. Then,

$$\varphi_{w_1}(v'_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, 1\} & \text{if } -v'_{w_1} \leq 1 \\ \emptyset & \text{if } -v'_{w_1} > 1. \end{cases}$$

This implies that $u_{w_1}(\varphi(v_{w_1}, v_{w_2})|v_{w_1}) \geq u_{w_1}(\varphi(v'_{w_1}, v_{w_2})|v_{w_1})$ for all $v_{w_1}, v'_{w_1} \in \mathbb{R}$.

The above argument shows that reporting truthfully is (weakly) dominant for w_1. Similarly, reporting truthfully is (weakly) dominant for w_2. Therefore, φ is a CE and SP mechanism while D_f violates GS.

This example also illustrates that the rural hospital theorem (Lemma 1) does not hold in the model with non-quasilinear utility: $(\{(f, w_1), (f, w_2)\}, (1-\epsilon, 0))$ and $(\{(f, w_2)\}, (1-\epsilon-\eta, \frac{1}{2+\epsilon}))$ are competitive equilibria at $(v_{w_1}, v_{w_2}) = (1-\epsilon-\eta, 0)$ with small $\eta > 0$.

5.2 Model with discrete transfers

Crawford and Knoer (1981) and Kelso and Crawford (1982) studied a *discrete market*, which differs from our model in that the transfers or the workers' salaries are discrete variables. The following example shows that Theorem 1 does not hold in a model with discrete transfers and quasilinear utility.

Example 8. Let $F = \{f\}$ and $W = \{w_1, w_2\}$. We assume that each agent has a quasilinear utility where possible valuations and salaries are given by integers. The valuation of f is give by $v_f(\{w_1, w_2\}) = 1$, $v_f(\{w_i\}) = 0$ for $i = 1, 2$. Clearly, v_f violates GS.

In this example, a mechanism is defined as a function φ that specifies an outcome $\varphi(v_{w_1}, v_{w_2})$ for each valuation profile $(v_{w_1}, v_{w_2}) \in \mathbb{Z}^2$ where v_{w_i} denotes w_i's valuation to f. Note that a mechanism is defined on $\mathbb{Z}^2$ rather than $\mathbb{R}^2$ since we assume that each worker has an integer valuation.

We define a mechanism as follows:

$$\varphi(v_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, 1), (f, w_2, 0)\} & \text{if } -v_{w_1} \leq 1, -v_{w_2} \leq 0, \\ \{(f, w_2, 0)\} & \text{if } -v_{w_1} > 1, -v_{w_2} \leq 0, \\ \{(f, w_1, 0)\} & \text{if } -v_{w_1} \leq 0, -v_{w_2} \geq 1, \\ \emptyset & \text{if } -v_{w_1}, -v_{w_2} \geq 1. \end{cases}$$

For $(v_{w_1}, v_{w_2}) \in \mathbb{Z}^2$, if $-v_{w_1} \leq 1$ and $-v_{w_2} \leq 0$, then $(\{(f, w_1), (f, w_2)\}, (1, 0))$ is a competitive equilibrium by $\{w_1, w_2\} \in D_f(1, 0)$, and if $-v_{w_1} > 1$ and $-v_{w_2} \leq 0$, then $(\{(f, w_2)\}, (-v_{w_1}, 0))$ is a competitive equilibrium by $\{w_2\} \in D_f(-v_{w_1}, 0)$. We can check that φ selects a competitive equilibrium outcome for the other cases. Thus, φ is a CE mechanism.

We show that φ is a SP mechanism. Fix any $v_{w_2} \in \mathbb{Z}$. Consider the following two cases.

(1) Suppose that $-v_{w_2} \leq 0$. Then,

$$\varphi_{w_1}(v'_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, 1)\} & \text{if } -v'_{w_1} \leq 1, \\ \emptyset & \text{if } -v'_{w_1} > 1. \end{cases}$$

This implies that $u_{w_1}(\varphi(v_{w_1}, v_{w_2})|v_{w_1}) \geq u_{w_1}(\varphi(v'_{w_1}, v_{w_2})|v_{w_1})$ for all $v_{w_1}, v'_{w_1} \in \mathbb{Z}$.

