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Ryan Tierney
University of Southern Denmark

Yu Zhou
Nagoya University

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Abstract

We study the auction model of selling multiple heterogenous objects in which (i) unit demand agents have utility functions accommodating wealth effects and (ii) prices can only be discretely adjusted. The minimum price equilibrium (MPE), a natural generalization of the Vickrey allocation to settings without assuming quasi-linearity, plays a central role in designing efficient and incentive-compatible auctions. Nevertheless, discrete prices do not always support the MPEs. We instead propose an efficient equilibrium notion, tight equilibrium, and calculate the upper and lower deviation bounds between any tight equilibrium price and the (unique) MPE price. We also develop a descending-price auction that finds a tight equilibrium in finitely many steps. We further introduce a new notion of incentive compatibility, compensating strategy-proofness, to measure the non-manipulability of our proposed auction in an approximate sense.

Keywords: Multiple-object auction, discrete pricing, wealth effects, minimum price equilibrium, tight equilibrium, deviation bound, approximate incentive

JEL Classification: C78, D44, D74

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[†]Department of Economics, University of Southern Denmark, Denmark. E-mail: ryanetierney@gmail.com

[‡]Graduate School of Economics, Nagoya University, Japan. E-mail: zhouyu_0105@hotmail.com

1 Introduction

Spectrum licence auctions and procurement auctions in OECD countries are the most successful applications of auction theory (Klemperer, 2004; Milgrom, 2004). They often generate enormous revenue and play an important role in public finance and telecommunications industrial development. For instance, in the 2000 British 3G spectrum license auctions, the revenue generated from selling five licenses amounted to 2.5% of the UK's GNP (Klemperer, 2004). In those auctions, winning bids are so large that *wealth effects* cannot be ignored. Moreover, *discrete pricing* is also widely adapted, i.e., prices are adjusted discretely in sizable increments.[1] In practical auction design, discrete pricing is desirable since auction speed is an important design consideration (Milgrom and Segal, 2017; Ausubel et al. 2017),[2] and it can be controlled via the size of the increment or decrement.[3] We study the problem of multi-object auction design when agents' preferences have wealth effects and are defined over continuous transfers, but bids must be discrete. Zhou and Serizawa (2023) revealed the potentially severe problems these features cause; we propose solutions.

Designing an *efficient* and *strategy-proof* auction is of great importance in both theory and practice. *Efficiency* requires that objects should be given to those who value them the most. However, information about agents' preferences for objects is only privately known in many cases. Thus, it is crucial for the auction to incentivize agents to reveal their true information to attain efficiency. *Strategy-proofness* requires that revealing true information is a dominant strategy.

In the auction model of selling multiple heterogeneous objects where prices can be continuously adjusted, agents have unit demand, and have *classical* utility functions that accommodate wealth effects, there is a *minimum price (Walrasian) equilibrium* (MPE) (Demange and Gale, 1985). The associated (direct) mechanism that selects an MPE for each utility profile is known as the *MPE mechanism.* In quasi-linear settings, the MPE mechanism coincides with the Vickrey mechanism

[1] In the year 2000 UK 3G spectrum auctions, the increment was set at first 5% and then 1.5% of the highest bid in the previous round. In the 2021 auction, it was £10 million (Myers, 2023). We study the latter case of fixed, additive increments in this paper. Fixed multiplicative increments can be handled easily via a logarithmic transformation, since we study classical preferences.

[2] If speed is the primary concern, the sealed-bid auction is also a useful choice, e.g., when selling financial assets (Klemperer, 2010). However, considering that our goals are primarily efficiency and incentives, the sealed-bid auction is no longer desirable in the presence of wealth effects. See Section 2 or Zhou and Serizawa (2023) for detailed discussions.

[3] Empirical works show that the size of bid increment has significant effects on bidders' bidding behaviors and auction performance (Bradlow and Park, 2007; Lacetera et al., 2016).

(Leonard, 1983) so it can be characterized by efficiency and strategy-proofness, together with some technical properties (Holmstrom, 1974). The same characterization result holds even when agents have classical utility functions (Morimoto and Serizawa, 2015). For this reason, designing auctions that target the MPE has been a focus of the literature; see Section 2 for further discussion.

We study the same auction model as above, but restrict prices to be discrete: They are integer multiples of some given price grid. The grid step can be interpreted as the increment in the auction so a sizable increment corresponds to a large grid. Agents' utility functions are still defined on the continuous consumption space and so, generically, induce strict rankings over objects priced on the grid. Moreover, MPE prices (and Walrasian prices more generally) will generically *not* fall on such a grid. Thus, proposing an efficient equilibrium notion compatible with discrete prices that also approximates the MPE—so strategy-proofness, at least in the approximate sense, can be guaranteed—is a natural direction to proceed.

In settings with discrete prices, a discrete equilibrium at which agents are assigned objects that approximately maximize their welfare can always be found.[4] There are discrete equilibria whose prices well approximate the MPE prices, e.g., those supported by the largest discrete prices no greater than the MPE prices. However, those equilibria may not be efficient, and they are hard to constructively obtain even in quasi-linear settings (Zhou and Serizawa, 2023). These weak points largely limit their practical applications.

Demange et al. (1986) (DGS) make the first attempt to identify and approximate the MPE in quasi-linear settings with discrete prices, doing so via two auction algorithms. However, they assumed that agents' quasi-linear valuations conformed to the price grid, and this causes their first auction to drastically overshoot the MPE. Their second, slower auction does not suffer this problem so greatly, but only approximates the MPE, and their approximation is invalid in the presence of wealth effects. We elaborate on these issues in Section 2 and discuss how variants of the DGS auctions inherit their deficits, which then necessitates our novel approach. This new approach allows us to improve upon the DGS auctions by providing a valid approximation in the presence of wealth effects *and* a tighter bound in their absence (i.e., in quasilinear domains).

Main results We propose a new equilibrium notion, *tight equilibrium*. This is a discrete equilibrium that satisfies two properties, "local tightness" and "no improvement cycles." The former says that each object is strictly preferred by

[4]Discrete equilibria and similar solution concepts have been studied by Demange et al. (1986), Roughgarden (2014), Zhou and Serizawa (2023), and Herings (2024).

some agent to their own assignment at the equilibrium. The latter says that there is no (Pareto)-improving object reassignment at the given price. Tight equilibrium satisfies efficiency, which, unlike the models studied by, see, e.g., Ergin (2002), is achieved without imposing any priority structure on objects.

We find tight bounds on the distance between any tight equilibrium price and the MPE price. We focus on the subclass of classical utility functions that satisfy Lipschitz continuity in payments.[5] We use the Lipschitz constant to parameterize the wealth effect and show how it can be estimated in some practical applications. In quasi-linear settings, the Lipschitz constant is equal to one. In more general settings, the larger the wealth effect is, the larger the Lipschitz constant is. We show that both the upper and lower price bounds are functions of the number of objects, the price grid, and a polynomial of the Lipschitz constant. In particular, the lower bound is higher than that in Demange et al. (1986).

Tight equilibrium can neither be identified by the approximate ascending auction of Demange et al. (1986) nor by the cumulative offer process of Hatfield and Milgrom (2005). We propose a "Sequential Descending (SD)" auction that finds a tight equilibrium in a finite number of steps. The SD auction sequentially reduces object prices and objects are tentatively assigned and reallocated among agents in the auction. The SD auction generates a monotonically decreasing price path and only requires agents to report partial information about their demands. Since there is no tight-equilibrium-selecting mechanism that satisfies strategy-proofness, we instead propose a weaker notion of strategy-proofness, compensating strategy-proofness. It says that the gain from misreporting, measured by the money surplus the manipulation outcome yields over getting the manipulation *object* at the true outcome utility level, should be bounded. Any tight-equilibrium-selecting mechanism satisfies efficiency and compensating strategy-proofness.

Novelty This paper makes the first contribution to the study of multi-object auction design with wealth effects subject to discrete price restrictions. The novelty of our results and analytical technique will be discussed in Section 2. We also provide a simple, graph-theoretic language to help visualize and understand the connections between tight equilibria and MPEs at the end of Section 7. Practitioners in particular can take away the following following core insights. Compared to the prediction in quasi-linear settings, the trade-off between the exact goals of auction design (efficiency and strategy-proofness) and practical concerns (the auction speed controlled via the size of increment) will be greatly exarcerbated in

[5]In a different setup, Che and Gale (1998) study the optimal mechanism design problem for selling one object with financially constrained buyers. They also impose a Lipschitz continuity assumption on buyers' cost functions.

the presence of wealth effects. That said, once the Lipschitz constant in a given environment can be estimated, if wealth effects are relatively small, a sizeable increment guarantees auction speed with limited loss of incentive compatibility. If wealth effects are relatively large, on the other hand, a small increment reduces agents' incentive to misreport, but at the expense of auction speed.

This paper is organized as follows. After discussing related literature in Section 2, we present the model in Section 3, define Walrasian equilibrium, and review the results of MPE mechanisms in continuous settings in Section 4. In Section 5, we move to settings with discrete prices and define discrete equilibrium. In Sections 6, 7, and 8, we define tight equilibrium, derive the bound between tight equilibrium prices and the MPE price, propose an auction for tight equilibrium, and discuss its incentive property, respectively. Section 9 is a discussion of the results.

2 Related literature

Our results are related to efficient and strategy-proof auction designs with quasi-linearity and those with wealth effects, as well as the study of the structural properties of equilibrium concepts in matching models with wealth effects.

Efficient and strategy-proof auction design with quasi-linearity Demange et al. (1986) propose two auctions with discrete price adjustments when agents have unit demand. The first one is the exact ascending auction. It finds an MPE under *the coincidence assumption*, i.e., valuations are multiples of increments. If this assumption is dropped—as in the present work—their auction can overshoot the MPE by an arbitrarily large distance, and cannot achieve efficiency and strategy-proofness in any approximate sense (Zhou and Serizawa, 2023). Any auctions whose operations rely on the coincidence assumption face the same problems as stated above. These auctions include (i) the variants of the exact ascending auction with unit demand agents (Mishra and Parkes, 2009; Andersson and Erlanson, 2013), and (ii) those with multi-unit demand agents, but essentially coinciding with the exact ascending auction when applied to the settings with unit demand agents (Gul and Stacchetii, 2000; Ausubel, 2006; Sun and Yang, 2009). Note that once the increment is fixed, even in quasi-linear settings, the coincidence assumption fails to hold almost surely.

The second one is the approximate ascending (AA) auction. Whereas the exact ascending auction proceeds by simultaneously incrementing a set of prices based on the declared demand of all buyers, the AA gives agents tentative assignment of objects and allows empty-handed agents to bid them away. It thus works without the coincidence assumption, but has the fault that items can get bid up one increment farther than necessary, as shown by our bounds, which are tight. Clearly then, the outcome of the AA is generally not a tight equilibrium. Finally, their deviation bounds are valid only in quasi-linear settings (Zhou and Serizawa, 2023).

If each object is owned by a seller who only cares about revenue, and moreover, sellers implement some tie-breaking rule in selecting agents with the same payment, our model can be reformulated as two equivalent classes of models (Echenique, 2012): the labor market of Crawford and Knoer (1981) and Kelso and Crawford (1982) and the matching with contracts model of Hatfield and Milgrom (2005). The adjustment procedures given in those two models obtain the same approximation result to the MPE as the AA auction in quasi-linear settings, but suffer from the same problem with wealth effects. Their outcomes may not be tight equilibria and may fail to satisfy efficiency.[6]

We remark that the continuous-time versions of the above two auctions in Demange et al. (1986) are not well-defined when agents have classical utility functions (Zhou and Serizawa, 2023).

The sealed-bid Vickrey auction with either unit demand agents or multi-unit demand agents is both efficient and strategy-proof. Nevertheless, when agents have preferences exhibiting wealth effects, agents' valuations are naturally replaced with their willingness to pay as if they get nothing and pay nothing. The associated generalized sealed-bid Vickrey auction is neither efficient nor strategy-proof in any approximate sense (Zhou and Serizawa, 2023).

Efficient and strategy-proof auction design with wealth effects Morimoto and Serizawa (2015) and Mishra et al. (2023) characterize the MPE mechanism by efficiency, strategy-proofness, and fairness when agents have unit demand. Malik and Mishra (2021) characterize the efficient and strategy-proof mechanism when agents have multi-unit demand, but their preferences are dichotomous and exhibit positive income effects.[7] In more general preference settings with multi-unit demand agents, Baisa (2020) shows that there is no efficient and strategy-proof mechanism. Note that prices are continuous variables in all these models. In the same model as ours, Sakai et al. (2023) assign a priority structure to objects and characterize the salary adjustment mechanism proposed by Crawford and Knoer

[6]See Example 5 and Corollary 2, together with associated discussion for details.

[7]An agent i has a dichotomous preference if there are a set of non-empty bundles $\mathcal{A}_i$ and a function w_i from $\mathbb{R} \to \mathbb{R}_{++}$ such that for each $t \in \mathbb{R}$, $u_i(A, t + w_i(t)) = u_i(\emptyset, t)$ if $A \in \mathcal{A}_i$ and $u_i(A, t) = u_i(\emptyset, t)$, otherwise. It moreover exhibits a positive income effect if $w_i(t)$ is non-increasing in t.

(1981) via efficiency and strategy-proofness. Our results focus on equilibrium implementation instead of mechanism characterization.

When agents have unit demand, the MPE can be obtained via complex computation procedures, see, e.g., Caplin and Leahy (2004), Alaei et al. (2016), and Zhou and Serizawa (2021). All these procedures (i) assume that prices can be continuously adjusted (so the MPE is always well-defined) and (ii) require agents to report a huge number of "indifference prices" (at least factorial to the number of objects or agents). In contrast to our results, none of them can tackle the discrete pricing constraints that may lead to the non-existence of MPEs, and none contain a monotonic price path. Moreover, our auction requires much less information revelation and has a simpler and clearer equilibrium price formation process.

Matching models with wealth effects The main themes here are investigations of the equivalence between Walrasian equilibrium and solution concepts from cooperative game theory such as stability and core, and structural properties of Walrasian equilibrium such as the lattice property and the rural hospitals theorem (see, e.g., Demange and Gale (1985), Fleiner et al. (2019), Schlegel (2022), and Herings (2024)). None of these works study how to approximate the extreme points of the equilibrium price lattice via a feasible adjustment process. We make the first attempt to provide a systematic study of equilibrium implementation subject to practical constraints when wealth effects are accommodated.

3 The Model

The model below builds on the classical multi-object auction model of Demange et al. (1986), but drops the quasi-linear assumption on agents' preferences.[8]

There is a finite set of agents N that contains n agents and a finite set of objects M that contains m objects. We assume $n > m$. Not receiving an object is called receiving the *null*, which is denoted by 0. Let $L = M \cup \{0\}$. Each agent has a unit demand, i.e., she either receives a single object or the null.

Agents have preferences on the consumption set $L \times \mathbb{R}$. We abuse language and identify a preference of agent i with her utility representation u_i.

Definition 1: A utility function $u_i : L \times \mathbb{R} \to \mathbb{R}$ is *classical* if:
(i) For each $l \in L$, $u_i(l, \cdot)$ is continuous and strictly decreasing in $\mathbb{R}$.
(ii) For each pair $l, l' \in L$, each $t \in \mathbb{R}$, there is $t' \in \mathbb{R}$ such that $u_i(l, t) = u_i(l', t')$.
(iii) For each $l \in M$, each $t \in \mathbb{R}$, $u_i(l, t) > u_i(0, t)$.

[8]This model is also known as the object assignment model with non-quasi-linear preferences studied by, see, e.g., Morimoto and Serizawa (2015).

