Multi-Dimensional Screening with Rich Consumer Data

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Motivation

Multi-dimensional screening

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- optimal mechanism: general characterization computationally difficult.

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- optimal mechanism: general characterization computationally difficult.
- mechanisms used in practice are often "simple"
  - although optimal mechanisms are more complicated.

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- Study multi-dimensional screening where seller observes **rich data** about consumer's type
  - e.g., technological advances in data collection/analysis.

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  - e.g., technological advances in data collection/analysis.
- Study **convergence rate** of mechanisms: how fast does the seller's revenue approximate first-best with rich data
  - natural efficiency measure of mechanisms in data-rich settings

Main Questions:

1. What is the **optimal** convergence rate?
2. Can simple mechanisms achieve the optimal convergence rate?

Main Results

1. Optimal convergence rate determined by **Fisher information**

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2. Optimal convergence rate achieved by **pure bundling**

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2. Optimal convergence rate achieved by **pure bundling**
   - but **not** by separate sales

Related literature (incomplete)

Multi-dimensional screening:

- Optimal mechanisms:
  - Wilson (1993), Armstrong (1996), Rochet and Chone (1998), Manelli and Vincent (2006), Daskalakis, Deckelbaum, Tzamos (2017)
  - Bundling mechanisms: Haghpanah and Hartline (2021), Ghili (2023), Yang (2023), etc.
- Robust optimality of simple mechanisms:
  - Carroll (2017), Deb and Roesler (2023), Che and Zhong (2021)
- Revenue guarantee/approximate optimality:
  - Hart and Nisan (2012), Cai et al. (2016), etc.
  - many products: Armstrong (1999), etc.

Model

Model

Seller:

- endowed with a finite set, G, of indivisible goods.

Single buyer of unknown type $\theta \in \Theta \subseteq \mathbb{R}^G_{++}$:

- Θ: compact set with non-empty interior,
- θ drawn according to prob. with density g,
- consuming goods $B \subseteq G$ with transfer t yields:

$$\mathbf{1}^B \cdot \theta - t.$$

- $\mathbf{1}^B_\ell := 1_{\ell \in B}$.

Seller's Information

Before sale, seller observes a sequence of n signals, $x^n = (x_1, \ldots, x_n)$:

- $x_1, \ldots, x_n$: drawn iid $P_\theta \in \Delta(X)$ with density $f(\cdot, \theta)$.
- n parametrizes richness of seller's data.

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Technical regularity assumptions on g and f throughout:

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After observing x^n, seller commits to a direct mechanism, (q, t):

- $q : \Theta \to \Delta(2^G)$,
- $t : \Theta \to \mathbb{R}$.

Seller's Problem

$$R^{SB}(x^n) := \sup_{(q,t)} \mathbb{E}\left[t(\theta) \mid x^n\right]$$

such that for all θ,

$$\sum_{B \subseteq G} q(B \mid \theta)\left(\mathbf{1}^B \cdot \theta\right) - t(\theta) = \max_{\theta' \in \Theta} \sum_{B \subseteq G} q(B \mid \theta')(\mathbf{1}^B \cdot \theta) - t(\theta'), \quad \text{(IC)}$$

$$\sum_{B \subseteq G} q(B \mid \theta)(\mathbf{1}^B \cdot \theta) - t(\theta) \geq 0. \quad \text{(IR)}$$

Simple Mechanisms

(Pure) bundling: post a single price, p_G, for **grand bundle**, G

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- $R^{\mathrm{bd}}(x^n)$: optimal revenue under bundling mechanisms

Separate sales: post a price, p_g, for each good $g \in G$

- $R^{\mathrm{sep}}(x^n)$: optimal revenue under separate sales mechanisms

Main Questions

Clear that

$$\mathbb{E}\left[R^{\mathrm{SB}}(x^n)\right], \mathbb{E}\left[R^{\mathrm{bd}}(x^n)\right], \mathbb{E}\left[R^{\mathrm{sep}}(x^n)\right] \to \underbrace{\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right]}_{\text{first best}}.$$

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1. What are the rates of convergence of the above?
2. How do they compare?

Main Results

Fisher Information

Informativeness of signals determines how much surplus seller can capture.

- What is the right measure of informativeness?