(2) Suppose that $-v_{w_2} > 0$. Then,

$$\varphi_{w_1}(v'_{w_1}, v_{w_2}) = \begin{cases} \{(f, w_1, 0)\} & \text{if } -v'_{w_1} \leq 0, \\ \emptyset & \text{if } -v'_{w_1} > 0. \end{cases}$$

This implies that $u_{w_1}(\varphi(v_{w_1}, v_{w_2})|v_{w_1}) \geq u_{w_1}(\varphi(v'_{w_1}, v_{w_2})|v_{w_1})$ for all $v_{w_1}, v'_{w_1} \in \mathbb{Z}$.

The above argument shows that reporting truthfully is (weakly) dominant for w_1.

Moreover, for any $v_{w_1} \in \mathbb{Z}$,

$$\varphi_{w_2}(v_{w_1}, v'_{w_2}) = \begin{cases} \{(f, w_2, 0)\} & \text{if } -v'_{w_2} \leq 0, \\ \emptyset & \text{if } -v'_{w_2} > 0, \end{cases}$$

which implies that $u_{w_2}(\varphi(v_{w_1}, v_{w_2})|v_{w_2}) \geq u_{w_2}(\varphi(v_{w_1}, v'_{w_2})|v_{w_2})$ for all $v_{w_2}, v'_{w_2} \in \mathbb{Z}$. Therefore, φ is a CE and SP mechanism while v_f violates GS.

This example also illustrates that Proposition 1 does not hold in the model with discrete transfers: While $(\{(f, w_1), (f, w_2)\}, (1, 0))$ and $(\{(f, w_1), (f, w_2)\}, (0, 1))$ are competitive equilibria at $(v_1, v_2) = (0, 0)$, any arrangement $(\{(f, w_1), (f, w_2)\}, (p_{(f,w_1)}, p_{(f,w_2)}))$ with $p_{(f,w_1)}, p_{(f,w_2)} \geq 1$ cannot be a competitive equilibrium at (v_1, v_2).

References

Athey, S., D. Coey, and J. Levin (2013): "Set-asides and subsidies in auctions," *American Economic Journal: Microeconomics*, 5, 1–27.

Baldwin, E., R. Jagadeesan, P. Klemperer, and A. Teytelboym (2023): "The equilibrium existence duality," *Journal of Political Economy*, 131, 1440––1476.

Baldwin, E. and P. Klemperer (2019): "Understanding preferences:"demand types", and the existence of equilibrium with indivisibilities," *Econometrica*, 87, 867–932.

Crawford, V. P. and E. M. Knoer (1981): "Job matching with heterogeneous firms and workers," *Econometrica*, 49, 437–450.

Danilov, V., G. Koshevoy, and K. Murota (2001): "Discrete convexity and equilibria in economies with indivisible goods and money," *Mathematical Social Sciences*, 41, 251–273.

Demange, G. (1982): "Strategyproofness in the assignment market game," *Working Paper*.

Demange, G. and D. Gale (1985): "The strategy structure of two-sided matching markets," *Econometrica*, 53, 873–888.

Fleiner, T., R. Jagadeesan, Z. Jankó, and A. Teytelboym (2019): "Trading networks with frictions," *Econometrica*, 87, 1633–1661.

Fujishige, S. and Z. Yang (2003): "A note on Kelso and Crawford's gross substitutes condition," *Mathematics of Operations Research*, 28, 463–469.

Gul, F. and E. Stacchetti (1999): "Walrasian equilibrium with gross substitutes," *Journal of Economic Theory*, 87, 95–124.

Hatfield, J. W. and F. Kojima (2010): "Substitutes and stability for matching with contracts," *Journal of Economic Theory*, 145, 1704–1723.

Hatfield, J. W., F. Kojima, and S. D. Kominers (2018): "Strategy-proofness, investment efficiency, and marginal returns: An equivalence," *Becker Friedman Institute for Research in Economics Working Paper*.