In Definition 1, Condition (i) states that for a given object, less payment gives higher welfare. Condition (ii) implies that no object is infinitely good or bad. Condition (iii) says that at any payment level, getting an object is better than getting nothing. Let $\mathcal{U}$ be the set of classical utility functions and $\mathcal{U}^n$ be the *classical domain*. Let $u = (u_i)_{i \in N} \in \mathcal{U}^n$ be a profile of utility functions.

Definition 2: A utility function $u_i \in \mathcal{U}$ is *quasi-linear* if there is a valuation function $v_i : L \to \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $l \in M$, $v_i(l) > 0$ and (iii) for each $(l,t), (l',t') \in L \times \mathbb{R}$, $u_i(l,t) \geq u_i(l',t')$ if and only if $v_i(l) - t \geq v_i(l') - t'$.

Let $\mathcal{U}^{QL}$ be the set of quasi-linear utility functions and $(\mathcal{U}^{QL})^n$ be the *quasi-linear domain*. It is obvious that $\mathcal{U}^{QL} \subsetneq \mathcal{U}$.

4 Minimum Price Equilibrium and Mechanism

In this section, prices can be adjusted continuously. We review the results of Walrasian equilibrium, with particular attention to the "minimum price equilibrium" and the mechanism that always selects a minimum price equilibrium.

For each agent $i \in N$, let $\mu_i \in L$ be her assigned object. An assignment $\mu = (\mu_i)_{i \in N} \in L^n$ is a list of assigned objects such that, except for the null, no two agents obtain the same object, i.e., if $\mu_i \neq 0$ and $i \neq j$, $\mu_i \neq \mu_j$. Let $\mathcal{M}$ be the set of assignments. Each agent will consume a bundle $z_i = (\mu_i, t_i)$ and an allocation $z = (\mu_i, t_i)_{i \in N} \in (L \times \mathbb{R})^n$ is a list of bundles such that $\mu \in \mathcal{M}$. Let Z be the set of allocations.

For each $l \in L$, let $p_l \in \mathbb{R}_+$ be the price of object l and $p = (p_l)_{l \in L} \in \mathbb{R}^{m+1}_+$ be a price (vector). Without loss of generality, we assume that the price of the null is zero, i.e., $p_0 = 0$, and the reserve prices of all the objects are zero. Agent $i's$ demand set at price $p \in \mathbb{R}^{m+1}$ is defined as $D_i(p) = \{l \in L : u_i(l, p_l) \geq u_i(l', p_{l'}), \forall l' \in L\}$.

Definition 3: A pair $(\mu, p) \in \mathcal{M} \times \mathbb{R}^{m+1}_+$ is a *Walrasian equilibrium* if:
(i) For each $i \in N$, $\mu_i \in D_i(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $\mu_i = l$.

In Definition 3, Condition (i) states that each agent i receives an object from her demand set, and pays its price. Condition (ii) states that an object with a positive price must be assigned. Equivalently, an unassigned object has zero price.

For each $u \in \mathcal{U}^n$, there is a Walrasian equilibrium for u (Alkan and Gale, 1990). Walrasian equilibrium always satisfies efficiency, as shown below.

For each $z \in Z$, let $Rev(z) = \sum_{i \in N} t_i$ be the revenue generated by z. An allocation $z \in Z$ is *efficient* for $u \in \mathcal{U}^n$ if there is no $z' \in Z$ such that (i) for each

$i \in N$, $u_i(z'_i) \geq u_i(z_i)$ with at least one strict inequality and (ii) $Rev(z') \geq Rev(z)$.

Fact 1 (Morimoto and Serizawa, 2015): Each Walrasian equilibrium allocation satisfies efficiency.

The set of Walrasian equilibrium prices forms a lower semi-lattice[9] so there is an equilibrium price which is component-wise-smallest among all the equilibrium prices (Demange and Gale, 1985). Let $p^{\min}(u)$ be the minimum equilibrium price for the utility profile u. An associated Walrasian equilibrium is called a *minimum price equilibrium* (MPE). Note that for each given utility profile, the associated MPE price is unique, but the corresponding assignment may not be unique since indifference in preferences is allowed; it follows that each agent is indifferent among all MPEs.

The following result shows the "demand connectedness property" of MPE. It says that each object with a positive price can be associated with a sequence of objects starting from the null, and each pair of adjacent objects in the sequence are connected by agents' demands.

Fact 2 (Morimoto and Serizawa, 2015): Let $u \in \mathcal{U}^n$ and $(\mu, p^{\min})$ be an MPE for u. For each $l \in M$ such that $p_l^{\min} > 0$, there is a sequence $\{i_k\}_{k=1}^{\Lambda}$ of Λ distinct agents such that:
(i) $\mu_{i_1} = 0$ and $\mu_{i_\Lambda} = l$.
(ii) For each $k \in \{2, \cdots, \Lambda - 1\}$, $\mu_{i_k} \in M$ and $p^{\min}_{\mu_{i_k}} > 0$.
(iii) For each $k \in \{1, \cdots, \Lambda - 1\}$, $\{\mu_{i_k}, \mu_{i_{k+1}}\} \subseteq D_{i_k}(p^{\min})$.

Let $\mathcal{D} \subseteq \mathcal{U}$. A *(direct) mechanism* f is a function from $\mathcal{D}^n$ to Z that maps each utility profile u to an allocation z. For each $i \in N$, let $f_i(u) = (\mu_i(u), t_i(u))$ where $\mu_i(u)$ is the object and $t_i(u)$ is the associated transfer recommended by f. Given a utility profile u, $(f(\cdot), u)$ forms a revelation game: agents report their utility functions, and the outcome of their reports is selected by $f(\cdot)$.

A mechanism is efficient if for each utility profile, it selects an efficient allocation.

Efficiency: A mechanism f is *efficient* on domain $\mathcal{D}^n$ if for each $u \in \mathcal{D}^n$, $f(u)$ is efficient for u.

Strategy-proofness says that no agent ever benefits from misreporting her utility function.

Strategy-proofness: A mechanism f is *strategy-proof* on domain $\mathcal{D}^n$ if for each $u \in \mathcal{D}^n$, each $i \in N$, and each $u'_i \in \mathcal{D}$, $u_i(f_i(u)) \geq u_i(f_i(u'_i, u_{-i}))$.

[9] A set X with a partial order $\succsim$ is a lower semi-lattice if for every non-empty subset $X' \subseteq X$, the greatest lower bound $\wedge_X X'$ exists in X.

Strategy-proofness implies that in the revelation game $(f(\cdot), u)$, truthfully reporting her utility function is a dominant strategy for each agent.

An MPE mechanism is a mechanism that maps to each utility profile an MPE allocation.

Fact 3 (Demange and Gale, 1985; Morimoto and Serizawa, 2015): The MPE mechanism on $\mathcal{U}^n$ is efficient and strategy-proof.

On the quasi-linear domain, the MPE mechanism coincides with the Vickrey mechanism (Leonard, 1983). The Vickrey mechanism can be characterized by efficiency, strategy-proofness, and individual rationality (Holmstrom, 1974). The MPE mechanism is a natural generalization of the Vickrey mechanism on the classical domain and can be characterized by the same axioms, as well as some fairness axioms (Morimoto and Serizawa, 2015; Mishra et al. 2023). Thus any auction that targets the MPE has nice incentive properties.

In the next section, we restrict prices to be discrete, and argue that the MPE, and Walrasian equilibrium more generally, is incompatible with discrete pricing constraints.

5 Discrete Equilibrium

Now we restrict attention to settings where prices can be only adjusted discretely. We model such a situation by assuming that the set of (admissible) prices is $(\varepsilon\mathbb{Z})^{m+1}$ for some $\varepsilon > 0$.[10] One interpretation of ε is as the increment in the auction and a sizable increment corresponds to a large ε.

Let $\mathcal{U}^* \subseteq \mathcal{U}$ be the set of utility functions u_i satisfying, for each pair (l, t) and (l', t'), both in $L \times \varepsilon\mathbb{Z}$, that $u_i(l, t) \neq u_i(l', t')$. Note that $\mathcal{U}^*$ is open and dense in $\mathcal{U}$. Here and henceforth, we fix $\varepsilon > 0$ and focus on $(\mathcal{U}^*)^n$.

The following example shows a case where no Walrasian equilibrium is supported by discrete prices.

Example 1: Let $N = \{1, 2\}$, $M = \{l\}$, and $\varepsilon = 1$. Two agents have quasi-linear utility functions as follows:

$$(v_1(0), v_1(l)) = (0, 3 - \delta_1).$$
$$(v_2(0), v_2(l)) = (0, 3 - \delta_2).$$

where $0 < \delta_1 < \delta_2 < 1$.

[10] For a positive real $d > 0$, let $d \cdot \mathbb{Z} = \{\cdots - d, 0, d, \cdots\}$ and $d \cdot \mathbb{N} = \{0, d, \cdots\}$.

For any price $p_l \geq 3$, no agent demands l, but l has a positive price. For any price $p_l \leq 2$, both agents demand l, but its supply is only one. Therefore, there is no Walrasian equilibrium compatible with $p_l \in \mathbb{N}$. $\triangle$

Discrete equilibrium is an alternative equilibrium notion for competitive analysis when prices are discrete. It allows agents to receive an object that *approximately* maximizes her welfare at the given price. For quasilinear preferences, it is reasonable to measure this directly from the utility function, but we need a more general approach. Thus, we adapt the classical rate-of-substitution concept. We say that an agent *discretely* demands an object l at prices p if, were she to get (l, p_l), she would not be willing to bid up p to the next grid step at any object. Formally, agent *i's discrete* demand set at price $p \in (\varepsilon\mathbb{Z})^{m+1}$ is defined as

$$D_i^\varepsilon(p) = \{l \in L : u_i(l, p_l) \geq u_i(0, 0) \text{ and } u_i(l, p_l) \geq u_i(l', p_{l'} + \varepsilon), \forall l' \in M\}.$$

Discrete equilibrium is then defined as follows.

Definition 4: A pair $(\mu, p) \in \mathcal{M} \times (\varepsilon\mathbb{Z})^{m+1}$ is a *discrete equilibrium* if:
(i) For each $i \in N$, $\mu_i \in D_i^\varepsilon(p)$.
(ii) For each $l \in M$, if $p_l > 0$, there is $i \in N$ such that $\mu_i = l$.

A Walrasian equilibrium is a discrete equilibrium, but the converse may not be true. In Example 1, agent 1 getting the object at price $p_l = 2$ while agent 2 getting the null with no payment is a discrete equilibrium, but not a Walarasian equilibrium. In contrast to Walrasian equilibrium prices, the set of discrete equilibrium prices is not a lower semi-lattice, see Appendix A.7 for a numerical illustration.

The existence of discrete equilibrium is implicitly shown by Demange et al. (1986) via their approximate ascending (AA) auction. This was done in a quasi-linear setting (Roughgarden, 2014), but is still valid in settings with classical utility functions (Zhou and Serizawa, 2023).

In discrete price settings, it is natural to study the discrete version of efficiency where some feasibility constraints over transfers are imposed on possible improvements, i.e., transfers should come from $\varepsilon\mathbb{Z}$, instead of $\mathbb{R}$ (Crawford and Knoer, 1981). Formally, let $Z^\varepsilon = Z \cap (L \times (\varepsilon\mathbb{Z}))^n$. An allocation $z \in Z^\varepsilon$ satisfies *discrete efficiency* for $u \in (\mathcal{U}^*)^n$ if there is no $z' \in Z^\varepsilon$ that dominates z: (i) for each $i \in N$, $u_i(z'_i) \geq u_i(z_i)$ with at least one strict inequality, and (ii) $Rev(z') \geq Rev(z)$.

A discrete equilibrium allocation may violate discrete efficiency, as illustrated below by Example 2.

Example 2: Let $N = \{1, 2, 3\}$, $M = \{l, l'\}$, and $\varepsilon = 1$. Agents have quasi-linear utility functions as follows:

$$
\begin{aligned}
(v_1(0), v_1(l), v_1(l')) &= (0, 3-\delta_1, 3-\delta_2)\\
(v_2(0), v_2(l), v_2(l')) &= (0, 2-\delta_2, 2-\delta_1)\\
(v_3(0), v_3(l), v_3(l')) &= (0, 1-\delta_4, 1-\delta_3)
\end{aligned}
$$

where $0 < \delta_1 < \delta_3 < \delta_4 < \delta_2 < 1$. Assigning l' to agent 1 at price 0, l to agent 2 at price 0, and the null to agent 3 is a discrete equilibrium allocation, but it violates discrete efficiency. We simply switch the bundles between agents 1 and 2. The reassignment makes agents 1 and 2 strictly better off, leaves agent 3's well being unchanged, and prices unchanged. $\triangle$

We remark that discrete equilibria whose prices well approximate the MPE prices, e.g., those supported by the largest discrete prices no greater than the MPE prices, are always well-defined. However, those equilibria may not satisfy discrete efficiency.[11] Moreover, as pointed out by Zhou and Serizawa (2023), they are hard to obtain constructively even in quasi-linear settings.

In the next sections, we focus on a subclass of discrete equilibrium that satisfies discrete efficiency, can approximate the MPE, and can be constructively obtained.

6 Tight Equilibrium: The Concept

Before giving the formal definition of tight equilibrium, we introduce two concepts.

Definition 5: A pair $(\mu, p) \in \mathcal{M} \times (\varepsilon\mathbb{Z})^{m+1}$ satisfies *local tightness* if for each $l \in M$, there is $i \in N$ such that $u_i(l, p_l) > u_i(\mu_i, p_{\mu_i})$.

Local tightness says that each object is strictly preferred by some agent to their own assignment at the current price.

Definition 6: A sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents ($\Lambda \geq 1$) forms *an improvement cycle* at $p \in (\varepsilon\mathbb{Z})^{m+1}$ from μ to μ' if
(i) For each $k = 1, \cdots, \Lambda$, $u_{i_k}(\mu'_{i_k}, p_{\mu'_{i_k}}) \geq u_{i_k}(\mu_{i_k}, p_{\mu_{i_k}})$ with at least one strict inequality.
(ii) For each $k = 1, \cdots, \Lambda - 1$, $\mu'_{i_k} = \mu_{i_{k+1}}$.
(iii) Either (1) $\mu'_{i_\Lambda} = \mu_{i_1}$ or (2) $\mu'_{i_\Lambda} \in L \backslash \{l : l = \mu_i$, for some $i \in N\}$ holds.

In an improvement cycle, there are two possible ways to improve agents' welfare. Consider the case of two agents. One possibility is that agent 1 gets agent

[11] In Example 2, the MPE price is $(0, 1-\delta_4, 1-\delta_3)$ where agent 3 gets the null at the MPE. The conclusion is reached via the discrete equilibrium discussed in that example.

2's object while agent 2 gets agent 1's object (Conditions (ii) and (iii-1)). The other possibility is that agent 1 gets agent 2's object while agent 2 gets another object that is unassigned at the original allocation (Conditions (ii) and (iii-2)).

Now we are ready to define tight equilibrium.

Definition 7: A discrete equilibrium $(\mu, p) \in \mathcal{M} \times (\varepsilon\mathbb{N})^{m+1}$ is a *tight equilibrium* if:
(i) (μ, p) satisfies local tightness.
(ii) There is no improvement cycle at p.

Note that the discrete equilibrium given in Example 2 violates Condition (ii), and that given in Example 4 below violates Condition (i). Thus, tight equilibrium is a strict refinement of discrete equilibrium.

We use the following example to illustrate the concept of tight equilibrium.

