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Abstract

We consider a one-to-one matching with contracts model in the presence of liquidity constraints on the buyers side. Liquidity constraints can be either soft or hard. Competitive equilibria do exist in economies with soft liquidity constraints, but not necessarily in the presence of hard liquidity constraints. The limit of a convergent sequence of competitive equilibria in economies with increasingly stringent soft liquidity constraints may fail to be a competitive equilibrium in the limit economy with hard liquidity constraints. We establish equivalence and existence results of two alternative notions of competitive equilibrium, quantity-constrained competitive equilibrium and expectational equilibrium, together with stable outcomes and core outcomes, in the economies with both types of liquidity constraints. We argue that these notions of equilibrium and stability do not suffer from discontinuity problems by showing appropriate limit results.

Keywords: Liquidity constraints, matching with contracts, competitive equilibrium, quantity-constrained competitive equilibrium, expectational equilibrium, equivalence result, limit result.

JEL Classification: C72, C78, D45, D52

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1 Introduction

Financial market imperfections often cause economic agents to be subject to liquidity constraints. Such constraints have important consequences for the saving behavior of consumers and investment and operational decisions by producers.[1] There is also a wide range of empirical and anecdotal evidence that highlights the practical importance of buyers' limited purchasing power in auction settings.[2] Understanding how liquidity constraints influence market outcomes is therefore of great importance.

Liquidity constraints come in two forms. A *soft* liquidity constraint refers to the case where an agent can always get some additional liquidity by paying appropriate borrowing costs. A *hard* liquidity constraint refers to the case where an agent is subject to a strict payment limit and cannot obtain any additional liquidity.

In matching theory, there are remarkable differences between models with soft liquidity constraints and those with hard liquidity constraints.[3] Conditions that guarantee the existence and nice properties of competitive equilibria with soft constraints are no longer sufficient to tackle models with hard constraints, in which competitive equilibria may not exist and those properties may not hold. Besides, these two types of liquidity constraints may also result in different monotone matching patterns among agents.

We introduce an integrated framework that encompasses both economies with soft and hard liquidity constraints and that smoothens the connection between them as economies with hard liquidity constraints can also be treated as the limit of economies with increasingly stringent soft liquidity constraints. We study various notions of equilibrium which do not suffer from non-existence problem in economies with different types of liquidity constraints. We further examine the behavior of those equilibria when going from economies with increasingly stringent soft to the limit economy with hard liquidity constraints. Indeed, studying the continuity of equilibria with respect to a continuous change in some primitive of the economy is an important and classical theme in general equilibrium and game theory (Hildenbrand and Mertens, 1972; Echenique, 2002; Balder and Yannelis, 2006). This line of research sheds light on the robustness of equilibrium notions in relation to perturbations of the market environment.

We consider a two-sided matching model with bilateral contracts in the sense of Hatfield and Milgrom (2005), Hatfield, Kominers, Nichifor, Ostrovsky, and Westkamp (2013), and Fleiner, Jagadeesan, Jankó, and Teytelboym (2019). We focus on the one-to-one setting, which does not subsume those models, but extend them by the accommodation of both soft

[1]See, e.g., Carroll (2001) and Blalock, Gertler, and Levine (2008).

[2]See, e.g., Che and Gale (1998).

[3]See, e.g., Legros and Newman (2007), Talman and Yang (2015), Herings and Zhou (2022), and Jagadeesan and Teytelboym (2023).

and hard liquidity constraints. Buyers and sellers choose from a finite set of possible trades. Each trade is bilateral and designates its buyer and its seller. A contract consists of a trade and an amount of money transferred from the buyer to the seller. Buyers and sellers have general utility functions, allowing for income effects. At a technical level, agents' utility functions are not continuous when going from increasingly stringent soft to hard liquidity constraints, which explains the observed anomalies.

With soft liquidity constraints, when the amount of money in a contract exceeds the buyer's initial budget, the buyer can get additional liquidity on financial markets after paying an appropriate interest rate.[4] A higher interest rate corresponds to a more stringent soft liquidity constraint. At the limit, when the interest rate approaches plus infinity, the buyer faces a hard liquidity constraint and any monetary transfer exceeding the initial budget is not feasible. By varying interest rates, our model covers the entire spectrum of liquidity constraints and allows for the coexistence of soft and hard liquidity constraints.

A competitive equilibrium always exists in the case of soft liquidity constraints, but may fail to do so in the presence of hard liquidity constraints. As a consequence, the limit of a convergent sequence of competitive equilibria in economies with increasingly stringent soft liquidity constraints may fail to be a competitive equilibrium in the limit economy with hard liquidity constraints.

We study alternative equilibrium notions that do not suffer from such discontinuity problems. The first two are alternative notions of competitive equilibrium introduced by Herings (2020) and Herings and Zhou (2022).

Herings and Zhou (2022) argue that the standard notion of competitive equilibrium is not appropriate and introduce a quantity-constrained competitive equilibrium (QCCE). A QCCE extends a standard competitive equilibrium by introducing endogenously determined expectations of buyers about the availability of trades. In particular, when there is a hard liquidity constraint and the price of a trade is equal to or above this constraint, a buyer is allowed to expect no supply of the trade, and therefore demands another trade that is expected to be supplied.