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Fisher information at θ:

$$I(\theta) := \left(-\mathbb{E}\left[\frac{\partial^2}{\partial\theta_g \partial\theta_{g'}} \ln f(x_1, \theta) \mid \theta \right] \right)_{g,g' \in G}$$

- differential analogue of Kullback-Leibler divergence:

$$\mathrm{KL}(P_\theta, P_{\theta'}) = (\theta - \theta') \cdot I(\theta)(\theta - \theta') + o\left(\|\theta - \theta'\|^2 \right)$$

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Assume henceforth that $I(\theta)$ is positive definite for all θ

- ensures complete learning about θ for large n.

Fisher Information and Bernstein-von Mises

Bernstein-von Mises Theorem:

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Thus,

$$\mathbf{1}^B \cdot \theta \approx N\left(\mathbf{1}^B \cdot \theta, \frac{\lambda^B(\theta)}{\sqrt{n}}\right).$$

Main Theorem

Theorem

Under both the optimal and bundling mechanisms, first-best gap vanishes at the same rate:

$$\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right] - \mathbb{E}\left[R_n^{\mathrm{SB}}(x^n)\right] = \mathbb{E}\left[\lambda^G(\theta)\right]\sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right);$$

$$\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right] - \mathbb{E}\left[R_n^{\mathrm{bd}}(x^n)\right] = \mathbb{E}\left[\lambda^G(\theta)\right]\sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right).$$

Main Theorem (Cont.): Separate Sales

Theorem

Under separate sales, first-best gap vanishes at a slower rate:

$$\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right] - \mathbb{E}\left[R_n^{\text{sep}}(x^n)\right] = \mathbb{E}\left[\sum_{g \in G} \lambda^g(\theta)\right] \sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right).$$

By triangle inequality,

$$\sum_{g \in G} \lambda_d(\theta) = \sum_{g \in G} \sqrt{\mathbf{1}^g \cdot I(\theta)^{-1} \cdot \mathbf{1}^g} \geq \sqrt{\mathbf{1}^G \cdot I(\theta)^{-1} \mathbf{1}^G} = \lambda^G(\theta).$$

Discussion

Optimal and pure bundling converge to first best at same rate.

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Discussion

Optimal and pure bundling converge to first best at same rate.

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To interpret, under pure bundling,

- for any $\varepsilon > 0$, εn extra signals outperforms SB for n suff. large.

In contrast, under separate sales,

- number of extra signals needed to outperform SB for n suff. large is at least

$$\underbrace{\left(\left(\frac{\mathbb{E}\left[\sum_{g \in G} \lambda^g(\theta)\right]}{\mathbb{E}\left[\lambda^G(\theta)\right]} \right)^2 - 1 \right)}_{>0} n.$$

Fisher Information: Gaussian Example

Suppose $|D| = 2$ and

$$x_i \sim N\left(\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}\right)$$

Inverse Fisher information:

$$I(\theta)^{-1} = \begin{pmatrix} \sigma^2 & \rho\sigma^2 \\ \rho\sigma^2 & \sigma^2 \end{pmatrix}$$

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Bundling and optimal mechanism:

$$\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right] - \mathbb{E}\left[R^{\text{bd}}(x^n)\right] \approx \underbrace{\sigma\sqrt{2(1+\rho)}}_{\lambda^G(\theta)}\sqrt{\frac{\ln n}{n}}.$$

Separate sales:

$$\mathbb{E}\left[\mathbf{1}^G \cdot \theta\right] - \mathbb{E}\left[R^{\text{sep}}(x^n)\right] \approx \underbrace{2\sigma}_{\lambda^1(\theta)+\lambda^2(\theta)}\sqrt{\frac{\ln n}{n}}.$$

Proof

Proof Outline

1. Reduction to normally distributed types;
2. Convergence rates of pure bundling and separate sales;
3. General mechanisms cannot improve the convergence rate.