Example 3: Consider the same setting as in Example 2. Let $\mu = (\mu_1, \mu_2, \mu_3) = (l, l', 0)$ and $p = (0, 0, 0)$. We argue that (μ, p) is a tight equilibrium. It is easy to verify that (μ, p) is a discrete equilibrium. Definition 7(i) follows from the fact that that $u_3(l', 0) > u_3(l, 0) > u_3(0, 0)$. At p, both agents 1 and 2 strictly prefer their own assignment to any other object. Thus Definition 7(ii) holds. $\triangle$

Tight equilibrium prices fail to form a lower semi-lattice, see Appendix A.7 for a numerical illustration. The existence of tight equilibrium will be constructively shown in Section 8.

Tight equilibrium is defined via "local" properties. Intuitively, Definition 7(i) requires a pair-wise comparison of an agent's own assignment and some other object. Definition 7(ii) yields the efficiency of the object allocation *conditional* on a given price.[12] It turns out that these two local properties imply that tight equilibrium satisfies two global properties, i.e., discrete efficiency and the connectedness property in Definition 8 below.

Theorem 1: A tight equilibrium allocation always satisfies discrete efficiency.

The proof of Theorem 1 is given in Appendix A.1. The following is a direct consequence of discrete efficiency.

Corollary 1: Objects are all assigned at a tight equilibrium.[13]

[12]This is similar to *constrained efficiency* of Andersson and Svensson (2014), but absent any rationing constraints.

[13]Let (μ, p) be a tight equilibrium. By contradiction, suppose that there is an unassigned object $l \in M$ at (μ, p). Since $n > m$, there is $i \in N$ such that $\mu_i = 0$. Consider the reallocation such that agent i gets l at price 0 while all other agents keep their same bundles as those at (μ, p). Such a relocation contradicts that (μ, p) satisfies discrete efficiency.

Now we introduce the connectedness property alluded to above.

Definition 8: A pair $(\mu, p) \in \mathcal{M} \times (\varepsilon\mathbb{Z})^{m+1}$ satisfies *discrete connectedness* if each $l \in M$, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents ($\Lambda \geq 2$) such that:
(i) $\mu_{i_1} = 0$ and $\mu_{i_\Lambda} = l$.
(ii) For each $\lambda \in \{2, \cdots, \Lambda\}$, $\mu_{i_\lambda} \neq 0$.
(iii) For each $\lambda \in \{1, \cdots, \Lambda - 1\}$, $u_{i_\lambda}(\mu_{i_{\lambda+1}}, p_{\mu_{i_{\lambda+1}}}) > u_{i_\lambda}(\mu_{i_\lambda}, p_{\mu_{i_\lambda}})$.

Proposition 1: (i) A tight equilibrium satisfies discrete connectedness.
(ii) A discrete equilibrium satisfying discrete connectedness and having no improvement cycles at its price is a tight equilibrium.

The proof of Proposition 1 is given in Appendix A.2. In discrete connectedness, each pair of objects in the sequence is connected via agents' discrete demands (Definition 6(iii)), whereas the demand connectedness property of MPE uses (continuous) demands (Fact 2(iii)). Note also that discrete connectedness allows agents in the sequence to get objects with zero prices, whereas the demand connectedness property of MPE requires that except for i_1, all other agents in the sequence get objects with positive prices (Fact 2(ii)).

Comparison to other equilibria First, the AA auction in Demange et al. (1986) may fail to find a tight equilibrium. Example 4 illustrates this point.

Example 4: Let $N = \{1, 2, 3, 4\}$, $M = \{l, l'\}$, and $\varepsilon = 1$. Agents' utility functions $u \in (\mathcal{U}^*)^4$ are given by

$$\text{Agent } 1: u_1(l', 0) > u_1(l, 0) > u_1(l', 1) > u_1(l, 1) > u_1(0, 0).$$
$$\text{Agent } 2: u_2(l, 0) > u_2(l', 0) > u_2(l, 1) > u_2(l', 1) > u_2(0, 0).$$
$$\text{Agent } 3: u_3(l', 0) > u_3(l, 0) > u_3(0, 0) > u_3(l', 1) > u_3(l, 1).$$
$$\text{Agent } 4: u_4(l, 0) > u_4(l', 0) > u_4(0, 0) > u_3(l, 1) > u_3(l', 1).$$

First, we operate the AA auction. The bidding order is that agent 3 bids first, 4 second, 1 third, and 2 fourth. First, agent 3 bids on l' and is tentatively assigned l' with price 0. Second, agent 4 bids on l and is tentatively assigned l with price 0. Then it is agent 1's turn to bid. If she bids on l, the price of l that 1 faces is 1, and if she bids on l', the price of l' that 1 faces is also 1. Therefore, agent 1 bids on l' and is tentatively assigned l' with price 1 meanwhile agent 4 exits the auction. Finally, it is agent 2's turn to bid. If she bids on l, the price of l that 2 faces is 1 and if she bids on l', the price of l' that 2 faces is 2. Therefore, agent 2 bids on l and is tentatively assigned l with price 1 meanwhile agent 3 exits the auction. The resulting allocation is not a tight equilibrium allocation because l and l' are not locally tight so Definition 7(i) fails to hold. $\triangle$

Second, if each object is owned by a seller who only cares about revenue, and moreover, sellers implement some tie-breaking rule in selecting agents with the same payment, our model can be reformulated as the labor market of Crawford and Knoer (1981) and Kelso and Crawford (1982) or the matching with contracts model of Hatfield and Milgrom (2005). These two types of models, and the associated salary adjustment process and cumulative offer process in each model, are indeed equivalent (Echenique, 2012). Taking the cumulative offer process, for instance, it may fail to identify a tight equilibrium, Example 5 illustrates this point.

Example 5: Consider the same setting as in Example 4. Now suppose that l and l' are owned by two pseudo sellers s_l and $s_{l'}$. For each pair $t, t' \in \mathbb{Z}$ such that $t > t'$, $u_{s_l}(\cdot, t) > u_{s_l}(\cdot, t')$ and $u_{s_{l'}}(\cdot, t) > u_{s_{l'}}(\cdot, t')$. To run the cumulative offer process, consider the following tie-breaking rule: For each $t \in \mathbb{Z}$, $u_{s_l}(4, t) > u_{s_l}(2, t) > u_{s_l}(1, t) > u_{s_l}(3, t)$ and $u_{s_{l'}}(3, t) > u_{s_{l'}}(1, t) > u_{s_{l'}}(2, t) > u_{s_{l'}}(4, t)$. The allocation obtained via the cumulative offers process proposed by the agents' side is the same allocation obtained via the AA auction so it is not a tight equilibrium. $\triangle$

Note that using the same reformulation as mentioned above, Sakai et al. (2023) show that the outcome of the cumulative offer process may fail to satisfy discrete efficiency. They argue that efficiency cannot be achieved unless sellers have a common and payment-independent tie-breaking rule like the Ergin acyclicity condition (Ergin, 2002). In contrast, the discrete efficiency of tight equilibrium is achieved without imposing any priority structure.

7 Tight Equilibrium: Deviation Bound Estimation

In this section, we parameterize the wealth effect by a single variable, and show that the deviation between a tight equilibrium price and the MPE price is bounded by a constant[14] that depends only on the number of objects m, the price increment ε, and the wealth effect parameter.

Parameterizing wealth effects We impose a stronger continuity assumption on utility functions. For each $i \in N$, each $(l, t) \in L \times \mathbb{R}$, and each $l' \in L$, let $v_i^{l'}(l, t) \in \mathbb{R}$ be such that $u_i(l, t) = u_i(l', v_i^{l'}(l, t))$. This $v_i^{l'}(l, t)$ is known as the *compensated valuation* of l' from the bundle (l, t). By Definition 1, $v_i^{l'}(l, t)$ is unique and $v_i^{l'}(l, \cdot)$ is increasing. A utility function $u_i \in \mathcal{U}$ satisfies *Lipschitz*

[14] Here we mean constant relative to preferences.

continuity if there is a constant $d > 0$ such that for each pair $l, l' \in L$, each $t \in \mathbb{R}$, each $t' \in \mathbb{R}$, $\left|v_i^{l'}(l,t) - v_i^{l'}(l,t')\right| \leq d \cdot |t - t'|$. Thus, we assert that the wealth effect is well measured by the differences in compensating valuations. A simple calculation yields that, that for quasilinear u_i, $v_i^{l'}(l,t) - v_i^{l'}(l,t') = t - t'$. The following result shows that the Lipschitz constant is always at least 1, and that the foregoing equation characterizes quasilinearity.

Proposition 2: Let $i \in N$ and $u_i \in \mathcal{U}$.
(i) Suppose that $u_i \in \mathcal{U}$ satisfies Lipschitz continuity. Then $d \geq 1$.
(ii) $u_i \in \mathcal{U}^{QL}$ if and only if u_i satisfies Lipschitz continuity and $d = 1$.

The proof of Proposition 2 is relegated to Appendix A.3. In many practical situations, d can be estimated via information that is accessible to the auctioneer or even publicly available. Example 6 illustrates this point by using utility functions that are not quasi-linear, but satisfy Lipschitz continuity.

Example 6: We show how to estimate d in position auctions and auction with soft financial constraints.

Position auctions (Blumrosen et al. 2008): Objects and agents are read as advertisement slots and advertisers, respectively. Each advertiser has an underlying, constant valuation for all slots, but different locations on the webpage may bring her different levels of revenue. Specifically, the utility of advertiser i derived from getting slot l with payment p_l is $u_i(l, p_l) = c_i^l(a_i^l v_i - p_l)$ where c_i^l is the click-through rate[15] of slot l for advertiser i, a_i^l is the conversion rate[16] of slot l for advertiser i, and v_i is the value of a slot. For $l = 0$, $a_i^l = 0$ and for each $l \in M$, $a_i^l > 0$.

For each $i \in N$, each $l, l' \in L$, each $p_l \in \mathbb{R}_+$, $v_i^{l'}(l, p_l) = a_i^{l'} v_i - \frac{c_i^l}{c_i^{l'}} a_i^l v_i + p_l \frac{c_i^l}{c_i^{l'}}$ so $\left|v_i^{l'}(l,p_l) - v_i^{l'}(l,p_l')\right| = \frac{c_i^l}{c_i^{l'}} \cdot |p_l - p_l'|$. Therefore, we can set $d = \max_{i \in N, l, l' \in M} \frac{c_i^l}{c_i^{l'}} + \alpha$ for some $\alpha > 0$. For example, as the auctioneer, Microsoft or Google has the historical data of click-through rates.

Auctions for heterogeneous objects with soft financial constraints: Each agent $i \in N$ is endowed with an amount of money $\overline{m}_i \in \mathbb{R}_+$. If the price of object l exceeds $\overline{m}_i$, agent i could borrow from the financial market at a common interest rate $\rho \geq 0$. Therefore, if $p_l \leq \overline{m}_i$, $u_i(l, p_l) = v_i(l) - p_l$ and if $p_l > \overline{m}_i$, $u_i(l, p_l) = v_i(l) - p_l - \rho(p_l - \overline{m}_i)$.

In such a model, $\left|v_i^{l'}(l,p_l) - v_i^{l'}(l,p_l')\right| \leq (1+\rho) \cdot |p_l - p_l'|$. Therefore, we can set $d = 1 + \rho$ and estimate it from publicly available financial market data. $\triangle$

[15]The probability that an advertisement is clicked.
[16]The probability that a click becomes a transaction.

Deviation bounds A utility profile $u \in \mathcal{U}^*$ is $d-$*bounded* if for each $i \in N$, d is a Lipschitz constant for u_i. Now we are ready to present the deviation-bound results.

Theorem 2: Let $u \in (\mathcal{U}^*)^n$ satisfy $d-$boundedness. Let $p^{\min} \in \mathbb{R}^{m+1}$ and $p \in (\varepsilon\mathbb{N})^{m+1}$ be the MPE price and a tight equilibrium price for u.
(i) (**Upper deviation bound**) For each $l \in M$,[17]

$$p_l - p_l^{\min} \leq \sum_{k=1}^{m-1} d^k \cdot \varepsilon$$

(ii) (**Lower deviation bound**) For each $l \in M$,

$$p_l^{\min} - p_l \leq \sum_{k=1}^{m} d^{k-1} \cdot \varepsilon$$

Theorem 2 quantifies how wealth effects amplify the deviation of tight equilibrium from MPE even if the price grid is small.

The bounds given by Theorem 2 are binding; there cannot even be an object-wise improvement. Formally, there are no $l \in M$ and $\delta \in \mathbb{R}$ such that for each $u \in (\mathcal{U}^*)^n$ satisfying $d-$boundedness, $p_l - p_l^{\min} \leq$ Upper deviation bound $-\,\delta$. The statement for the lower deviation bound is symmetric. The formal analysis to support these statements is given in Appendix A.7.

The proof of Theorem 2 is relegated to Appendix A.4. The main challenges are to (i) build a connection between an MPE assignment and a given tight equilibrium assignment, and (ii) based on such a connection, study the object-wise price discrepancy between two equilibria with wealth effects. It turns out that Part (i) and Part (ii) require different analytical approaches, sketched below. In both cases, we use one of the distinct connectedness properties of MPE (Fact 2(iii)) and tight equilibrium to construct a sequence of objects ending at the null. This allows us to bound, via Lipschitz continuity, the price deviation at one object given the deviation of the object it is connected to. This then is inductively applied along the sequence, where we know that there can be no deviation at the the null.
Part (i): Begin by fixing an MPE assignment. Say objects l and l' are adjacent in the sequence if and only if there is $i \in N$ such that either (i) or (ii) below is true. (i) If l is both agent $i's$ MPE assignment and her tight equilibrium assignment,

[17] In the case of $m = 1$, let $\sum_{k=1}^{0} d^k = 0$.

then l' is an object that i strictly prefers to l at tight equilibrium. (ii) If l is agent $i's$ MPE assignment, but not her tight equilibrium assignment, then either l' is agent i's tight equilibrium assignment or agent i strictly prefers l' to her own assignment at the tight equilibrium. Note that the discrete connectedness of tight equilibrium ensures we will be able to find enough instances of either (i) or (ii) to build the sequence we need. Next, we show the discrepancy between $p_{l'}$ and $p_{l'}^{\min}$ is bounded above by d times the discrepancy between p_l and $p_l^{\min}$ plus an additional term, $d \cdot \varepsilon$. Third, we argue that there is an object whose tight equilibrium price is smaller than its MPE price. Putting the above three steps together, starting from the given object, iterating the price discrepancy of two adjacent objects along the sequence to the null, gives the upper deviation bound.
Part (ii): In this case, begin by fixing a tight equilibrium. Then, l and l' are adjacent in the sequence if and only if there is $i \in N$ such that l is i's object at the tight equilibrium and l' is either i's MPE object *or* an object that i is indifferent to given p^{min}. In other words, l' is in i's *continuous* demand set at p^{min}. Here it is the demand connectedness property of MPE that allows us to build the sequence we need. Then, in contrast to the counterpart step in Part (i), we show that the discrepancy between $p_{l'}^{\min}$ and $p_{l'}$ is bounded above by d times the discrepancy between $p_l^{\min}$ and p_l, plus an additional term ε. Finally, iterating the price discrepancy of two adjacent objects along the sequence up to the null gives the result.

When we apply Theorem 2 to *generic* quasi-linear settings where $\mathcal{U}^{QL*} = \mathcal{U}^{QL} \cap \mathcal{U}^*$ (Recall $d = 1$ by Proposition 2), the following result holds.

Corollary 2: Let $u \in (\mathcal{U}^{QL*})^n$. Let $p^{\min} \in \mathbb{R}^{m+1}$ and $p \in (\varepsilon\mathbb{N})^{m+1}$ be the MPE price and a tight equilibrium price for u. For each $l \in M$, $-m \cdot \varepsilon \leq p_l - p_l^{\min} \leq (m-1) \cdot \varepsilon$.