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013): "Stability and competitive equilibrium in trading networks," *Journal of Political Economy*, 121, 966–1005.

Hatfield, J. W. and P. R. Milgrom (2005): "Matching with contracts," *American Economic Review*, 95, 913–935.

Hirata, D. and Y. Kasuya (2017): "On stable and strategy-proof rules in matching markets with contracts," *Journal of Economic Theory*, 168, 27–43.

Jagadeesan, R., S. D. Kominers, and R. Rheingans-Yoo (2018): "Strategy-proofness of worker-optimal matching with continuously transferable utility," *Games and Economic Behavior*, 108, 287–294.

——— (2020): "Lone wolves in competitive equilibria," *Social Choice and Welfare*, 55, 215–228.

Kasuya, Y. (2021): "Unilateral substitutability is necessary for doctor-optimal stability," *Economics Letters*, 207, 110047.

Kazumura, T., D. Mishra, and S. Serizawa (2020): "Strategy-proof multi-object mechanism design: Ex-post revenue maximization with non-quasilinear preferences," *Journal of Economic Theory*, 188, 105036.

Kelso, A. S. and V. P. Crawford (1982): "Job matching, coalition formation, and gross substitutes," *Econometrica*, 1483–1504.

Kojima, F., N. Sun, and N. N. Yu (2024): "Job matching with subsidy and taxation," *Review of Economic Studies*, 91, 372–402.

Kominers, S. D. and T. Sönmez (2016): "Matching with slot-specific priorities: Theory," *Theoretical Economics*, 11, 683–710.

Leonard, H. B. (1983): "Elicitation of honest preferences for the assignment of individuals to positions," *Journal of Political Economy*, 91, 461–479.

Miyake, M. (1998): "On the incentive properties of multi-item auctions," *International Journal of Game Theory*, 27, 1–19.

Morimoto, S. and S. Serizawa (2015): "Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule," *Theoretical Economics*, 10, 445–487.

Murota, K. (2016): "Discrete convex analysis: A tool for economics and game theory," *Journal of Mechanism and Institution Design*, 1, 151–273.

Murota, K. and A. Shioura (1999): "M-convex function on generalized polymatroid," *Mathematics of Operations Research*, 24, 95–105.

Nguyen, T. and R. Vohra (2022): "(Near) Substitute Preferences and Equilibria with Indivisibilities," *Working Paper*.

Reijnierse, H., A. van Gellekom, and J. A. Potters (2002): "Verifying gross substitutability," *Economic Theory*, 20, 767–776.

Schlegel, J. C. (2022): "The structure of equilibria in trading networks with frictions," *Theoretical Economics*, 17, 801–839.

Shinozaki, H. and S. Serizawa (2024): "Efficiency, fairness, and strategy-proofness in constrained object allocation problems with money," *Working Paper*.

Sun, N. and Z. Yang (2006): "Equilibria and indivisibilities: Gross substitutes and complements," *Econometrica*, 74, 1385–1402.

Teytelboym, A. (2014): "Gross substitutes and complements: A simple generalization," *Economics Letters*, 123, 135–138.

Zhou, Y. and S. Serizawa (2018): "Strategy-proofness and efficiency for non-quasi-linear and common-tiered-object preferences: Characterization of minimum price rule," *Games and Economic Behavior*, 109, 327–363.

A Necessity of Matroid

We assume that $v_f(\emptyset) = 0$ and thus $\emptyset \in \text{dom } v_f$ for all $f \in F$.

Lemma 3. Suppose that there exists a CE and SP mechanism. Then, dom v_f satisfies the following property for all $f \in F$;

(a) if $W' \in \text{dom } v_f$ and $W'' \subseteq W'$, then $W'' \in \text{dom } v_f$.