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Key Idea: Bernstein-von Mises Theorem

Define $R^{\mathrm{SB}}(\mu, \Sigma), R^{\mathrm{bd}}(\mu, \Sigma), R^{\mathrm{sep}}(\mu, \Sigma)$:

- corresponding optimal revenues when $\theta' \sim N(\mu, \Sigma)$

Lemma

For any $\theta \in \Theta$ and for all $i \in \{\mathrm{SB}, \mathrm{bd}, \mathrm{sep}\}$,

$$\mathbb{E}\left[R_n^i(x^n) \mid \theta\right] - R^i\left(\theta, \frac{1}{n} I(\theta)^{-1}\right) = o\left(\sqrt{\frac{\ln n}{n}}\right).$$

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Convergence Rates of Simple Mechanisms

By Lemma, suffices to show for each θ^*:

$$\mathbf{1}^G \cdot \theta^* - R^{\mathrm{bd}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) = \lambda^G(\theta^*)\sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right),$$

$$\mathbf{1}^G \cdot \theta^* - R^{\mathrm{sep}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) = \left(\sum_{g \in G} \lambda^g(\theta^*)\right)\sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right).$$

Warm-Up Exercise: Single-good Monopoly

Pure bundling & separate selling: single-good mech design

- single posted price optimal.

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Consider a type distribution, F_n, for a **single** good: $N(\mu, \sigma^2/n)$.

- assume $\mu > 0$.

Warm-Up Exercise: Single-good Monopoly

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- single posted price optimal.

Consider a type distribution, F_n, for a **single** good: $N(\mu, \sigma^2/n)$.

- assume $\mu > 0$.

What is the optimal profit, Π_n^*, under uniform monopoly pricing?

$$\Pi_n^* = \max_p p(1 - F_n(p)), \; p_n^* = \arg\max_p p(1 - F_n(p)).$$

Warm-up Exercise: Single-good Monopoly

Proposition (Single-Good Monopoly)

$$\mu - \Pi_n^* = \sigma \sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right).$$

Moreover,

$$p_n^* = \mu - \sigma \sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right), F_n(p_n^*) = o\left(\sqrt{\frac{\ln n}{n}}\right)$$

Proof of Proposition

$$\mu - \Pi_n^* = \underbrace{\mu - p_n^*}_{\text{intensive margin}} + \underbrace{\mu F_n(p_n^*)}_{\text{extensive margin}} - \underbrace{(\mu - p_n^*) F_n(p_n^*)}_{\text{smaller order terms}}.$$

Key idea: suppose that $p_n = \mu - \alpha\sqrt{\frac{\ln n}{n}}$ for some $\alpha > 0$.

$$\mu - \Pi_n^* \approx \underbrace{\alpha\sqrt{\ln n} \cdot n^{-\frac{1}{2}}}_{\text{intensive margin}} + \underbrace{\frac{\mu}{\sqrt{2\pi}} \frac{\sigma}{\alpha\sqrt{\ln n}} \cdot n^{-\frac{\alpha^2}{2\sigma^2}}}_{\text{extensive margin}}$$

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Moreover, $\alpha = \sigma$ is optimal because,

1. If $\alpha > \sigma$, intensive margin increases.
2. If $\alpha < \sigma$, extensive margin dominates.

Implications of Proposition

Single-good monopoly proposition implies:

$$\mathbf{1}^G \cdot \theta^* - R^{\mathrm{bd}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) = \underbrace{\sqrt{\mathbf{1}^G \cdot I(\theta^*)^{-1}\mathbf{1}^G}}_{\lambda^G(\theta)} \sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right)$$

Moreover,

$$\underbrace{\mathbf{1}^G \cdot \theta^* - p_n^*}_{\text{intensive margin}} \approx \lambda^G(\theta^*)\sqrt{\frac{\ln n}{n}}, \quad \underbrace{F_n(p_n^*)}_{\text{extensive margin}} = o\left(\sqrt{\frac{\ln n}{n}}\right)$$

- Gap to first-best under optimal bundling is of the same order as the intensive margin.