In quasi-linear settings, Demange et al. (1986) estimate the deviation between the MPE price and the outcome price of their AA auction as $-m \cdot \varepsilon \leq p_l - p_l^{\min} \leq m \cdot \varepsilon$.[18] As pointed out by Zhou and Serizawa (2023), in settings with wealth effects, neither the upper deviation bound nor the lower deviation bound of their estimation result holds.

If we reformulate our model as the labor market of Crawford and Knoer (1981) and the matching with contracts model of Hatfield and Milgrom (2005) (See Example 5) and focus on quasi-linear settings, the adjustment processes studied in those models approximate the MPE with the same deviation bound as the AA auction (Roughgarden, 2014). One may easily verify that they suffer from the

[18]Their bound is actually $\min\{m, n\} \cdot \varepsilon$. In our setting, $n > m$ so it is $m \cdot \varepsilon$.

same problems as the AA auction when there are wealth effects.

In terms of analytical techniques, Demange et al. (1986) assert their deviation bounds and proceed by contradiction. They show the upper bound by induction and use the connectedness property of MPE only for the lower bound. Notably, the whole analysis crucially depends on the quasilinearity assumption. Our approach has the twin virtues of being constructive and discarding quasilinearity. In particular, our proof elucidates how the connectedness properties of tight equilibrium yield the bounds we derive. Moreover, tight equilibrium has further properties, of interest independent of any algorithm.

A graph theoretic visualization It is well-known that assignment models in the presence of wealth effects cannot be formulated as linear programming problems.[19] We give a brief discussion in the terminology of directed graphs on how to (i) represent MPE $(\mu^*, p^{\min})$ and tight equilibrium (μ, p) and (ii) build the connections between μ^* and μ that are so useful in the proof of Theorem 2.[20]

We begin by making n copies of the null such that each agent only gets access to one copy and each copy is only available to one agent. Then we use a directed forest to describe the equilibria. A directed tree is a directed, acyclic graph with a unique *root* vertex having no incoming paths. A directed forest is a collection of directed trees. By the demand connectedness property of MPE, any MPE can be associated with a directed forest such that (i) the roots are null copies, (ii) each vertex corresponds to some agent's MPE assignment, and (iii) each arc, say $l \to l'$, represents an indifference relation in which the agent i who is assigned $(l, p_l^{\min})$ is indifferent to $(l', p_{l'}^{\min})$.[21]

By the discrete connectedness property, any tight equilibrium can be associated with a directed forest such that (i) and (ii) are analogous to the above and (iii') each arc $l \to l'$ represents a strict relation in which the agent i who is assigned (l, p_l) strictly prefers (l', p_l').

Given $(\mu', p') \in \{(\mu, p), (\mu^*, p^{\min})\}$, each graph above is a sub-graph of the following graph: $l \to l'$ is an arc if there is i for whom $\mu_i = l$ and i finds (l', p_l') at least as good as (l, p_l'). In the discrete case, $u \in \mathcal{U}^*$ ensures that each arc in this graph is a strict relation. Moreover, the no-improvement-cycle condition of tight

[19]In quasi-linear settings, the set of assignments compatible with efficient allocations coincide with those assignments that maximize the sum of agents' valuations of objects. In settings with wealth effects, such a result no longer holds.

[20]Formal analysis available upon request.

[21]If moreover a weight defined as $v_i^{l'}(l, p_l^{\min})$ is assigned to the arc $l \to l'$, then the MPE assignment solves the problem of minimizing the sum of compensated valutions among directed forests induced by all possible assignments (Caplin and Leady, 2012). This is not true for discretely efficient discrete equilibria, and we do not need weighted arcs in our approach.

equilibrium ensures that the graph is acyclic to begin with.

Now fix a directed forest compatible with (μ, p). We show how the sequence given in Part (i) in the proof of Theorem 2 is constructed. Assign each agent her MPE assignment. For agent i, let $\mu_i = l$. If $l = \mu_i^*$, do nothing. If $l \neq \mu_i^*$, we add arc $\mu_i^* \to l$, and replace each arc of the form $l \to l'$, by one of the form $\mu_i^* \to l'$. After this operation, it remains that each node can be reached via a directed path starting from some root; the sequence in the proof is one of these.

For Part (ii), we simply begin with the empty graph. Then, the agent i who is assigned $l \in L$ at (μ, p) points to all objects in her continuous (MPE) demand set. Note that if i is assigned exactly $(\mu_i^*, p_{\mu_i^*}^{min})$ at tight equilibrium, and if μ_i^* is her *unique* demanded object at MPE, then she points to nothing. Nonetheless, when each "point" corresponds to an arc, this again results in a graph with a sub-forest spanning all objects and with roots that are copies of the null.

Like the Edgeworth box in the study of exchange economies, we hope readers find this formulation useful in analyzing matching problems with wealth effects in a visualizable way (See also Caplin and Leady, 2012). The directed graph representation, with appropriate modification, could be used to understand equilibrium structures in other settings where wealth effects matter.

8 Tight Equilibrium: The Auction and Incentives

In this section, we propose an auction to find a tight equilibrium and then analyze the incentives of the associated (direct) mechanism. Note that we defer analysis of the *dynamic* incentives of the auction to future work and, presently, view it as a computational device only.

The auction proceeds in two stages. The goal of the first stage is simply to find *some* discrete equilibrium. From there, Stage 2 proceeds by incrementally raising agent welfare, either by lowering a price when possible or by executing preference cycles when they appear.

The Sequential Descending (SD) Auction

Set the increment to $\varepsilon > 0$. The price of the null is fixed at zero. The auction starts from Stage 1.

Stage 1: Each object $l \in M$ sets a high starting price $p_{li}^0 \in \varepsilon\mathbb{N}$ for each agent i such that no agent demands l.

Step $k(\geq 0)$: Each object $l \in M$ is in one of the following cases:

If $p^k_{li} < 0$ for each agent i, then object l exits Stage 1.
Else if object l points to some agent in the following way:

If l is unassigned, then it arbitrarily points to an agent, say, agent i, who has the highest price p^k_{li} across all the agents.

Else if l is assigned to some agent, object l only points to that agent.

The null points to all the agents. Among all the objects pointing to her and the null, agent i arbitrarily chooses one that maximizes her welfare. If object l' is chosen by some agent i', then l' is tentatively assigned to agent i' at price $p^k_{l'i'}$. Set $p^{k+1}_{l'} = p^k_{l'}$. If object l' is rejected by agent i', then set $p^{k+1}_{l'i'} = p^k_{l'i'} - \varepsilon$ and $p^{k+1}_{l'j'} = p^k_{l'j'}$ for $i' \neq j$.

Except for the exiting objects, if there is some unassigned object, then go to **Step** $k+1$. Otherwise, **Stop** at **Step** k. Now set a single stopping price $\overline{p}$ for each object in a way that the exiting object has zero stopping price and the assigned object has its current trading price as the stopping price. Let $(\overline{\mu}, \overline{p})$ be the resulting outcome where $\overline{\mu}$ is the assignment at Step k. Then go to **Stage 2**.

Stage 2: All the objects are available in Stage 2. Let $(\mu^0, p^0) = (\overline{\mu}, \overline{p})$.
Step $k(\geq 0)$: Let $O_i(p^k) = \{l \in L : u_i(l, p^k_l) > u_i(\mu^k_i, p^k_{\mu^k_i})\}$ be the set of objects that agent i prefers to her assignment at (μ^k, p^k). Each agent i reports $O_i(p^k)$.
If (μ^k, p^k) satisfies local tightness and there is no improvement cycle at p^k, then **Stop** at (μ^k, p^k).
Else if, operate the following adjustments at (μ^k, p^k).

If there are an agent i and an object l such that $\mu^k_i = l$, $p^k_l > 0$, and $l \notin \cup_{j \in N \setminus \{i\}} O_i(p^k)$, then arbitrarily select l, and set $p^{k+1}_l = p^k_l - \varepsilon$ and $p^{k+1}_{l'} = p^k_{l'}$ for $l' \neq l$, and $\mu^{k+1} = \mu^k$. Then go to **Step** $k+1$.

Else if there is an improvement cycle, execute it by transferring each object to the agent pointing at it. This leads to a new assignment μ^{k+1} at p^k and let $p^{k+1} = p^k$. Then go to **Step** $k+1$.

In Stage 1, each object starts from a personalized price but ends at a single price. Stage 1 is in the spirit of an object-proposing auction, which can be treated as the dual procedure to the AA auction. In Stage 2, say at Step k, each agent i needs to reveal $O_i(p^k)$. Such information is essential to verify the object whose price should be reduced and help eliminate improvement cycles. We highlight that the SD auction has a monotonically decreasing price path.

Theorem 3: For each $u \in (\mathcal{U}^*)^n$, the SD auction finds a tight equilibrium for u in a finite number of steps.

The proof of Theorem 3 is relegated to Appendix A.5.

In the following, we investigate the efficiency and incentives of the mechanism associated with the SD auction.

The first property is discrete efficiency, which requires that for each utility profile, the mechanism always selects a discretely efficient allocation.

Discrete Efficiency: A mechanism f is *discretely efficient* on domain $\mathcal{D}^n$ if for each $u \in \mathcal{D}^n$, $f(u)$ satisfies discrete efficiency for u.

Moving to incentives, we first observe that no mechanism that selects tight equilibrium is strategy-proof.

Example 7: Consider the same setting as in Example 1. There are two tight equilibria. The first one assigns l to agent 1 at price 2 while the second one assigns l to agent 2 at price 2.

If the mechanism selects the first tight equilibrium, then agent 2 reports $(v_2'(0), v_2'(l)) = (0, 8 - \delta_2)$, which results in a unique tight equilibrium for (v_1, v_2') that assigns l to agent 2 at price 2. Thus agent 2 benefits from misreporting. If the mechanism selects the second tight equilibrium, then agent 1 reports $(v_1'(0), v_1'(l)) = (0, 8 - \delta_1)$, which results in a unique tight equilibrium for (v_1', v_2) that assigns l to agent 1 at price 2. Thus agent 2 benefits from misreporting. Thus no mechanism that selects tight equilibrium is strategy-proof. $\triangle$

We instead consider a weaker notion of strategy-proofness. The idea is to allow agents to gain from misreporting, but restricted to a certain level measured by money. Suppose that under truthful report u_i agent i gets (l, t), and under the manipulation u_i', she gets (l', t'). Recall that agent i is indifferent between (l, t) and $(l', v_i^{l'}(l, t))$. By misreporting, the actual payment for l' is t', so agent's money surplus from misreporting is $v_i^{l'}(l, t) - t'$. Thus the benefit from misreporting, measured via money is $\lambda_i((l, t), (l', t')) = \max\{0, v_i^{l'}(l, t) - t'\}$. For strategy-proofness, we have that $v_i^{l'}(l, t) \leq t'$. Thus $\lambda_i((l, t), (l', t')) = 0$ and agent i cannot gain any money surplus by misreporting.

Now we are ready to propose the following notion of strategy-proofness.

$k-$**bounded Compensating Strategy-proofness**: A mechanism f is $k-$*bounded compensating strategy-proof* on domain $\mathcal{D}^n$ if there is $k \in \mathbb{R}_+$ such that for each $u \in D^n$, each $i \in N$, and each $u_i' \in D$, $\lambda_i(f_i(u), f_i(u_i', u_{-i})) \leq k$.

Clearly the case $k = 0$ coincides with strategy-proofness. In quasi-linear settings where money and utility are perfectly transferable, $\lambda_i((l, t), (l', t')) \leq k$ is equivalent to a bounded welfare gain from misreporting, i.e., $|u_i(f_i(u)) - u_i(f_i(u_i', u_{-i}))| \leq k$. In spirit, it is in line with the approximate strategy-proofness notions often defined via bounded welfare gain in such settings, see, e.g., Roughgarden (2014).

However, when agents have classical utility functions, money and utility are no longer perfectly transferable, and it is intractable to consider the gain from misreporting via welfare changes.

A *tight equilibrium mechanism* $f^{TE}(\cdot)$ is defined as a function from $(\mathcal{U}^*)^n$ to Z that maps to each utility profile u a tight equilibrium allocation z.

Let $(\mathcal{U}^{*d})^n \subseteq (\mathcal{U}^*)^n$ be the class of utility profiles that satisfy $d-$boundedness.

Theorem 4: Let $d > 0$ be given and let $d^* = \sum_{k=1}^{m} d^{k-1} + \sum_{k=1}^{m-1} d^k + 1$. Each tight equilibrium mechanism f^{TE} on $(\mathcal{U}^{*d})^n$ satisfies discrete efficiency and $d^* \cdot \varepsilon-$bounded compensating strategy-proofness.[22]

The proof of Theorem 4 is relegated to Appendix A.6. Discrete efficiency follows from Theorem 1. For the incentives, assume that under truthful reporting u_i, agent i gets l at price p_l and under the misreporting u_i', she gets l' at price p_l'. We show that $v_i^{l'}(l, p_l)$ is upper-bounded by

$$p_{l'}^{\min}(u) + \text{ Upper deviation bound (Theorem 2(i)) } + \varepsilon.$$

Moreover, p_l' is lower bounded by

$$p_{l'}^{\min}(u) - \text{ Lower deviation bound (Theorem 2(ii))}.$$

Thus, the discrepancy between $v_i^{l'}(l, p_l) - p_l'$ is bounded by $d^* \cdot \varepsilon$, as desired.

Showing the lower bound of p_l' is essential in the proof. Since agent i gets l' at $f^{TE}(u_i', u_{-i})$, we first construct $\widehat{u}_i$ such that l' is agent $i's$ favorite object so $f^{TE}(u_i', u_{-i})$ is also a discrete equilibrium allocation for $(\widehat{u}_i, u_{-i})$. We then construct a tight equilibrium for $(\widehat{u}_i, u_{-i})$ at which the price of l', say, $\widetilde{p}_{l'}$, satisfies $\widetilde{p}_{l'} \leq p_{l'}'$. Furthermore, we show $p_{l'}^{\min}(\widehat{u}_i, u_{-i}) \geq p_{l'}^{\min}(u)$. Thus, the discrepancy between $p_{l'}^{\min}(u)$ and $p_{l'}'$ is less than or equal to the discrepancy between $p_{l'}^{\min}(\widehat{u}_i, u_{-i})$ and $\widetilde{p}_{l'}$. The latter one is bounded above by the deviation bound given by Theorem 2(ii) so the conclusion follows.

In generic quasi-linear settings, we have the following result.

Corollary 3: Each tight equilibrium mechanism f^{TE} on $(\mathcal{U}^{QL*})^n$ satisfies discrete efficiency and $2m \cdot \varepsilon-$bounded compensating strategy-proofness.

In quasi-linear settings, since money and utility are perfectly transferable, Corollary 3 implies that tight equilibrium mechanisms are strategy-proof up to

[22]An alternative way is to measure the money surplus regarding the price of the initially assigned object, i.e., $\lambda_i(f_i(u_i', u_{-i}), f_i(u)) = \max\{0, v_i^l(l', t) - t\}$ where $f_i(u_i', u_{-i}) = (l', t')$ and $f_i(u) = (l, t)$. Note that by Lipschitz continuity, $\lambda_i(f_i(u_i', u_{-i}), f_i(u)) \leq d \cdot \lambda_i(f_i(u), f_i(u_i', u_{-i})) \leq d \cdot d^* \cdot \varepsilon$ so Theorem 4 holds with a very minor modification of the bound.

error $2m \cdot \varepsilon$ in welfare. Thus, the SD auction has the same incentive property as the AA auction in Demange et al. (1986) (as well as the processes studied in the labor market and matching with contracts mentioned earlier when applied to our settings) shown by Roughgarden (2014). Note that their proof crucially depends on the assumption that utility is perfectly transferable via money and so it cannot be extended to establish Theorem 4.