A more fundamental notion of competitive equilibrium is expectational equilibrium as introduced in Herings (2020). The concept is formulated for general many-to-one matching models. At an expectational equilibrium, agents have endogenously determined expectations about the tradability of contracts. Agents hold expectations about the contracts that are to be supplied by agents on the other side of the market. Rationing constraints are used to express that certain contracts are not expected to be supplied. At equilibrium, agents choose optimal contracts subject to the rationing constraints, and for each contract

[4] In the object assignment model with monetary transfers, Saitoh and Serizawa (2008) and Morimoto and Serizawa (2015) mention the idea of using an interest rate for buyers when borrowing money. They use this to motivate their setting with preferences exhibiting income effects.

at least one side is not rationed.

We provide a novel equivalence between QCCE outcomes and expectational equilibrium outcomes. Herings and Zhou (2022) show the existence and coincidence of QCCE outcomes, stable outcomes, and core outcomes in matching models with hard liquidity constraints. We argue that such a result also holds with possible coexistence of both types of liquidity constraint. Putting them together, we establish a general equivalence and existence result of the above-mentioned four concepts in our model.

Then we present the limit results for those four concepts, with particular attention to the limit result of QCCEs. Consider a sequence of economies with increasingly stringent soft liquidity constraints. We show that a corresponding sequence of QCCE prices and outcomes always contains a convergent subsequence, and every convergent subsequence always leads to prices and outcome that are compatible with a QCCE of the limit economy. Notice that such a result does not hold for competitive equilibria. The fact that, contrary to competitive equilibria, QCCEs behave continuously when going from increasingly stringent soft to hard liquidity constraints, reinforces the view that QCCE is the appropriate notion of competitive equilibrium. The corresponding limit results of expectational equilibrium outcomes, stable outcomes, and core outcomes follow from the equivalence result.

We demonstrate that like competitive equilibrium, strongly stable outcomes and strict core outcomes suffer from the same discontinuity problem. Moreover, we show that our results can be extended to settings where liquidity constraints correspond to more general forms of borrowing costs.

Notions like competitive equilibrium, stable outcome, and core outcome, and their structural properties have been intensively studied in all sorts of matching models when agents have general utility functions, see, e.g., Crawford and Knoer (1981), Quinzii (1984), Demange and Gale (1985), Legros and Newman (2007), Fleiner et al. (2019), and Schlegel (2022). A matching model with general utility functions can be used to analyze matching models with soft liquidity constraints and results established in a model with general utility functions carry over to models with soft liquidity constraints. However, such models cannot be used to analyze matching models with hard liquidity constraints and results established in models with general utility functions may fail to hold in matching models with hard liquidity constraints. Therefore, these two types of models are fundamentally different, see, e.g, Herings and Zhou (2022) and Section 3 for detailed discussions.

In the assignment model with unit-demand agents and hard liquidity constraints, Talman and Yang (2015) propose an auction that constructively finds a core outcome. In one-to-one matching models with imperfectly transferable utility and unobserved heterogeneity in tastes, Galichon, Kominers, and Weber (2019) define the notion of aggregate equilibrium. In many-to-one matching models with general constraints, Herings (2020) proposes expectational equilibrium and discusses its equivalence with standard notions of competitive equilibrium and stable outcomes in a large variety of settings. In one-to-one matching models with hard liquidity constraints, Herings and Zhou (2022) propose the notion of a QCCE. They study the formation of QCCEs in decentralized markets and the structural properties of QCCEs. In many-to-many matching models with liquidity constraints, Jagadeesan and Teytelboym (2023) show the existence of stable outcomes under the net substitutability condition and analyze various properties of stable outcomes. All these results are different from ours as none of them explicitly studies a model with the coexistence of different types of liquidity constraints and analyzes the continuity property of equilibrium notions by relating economies with hard liquidity constraints to those with increasingly stringent soft liquidity constraints.[5]

We contribute to the works mentioned above by formulating an integrated model that allows for different types of liquidity constraints and by throwing light on the continuity properties of various equilibrium notions via their equivalence, existence, and limit results. Our analysis provides a new connection between equilibria in matching models with soft liquidity constraints and those with hard liquidity constraints.

The remaining part of the paper is organized as follows. Section 2 presents the model and Section 3 shows the problematic behavior of competitive equilibria in limit economies. Section 4 shows the coincidence and existence of QCCEs, expectational equilibria, stable outcomes, and core outcomes. Section 5 presents the limit results of the concepts studied in Section 4. Section 6 discusses the robustness of our results. Section 7 contains the conclusion.

2 The Matching Model with Liquidity Constraints

There is a finite set of buyers B and a finite set of sellers S. Buyers and sellers participate in bilateral trades in a finite set Ω. Each trade $\omega \in \Omega$ is associated with a buyer $b(\omega) \in B$ and a seller $s(\omega) \in S$. A trade specifies the precise contractual terms of the delivery of a good or a service from the seller to the buyer, with the exception of the price against which such a transaction occurs. We denote the set of trades in Ω involving buyer $b \in B$ and seller $s \in S$, respectively, by

$$\Omega^b = \{\omega \in \Omega \mid b(\omega) = b\},$$
$$\Omega^s = \{\omega \in \Omega \mid s(\omega) = s\}.$$

[5]Dupuy, Galichon, Jaffe, and Kominers (2020) study how changes in taxes influence the firm-worker sorting patterns and efficiency. Their results provide a link between matching models with and without transfers. In contrast to the hard liquidity constraints, the introduction of taxes does not influence equilibrium existence.