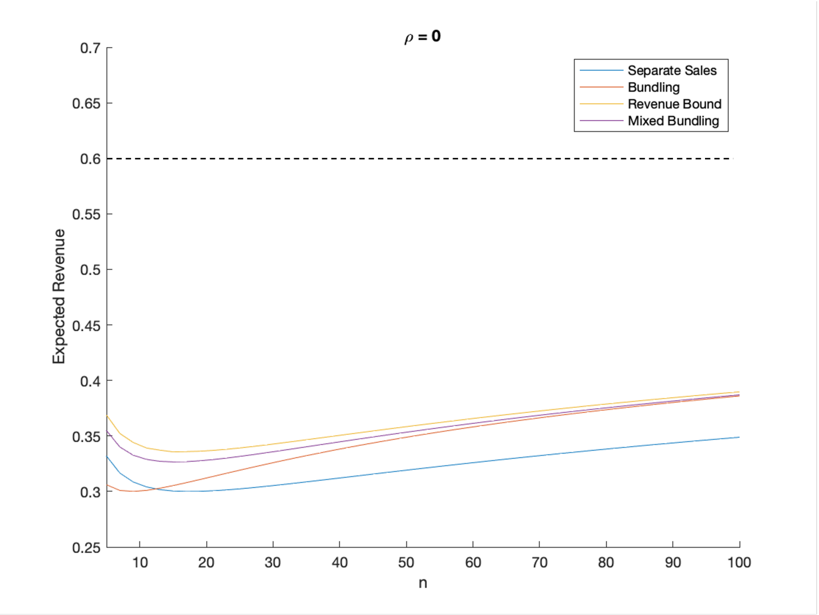
Implications of Proposition (Cont.)

Single-good monopoly proposition also implies:

$$\mathbf{1}^G \cdot \theta^* - R^{\mathrm{sep}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) = \sum_{g \in G} \underbrace{\sqrt{\mathbf{1}^g \cdot I(\theta^*)^{-1} \mathbf{1}^g}}_{\lambda^g(\theta^*)} \sqrt{\frac{\ln n}{n}} + o\left(\sqrt{\frac{\ln n}{n}}\right)$$

- $\Rightarrow$ suboptimality of separate sales.

Numerical Examples



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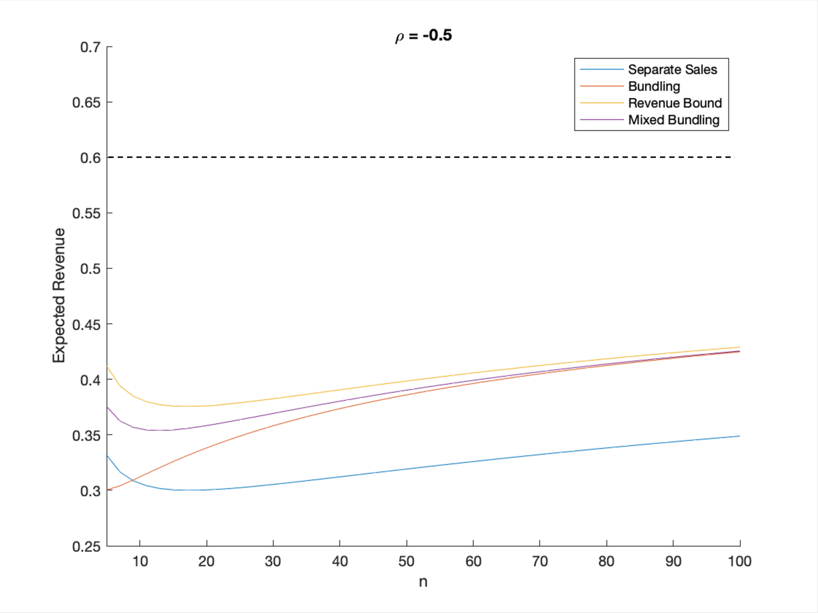

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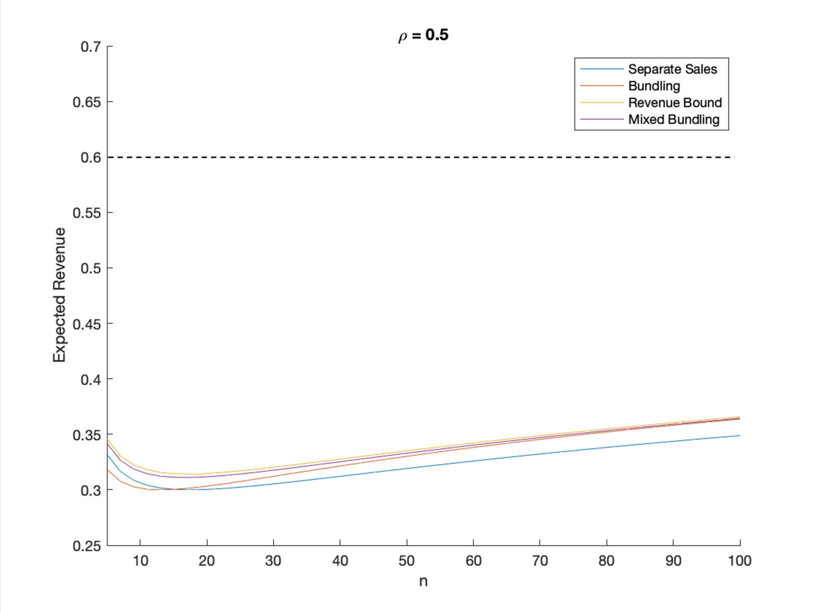