9 Discussion

We conclude by giving some further discussion.

First, our results and insights carry over when we drop the assumption of $n > m$ or consider an even more general class of utility functions by dropping Condition (iii) in Definition 1. Nevertheless, imposing these two assumptions makes the current exposition nested and proofs short.

Second, the concept of a "core allocation" plays a central role in multi-object auction design. In continuous settings, the Walrasian equilibrium allocation is a core allocation (equivalently, a weak core allocation). In discrete settings, we have discrete versions of the foregoing (Crawford and Knoer, 1981).[23] Nevertheless, the set of discrete core allocations could be empty.[24] It is easy to verify that a discrete equilibrium allocation, and so a tight equilibrium allocation, is a discrete weak core allocation. Therefore, the SD auction is a weak-core-selecting auction endowed with an approximate incentive compatibility property.

There are different ways to study discrete pricing. For example, the price grids could be different for different objects, e.g., $\varepsilon, 2\varepsilon, 3\varepsilon, \cdots$ for object l and $\varepsilon', 2\varepsilon', 3\varepsilon', \cdots$ for object l'. Price grid could also be proportional to the previous price, e.g., 1, $(1+\varepsilon), (1+\varepsilon)^2 \ldots$. Moreover, objects can be sold via discriminatory pricing based on agents' identities. We are optimistic that the equilibrium notion and analytical techniques used in the current context, with appropriate modifications, can carry over. We leave such extensions open for future research.

Appendix

A.1 Proof of Theorem 1

[23] An allocation $z \in Z$ is *a discrete weak core allocation* for $u \in \mathcal{U}^n$ if there is no set of agents $N' \subseteq N$ and no allocation $z' \in Z^{\varepsilon}$ such that (i) for each $i \in N'$, $u_i(z'_i) > u_i(z_i)$, (ii) for each $j \in N \backslash N'$, $z'_j = (0,0)$, and (iii) $Rev(z) < Rev(z')$. An allocation $z \in Z$ is *a discrete core allocation* for $u \in \mathcal{U}^n$ if there is no set of agents $N' \subseteq N$ and no allocation $z' \in Z^{\varepsilon}$ such that (i) and (iii) together require at least one strict inequality and (ii) remains the same.

[24] One may verify this statement using the same settings as Example 1.

By contradiction, suppose that (μ, p) is a tight equilibrium, but violates discrete efficiency. Then there is an allocation $z' = (\mu'_i, t'_i)_{i\in N} \in Z^\varepsilon$ such that
(i) for each $i \in N$, $u_i(\mu'_i, t'_i) \geq u_i(\mu_i, p_{\mu_i})$ with at least one strict inequality and
(ii) $\sum_{i\in N} t'_i \geq \sum_{i\in N} p_{\mu_i}$.

Since (μ, p) is a tight equilibrium and $u_i \in \mathcal{U}^*$, it holds that for each $i \in N$, $u_i(\mu_i, p_{\mu_i}) > u_i(\mu'_i, p_{\mu'_i} + \varepsilon)$. By (i), $u_i(\mu'_i, t'_i) \geq u_i(\mu_i, p_{\mu_i}) > u_i(\mu'_i, p_{\mu'_i} + \varepsilon)$ so $t'_i < p_{\mu'_i} + \varepsilon$. Since $p, t' \in (\varepsilon\mathbb{Z})^{m+1}$, then $t'_i \leq p_{\mu'_i}$. If there is $i \in N$ such that $t'_i < p_{\mu'_i}$, then $\sum_{i\in N} t'_i < \sum_{i\in N} p_{\mu'_i} \leq \sum_{l\in L} p_l = \sum_{i\in N} p_{\mu_i}$, contradicting (ii). Thus for each $i \in N$, we have that $t'_i = p_{\mu'_i}$. By (i), there is an improvement cycle at p, contradicting that (μ, p) be a tight equilibrium.

A.2 Proof of Proposition 1

Part (ii) is straightforward. We only show Part (i) below.

Let (μ, p) be a tight equilibrium. By Corollary 1, objects are fully assigned. Consider $i \in N$ and $l \in M$ such that $\mu_i = l$. Suppose that $p_l = 0$. By $n > m$, there is $j \in N$ such that $\mu_j = 0$. By $u_i \in \mathcal{U}^*$, $u_i(0, 0) < u_i(l, 0)$. Thus discrete connectedness holds for l. Now suppose that $p_l > 0$. Then by local tightness, there is $i_1 \in N\backslash\{i\}$ with $\mu_{i_1} = l_1$ such that $u_{i_1}(l, p_l) > u_{i_1}(l_1, p_{l_1})$. If $p_{l_1} = 0$, we are done. If $p_{l_1} > 0$, by local tightness, there is $i_2 \in N\backslash\{i_1\}$ with $\mu_{i_2} = l_2$ such that $u_{i_2}(l_1, p_{l_1}) > u_{i_2}(l_2, p_{l_2})$. By no improvement cycle at p, $i_2 \neq i$. Otherwise, i.e., $i_2 = i$, i_1 and i_2 form an improvement cycle by switching their bundles. Thus $i_2 \in N\backslash\{i, i_1\}$. If $p_{l_2} = 0$, we are done. If $p_{l_2} > 0$, by the analogous argument, there is $i_3 \in N\backslash\{i, i_1, i_2\}$ with $\mu_{i_3} = l_3$ such that $u_{i_3}(l_2, p_{l_2}) > u_{i_3}(l_3, p_{l_3})$. Repeat the above argument and since the number of agents is finite, we conclude that l satisfies discrete connectedness.

A.3 Proof of Proposition 2

Part (i): Let $i \in N$, $l, l' \in L$, and $t', t \in \mathbb{R}$ such that $t > t'$. By contradiction, suppose that $d < 1$. Let $\Delta_t = t - t'$ and $\Delta_v = v_i^{l'}(l, t) - v_i^{l'}(l, t')$. Since $t > t'$ and $v_i^{l'}(l, \cdot)$ is increasing in $\cdot$, then $\Delta_t > 0$ and $\Delta_v > 0$. By Lipschitz continuity, $\Delta_v \leq d \cdot \Delta_t < \Delta_t$. Therefore,

$$u_i(l, t) = u_i(l', v_i^{l'}(l, t') + \Delta_v) > u_i(l', v_i^{l'}(l, t') + \Delta_t). \qquad *$$

Recall that $t' = v_i^l(l', v_i^{l'}(l, t'))$. Let Δ' be such that $v_i^l(l', v_i^{l'}(l, t') + \Delta_t) = t' + \Delta'$. Since $\Delta_t > 0$, then $\Delta' > 0$. Therefore,

$$0 < t' + \Delta' - t' = v_i^l(l', v_i^{l'}(l, t') + \Delta_t) - v_i^l(l', v_i^{l'}(l, t')) \underset{\text{Lipschitz continuity}}{\leq} d \cdot \Delta_t < \Delta_t$$

so $0 < \Delta' < \Delta_t$. Therefore,

$$u_i(l', v_i^{l'}(l,t') + \Delta_t) = u_i(l, t' + \Delta') > u_i(l, t' + \Delta_t) = u_i(l,t),$$

contradicting $(*)$. Thus $d \geq 1$.

Part (ii): It is straightforward that $u_i \in \mathcal{U}^{QL} \Longrightarrow u_i$ satisfies Lipschitz continuity and $d = 1$. In the following, we show the opposite direction "$\Longleftarrow$."

Let $i \in N$, $l, l' \in L$, and $t', t \in \mathbb{R}$ with $t \geq t'$. Let $\Delta_t = t - t'$ and $\Delta_v = v_i^{l'}(l,t) - v_i^{l'}(l,t')$. It is easy to see that $\Delta_t \geq 0$ and $\Delta_v \geq 0$. By $d = 1$ and Lipschitz continuity, we have that $0 \leq \Delta_v \leq d \cdot \Delta_t = \Delta_t$. By $t = v_i^l(l', v_i^{l'}(l,t))$, $t' = v_i^l(l', v_i^{l'}(l,t'))$, and $\Delta_t = t - t'$, we have that

$$0 \leq \Delta_t = v_i^l(l', v_i^{l'}(l,t)) - v_i^l(l', v_i^{l'}(l,t')) \underset{\text{Lipschitz continuity}}{\leq} d \cdot (v_i^{l'}(l,t) - v_i^{l'}(l,t')) \underset{d=1}{=} \Delta_v.$$

Together with $\Delta_v \leq \Delta_t$, we have $\Delta_t = \Delta_v$. Thus, for each $i \in N$, each $l, l' \in L$, and each $t', t \in \mathbb{R}$ with $t \geq t'$, it holds that

$$v_i^{l'}(l,t) - t = v_i^{l'}(l,t') - t'. \qquad **$$

To conclude $u_i \in \mathcal{U}^{QL}$, we construct $v_i : L \to \mathbb{R}_+$ below that satisfies three conditions in Definition 2.

Let $v_i(0) = 0$ so Condition (i) holds. For each $l \in M$, let $v_i(l) = v_i^l(0,0)$.

For each $(l,t), (l',t') \in L \times \mathbb{R}$, by $(**)$, we have that $v_i^l(0, v_i^0(l,t)) - v_i^0(l,t) = v_i^l(0,0) = v_i(l)$. Since $v_i^l(0, v_i^0(l,t)) = t$, then $t - v_i(l) = v_i^0(l,t)$. Analogously, we can show that $t' - v_i(l') = v_i^0(l',t')$. Thus $v_i^0(l,t) \leq v_i^0(l',t')$ if and only if $v_i(l) - t \geq v_i(l') - t'$. Since $v_i^0(l,t) \leq v_i^0(l',t')$ if and only if $u_i(l,t) \geq u_i(l',t')$, it follows that Condition (iii) holds.

Since $u_i \in \mathcal{U}$, Condition (iii) in Definition 1 implies that for each $l \in M$, $v_i(l) - t > v_i(0) - t$. Thus Condition (ii) holds.

A.4 Proof of Theorem 2

Let (μ, p) be a tight equilibrium and $(\mu^*, p^{\min})$ be an MPE for u. For $t \in \mathbb{R}$, let $(t)^+ = \max\{0, t\}$.

We begin with the following observations.

Observation (a): Each $l \in M$ has $p_l^{\min} > 0$ and is assigned at $(\mu^*, p^{\min})$. Suppose that there is $l \in M$ such that $p_l^{\min} = 0$. Since $n > m$, there is $i \in N$ such that $\mu_i^* = 0$. By $u_i \in \mathcal{U}$, $u_i(l,0) > u_i(0,0)$ so $0 \notin D_i(p^{\min})$, contradicting that $(\mu^*, p^{\min})$ is an MPE. Together with Definition 3(ii), we get the desired result.

Observation (b): Let $t, t' \in \mathbb{R}$. If $t \leq t'$, then $t^+ \leq t'^+$. In case $t \geq 0$, $t^+ = t \leq t' = t'^+$. In case $t < 0$, $t^+ = 0 \leq t'^+$.

Part (i): Let $N_0 = \{i \in N : \mu_i^* = 0\}$ and for each $k \geq 1$, let $N_k = \{i \in N \backslash (\cup_{s=0}^{k-1} N_s) :$ there is $i' \in N_{k-1}$ such that $\mu_i^* = \mu_{i'}$ or $u_{i'}(\mu_i^*, p) > u_{i'}(\mu_{i'}, p)\}$. For each $k \geq 0$, let $L_k = \{l \in L : l = \mu_i^*$ for some $i \in N_k\}$.

Step 1: There is some $T \in \mathbb{N}$ such that (1) for each $s \leq T$, $N_s \neq \emptyset$, and $N_{T+1} = \emptyset$ and (2) $N = \cup_{s=0}^{T} N_s$ and $L = \cup_{s=0}^{T} L_s$.

Since $n > m$, $N_0 \neq \emptyset$. Since the numbers of agents and objects are finite, there is some $T \in \mathbb{N}$ satisfying (1).

To show (2), by contradiction, suppose that $N \supsetneq \cup_{s=0}^{T} N_s$. Let $\overline{N} = N \backslash \cup_{s=0}^{T} N_s$ and $\overline{L} = L \backslash \cup_{s=0}^{T} L_s$. We claim that for each $i \in \cup_{s=0}^{T} N_s$, $\mu_i \notin \overline{L}$. By contradiction, suppose that there is $i \in \cup_{s=0}^{T} N_s$ such that $\mu_i \in \overline{L}$. By construction of $\cup_{s=0}^{T} L_s$, we have $\mu_i^* \in \cup_{s=0}^{T} L_s$ so $\mu_i^* \neq \mu_i$. By observation (a), μ_i is assigned to some agent k at $(\mu^*, p^{\min})$, i.e., $\mu_k^* = \mu_i$. By the construction of $\cup_{s=0}^{T} N_s$, we have that $k \in \cup_{s=0}^{T} N_s$ so $\mu_k^* \in \cup_{s=0}^{T} L_s$, contradicting $\mu_k^* = \mu_i \in \overline{L}$. Thus we have that for each $i \in \cup_{s=0}^{T} N_s$, $\mu_i \in \cup_{s=0}^{T} L_s$. By Corollary 1, objects in $\overline{L} \subseteq M$ are fully assigned at (μ, p). Thus for each $i \in \overline{N}$, $\mu_i \in \overline{L}$ and $|\overline{N}| = |\overline{L}|$. Recall that for each $i \in \cup_{s=0}^{T} N_s$, we have $\mu_i^* \in \cup_{s=0}^{T} L_s$. Thus, there are no $i' \in \cup_{s=0}^{T} N_s$ and $l \in \overline{L}$ such that $u_{i'}(l, p) > u_{i'}(\mu_{i'}, p)$. Since $0 \in \cup_{s=0}^{T} L_s$, no object in $\overline{L}$ satisfies discrete connectedness, by Proposition 1, contradicting that (μ, p) is a tight equilibrium. Thus $N = \cup_{s=0}^{T} N_s$, and by observation (a), $L = \cup_{s=0}^{T} L_s$.

In the following three steps, i.e., Step 2, Step 3, and Step 4, let $i \in N$, $\mu_i = l$. and $\mu_i^* = l^*$. Suppose that there is $l' \in M$ such that $u_i(l, p_l) < u_i(l', p_{l'})$.

Step 2: Suppose that $l = l^*$. Then $(p_{l'} - p_{l'}^{\min})^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+$.

In the following, we use l^* instead of l and obviously, $u_i(l^*, p_{l^*}) < u_i(l', p_{l'})$.

Step 2-1: $(p_{l'} - p_{l'}^{\min})^+ \leq (v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+$.

Since $u_i(l^*, p_{l^*}) < u_i(l', p_{l'})$, we have that $v_i^{l'}(l^*, p_{l^*}) > p_{l'}$. Since $(\mu^*, p^{\min})$ is an MPE, then $v_i^{l'}(l^*, p_{l^*}^{\min}) \leq p_{l'}^{\min}$. Combining these two inequalities, we have that $p_{l'} - p_{l'}^{\min} \leq v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min})$. By observation (b), Step 2-1 holds.

Step 2-2: $(v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+$.

If $p_{l^*}^{\min} \geq p_{l^*}$, then $v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}) \leq 0 = (v_i^{l'}(l^*, p) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+$.

If $p_{l^*}^{\min} < p_{l^*}$, then $v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}) > 0$ so

$$0 < (v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+ = \left| v_i^{l'}(l^*, p_{l^*}) - v_i^{l'}(l^*, p_{l^*}^{\min}) \right| \underset{\text{Lipschitz continuity}}{\leq} d \cdot \left| p_{l^*} - p_{l^*}^{\min} \right| \underset{p_{l^*}^{\min} < p_{l^*}}{=} d \cdot (p_{l^*} - p_{l^*}^{\min})^+.$$

Combining Step 2-1 and Step 2-2, Step 2 holds.