Transactions take place by signing contracts. A contract $c = (\omega, t) \in \Omega \times \mathbb{R}$ specifies a trade $\omega \in \Omega$ and a payment $t \in \mathbb{R}$ that is transferred from buyer $b(\omega)$ to seller $s(\omega)$. For every contract $c \in \Omega \times \mathbb{R}$, let $\omega(c)$ and $t(c)$ be the corresponding trade and payment, respectively. For a set of contracts $Y \subseteq \Omega \times \mathbb{R}$, $\omega(Y) = \{\omega(c) \in \Omega \mid c \in Y\}$ corresponds to the set of trades related to contracts in Y.

We consider a one-to-one matching set-up. A buyer signs a contract with at most one seller and a seller signs a contract with at most one buyer. An agent $i \in B \cup S$ who does not sign any contract receives the *no-trade* option o^i.

For every buyer $b \in B$, the utility function $u^b : (\Omega^b \times \mathbb{R}) \cup \{o^b\} \to \mathbb{R}$ is such that, (b-i) for every $\omega \in \Omega^b$, $u^b(\omega, \cdot)$ is continuous and strictly decreasing on $\mathbb{R}$ and, (b-ii) for every $\omega \in \Omega^b$, the range of $u^b(\omega, \cdot)$ is all of $\mathbb{R}$. Condition (b-i) says that given a trade, a lower payment improves the buyer's utility. Condition (b-ii) says that there is no trade ω that is infinitely good or bad. That is, for every trade $\omega \in \Omega^b$, there is $\bar{t}^b_\omega \in \mathbb{R}$ such that $u^b(\omega, \bar{t}^b_\omega) = u^b(o^b)$. Moreover, for every contract $c \in \Omega^b \times \mathbb{R}$, for every trade $\omega' \in \Omega^b$, there is $t' \in \mathbb{R}$ such that $u^b(c) = u^b(\omega', t')$. For every $\omega \in \Omega^b$, we define $u^b(\omega, +\infty) = -\infty$.

For every seller $s \in S$, the utility function $u^s : (\Omega^s \times \mathbb{R}) \cup \{o^s\} \to \mathbb{R}$ is such that, (s-i) for every $\omega \in \Omega^s$, $u^s(\omega, \cdot)$ is continuous and strictly increasing on $\mathbb{R}$ and, (s-ii) for every $\omega \in \Omega^s$, the range of $u^s(\omega, \cdot)$ is all of $\mathbb{R}$. The economic explanation for (s-i) and (s-ii) is analogous to (b-i) and (b-ii). Thus, for every trade $\omega \in \Omega^s$, there is $\underline{t}^s_\omega \in \mathbb{R}$ such that $u^s(\omega, \underline{t}^s_\omega) = u^s(o^s)$. Moreover, for every contract $c \in \Omega^s \times \mathbb{R}$, for every trade $\omega' \in \Omega^s$, there is $t' \in \mathbb{R}$ such that $u^s(c) = u^s(\omega', t')$.

Assumptions (b-ii) and (s-ii) are standard in the matching literature when agents have general utility functions, see, e.g., Demange and Gale (1985). The commonly used quasi-linear utility functions satisfy these assumptions.

We denote the profile of utility functions by $u = (u^i)_{i \in B \cup S}$.

Given a set of contracts $Y \subseteq \Omega \times \mathbb{R}$, Y^b denotes the set of contracts involving buyer $b \in B$ and Y^s denotes the set of contracts involving seller $s \in S$, so

$$Y^b = \{c \in Y \mid b(\omega(c)) = b\},$$
$$Y^s = \{c \in Y \mid s(\omega(c)) = s\}.$$

Notice that in case agent $i \in B \cup S$ is not part of a contract in Y, then $Y^i = \emptyset$.

The consumption set of agent $i \in B \cup S$ is equal to

$$X^i = \{Y \subseteq \Omega^i \times \mathbb{R} \mid |Y| \leq 1\}.$$

Thus, X^i consists of the singleton subsets of $\Omega^i \times \mathbb{R}$ and the empty set.

A set of contracts $A \subseteq \Omega \times \mathbb{R}$ is an *outcome* if, for every $i \in B \cup S$, $A^i \in X^i$. Let $\mathcal{A}$ be the collection of outcomes, i.e.,

$$\mathcal{A} = \{A \subseteq \Omega \times \mathbb{R} \mid \text{for every } i \in B \cup S,\ A^i \in X^i\}.$$

Notice that $\emptyset \in \mathcal{A}$, so $\emptyset$ is an outcome. More generally, for every $i \in B \cup S$, any element of X^i is an outcome as well.

In the following, we introduce soft and hard liquidity constraints. Every buyer $b \in B$ is endowed with an amount of money $M^b \in [0, +\infty)$. If a payment exceeds M^b, then buyer b can borrow money on the financial markets against an interest rate $r^b \in [0, +\infty]$.[6] In case $r^b \in [0, +\infty)$, buyer b faces a *soft liquidity constraint*. A special case of a soft liquidity constraint is the absence of a liquidity constraint if $r^b = 0$. In case $r^b = +\infty$, buyer b never chooses to make a payment exceeding M^b and is subject to a *hard liquidity constraint*.

Depending on their amount of collateral or their social network, different buyers may be able to acquire additional liquidity against different interest rates. Thus, we allow interest rates to be buyer-dependent. Let $M = (M^b)_{b \in B}$ be the profile of monetary endowments and $r = (r^b)_{b \in B}$ be the profile of interest rates.