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Pure Bundling vs. "Simple" Mixed Bundling Mechanisms

First consider deterministic mechanisms:

- Offer a collection of deterministic bundles:

$$S_1 = G, S_2, \ldots, S_m \subsetneq G.$$

Pure Bundling vs. “Simple” Mixed Bundling Mechanisms

Consider first the optimal bundling mechanism with price p_G^*.

Pure Bundling vs. "Simple" Mixed Bundling Mechanisms

Consider first the optimal bundling mechanism with price p_G^*.

Recall:

1. First-best gap from optimal bundling mechanism:

$$\mathbf{1}^G \cdot \theta^* - R^{\mathrm{bd}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) \approx \underbrace{\lambda^G(\theta^*)\sqrt{\frac{\ln n}{n}}}_{\text{intensive margin}}.$$

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2. Extensive margin:

$$F_n(p_G^*) = o\left(\sqrt{\frac{\ln n}{n}}\right).$$

Pure Bundling vs. "Simple" Mixed Bundling Mechanisms

Keeping fixed p_G^*, benefit of additionally offering $S_2, \ldots, S_m$ only materializes if G is rejected.

- but this benefit is small: $F_n(p_G^*) = o\left(\sqrt{\frac{\ln n}{n}}\right)$.

Pure Bundling vs. "Simple" Mixed Bundling Mechanisms

Keeping fixed p_G^*, benefit of additionally offering $S_2, \ldots, S_m$ only materializes if G is rejected.

- but this benefit is small: $F_n(p_G^*) = o\left(\sqrt{\frac{\ln n}{n}}\right)$.

Only way to improve profits substantially is to raise $p_G \gg p_G^*$.

- But this makes the extensive margin too large: $F_n(p_G) \gg \sqrt{\frac{\ln n}{n}}$.
- Any surplus from sales of $S_2, \ldots, S_m$ bounded away from $\mathbf{1}^G \cdot \theta^*$.

General Mechanisms

Using single-good monopoly proposition, straightforward to compare convergence rates for "simple" mechanisms.

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However, general mechanisms can be substantially more complex:

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- Offer continuum of random bundles.

Question: How to bound the convergence rate for all mechanisms?

- **Challenge:** solving for optimal mechanism for n is intractable.

Key Idea: Relaxed Problem

Fix some $(|D|-1) \times |D|$ matrix, A, with full row rank.

Relaxed Problem:

$$\bar{R}^{\mathrm{SB}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) := \sup_{(q,t)} \mathbb{E}\left[t(\theta)\right]$$

such that for all θ,

$$\sum_{B \subseteq G} q(B \mid \theta)\left(\mathbf{1}^B \cdot \theta\right) - t(\theta) = \max_{\theta' : A\theta' = A\theta} \sum_{B \subseteq G} q(B \mid \theta')(\mathbf{1}^B \cdot \theta) - t(\theta'), \quad \text{(IC)}$$

$$\sum_{B \subseteq G} q(B \mid \theta)(\mathbf{1}^B \cdot \theta) - t(\theta) \geq 0. \quad \text{(IR)}$$

Rewriting Relaxed Problem

Relaxed Problem:

$$\bar{R}^{\mathrm{SB}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1} \mid y\right) = \sup_{(q,t)} \mathbb{E}\left[t(\theta) \mid A\theta = y\right]$$