Step 3: Suppose that $l \neq l^*$. Then $(p_{l'} - p_{l'}^{\min})^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon$.

Let $\Delta = (v_i^{l'}(l^*, v_i^{l^*}(l, p_l)) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+$ and $\Delta' = (v_i^{l'}(l^*, p_{l^*} + \varepsilon) - v_i^{l'}(l^*, p_{l^*}^{\min}))^+$. Since (μ, p) is a discrete equilibrium, then $u_i(l, p_l) \geq u_i(l^*, p_{l^*} + \varepsilon)$. Thus, $v_i^{l^*}(l, p_l) \leq p_{l^*} + \varepsilon$, which implies that $\Delta' \geq \Delta$.

Step 3-1: $\Delta' \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon$

If $p_{l^*} + \varepsilon \geq p_{l^*}^{\min}$, then $\Delta' = v_i^{l'}(l^*, p_{l^*} + \varepsilon) - v_i^{l'}(l^*, p_{l^*}^{\min}) \geq 0$ and moreover

$$\begin{aligned}\Delta' &\underset{\text{Lipschitz continuity}}{\leq} d \cdot \left|p_{l^*} + \varepsilon - p_{l^*}^{\min}\right| \\ &= d \cdot (p_{l^*} + \varepsilon - p_{l^*}^{\min}) \underset{\text{observation (b)}}{\leq} d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon.\end{aligned}$$

If $p_{l^*} + \varepsilon < p_{l^*}^{\min}$, then $v_i^{l'}(l^*, p_{l^*} + \varepsilon) - v_i^{l'}(l^*, p_{l^*}^{\min}) < 0$. Thus $\Delta' = 0 \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon$.

Step 3-2: $(p_{l'} - p_{l'}^{\min})^+ \leq \Delta$.

By $u_i(l, p_l) < u_i(l', p_{l'})$, we have that $v_i^{l'}(l, p_l) > p_{l'}$. Since $u_i(l^*, v_i^{l^*}(l, p_l)) = u_i(l, p_l)$, it holds that $v_i^{l'}(l^*, v_i^{l^*}(l, p_l)) = v_i^{l'}(l, p_l)$ so $v_i^{l'}(l^*, v_i^{l^*}(l, p_l)) > p_{l'}$. Since $(\mu^*, p^{\min})$ is an MPE, $v_i^{l'}(l^*, p_{l^*}^{\min}) \leq p_{l'}^{\min}$. Thus

$$p_{l'} - p_{l'}^{\min} \leq v_i^{l'}(l^*, v_i^{l^*}(l, p_l)) - v_i^{l'}(l^*, p_{l^*}^{\min}) \leq \Delta.$$

By observation (b) and $\Delta = \Delta^+$, we have Step 3-2.

Combining Step 3-1, Step 3-2, and $\Delta \leq \Delta'$, Step 3 holds.

Step 4: $(p_l - p_l^{\min})^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon$.

In either case of $l = l^*$ or the case of $p_l < p_l^{\min}$ (so $(p_l - p_l^{\min})^+ = 0$), it is straightforward that Step 4 holds. Now assume $l \neq l^*$ and $p_l \geq p_l^{\min}$.

Case 1: $u_i(l, p_l) \geq u_i(l^*, p_{l^*})$. In such a case, we have $v_i^l(l^*, p_{l^*}) \geq p_l$. Since $(\mu^*, p^{\min})$ is an MPE, then $v_i^l(l^*, p_{l^*}^{\min}) \leq p_l^{\min}$. Thus, $p_l - p_l^{\min} \leq v_i^l(l^*, p_{l^*}) - v_i^l(l^*, p_{l^*}^{\min})$. Since $p_l \geq p_l^{\min}$, then $v_i^l(l^*, p_{l^*}) - v_i^l(l^*, p_{l^*}^{\min}) \geq 0$. Thus $p_{l^*} \geq p_{l^*}^{\min}$. By Lipschitz continuity, we have that $v_i^l(l^*, p_{l^*}) - v_i^l(l^*, p_{l^*}^{\min}) \leq d \cdot (p_{l^*} - p_{l^*}^{\min})$. Thus $p_l - p_l^{\min} \leq d \cdot (p_{l^*} - p_{l^*}^{\min})$. By observation (b), $(p_l - p_l^{\min})^+ \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+$ so the desired result holds.

Case 2: $u_i(l, p_l) < u_i(l^*, p_{l^*})$. Since (μ, p) is a tight equilibrium, $u_i(l, p_l) \geq u_i(l^*, p_{l^*} + \varepsilon)$ so $v_i^l(l^*, p_{l^*} + \varepsilon) \geq p_l$. Since $(\mu^*, p^{\min})$ is an MPE, $v_i^l(l^*, p_{l^*}^{\min}) \leq p_l^{\min}$. Thus, $p_l - p_l^{\min} \leq v_i^l(l^*, p_{l^*} + \varepsilon) - v_i^l(l^*, p_{l^*}^{\min})$. Since $p_l \geq p_l^{\min}$, then $v_i^l(l^*, p_{l^*} + \varepsilon) - v_i^l(l^*, p_{l^*}^{\min}) \geq 0$. Thus $p_{l^*} + \varepsilon \geq p_{l^*}^{\min}$. Thus

$$\begin{aligned}p_l - p_l^{\min} = (p_l - p_l^{\min})^+ &\leq v_i^l(l^*, p_{l^*} + \varepsilon) - v_i^l(l^*, p_{l^*}^{\min}) \\ &\underset{\text{Lipschitz continuity}}{\leq} d \cdot (p_{l^*} + \varepsilon - p_{l^*}^{\min}) \leq d \cdot (p_{l^*} - p_{l^*}^{\min})^+ + d \cdot \varepsilon.\end{aligned}$$

Step 5: There is $l \in M$ such that $p_l - p_l^{\min} \leq 0$.

By contradiction, suppose that for each $l \in M$, $p_l > p_l^{\min}$. Since $n > m$, there is $i \in N$ such that $\mu_i^* = 0$. Thus, for each $i \in N$ with $\mu_i^* = 0$, we have that for each $l \in M$, $v_i^l(0,0) \leq p_l^{\min} < p_l$ and so $\mu_i = 0$. Together with observations (a) and (b), $\{i \in N : \mu_i^* \in M\} = \{i \in N : \mu_i \in M\}$. Thus, objects in M violate discrete connectedness at (μ, p), contradicting that (μ, p) is a tight equilibrium.

Completion of proof: Let $l \in M$. By Step 1, there is a sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$ of Λ distinct agents ($\Lambda \geq 2$) such that:
(i) $\mu_{i_1}^* = 0$, $\mu_{i_\Lambda}^* = l$, and for each $\lambda \in \{2, \cdots, \Lambda - 1\}$, $\mu_{i_\lambda}^* \in M$.
(ii) for each $\lambda \in \{1, \cdots, \Lambda - 1\}$, either $u_{i_\lambda}(\mu_{i_\lambda}, p) < u_{i_\lambda}(\mu_{i_{\lambda+1}}^*, p)$ or $\mu_{i_\lambda} = \mu_{i_{\lambda+1}}^*$.

First consider agent $i_{\Lambda-1}$. Recall that $\mu_{i_\Lambda}^* = l$. If $u_{i_{\Lambda-1}}(\mu_{i_{\Lambda-1}}, p) < u_{i_{\Lambda-1}}(l, p)$, then by Steps 2 and 3, we have

$$(p_l - p_l^{\min})^+ \leq d \cdot (p_{\mu_{i_{\Lambda-1}}^*} - p_{\mu_{i_{\Lambda-1}}^*}^{\min})^+ + d \cdot \varepsilon. \qquad *$$

If $\mu_{i_\lambda} = l$, $*$ holds by Step 4.

Next consider agent $i_{\Lambda-2}$. If $u_{i_{\Lambda-2}}(\mu_{i_{\Lambda-2}}, p) < u_{i_{\Lambda-1}}(\mu_{i_{\Lambda-1}}^*, p)$, then by Steps 2 and 3,

$$(p_{\mu_{i_{\Lambda-1}}^*} - p_{\mu_{i_{\Lambda-1}}^*}^{\min})^+ \leq d \cdot (p_{\mu_{i_{\Lambda-2}}^*} - p_{\mu_{i_{\Lambda-2}}^*}^{\min})^+ + d \cdot \varepsilon \qquad **$$

If $\mu_{i_{\Lambda-2}} = \mu_{i_{\Lambda-1}}^*$, by Step 4, $**$ also holds. Replacing $(p_{\mu_{i_{\Lambda-1}}^*} - p_{\mu_{i_{\Lambda-1}}^*}^{\min})^+$ in $*$ with $**$, we have

$$\begin{aligned}(p_l - p_l^{\min})^+ &\leq d \cdot [d \cdot (p_{\mu_{i_{\Lambda-2}}^*} - p_{\mu_{i_{\Lambda-2}}^*}^{\min})^+ + d \cdot \varepsilon] + d \cdot \varepsilon \\ &= d^2 \cdot (p_{\mu_{i_{\Lambda-2}}^*} - p_{\mu_{i_{\Lambda-2}}^*}^{\min})^+ + d^2 \cdot \varepsilon + d \cdot \varepsilon\end{aligned}$$

Repeatedly applying the above argument, we can get $p_l - p_l^{\min} \leq (p_l - p_l^{\min})^+ \leq \sum_{k=1}^{\Lambda-1} d^k \cdot (p_{\mu_{i_1}^*} - p_{\mu_{i_1}^*}^{\min})^+ + \sum_{k=1}^{\Lambda-1} d^k \cdot \varepsilon$. Since $\mu_{i_1}^* = 0$, then $p_{\mu_{i_1}^*} = p_{\mu_{i_1}^*}^{\min} = 0$. Thus $p_l - p_l^{\min} \leq \sum_{k=1}^{\Lambda-1} d^k \cdot \varepsilon$.

In the following, we show that for each $l \in M$, we can even have $p_l - p_l^{\min} \leq \sum_{k=1}^{m-1} d^k \cdot \varepsilon$. In the case of $\Lambda \leq m$, the conclusion holds vacuously. Now assume that $\Lambda = m + 1$. By Step 5, there is an agent, say, agent i_q, such that $p_{\mu_{i_q}^*} - p_{\mu_{i_q}^*}^{\min} \leq 0$, in the sequence $\{i_\lambda\}_{\lambda=1}^{\Lambda}$. By repeatedly using $**$ until agent i_{q+1}, we have

$$(p_l - p_l^{\min})^+ \leq \sum_{k=1}^{\Lambda-(k+1)} d^{\Lambda-(k+1)} \cdot (p_{\mu_{i_{k+1}}^*} - p_{\mu_{i_{k+1}}^*}^{\min})^+ + \sum_{k=1}^{\Lambda-1} d^{\Lambda-(k+1)} \cdot \varepsilon.$$

By Steps 2, 3, and 4, we have

$$(p_{\mu^*_{i_{q+1}}} - p^{\min}_{\mu^*_{i_{q+1}}})^+ \le d \cdot (p_{\mu^*_{i_q}} - p^{\min}_{\mu^*_{i_q}})^+ + d \cdot \varepsilon.$$

Since $p_{\mu^*_{i_q}} - p^{\min}_{\mu^*_{i_q}} \le 0$, it holds that $p_{\mu^*_{i_q}} - p^{\min}_{\mu^*_{i_q}} \le (p_{\mu^*_{i_{q-1}}} - p^{\min}_{\mu^*_{i_{q-1}}})^+$ so

$$(p_{\mu^*_{i_{q+1}}} - p^{\min}_{\mu^*_{i_{q+1}}})^+ \le d \cdot (p_{\mu^*_{i_{q-1}}} - p^{\min}_{\mu^*_{i_{q-1}}})^+ + d \cdot \varepsilon.$$

By repeatedly applying the recursive process the same as $**$ with respect to agents $i_{q-1}, \cdots, i_1$, we have

$$p_l - p^{\min}_l \le (p_l - p^{\min}_l)^+ \le \sum_{k=1}^{\Lambda-2} d^k \cdot (p_{\mu^*_{i_1}} - p^{\min}_{\mu^*_{i_1}})^+ + \sum_{k=1}^{\Lambda-2} d^k \cdot \varepsilon.$$

Recall that $\mu^*_{i_1} = 0$ (so $p_{\mu^*_{i_1}} = p^{\min}_{\mu^*_{i_1}} = 0$) and $\Lambda = m + 1$. We conclude as desired.

Part (ii): Let $N_0 = \{i \in N : \mu_i = 0\}$ and for each $k \ge 1$, let $N_k = \{i \in N \backslash (\cup_{s=0}^{k-1} N_s) : \mu_i \ne 0 \text{ and } u_{i'}(\mu_i, p^{\min}) = u_{i'}(\mu^*_{i'}, p^{\min}) \text{ for some } i' \in N_{k-1}\}$. For each $k \ge 0$, let $L_k = \{l \in L : l = \mu_i \text{ for some } i \in N_k\}$.

Step 1: There is some $T \in \mathbb{N}$ such that (1) for each $s \le T$, $N_s \ne \emptyset$, and $N_{T+1} = \emptyset$ and (2) $N = \cup_{s=0}^T N_s$ and $L = \cup_{s=0}^T L_s$.

Since $n > m$, $N_0 \ne \emptyset$. Since the numbers of agents and objects are finite, there is some $T \in \mathbb{N}$ satisfying (1).

To show (2), by contradiction, suppose that $N \supsetneq \cup_{s=0}^T N_s$. Let $\overline{N} = N \backslash \cup_{s=0}^T N_s$ and $\overline{L} = L \backslash \cup_{s=0}^T L_s$. By observation (a), we have: (a*) for each $l \in \overline{L}$, $l \in M$ and $p^{\min}_l > 0$. By Corollary 1, objects in $\overline{L}$ are fully assigned at (μ, p). Moreover, by the construction of $\cup_{s=0}^T N_s$, $\overline{L}$ can be only assigned to agents in $\overline{N}$. Thus we have: (b*) for each $i \in \overline{N}$, $\mu_i \in \overline{L}$ and $|\overline{N}| = |\overline{L}|$.

We show that: (c*) for each $i \in \overline{N}$, $\mu^*_i \in \overline{L}$. By contradiction, suppose that there is $i \in \overline{N}$ such that $\mu^*_i \notin \overline{L}$, i.e., $\mu^*_i \in \cup_{s=0}^T L_s$. By (a*), there is $j \in \cup_{s=0}^T N_s$ such that $\mu^*_j \in \overline{L}$. By construction of $\cup_{s=0}^T L_s$, $\mu_j \in \cup_{s=0}^T L_s$. By (b*), u^*_j is assigned to some agent $k \in \overline{N}$ at (μ, p) so $\mu_k = \mu^*_j$. By the construction of $\cup_{s=0}^T N_s$ and $u_j(\mu_k, p^{\min}) = u_j(\mu^*_j, p^{\min})$, we have that $k \in \cup_{s=0}^T N_s$, contradicting $k \in \overline{N}$.

By (a*), (c*), and Fact 2, there are $i \in \cup_{s=0}^T N_s$ and $j \in \overline{N}$ such that $u_i(\mu^*_j, p^{\min}) = u_i(\mu^*_i, p^{\min})$. By (b*), $\mu^*_j \in \overline{L}$ is assigned to some agent $k' \in \overline{N}$ at (μ, p), i.e., $\mu_k = \mu^*_j$. Thus $u_i(\mu_k, p^{\min}) = u_i(\mu^*_i, p^{\min})$. By the construction of $\cup_{s=0}^T N_s$, we conclude that $k \in \cup_{s=0}^T N_s$, contradicting $k \in \overline{N}$.