For every buyer $b \in B$, for every contract $c \in \Omega^b \times \mathbb{R}$, the transfer plus borrowing cost for contract c is given by

$$t^+(c; r^b) = \begin{cases} t(c), & \text{if } t(c) \leq M^b, \\ t(c) + r^b(t(c) - M^b), & \text{if } t(c) > M^b. \end{cases}$$

The amount $t^+(c; r^b)$ is larger than or equal to $t(c)$. In case $t(c) \leq M^b$, buyer b does not need to borrow and $t^+(c; r^b) = t(c)$. In case $t(c) > M^b$, buyer b pays a borrowing cost $r^b(t(c) - M^b)$ and $t^+(c; r^b) = t(c) + r^b(t(c) - M^b)$. If buyer b faces a hard liquidity constraint, then $t(c) > M^b$ implies $t^+(c; r^b) = +\infty$. By our assumptions on utility functions, such a contract is never chosen by the buyer, since it is inferior to the no-trade option.

The utility function u^b of buyer $b \in B$ over contracts induces the utility function U^b over outcomes. For every profile of interest rates $r \in [0, +\infty]^B$, for every $A \in \mathcal{A}$, in case $A^b = \emptyset$, $U^b(A; r) = u^b(o^b)$, and in case $A^b = \{c\}$, $U^b(A; r) = u^b(\omega(c), t^+(c; r^b))$. In the case of soft liquidity constraints, assumption (b-i) on u^b implies that for a given trade $\omega \in \Omega^b$, $U^b(\{(\omega, \cdot)\}; r)$ is continuous and strictly decreasing in transfers. In the case of hard liquidity constraints, the same assumption implies that $U^b(\{(\omega, \cdot)\}; r)$ exhibits a discontinuity at a transfer equal to the monetary endowment, $U^b(\{(\omega, \cdot)\}; r)$ is equal to $-\infty$ at transfers exceeding the monetary endowment, and $U^b(\{(\omega, \cdot)\}; r)$ is continuous at transfers lower than the monetary endowment.[7]

The utility function u^s of seller $s \in S$ over contracts induces the utility function U^s over outcomes. For every $A \in \mathcal{A}$, in case $A^s = \emptyset$, $U^s(A) = u^s(o^s)$, and in case $A^s = \{c\}$, $U^s(A) = u^s(c)$.

[6] All our results hold even when the interest rate could be trade-dependent, i.e., for each trade $\omega \in \Omega$, buyer $b(\omega)$ can borrow money against an interest rate $r^\omega \in [0, +\infty]$. To do so would require additional notation, which complicates the exposition, while offering few new insights.

[7] The discontinuity of $U^b(\{(\omega, \cdot)\}; r)$ is illustrated via a numerical example in Appendix A.3.

The primitives of the economy are summarized by $\mathcal{E} = (B, S, \Omega, u, M, r)$.

Our model subsumes the following models as special cases:

• Matching models with no liquidity constraints, see, e.g., Crawford and Knoer (1981), Demange and Gale (1985), and Alkan and Gale (1990): The set of trades $\Omega = B \times S$ corresponds to the possible matches between agents in B and S, the no-trade option corresponds to being unmatched, and $r = (0, \dots, 0)$.

• Matching models with hard liquidity constraints, see, e.g., Herings and Zhou (2022). All other primitives are the same except for the profile of interest rates r, i.e., for each $b \in B$, $r^b = 0$ or $+\infty$.

• Auction models with hard liquidity constraints, see, e.g., Talman and Yang (2015): The set of trades is equal to $\Omega = B \times S$. For every $b \in B$, $r^b = 0$ or $+\infty$, $u^b(o^b) = 0$, and for every $\omega \in \Omega^b$, for every $t \in \mathbb{R}$, $u^b(\omega, t) = V^b(\omega) - t$ where $V^b(\omega) \in \mathbb{R}$ is the value buyer b assigns to trade ω. For every seller $s \in S$, for every $\omega, \omega' \in \Omega^s$, $u^s(\omega, \cdot) = u^s(\omega', \cdot)$, $\underline{t}^s_\omega = \underline{t}^s_{\omega'} \geq 0$, and $u^s(o^s) = 0$.

• Auction models with soft liquidity constraints, see, e.g., Saitoh and Serizawa (Example 1, 2008): For every buyer $b \in B$, for every $\omega \in \Omega^b$, for every $t \in \mathbb{R}$, $r^b \in (0, +\infty)$, $u^b(o^b) = 0$, and $u^b(\omega, t) = V^b(\omega) - t$, and for every $\omega, \omega' \in \Omega^b$, $V^b(\omega) = V^b(\omega')$ (identical trades). For every seller $s \in S$, $u^s(o^s) = 0$ and for every $\omega \in \Omega^s$, $\underline{t}^s_\omega = 0$.

Varying the interest rates enables us to integrate matching models with soft liquidity constraints and those with hard liquidity constraints. Appendix A.1 provides an illustrative example.

3 Competitive Equilibrium

In the competitive analysis of matching models, each trade $\omega \in \Omega$ is assigned a price $p_\omega \in \mathbb{R}$, which results in a price vector $p \in \mathbb{R}^\Omega$. Prices are allowed to be personalized. For instance, a seller $s \in S$ can sell an identical commodity at different prices to different buyers. Personalized prices have been used before in competitive settings, see, e.g., Hatfield et al. (2013) and Fleiner et al. (2019).