such that for all $\theta \in A^{-1}(y)$,

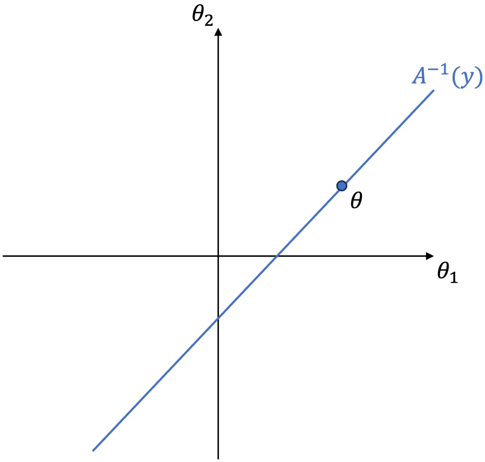
$$\sum_{B \subseteq G} q(B \mid \theta)\left(\mathbf{1}^B \cdot \theta\right) - t(\theta) = \max_{\theta' \in A^{-1}(y)} \sum_{B \subseteq G} q(B \mid \theta')(\mathbf{1}^B \cdot \theta) - t(\theta'), \quad \text{(IC)}$$

$$\sum_{B \subseteq G} q(B \mid \theta)(\mathbf{1}^B \cdot \theta) - t(\theta) \geq 0. \quad \text{(IR)}$$

Note:

$$\bar{R}^{\mathrm{SB}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1}\right) = \mathbb{E}\left[R^{\mathrm{SB}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1} \mid y\right)\right].$$

Relaxed Problem



Proof Overview

1. In the relaxed problem, $\bar{R}^{\mathrm{SB}}\left(\theta^*, \frac{1}{n} I(\theta^*)^{-1} \mid y\right)$, type space is one-dimensional.
   - We prove that there exist optimal mechanisms that offer only deterministic bundles or a single random bundle.
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2. Similar arguments to comparison of pure bundling to deterministic mixed bundling $\Rightarrow$ approx. optimality of pure bundling after all y.
3. $\Rightarrow$ pure bundling after all y is approximately optimal in relaxed problem.

How to Choose A?

Can choose A such that $y = A\theta$ is orthogonal to $\mathbf{1}^G \cdot \theta$:

$$(\mathbf{1}^G \cdot \theta \mid A\theta = y) \sim (\mathbf{1}^G \cdot \theta) \; \forall y.$$

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Ensures that optimal bundling mechanism in relaxed problem is the same as the optimal bundling mechanism in the original problem.

- Set price at original grand bundle price after all y.

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Can choose A such that $y = A\theta$ is orthogonal to $\mathbf{1}^G \cdot \theta$:

$$(\mathbf{1}^G \cdot \theta \mid A\theta = y) \sim (\mathbf{1}^G \cdot \theta) \; \forall y.$$

Ensures that optimal bundling mechanism in relaxed problem is the same as the optimal bundling mechanism in the original problem.

- Set price at original grand bundle price after all y.

$\therefore$ optimal mechanism does not improve (in terms of convergence rate) on pure bundling. □

Concluding Remarks

Optimal convergence rate to first best achieved by pure bundling:

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Ongoing work:

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Approach of analyzing convergence rates seems fruitful in other applications:

- Large markets: Rustichini, Satterthwaite, and Williams (1994), Satterthwaite and Williams (2002), Hong and Shum (2004)
- Moral hazard contracts: Frick, Iijima, Ishii (2023)

Thank you!

Regularity Assumptions

1. g is strictly positive and locally Lipschitz continuous.
2. For each $x \in X$, $f(x, \cdot) > 0$ and twice-differentiable in θ.
3. There exists L such that

$$\sup_{\theta, g, g', x} \left| \frac{\partial^2 \log f(x, \theta)}{\partial \theta_g \partial \theta_{g'}} \right| \leq L,$$

$$\sup_{\theta, \theta', (d, d'), x} \left| \frac{\partial^2 \log f(x, \theta)}{\partial \theta_g \partial \theta_{g'}} - \frac{\partial^2 \log f(x, \theta')}{\partial \theta_g \partial \theta_{g'}} \right| \leq L \|\theta - \theta'\|.$$

4. P_θ is continuous in θ with respect to Total-variation metric.
5. $\sup_\theta \mathbb{E} \left[\left(\sup_{\theta' \in \Theta} \log f(x, \theta') \right)^2 \mid \theta \right] < \infty$.
6. **Fisher information** matrix $I(\theta)$ as defined by

$$I(\theta) := - \left(\mathbb{E} \left[\frac{\partial^2}{\partial \theta_g \theta_{g'}} \log f(x, \theta) \mid \theta \right] \right)_{g, g' \in G}$$

is positive definite for each θ.

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