In conclusion, $N = \cup_{s=0}^T N_s$, and together with Corollary 1, $L = \cup_{s=0}^T L_s$.

Step 2: Let $i \in N$ be such that $\mu_i = l$. Let $l^* \in M$ be such that $u_i(l^*, p^{\min}) = u_i(\mu_i^*, p^{\min})$. Let $L_i = v_i^{l^*}(l, p^{\min}) - p_{l^*}^{\min}$. Then (1) $L_i \geq 0$ and (2) $p_{l^*}^{\min} - p_{l^*} \leq (p_{l^*}^{\min} - p_{l^*})^+ \leq (\varepsilon - L_i)^+ + d \cdot (p_l^{\min} - p_l)^+$.

Since $(\mu^*, p^{\min})$ is an MPE, it holds that $u_i(l^*, p^{\min}) = u_i(\mu_i^*, p^{\min}) \geq u_i(l, p^{\min})$. Thus $v_i^{l^*}(l, p^{\min}) \geq p_{l^*}^{\min}$, i.e., $L_i \geq 0$. Thus (1) holds.

Since (μ, p) be a tight equilibrium, we have that $v_i^{l^*}(l, p_l) \leq p_{l^*} + \varepsilon$.

First, consider the case of $p_l \geq p_l^{\min}$. Recall that $v_i^{l^*}(l, \cdot)$ is increasing in $\cdot$. Thus $v_i^{l^*}(l, p^{\min}) \leq v_i^{l^*}(l, p_l) \leq p_{l^*} + \varepsilon$. Since $v_i^{l^*}(l, p^{\min}) = L_i + p_{l^*}^{\min}$, we have that $L_i + p_{l^*}^{\min} \leq v_i^{l^*}(l, p_l) \leq p_{l^*} + \varepsilon$. Thus $p_{l^*}^{\min} - p_{l^*} \leq \varepsilon - L_i$. If $p_{l^*}^{\min} - p_{l^*} \geq 0$, then

$$p_{l^*}^{\min} - p_{l^*} = (p_{l^*}^{\min} - p_{l^*})^+ \leq \varepsilon - L_i = (\varepsilon - L_i)^+.$$

By $d \cdot (p_l^{\min} - p_l)^+ \geq 0$, (2) holds. If $p_{l^*}^{\min} - p_{l^*} < 0$, (2) follows from $p_{l^*}^{\min} - p_{l^*} < 0 = (p_{l^*}^{\min} - p_{l^*})^+ \leq (\varepsilon - L_i)^+ + d \cdot (p_l^{\min} - p_l)^+$.

Then consider the case of $p_l < p_l^{\min}$. Since $v_i^{l^*}(l, p^{\min}) = L_i + p_{l^*}^{\min}$ and $v_i^{l^*}(l, p_l) \leq p_{l^*} + \varepsilon$, we have

$$L_i + p_{l^*}^{\min} - p_{l^*} - \varepsilon \leq v_i^{l^*}(l, p^{\min}) - v_i^{l^*}(l, p) \underset{\text{Lipschitz continuity}}{\leq} d \cdot (p_l^{\min} - p_l) \leq d \cdot (p_l^{\min} - p_l)^+,$$

and the rearrangement of the first term and last term implies that

$$p_{l^*}^{\min} - p_{l^*} \leq \varepsilon - L_i + d \cdot (p_l^{\min} - p_l)^+ \leq (\varepsilon - L_i)^+ + d \cdot (p_l^{\min} - p_l)^+.$$

If $p_{l^*}^{\min} - p_{l^*} \geq 0$, then $p_{l^*}^{\min} - p_{l^*} = (p_{l^*}^{\min} - p_{l^*})^+$ so we have (2). If $p_{l^*}^{\min} - p_{l^*} < 0$, then $p_{l^*}^{\min} - p_{l^*} < (p_{l^*}^{\min} - p_{l^*})^+ = 0 \leq (\varepsilon - L_i)^+ + d \cdot (p_l^{\min} - p_l)^+$, as desired.

Completion of proof: Let $l \in L_s$ for some $s \geq 1$. By Step 1, there is a sequence of distinct agents $\{i_\lambda\}_{\lambda=0}^s$ with $s \geq 1$ such that:
(i) $\mu_{i_s} = l \in L_s$ and $\mu_{i_0} = 0 \in L_0$.
(ii) for each $1 \leq k \leq s-1$, $\mu_{i_k} \in L_k \subseteq M$.
(iii) for each $1 \leq k \leq s$, $u_{i_{k-1}}(\mu_{i_k}, p^{\min}) = u_{i_{k-1}}(\mu_{i_{k-1}}^*, p^{\min})$.

To simplify the notation, we write $p_{i_\lambda}^{\min}$ and p_{i_λ} to represent $p_{\mu_{i_\lambda}}^{\min}$ and $p_{\mu_{i_\lambda}}$.

First, consider i_0. Since $\mu_{i_0} = 0$, then $p_{i_0}^{\min} = p_{i_0} = 0$. By Step 2, we have

$$p_{i_1}^{\min} - p_{i_1} \leq (p_{i_1}^{\min} - p_{i_1})^+ \leq (\varepsilon - L_{i_0})^+. \tag{$*$}$$

Then consider i_1. By Step 2, we have

$$\begin{aligned} p_{i_2}^{\min} - p_{i_2} \leq (p_{i_2}^{\min} - p_{i_2})^+ &\leq (\varepsilon - L_{i_1})^+ + d \cdot (p_{i_1}^{\min} - p_{i_1})^+ \\ &\underset{(*)}{\leq} (\varepsilon - L_{i_1})^+ + d \cdot (\varepsilon - L_{i_0})^+. \end{aligned} \tag{$**$}$$

Next consider i_2. By Step 2, we have

$$p_{i_3}^{\min} - p_{i_3} \leq (p_{i_3}^{\min} - p_{i_3})^+ \leq (\varepsilon - L_{i_2})^+ + d \cdot (p_{i_2}^{\min} - p_{i_2})^+.$$
$$\underset{(**)}{\leq} (\varepsilon - L_{i_2})^+ + d \cdot (\varepsilon - L_{i_1})^+ + d^2 \cdot (\varepsilon - L_{i_0})^+.$$

Repeating the above argument, we have

$$p_{i_s}^{\min} - p_{i_s} \leq \sum_{k=0}^{s-1} d^{s-1-k} \cdot (\varepsilon - L_{i_k})^+ \underset{\text{Step } 2(L_{i_k} \geq 0)}{\leq} \sum_{k=0}^{s-1} d^k \cdot \varepsilon = \sum_{k=1}^{s} d^{k-1} \cdot \varepsilon. \qquad (***)$$

The maximum number of agents in the sequence generated by Step 1 with respect to an arbitrary $l \in M$ is $m+1$. Let $s = m$ in $(***)$. Then we have Part (ii).

A.5 Proof of Theorem 3

Let T be the termination step of the SD auction and (μ^T, p^T) be the outcome of Stage 2. Since $\varepsilon > 0$, the price of each object for each agent is bounded below by $-\varepsilon$ in Stage 1, and since the numbers of agents and objects are finite, it is easily seen that both stages will terminate in a finite number of steps. In sum, the SD auction terminates in a finite number of steps, i.e., $T < +\infty$. In the following, we show that (μ^T, p^T) is a tight equilibrium.

Step 1: $(\overline{\mu}, \overline{p})$, the allocation output from Stage 1 of the SD auction, is a discrete equilibrium with no unassigned objects.

It is straightforward that $\overline{\mu}$ is an assignment. We first show that all the objects are assigned. By contradiction, suppose that there is $l \in M$ that is unassigned at $\overline{\mu}$. Since $n > m$, there is $i \in N$ such that $\overline{\mu}_i = 0$. By the definition of Stage 1, l points to i at some step with prize 0, but is rejected by i. By $u_i \in \mathcal{U}^*$, when i rejects l at price 0, she must be assigned to an object which generates a strictly higher welfare than $u_i(0,0)$ at that step. Since the agents' welfare in Stage 1 is non-decreasing, then agent i's welfare at $(\overline{\mu}, \overline{p})$ must be strictly higher than $u_i(0,0)$, contradicting that she gets $(0,0)$. Thus Definition 4(ii) holds vacuously.

To show Definition 4(i), we proceed by contradiction. Suppose that there is $i \in N$ such that $\overline{\mu}_i \notin D_i^\varepsilon(\overline{p})$. Since the null is always available for i to choose, we have that $u_i(\overline{\mu}_i, p_{\overline{\mu}_i}) \geq u_i(0,0)$. Thus $\overline{\mu}_i \notin D_i^\varepsilon(p)$ implies that there is $l \in M$ such that $u_i(l, \overline{p}_l + \varepsilon) > u_i(\overline{\mu}_i, p_{\overline{\mu}_i})$. By the definition of Stage 1, at some step k, object l with $p_{li}^k = \overline{p}_l + \varepsilon$ points to i and i rejects object l (recall that objects point to agents with the highest personalized price for them). This means that agent i chooses another object l' and by $u_i \in \mathcal{U}_i^*$, $u_i(l', p_{l'i}^k) > u_i(l, p_{li}^k)$. Since the agents' welfare in Stage 1 is non-decreasing, we have that $u_i(\overline{\mu}_i, p_{\overline{\mu}_i}) \geq u_i(l', p_{l'i}^k) > u_i(l, p_{li}^k) = u_i(l, \overline{p}_l + \varepsilon)$, contradicting that $u_i(l, \overline{p}_l + \varepsilon) > u_i(\overline{\mu}_i, p_{\overline{\mu}_i})$. Thus for each $i \in N$, $\overline{\mu}_i \in D_i^\varepsilon(\overline{p})$ and we have the desired conclusion.

Step 2: For each $k \leq T$, (μ^k, p^k) is a discrete equilibrium with no unassigned objects.

Let $0 \leq s < T$. We inductively show Step 2. Recall that $(\mu^0, p^0) = (\overline{\mu}, \overline{p})$. By Step 1, the induction base, $s = 0$, holds. Now suppose that (μ^s, p^s) is a discrete equilibrium. We show that (μ^{s+1}, p^{s+1}) is a discrete equilibrium. Assume that $(\mu^{s+1}, p^{s+1}) \neq (\mu^s, p^s)$. It is straightforward that μ^{s+1} is an assignment. In Stage 2, each agent will have non-decreasing welfare change so no agent will be reassigned to the null if she does not get the null at (μ^0, p^0). So agents who get the null at (μ^0, p^0) are exactly those who get the null at (μ^{s+1}, p^{s+1}). Thus, together with Step 1, all the objects are assigned at at (μ^{s+1}, p^{s+1}). Thus Definition 4(ii) holds vacuously.

We show that (μ^{s+1}, p^{s+1}) satisfies Definition 4(i) via the following two cases.

Case 1: There are an agent i and an object l such that $\mu_i^s = l$, $p_l^s > 0$, and $l \notin \cup_{j \in N \setminus \{i\}} O_j(p^s)$. Then we simply set $p_l^{s+1} = p_l^s - \varepsilon$, $p_{l'}^{s+1} = p_{l'}^s$, for $l' \neq l$, and $\mu^{s+1} = \mu^s$. Since $\mu_i^{s+1} = \mu_i^s = l$, $p_l^{s+1} < p_l^s$, and $p_{l'}^{s+1} = p_{l'}^s$, for $l' \neq l$, then $l \in D_i^\varepsilon(p^s)$ implies that $l \in D_i^\varepsilon(p^{s+1})$. Since $l \notin \cup_{j \in N \setminus \{i\}} O_i(p^s)$, it holds that $u_j(\mu_j^s, p^s_{\mu_j^s}) > u_j(l, p_l^s)$ for $j \neq i$, where the strictness of the relation is because $u \in (U^*)^n$. By $p_l^{s+1} = p_l^s - \varepsilon$, $u_j(\mu_j^s, p^s_{\mu_j^s}) > u_j(l, p_l^{s+1} + \varepsilon)$. Together with $p_{l'}^{s+1} = p_{l'}^s$, for $l' \neq l$, and $\mu^{s+1} = \mu^s$, it holds that $\mu_j^{s+1} = \mu_j^s \in D_j^\varepsilon(p^{s+1})$. Thus Definition 4(i) holds.

Case 2: There is an improvement cycle at (μ^s, p^s). Let $N' = \{j \in N : \mu_j^s \neq \mu_j^{s+1}\}$ be the set of agents who update their assigned objects. For each $i \in N \setminus N'$, since $p^{s+1} = p^s$ and $(\mu_i^{s+1}, p^{s+1}_{\mu_i^{s+1}}) = (\mu_i^s, p^s_{\mu_i^s})$, it holds that $\mu_i^{s+1} \in D_i^\varepsilon(p^{s+1})$. For each $j \in N'$, by $u \in (\mathcal{U}^*)^n$, it holds that $u_j(\mu_j^{s+1}, p^{s+1}_{\mu_j^{s+1}}) > u_j(\mu_j^s, p^s_{\mu_j^s})$. Together with $p^{s+1} = p^s$ and $\mu_j^s \in D_j^\varepsilon(p^s)$, we have $\mu_j^{s+1} \in D_j^\varepsilon(p^{s+1})$. Thus, Definition 4(i) holds.

Step 3: (μ^T, p^T) is a tight equilibrium.

No improvement cycle at p^T is a necessary condition for the SD auction to terminate so Definition 7(ii) holds. In the following, we show that (μ^T, p^T) satisfies local tightness. Then the conclusion follows.

Consider $i \in N$ and $l \in M$ such that $\mu_i^T = l$. Suppose that $p_l^T = 0$. By $n > m$, there is $i \in N$ such that $\mu_i^0 = 0$. By $u_i \in \mathcal{U}^*$, $u_i(0, 0) < u_i(l, 0)$. Thus the local tightness holds for l. Now suppose that $p_l^T > 0$. Then there is $i_1 \in N \setminus \{i\}$ such that $l \in O_{i_1}(p^T)$. If not, the price of l will be further reduced, contradicting that T is the termination step. Thus the local tightness holds in such a case as well.

A.6 Proof of Theorem 4

Discrete efficiency follows from Theorem 1. Let $i \in N$, $u_i \in \mathcal{U}^{*d}$ be agent i's true utility function, and $f_i^{TE}(u) = (l, p_l)$. If agent i misreports via $u'_i \in \mathcal{U}^{*d}$, suppose that she can get $f_i^{TE}(u') = (l', p'_{l'})$ where $u' = (u'_i, u_{-i}) \in (\mathcal{U}^{*d})^n$.

First suppose that $l' = 0$. Then $p_{l'} = p'_{l'} = 0$. Since f^{TE} is a discrete equilibrium allocation, it holds that $u_i(l, p_l) \geq u_i(l', p_{l'}) = u_i(0, 0)$ so $v_i^{l'}(l, p_l) \leq 0$. Thus, $\lambda_i((l, p_l), (l', p'_{l'})) = \max\{0, v_i^{l'}(l, p_l) - p'_{l'}\} = 0 \leq d^* \cdot \varepsilon$.

In the following, let $l' \in M$. Let $\widehat{u}_i \in \mathcal{U}^{*d} \cap \mathcal{U}^{QL}$ be such that
(i) $\widehat{v}_i^{l'}(0,0) > \max_{j \in N} v_j^{l'}(0,0) + v_i'^{l'}(0,0) + \varepsilon$, and
(ii) for each $l'' \in M \backslash \{l'\}$, $\widehat{v}_i^{l''}(0,0) < \min\{\min_{j \in N} v_j^{l''}(0,0), v_i'^{l''}(0,0), \varepsilon\}$.[25]

In words, agent i strictly prefers object l' to any other object l'' at $\widehat{u}_i$. Recall that $p^{\min}(\cdot) \in \mathbb{R}^{m+1}$ is the MPE price (function), and the MPE prices are not restricted to discrete prices.