The *budget set* of agent $i \in B \cup S$ is given by

$$\gamma^i(p) = \{A^i \in X^i \mid \forall c \in A^i,\ t(c) = p_{\omega(c)}\}, \quad p \in \mathbb{R}^\Omega.$$

The budget set of agent i contains those contracts in the consumption set of agent i with transfers equal to prices. The budget set always contains $\emptyset$ and is therefore non-empty.

In a competitive equilibrium, buyers take prices $p \in \mathbb{R}^\Omega$ as given and choose optimal contracts. The *demand set* of buyer $b \in B$ is given by

$$\delta^b(p; r) = \operatorname*{arg\,max}_{A^b \in \gamma^b(p)} U^b(A^b; r), \quad p \in \mathbb{R}^\Omega.$$

Sellers take prices $p \in \mathbb{R}^{\Omega}$ as given and choose optimal contracts. The *demand set* of seller $s \in S$ is given by

$$\delta^s(p) = \underset{A^s \in \gamma^s(p)}{\arg\max}\ U^s(A^s), \quad p \in \mathbb{R}^{\Omega}.$$

We now define the notion of a competitive equilibrium.

Definition 3.1: An element $(p, A) \in \mathbb{R}^{\Omega} \times \mathcal{A}$ is a *competitive equilibrium* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:
(i) For every $b \in B$, $A^b \in \delta^b(p; r)$.
(ii) For every $s \in S$, $A^s \in \delta^s(p)$.

In a competitive equilibrium, agents receive their demanded contracts. The usual condition that demand equals supply is implicitly incorporated in the definition of an outcome.

When buyers face no liquidity constraints, there is a competitive competitive, as shown by, see, e.g., Herings (2018).[8]

Proposition 3.2: Let $\mathcal{E} = (B, S, \Omega, u, M, r)$ be an economy such that, for every $b \in B$, $r^b = 0$. A competitive equilibrium exists.

We next generalize Proposition 3.2 by showing the existence of competitive equilibrium in the presence of arbitrary soft liquidity constraints.

Proposition 3.3: Let $\mathcal{E} = (B, S, \Omega, u, M, r)$ be an economy such that $r \in [0, +\infty)^B$. A competitive equilibrium exists.

Proof: For every $b \in B$, we define $\widetilde{u}^b : (\Omega^b \times \mathbb{R}) \cup \{o^b\} \to \mathbb{R}$ by $\widetilde{u}^b(o^b) = u^b(o^b)$ and, for every $c \in \Omega^b \times \mathbb{R}$, $\widetilde{u}^b(c) = u^b(\omega(c), t^+(c; r^b)) = U^b(\{c\}; r)$. Since $r \in [0, +\infty)^B$, it is easy to

[8] Crawford and Knoer (1981) prove the existence of a strict core outcome for the case with quasi-linear utility functions, a single trade for each buyer-seller pair, and no liquidity constraints. They also remark that such a strict core outcome is equivalent to a competitive equilibrium when appropriately defined. They argue that the proof of the existence of a strict core outcome does not depend on the assumption of quasi-linearity so that this result can be extended to settings with general utility functions and to the case with multiple trades between a given buyer-seller pair. A direct proof of existence of stable outcomes in the extended model with general utility functions and a single trade for each buyer-seller pair as suggested by Crawford and Knoer (1981) can be found in Alkan and Gale (1990). However, for the case with multiple possible trades between a given buyer-seller pair, the equivalence between a stable outcome (or a strict core outcome) and a competitive equilibrium is not straightforward as a stable outcome does not specify the prices of trades that are not part of the stable outcome. To the best of our knowledge, the earliest reference that contains a proof for the existence of a competitive equilibrium in the general specification of our model without liquidity constraints is Herings (2018).

see that $\widetilde{u}^b$ satisfies (b-i) and (b-ii). Let $\widetilde{u} = ((\widetilde{u}^b)_{b \in B}, (u^s)_{s \in S})$.

Now consider the economy without liquidity constraints $\widetilde{\mathcal{E}} = (B, S, \Omega, \widetilde{u}, M, \widetilde{r})$, where, for every $b \in B$, $\widetilde{r}^b = 0$. The utility function $\widetilde{u}^b$ of buyer $b \in B$ over contracts induces the utility function $\widetilde{U}^b$ over outcomes. By construction of $\widetilde{u}^b$, in case $A^b = \emptyset$, $\widetilde{U}^b(A^b; \widetilde{r}) = \widetilde{u}^b(o^b) = u^b(o^b) = U^b(A^b; r)$, and in case $A^b = \{c\}$, $\widetilde{U}^b(\{c\}; \widetilde{r}) = \widetilde{u}^b(c) = U^b(\{c\}; r)$. Thus, for every $b \in B$, for every $p \in \mathbb{R}^\Omega$, it holds that $\widetilde{\delta}^b(p; \widetilde{r}) = \delta^b(p; r)$.

We conclude from the above paragraph that the competitive equilibria of $\widetilde{\mathcal{E}}$ coincide with those of $\mathcal{E}$. By Proposition 3.2, there is a competitive equilibrium of $\widetilde{\mathcal{E}}$ and so $\mathcal{E}$ has a competitive equilibrium as well. **Q.E.D.**

In the proof of Proposition 3.3 an economy with soft liquidity constraints is transformed into an economy without liquidity constraints in such a way that for each contract, the involved buyer is indifferent between signing it but possibly paying the borrowing cost in the economy with soft liquidity constraints and signing it but without paying any borrowing cost in the economy without liquidity constraints. Therefore, at each price, buyers' demand sets remain the same in both the original economy and the transformed economy. Therefore, competitive equilibria in the transformed economy without liquidity constraints coincide with competitive equilibria in the original economy. Equilibrium existence in the original economy then follows from Proposition 3.2. The proposed transformation provides an intuition as to why a matching model with general utility functions can be used to analyze matching models with soft liquidity constraints. We give an example that illustrates the notion of competitive equilibrium in the presence of soft liquidity constraints in Appendix A.2.