Suppose that there exists $f \in F$ such that dom v_f does not satisfy (a). Then, there exist $W', W'' \subseteq W$ such that $W'' \subseteq W'$, $W' \in \text{dom } v_f$, and $W'' \notin \text{dom } v_f$. Let X be a minimum set satisfying $W'' \subseteq X$ and $X \in \text{dom } v_f$, that is, there exists no $X' \subsetneq X$ such that $W'' \subseteq X'$ and $X' \in \text{dom } v_f$. Note that $X \setminus W'' \neq \emptyset$ by $W'' \neq X$ and $W'' \subseteq X$. Fix any $\hat{w} \in X \setminus W''$. Then, $X \setminus \{\hat{w}\} \notin \text{dom } v_f$ by the minimality of X.

Let $\alpha \in \mathbb{R}$ be a sufficiently small number so that

$$\min\{v_f(Y) - v_f(Z) \mid Z \subseteq Y \subseteq X, Y, Z \in \text{dom } v_f\} > \alpha.$$

Note that $\alpha < 0$ holds. Let $\epsilon > 0$ be a sufficiently large number so that

$$\epsilon > v_f(X) - |X|\alpha.$$

Consider workers' valuation profile as follows,

$$v_w(f) > -\alpha + \epsilon \text{ for all } w \in X,$$
$$-v_w(f) > \max\{v_f(Z) \mid Z \in \text{dom } v_f\} - |X|(\alpha - \epsilon) \text{ for all } w \in W \setminus X, \text{ and}$$
$$-v_w(f') > \max\{v_{f'}(Z) \mid Z \in \text{dom } v_{f'}\} \text{ for all } w \in W \text{ and } f' \in F \setminus \{f\}.$$

Note that $v_w(f) > 0$ for all $w \in X$, $v_w(f) < 0$ for all $w \in W \setminus X$. In addition, $v_w(f') < 0$ for all $w \in W$ and all $f' \in F \setminus \{f\}$.

We show that (μ, p) is a CE where μ is a matching with $\mu_f = X$ and $\mu_{f'} = \emptyset$ for all $f' \in F \setminus \{f\}$ and p is a price vector defined by

$$p_{(f,w)} = \begin{cases} \alpha & \text{if } w \in X, \\ -v_w(f) & \text{if } w \in W \setminus X, \end{cases}$$

and $p_{(f',w)} = -v_w(f')$ for all $w \in W$ and $f' \in F \setminus \{f\}$. Clearly, $\{f\} \in D_w(p|v_w)$ for all $w \in X$ and $\varnothing \in D_w(p|v_w)$ for all $w \in X \setminus W$. Moreover, $u_{f'}(\emptyset, p) \geq u_{f'}(Y, p)$ for all $Y \subseteq W$ with $Y \in \text{dom } v_{f'}$ and $f' \in F \setminus \{f\}$ by the definition of $(v_{f'}(w))_{w \in W}$. Thus, $\mu_{f'} = \emptyset \in D_{f'}(p)$ for all $f' \in F \setminus \{f\}$. It remains to show $X \in D_f(p)$. Pick any $Y \subseteq W$ with $Y \in \text{dom } v_f$. When $Y \setminus X \neq \emptyset$, we have $u_f(\emptyset, p) > u_f(Y, p)$ since for each $w' \in Y \setminus X$,

$$\begin{aligned} u_f(Y,p) &= v_f(Y) - \sum_{w \in Y \cap X} p(f,w) + \sum_{w \in Y \setminus X} v_w(f), \\ &\leq v_f(Y) - \sum_{w \in Y \cap X} (\alpha - \epsilon) + \sum_{w \in Y \setminus X} v_w(f), \\ &\leq v_f(Y) - |X|(\alpha - \epsilon) + v_{w'}(f), \\ &< 0, \end{aligned}$$

where the second line follows from $p_{(f,w)} \geq \alpha - \epsilon$ for all $w \in X$, the third line follows from $|X| \geq |X \cap Y|$, $\alpha - \epsilon < 0$, and $v_w(f) < 0$ for all $w \in W \setminus X$, and the last line follows from the definition of $v_{w'}(f)$. When $Y \subseteq X$, we have $u_f(X, p) \geq u_f(Y)$ by the definition of α. Thus, $X \in D_f(p)$. Therefore, (μ, p) is a CE.