Step 1: $v_i^{l'}(l, p_l) \leq p_{l'}^{\min}(u) + \sum_{k=1}^{m-1} d^k \cdot \varepsilon + \varepsilon$.

Since $f^{TE}(u)$ is a discrete equilibrium allocation, we have that

$$u_i(l', p_{l'} + \varepsilon) \leq u_i(l, p_l) = u_i(l', v_i^{l'}(l, p_l)).$$

Thus $v_i^{l'}(l, p_l) \leq p_{l'} + \varepsilon$. By Theorem 2(i), $p_{l'} \leq p_{l'}^{\min}(u) + \sum_{k=1}^{m-1} d^k \cdot \varepsilon$. Thus $v_i^{l'}(l, p_l) \leq p_{l'}^{\min}(u) + \sum_{k=1}^{m-1} d^k \cdot \varepsilon + \varepsilon$, as desired.

Step 2: $f^{TE}(u')$ is a discrete equilibrium allocation for $\widehat{u}$.

For each agent $j \neq i$, their utility function remains the same at u' and $\widehat{u}$. Now consider agent i. Since $f^{TE}(u')$ is a tight equilibrium, $u'_i(l', p'_{l'}) \geq u'_i(0, 0)$, i.e., $p'_{l'} \leq v_i'^{l'}(0,0)$. By (i) in the construction of $\widehat{u}_i$, $p'_{l'} < \widehat{v}_i^{l'}(0,0)$ so $\widehat{u}_i(l', p'_{l'}) \geq \widehat{u}_i(0,0)$. By the construction of $\widehat{u}_i$, in particular, (ii) and $\widehat{u}_i \in \mathcal{U}^{QL}$, we have that for each $l'' \in M \backslash \{l'\}$, $\widehat{u}_i(l', p'_{l'}) \geq \widehat{u}_i(l'', 0) \geq \widehat{u}_i(l'', p'_{l''})$. Thus Definition 4(i) holds. Unassigned objects remain unassigned at price zero so Definition 4(ii) holds.

Step 3: There is a tight equilibrium $(\widetilde{\mu}, \widetilde{p})$ at $\widehat{u}$ such that $\widetilde{p}_{l'} \leq p'_{l'}$.

Let (μ', p') be the discrete equilibrium associated with $f^{TE}(u')$. By Step 2, (μ', p') is a discrete equilibrium for $\widehat{u}$. Now let $(\mu^0, p^0) = (\mu', p')$ and run Stage 2 of the SD auction with respect to (μ', p') and let $(\widetilde{\mu}, \widetilde{p})$ be the associated outcome. Following the same arguments as in Steps 2 and 3 in the proof of Theorem 3, we can show that $(\widetilde{\mu}, \widetilde{p})$ is a tight equilibrium for $\widehat{u}$. The price for each object is non-increasing in Stage 2 so $\widetilde{p}_{l'} \leq p'_{l'}$.

[25]Since $\widehat{u}_i \in \mathcal{U}$, Condition (iii) of Definition 1 implies $\widehat{v}_i^{l''}(0,0) > 0$.

Step 4: $p^{\min}_{l'}(\widehat{u}) \geq p^{\min}_{l'}(u)$.

Let $(\mu, p^{\min}(u))$ and $(\widehat{\mu}, p^{\min}(\widehat{u}))$ be the MPEs for u and $\widehat{u}$, respectively. By contradiction, suppose that $p^{\min}_{l'}(\widehat{u}) < p^{\min}_{l'}(u)$.

Step 4-1: Agent i gets l' at $(\widehat{\mu}, p^{\min}(\widehat{u}))$.

By contradiction, suppose that $\widehat{\mu}_i \neq l'$. Since $\widehat{\mu}_i \in D_i(p^{\min}(\widehat{u}))$, $p^{\min}_{l'}(\widehat{u}) \geq \widehat{v}^{l'}_i(\widehat{\mu}_i, p^{\min}_{\widehat{\mu}_i}(\widehat{u})) \geq \widehat{v}^{l'}_i(\widehat{\mu}_i, 0)$ and by the construction of $\widehat{u}_i$, in particular, (ii) and $\widehat{u}_i \in \mathcal{U}^{QL}$, it holds that $p^{\min}_{l'}(\widehat{u}) \geq \widehat{v}^{l'}_i(\widehat{\mu}_i, 0) > \max_{j \in N} v^{l'}_j(0,0) > 0$. Thus l' must be assigned to some agent j' at $p^{\min}(\widehat{u})$. Nevertheless, $v^{l'}_{j'}(0,0) \leq \max_{j\in N} v^{l'}_j(0,0) < p^{\min}_{l'}(\widehat{u})$, contradicting $l' \in D_{j'}(p^{\min}(\widehat{u}))$.

Step 4-2: Let $\widetilde{u}_i \in \mathcal{U}$ be such that for each $l \in M$, $\widetilde{v}^l_i(0,0) = p^{\min}_l(u)$. Then $(\mu, p^{\min}(u))$ is an MPE for $(\widetilde{u}_i, u_{-i})$.

First we introduce Lemma MS. A non-empty set of objects $M' \subseteq M$ is *overdemanded at p* if $|\{i \in N : D_i(p) \subseteq M'\}| > |M'|$. A set $M' \subseteq M$ of objects is *weakly underdemanded* at p if $[\forall l \in M',\, p_l > 0] \Rightarrow |\{i \in N : D_i(p) \cap M' \neq \emptyset\}| \leq |M'|$.

Lemma MS (Morimoto and Serizawa, 2015): Let $u \in \mathcal{U}^n$. Then p is an MPE price for u if and only if no set of objects is overdemanded and no set of objects is weakly underdemanded at p for u.

For agent i, by construction, $\mu_i \in \widetilde{D}_i(p^{\min}(u))$ and for each agent $j \neq i$, her utility function and assignment does not change so Definition 3(i) holds. Definition 3(ii) holds vacuously. Thus $(\mu, p^{\min}(u))$ is a Walrasian equilibrium for $(\widetilde{u}_i, u_{-i})$.

Since $(\mu, p^{\min}(u))$ is an MPE for u, by Lemma MS, we have that (1) for each non-empty set of objects $M' \subseteq M$, $|\{i \in N : D_i(p) \subseteq M'\}| \leq |M'|$ and (2) for each $M'' \subseteq M$ such that for each $l \in M''$, $p_l > 0$, $|\{i \in N : D_i(p) \cap M'' \neq \emptyset\}| > |M''|$. By the construction of $\widetilde{u}_i$, $D_i(p^{\min}(u)) \subseteq \widetilde{D}_i(p^{\min}(u))$. Thus $|\{i \in N : D_i(p) \subseteq M'\}|$ is non-increasing while $|\{i \in N : D_i(p) \cap M'' \neq \emptyset\}|$ is non-decreasing at $(\widetilde{u}_i, u_{-i})$. Thus, (1) and (2) hold at $p^{\min}(u)$ for at $(\widetilde{u}_i, u_{-i})$. By Lemma MS, $p^{\min}(u)$ is an MPE price for $(\widetilde{u}_i, u_{-i})$ so $(\mu, p^{\min}(u))$ is an MPE for $(\widetilde{u}_i, u_{-i})$.

By Step 4-2 and the construction of $\widetilde{u}_i$, agent i'*s* utility at the MPE for $(\widetilde{u}_i, u_{-i})$ is equal to $\widetilde{u}_i(l', p^{\min}_{l'}(u))$. By $p^{\min}_{l'}(\widehat{u}) < p^{\min}_{l'}(u)$ and Step 4-1, agent i benefit from misreporting $\widehat{u}_i$ when her true utility function is $\widetilde{u}_i$, contradicting that the MPE mechanism is not strategy-proof on $\mathcal{U}^n$ (Fact 3). Thus Step 4 holds.

Step 5: $p'_{l'} \geq p^{\min}_{l'}(u) - \displaystyle\sum_{k=1}^{m} d^{k-1} \cdot \varepsilon$.

The following inequality establishes Step 5:

$$p'_{l'} \underset{\text{Step 3}}{\geq} \widetilde{p}_{l'} \underset{\text{Theorem2(ii) for } \widehat{u}}{\geq} p^{\min}_{l'}(\widehat{u}) - \sum_{k=1}^{m} d^{k-1} \cdot \varepsilon \underset{\text{Step 4}}{\geq} p^{\min}_{l'}(u) - \sum_{k=1}^{m} d^{k-1} \cdot \varepsilon$$

The completion of the proof follows from that

$$\lambda_i((l,p_l),(l',p'_{l'})) \le \left| v_i^{l'}(l,p_l) - p'_{l'} \right| \underset{\text{Steps 1 and 5}}{\le} \sum_{k=1}^{m} d^{k-1}\cdot\varepsilon + \sum_{k=1}^{m-1} d^{k}\cdot\varepsilon + \varepsilon = d^*\cdot\varepsilon.$$

A.7 Supplementary Examples

Prices of discrete equilibria and tight equilibria: Let $N=\{1,2,3,4\}$, $M=\{l,l',l''\}$, and $\varepsilon=1$. Agents' utility functions $u\in(\mathcal{U}^*)^4$ are given by

$$\begin{aligned}
&\text{Agent 1}: u_1(l,1.5)=u_1(l',0.8)=u_1(l'',1.2)=u_1(0,0);\\
&\qquad u_1(l'',1)>u_1(l',-10)>u_1(l,-10).\\
&\text{Agent 2}: u_2(0,0)=u_2(l,0.5)=u_2(l',1.5)=u_2(l'',1.8),\\
&\qquad u_2(l'',1)=u_2(l',0.5)=u_2(l,0)>u_2(0,-50).\\
&\text{Agent 3}: u_3(l',2)>u_3(l,-100)>u_3(l'',-100)>u_3(0,0).\\
&\text{Agent 4}: u_4(l,2)>u_4(l',-120)>u_4(l'',-120)>u_4(0,0).
\end{aligned}$$

Let $\mu=(\mu_1,\mu_2,\mu_3,\mu_4)=(0,l'',l',l)$ and $p=(p_0,p_l,p_{l'},p_{l''})=(0,1,0,1)$. Let $\mu'=(\mu'_1,\mu'_2,\mu'_3,\mu'_4)=(l'',0,l',l)$ and $p'=(p'_0,p'_l,p'_{l'},p'_{l''})=(0,0,1,1)$. Then both (μ,p) and (μ',p') are tight equilibria. Now we let $p\wedge p'=(\min\{p_l,p'_l\})_{l\in L}=(0,0,0,1)$. Observe that there is no discrete equilibrium compatible with price $p\wedge p'$. Thus, we can conclude that neither the set of discrete equilibrium prices nor the set of tight equilibrium prices forms a lower semi-lattice.

Binding deviation bounds: First we show that the upper deviation bound given by Theorem 2(i) is binding. By contradiction, suppose that there are $\widehat{l}\in M$ and $\delta\in\mathbb{R}$ such that for each $u\in(\mathcal{U}^*)^n$ satisfying $d-$boundedness, $p_{\widehat{l}}-p_{\widehat{l}}^{\min}\le \sum_{k=1}^{m-1} d^k\cdot\varepsilon-\delta$.

Let $N=\{1,2,3\}$, $M=\{l,l'\}$ and $\varepsilon=1$. First consider the case of $\widehat{l}=l'$. Let three agents have the following quasi-linear utility functions:

$$\begin{aligned}
(v_1(0),v_1(l),v_1(l'))&=(0,3+k,5+2k)\\
(v_2(0),v_2(l),v_2(l'))&=(0,1.8,6.1)\\
(v_3(0),v_3(l),v_3(l'))&=(0,6.1,8.1+0.5)
\end{aligned}$$

where $0<2k<\min\{1,\delta\}$.

In such a case, $d=1$ and $m=2$ so $\sum_{k=1}^{m-1} d^{k-1}\cdot\varepsilon=1$. The MPE is as follows: $p^{\min}=(0,p_l^{\min},p_{l'}^{\min})=(0,3+k,5+2k)$ and agent 1 gets the null, agent 2 gets l' and agent 3 gets l. Now consider the tight equilibrium where $p = (0, p_l, p_{l'}) = (0, 3, 6)$, and agent 1 gets l, agent 2 gets the null, and agent 3 gets l'. Note that

$$p_{l'} - p_{l'}^{\min} = 6 - (5 + 2k) > \sum_{k=1}^{m-1} d^k \cdot \varepsilon - \delta = 1 - \delta$$

contradicting that $p_{l'} - p_{l'}^{\min} \leq \sum_{k=1}^{m-1} d^k \cdot \varepsilon - \delta$.

Now consider the case of $\widehat{l} = l$. For each $i \in N$, let $v'_i(0) = v_i(0)$, $v'_i(l) = v_i(l')$, and $v'_i(l') = v_i(l)$. The same reasoning for l' as above works for l. In sum, the upper deviation bound is binding.

Next we show that the lower deviation bound given by Theorem 2(ii) is binding. By contradiction, suppose that are $\widehat{l} \in M$ and $\delta \in \mathbb{R}$ such that $p_{\widehat{l}}^{\min} - p_{\widehat{l}} \leq$ Lower deviation bound $-\delta$.

Consider the same settings as Example 1 with additionally assuming $0 < \delta_1 < \delta_2 < \min\{1, \delta\}$. In such a case, $\widehat{l} = l$, $d = 1$ and $m = 1$ so $\sum_{k=1}^{m} d^{k-1} \cdot \varepsilon = 1$. The MPE price of l is $p_l^{\min} = 3 - \delta_2$ and agent 1 gets l. There is a unique tight equilibrium price, i.e., $p_l = 2$, compatible with two equilibrium assignments, i.e., either agent 1 gets l or agent 2 gets l. Note that

$$p_l^{\min} - p_l = 3 - \delta_2 - 2 > \sum_{k=1}^{m} d^{k-1} \cdot \varepsilon - \delta = 1 - \delta$$

contradicting that $p_l^{\min} - p_l \leq \sum_{k=1}^{m} d^{k-1} \cdot \varepsilon - \delta$.

No object-wise deviation bounds: We argue that no two objects can have different binding deviation bounds. We show the statement holds for the upper deviation bound. Analogous reasoning works for the lower deviation bound. By contradiction, suppose that there are $l, l' \in M$ such that $p_l - p_l^{\min} \leq \Delta_l$ and $p_{l'} - p_{l'}^{\min} \leq \Delta_{l'}$ with $\Delta_l \neq \Delta_{l'}$. Without loss of generality, assume $\Delta_l > \Delta_{l'}$. Then there is $\delta > 0$ such that $\Delta_l \geq \Delta_{l'} + \delta$. Since Δ_l is binding, there is $u \in (\mathcal{U}^*)^n$ satisfying $d-$boundedness such that $\Delta_l - \delta < p_l - p_l^{\min}$. Now consider $u' \in (\mathcal{U}^*)^n$ such that for each $i \in N$, $u_i(l, \cdot) = u'_i(l', \cdot)$, $u_i(l', \cdot) = u'_i(l, \cdot)$, and $u_i(\widehat{l}, \cdot) = u'_i(\widehat{l}, \cdot)$ for $\widehat{l} \neq l, l'$. Then $p_{l'}^{\min}(u') = p_l^{\min}(u)$ and moreover, $p'_{l'} = p_l$ is a tight equilibrium price of l' for u'. Then $\Delta_{l'} \leq \Delta_l - \delta < p_l - p_l^{\min} = p'_{l'} - p_{l'}^{\min}(u')$, contradicting that $\Delta_{l'}$ is an upper deviation bound of l'.

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