Unfortunately, there is no analogue of Proposition 3.3 in the presence of hard liquidity constraints. The literature has provided various examples of matching models with hard liquidity constraints where a competitive equilibrium fails to exist (Talman and Yang, 2015; Herings and Zhou, 2022; Jagadeesan and Teytelboym, 2023). For the sake of completeness, we present such an example in Appendix A.3.

4 Alternative Solution Concepts

In this section, we study alternative notions of equilibrium and stability, which turn out not to suffer from non-existence problems in economies with hard liquidity constraints.

We first consider two alternative competitive equilibrium notions, "quantity-constrained competitive equilibrium," and "expectational equilibrium."

Herings and Zhou (2022) study models with hard liquidity constraints and propose the notion of a quantity-constrained competitive equilibrium. Buyers form expectations about the availability of trades. A buyer may expect that a trade is not available or, equivalently, a buyer may expect a binding quantity constraint, if the buyer faces a hard liquidity constraint and the price of the trade is equal to or above the buyer's monetary endowment. The reason is that these are the only circumstances under which it is impossible for the buyer to offer a higher transfer to the seller in case the seller makes another trade. For every $\omega \in \Omega$, let $q_\omega \in \{0, 1\}$ be the quantity constraint of trade ω, where $q_\omega = 1$ means that buyer $b(\omega)$ expects trade ω to be available and $q_\omega = 0$ means that buyer $b(\omega)$ expects trade ω not to be available. We denote the vector of quantity constraints by $q = (q_\omega)_{\omega \in \Omega}$.

The *constrained budget set* of buyer $b \in B$ is given by

$$\gamma^b(p, q) = \{A^b \in X^b \mid \forall c \in A^b,\ t(c) = p_{\omega(c)} \text{ and } q_{\omega(c)} = 1\},\ (p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega.$$

The constrained budget set of buyer b contains those contracts that involve buyer b, i.e., $\{c\} \in X^b$, buyer b expects them to be supplied, i.e., $q_{\omega(c)} = 1$, and the payment is equal to the price, i.e., $t(c) = p_{\omega(c)}$. The set $\gamma^b(p, q)$ always contains $\emptyset$ and is therefore non-empty.

The *constrained demand set* of buyer $b \in B$ is given by

$$\delta^b(p, q; r) = \underset{A^b \in \gamma^b(p,q)}{\arg\max}\ U^b(A^b; r), \quad (p, q) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega.$$

The constrained demand set of buyer b equals the set of contracts which maximize buyer b's utility over the contracts in the constrained budget set.

Since a seller does not face liquidity constraints, the seller's decision problem is the same as before. A quantity-constrained competitive equilibrium is defined as follows.

Definition 4.1: An element $(p, q, A) \in \mathbb{R}^\Omega \times \{0, 1\}^\Omega \times \mathcal{A}$ is a *quantity-constrained competitive equilibrium (QCCE)* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:
(i) For every $b \in B$, $A^b \in \delta^b(p, q; r)$.
(ii) For every $s \in S$, $A^s \in \delta^s(p)$.
(iii) For every $\omega \in \Omega$, if $q_\omega = 0$, then $r^{b(\omega)} = +\infty$ and $p_\omega \geq M^{b(\omega)}$.

The first two conditions of Definition 4.1 correspond to optimization by buyers that take p and q as given and optimization by sellers that take p as given. These two conditions also imply equality of demand and supply at a QCCE. The third condition reflects that when buyer $b(\omega)$ expects trade ω not to be supplied, then buyer $b(\omega)$ faces a hard liquidity constraint, i.e., $r^{b(\omega)} = +\infty$, and the price is larger than or equal to the buyer's monetary endowment, i.e., $p_\omega \geq M^{b(\omega)}$. Since this condition only applies in case of a hard liquidity constraint, a QCCE coincides with a competitive equilibrium for models with soft liquidity constraints. Both p and q are endogenously determined in a QCCE. A numerical illustration of the concept of a QCCE is given in Appendix A.4.

QCCE is an equilibrium concept that is rooted in general equilibrium theory. Buyers take prices and quantity constraints as given and optimize accordingly. They do not need to form expectations about the behavior of other buyers and sellers. Since sellers do not face liquidity constraints, their decision problem is the usual one, coinciding with the one used in the definition of a competitive equilibrium. Prices and quantity constraints are in equilibrium if optimization by buyers and sellers leads to equality of supply and demand, i.e., markets clear.

In Definition 4.1, if for every $\omega \in \Omega$, $q_\omega = 1$, then a QCCE reduces to a competitive equilibrium, also in the presence of hard liquidity constraints.

Herings (2020) proposes the notion of expectational equilibrium. An expectational equilibrium does not explicitly depend on prices but rather on expectations related to tradable contracts. The concept unifies all the existing approaches of competitive equilibrium that have been proposed in the literature so far and, in particular, can be applied to both settings with and settings without monetary transfers.