We next show that (μ, q) is also a CE where q is a price vector defined by

$$q_{(f,w)} = \begin{cases} \alpha + \epsilon & \text{if } w = \hat{w}, \\ \alpha - \epsilon & \text{if } w \in X \setminus \{\hat{w}\}, \\ -v_w(f) & \text{if } w \in W \setminus X, \end{cases}$$

and $q_{(f',w)} = -v_w(f')$ for all $w \in W$ and $f' \in F \setminus \{f\}$. Clearly, $\{f\} \in D_w(q|v_w)$ for all $w \in X$ and $\varnothing \in D_w(q|v_w)$ for all $w \in X \setminus W$. Moreover, $u_{f'}(\emptyset, q) \geq u_{f'}(Y, q)$ for all $Y \subseteq W$ with $Y \in \text{dom } v_{f'}$ and $f' \in F \setminus \{f\}$ by the definition of $(v_{f'}(w))_{w \in W}$. It remains to show $X \in D_f(p)$. Pick any $Y \subseteq W$ with $Y \in \text{dom } v_f$. When $Y \setminus X \neq \emptyset$, we can show that $u_f(\emptyset, q) > u_f(Y, q)$ by the same argument above. Thus, we assume that $Y \subseteq X$. Suppose that $\hat{w} \notin X \setminus Y$. Then,

$$\begin{aligned} u_f(X,p) - u_f(Y,p) &= v_f(X) - v_f(Y) - \sum_{w \in X \setminus Y} (\alpha - \epsilon) \\ &\geq v_f(X) - v_f(Y) - \sum_{w \in X \setminus Y} \alpha \\ &\geq u_f(X,p) - u_f(Y,p) \\ &\geq 0, \end{aligned}$$

where the last line follows from $X \in D_f(p)$. Suppose that $\hat{w} \in X \setminus Y$. Note that $X \setminus Y \neq \{\hat{w}\}$ since otherwise we have $Y = X \setminus \{\hat{w}\}$, contradicting $Y \in \text{dom } v_f$. Thus, $X \setminus (Y \cup \{\hat{w}\}) \neq \emptyset$. Then,

$$\begin{aligned} u_f(X,p) - u_f(Y,p) &= v_f(X) - v_f(Y) - (\alpha + \epsilon) - \sum_{w \in X \setminus (Y \cup \{\hat{w}\})} (\alpha - \epsilon), \\ &= v_f(X) - v_f(Y) - |X \setminus Y|\alpha + (|X \setminus (Y \cup \{\hat{w}\})| - 1)\epsilon, \\ &\geq v_f(X) - v_f(Y) - \alpha, \\ &> 0, \end{aligned}$$

where the third line follows from $|X \setminus Y| \geq 1$, $\alpha < 0$, and $|X \setminus (Y \cup \{\hat{w}\})| - 1 \geq 0$ and the last line follows from the definition of α. Thus, $X \in D_f(q)$.

We next show that a worker-optimal competitive equilibrium does not exist. Suppose not. By Lemma 2, there exists a CE (μ, r) such that $u_w(\mu, r) \geq u_w(\mu, p)$ and $u_w(\mu, r) \geq u_w(\mu, q)$ for all $w \in X$. Thus, $r_{(f,\hat{w})} \geq \alpha + \epsilon$ and $r_{(f,w)} \geq \alpha$ for all $w \in X \setminus \{\hat{w}\}$. Then,

$$\begin{aligned} u_f(X,r) &\leq v_f(X) - (\alpha + \epsilon) - (|X| - 1)\alpha, \\ &= v_f(X) - |X|\alpha - \epsilon, \\ &< 0, \end{aligned}$$

where the last line follows from the definition of ϵ. Thus, $X \notin D_f(r)$, contradicting that (μ, r) is a CE. By Proposition 1, a CE and SP mechanism does not exist.