Let $\overline{Y} = \Omega \times \mathbb{R}$ be the set of all possible contracts and $2^{\overline{Y}}$ be the power set of $\overline{Y}$. Let $Q \subseteq \overline{Y}$ denote a set of rationing constraints on the buyers side. For every $b \in B$, Q^b represents the set of contracts for which buyer b expects no supply from the sellers side. Similarly, Let $R \subseteq \overline{Y}$ denote a set of rationing constraints on the sellers side. For every $s \in S$, R^s is the set of contracts for which seller s expects no demand from the buyers side. An expectational equilibrium corresponds to an endogenously determined profile of rationing constraints (Q, R) together with an outcome A.

Given a set of rationing constraints Q, the *rationed budget set* of buyer $b \in B$ is given by

$$\gamma^b(Q) = \{A^b \in X^b \mid A^b \cap Q^b = \emptyset\}$$

and the *rationed demand set* of buyer $b \in B$ is given by

$$\delta^b(Q; r) = \underset{A^b \in \gamma^b(Q)}{\arg\max}\, U^b(A^b; r).$$

The rationed budget set of buyer b consists of all contracts for which the buyer does not expect rationing together with the no-trade option. The rationed demand set of buyer b collects all choices that maximize buyer b's utility over the rationed budget set.

Given a set of rationing constraints R, the *rationed budget set* of seller $s \in S$ is given by

$$\gamma^s(R) = \{A^s \in X^s \mid A^s \cap R^s = \emptyset\}$$

and the *rationed demand set* of seller $s \in S$ is given by

$$\delta^s(R) = \underset{A^s \in \gamma^s(R)}{\arg\max}\, U^s(A^s).$$

The definition of an expectational equilibrium is as follows.

Definition 4.2: An element $(A, Q, R) \in \mathcal{A} \times 2^{\overline{Y}} \times 2^{\overline{Y}}$ is an *expectational equilibrium* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:
(i) For every $b \in B$, $A^b \in \delta^b(Q; r)$.
(ii) For every $s \in S$, $A^s \in \delta^s(R)$.
(iii) $Q \cap R = \emptyset$.

The first two conditions correspond to optimization by buyers and sellers given the profile of rationing constraints. A buyer $b \in B$ demands the best contract outside Q^b and a seller $s \in S$ chooses the best contract outside R^s. The third condition expresses that markets are transparent. For a given contract, it cannot be the case that both sides expect to be rationed at the same time. Moreover, the third condition also reflects that the expectations of buyers and sellers are rational. As an example, consider a contract c that involves a buyer b and a seller s. If buyer b expects no supply of such a contract, i.e., $c \in Q^b$, then $c \notin R^s$, so seller s is able to supply c, but chooses not to do so, since c does not belong to outcome A. Therefore, buyer b holds rational expectations regarding the absence of supply of c. A similar argument shows that expectations of sellers are rational. A numerical illustration of the concept of an expectational equilibrium is given in Appendix A.5.

Next, we consider solution concepts that come from cooperative game theory, "stable outcome" and "core outcome." They are widely used in matching theory, see, e.g., Crawford and Knoer (1981), Demange and Gale (1985), Hatfield and Milgrom (2005), Hatfield et al. (2013), and Fleiner et al. (2019).

For every $Y \subseteq \Omega \times \mathbb{R}$, the sets of optimal choices of buyer $b \in B$ and seller $s \in S$ within the set of contracts Y are defined as

$$\begin{aligned} C^b(Y) &= \underset{\{A^b \in X^b | A^b \subseteq Y^b\}}{\arg\max} \; U^b(A^b; r), \\ C^s(Y) &= \underset{\{A^s \in X^s | A^s \subseteq Y^s\}}{\arg\max} \; U^s(A^s). \end{aligned}$$

Notice that for agent $i \in B \cup S$, $C^i(Y) = \{\emptyset\}$ means that the no-trade option is strictly preferred to any choice in Y.

The definition of stable outcome is as follows.

Definition 4.3: An outcome $A \in \mathcal{A}$ is *stable* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if:
(i) For every $i \in B \cup S$, $A^i \in C^i(A)$.
(ii) There is no $c = (\omega, t) \in \overline{Y}$ such that $U^{b(\omega)}(\{c\}; r) > U^{b(\omega)}(A; r)$ and $U^{s(\omega)}(\{c\}) > U^{s(\omega)}(A)$.

A stable outcome is a set of contracts that contains at most one contract for each agent. The first condition says that every agent involved in a contract prefers this contract weakly to the no-trade option. The second condition says that there are no two agents who can sign a contract that makes both of them strictly better off.

Next, we define a core outcome.

Definition 4.4: An outcome $A \in \mathcal{A}$ is *a core outcome* for the economy $\mathcal{E} = (B, S, \Omega, u, M, r)$ if there is no outcome $A' \in \mathcal{A}$ and a set of agents $N \subseteq B \cup S$ such that:
(i) For every $i \in (B \cup S) \backslash N$, $(A')^i = \emptyset$.
(ii) For every $b \in N \cap B$, $U^b(A'; r) > U^b(A; r)$.
(iii) For every $s \in N \cap S$, $U^s(A') > U^s(A)$.

An outcome is a core outcome if no coalition of agents can propose a better outcome satisfying the following two conditions: Agents outside the coalition obtain the no-trade option, as shown in the first condition, and each agent in the coalition is strictly better off, as shown in the second and third condition.

In what comes next, we show the equivalence between the four concepts defined in this section and show their existence for both types of liquidity constraints.

Theorem 4.5: Let $\mathcal{E} = (B, S, \Omega, u, M, r)$ be an economy such that $r \in [0, +\infty]^B$. The sets of QCCE outcomes, expectational equilibrium outcomes, stable outcomes, and core outcomes coincide and are non-empty.

Proof: We first prove the equivalence part and then the existence part.

Equivalence: We first show that the sets of QCCE outcomes and expectational equilibrium outcomes coincide. The proof consists of the following two steps.

Step 1: If (p, q, A) is a QCCE of $\mathcal{E}$, there is $(Q, R) \in 2^{\overline{Y}} \times 2^{\overline{Y}}$ such that (A, Q, R) is an expectational equilibrium.

For every $b \in B$, we define

$$Q^b = \{(\omega, t) \in \Omega^b \times \mathbb{R} \mid \omega \in \omega(A) \ \&\ t < p_\omega\} \cup \{(\omega, t) \in \Omega^b \times \mathbb{R} \mid \omega \in \Omega \backslash \omega(A) \ \&\ t \leq p_\omega\}.$$

For every $s \in S$, we define

$$R^s = \{(\omega, t) \in \Omega^s \times \mathbb{R} \mid t > p_\omega\}.$$

By construction, we have that $Q \cap R = \emptyset$ and so Condition (iii) of Definition 4.2 holds for (A, Q, R). We show below that Conditions (i) and (ii) of Definition 4.2 hold for (A, Q, R).

Let $b \in B$. By construction of Q^b, we have

$$\gamma^b(Q) = \{\{(\omega, t)\} \in X^b \mid \omega \in \omega(A) \ \&\ t \geq p_\omega\} \cup \{\{(\omega, t)\} \in X^b \mid \omega \in \Omega \backslash \omega(A) \ \&\ t > p_\omega\} \cup \{\emptyset\}.$$

By Condition (i) of Definition 4.1, we have that $U^b(A^b;r) \geq U^b(\emptyset)$. Let $\{c\} \in \gamma^b(Q)$ be such that $\omega(c) \in \Omega\backslash\omega(A)$. In case $q_{\omega(c)} = 1$, by Condition (i) of Definition 4.1, it holds that $U^b(A^b;r) \geq U^b(\{(\omega, p_\omega)\};r) \geq U^b(\{c\};r)$. In case $q_{\omega(c)} = 0$, by Condition (iii) of Definition 4.1, it holds that $r^b = +\infty$ and $t(c) > p_{\omega(c)} \geq M^b$. Thus $U^b(A^b;r) > U^b(\{c\};r)$. If $A^b = \emptyset$, we are done. If $A^b = \{(\omega', p_{\omega'})\}$, it holds that $U^b(A;r) \geq U^b(\{(\omega', t')\};r)$ for every $t' \geq p_{\omega'}$. Consequently, Condition (i) of Definition 4.2 holds.

Let $s \in S$. By the construction of R^s, $\gamma^s(R) = \{\{(\omega,t)\} \in X^s \mid t \leq p_\omega\} \cup \{\emptyset\}$. Since (p,q,A) is a QCCE, by Condition (ii) of Definition 4.1, we have that $U^s(A) \geq U^s(\emptyset)$, and for every $\{(\omega,t)\} \in \gamma^s(R)$, it holds that $U^s(A) \geq u^s(\omega, p_\omega) \geq u^s(\omega,t)$. Thus Condition (ii) of Definition 4.2 holds.

Step 2: If (A,Q,R) is an expectational equilibrium of $\mathcal{E}$, there is $(p,q) \in \mathbb{R}^\Omega \times \{0,1\}^\Omega$ such that (p,q,A) is a QCCE of $\mathcal{E}$.

For every $s \in S$, for every $\omega \in \Omega^s$, there is $p_\omega \in \mathbb{R}$ such that $u^s(A) = u^s(\omega, p_\omega)$. Let $p_\omega \in \mathbb{R}$ be the price of trade ω. For every $\omega \in \Omega$, if $r^{b(\omega)} = +\infty$ and $p_\omega \geq M^{b(\omega)}$, $q_\omega = 0$, and if $r^{b(\omega)} < +\infty$ or $p_\omega < M^{b(\omega)}$, $q_\omega = 1$. By the construction of $(p,q) \in \mathbb{R}^\Omega \times \{0,1\}^\Omega$, it is easily seen that Conditions (ii) and (iii) of Definition 4.1 hold for (p,q,A). We show next that Condition (i) of Definition 4.1 holds for (p,q,A).

By contradiction, suppose Condition (i) of Definition 4.1 fails to hold. Then there is $b \in B$ such that $A^b \notin \delta^b(p,q;r)$. Since (A,Q,R) is an expectational equilibrium, by Condition (i) of Definition 4.2, $U^b(A^b;r) \geq U^b(\emptyset)$ and so there is $\{(\omega, p_\omega)\} \in \delta^b(p,q;r)$ such that $U^b(A^b;r) < U^b(\{(\omega,p_\omega)\};r)$ and $q_\omega = 1$. By the construction of q, either $r^{b(\omega)} < +\infty$ or $p_\omega < M^{b(\omega)}$ holds. In either case, there is $t > p_\omega$ such that $U^b(A^b;r) < U^b(\{(\omega,t)\};r)$ and $U^{s(\omega)}(A) = U^{s(\omega)}(\{(\omega,p_\omega)\}) < U^{s(\omega)}(\{(\omega,t)\})$. Since (A,Q,R) is an expectational equilibrium, by Conditions (i) and (ii) of Definition 4.2, we have that $\{(\omega,t)\} \in Q^b$ and $\{(\omega,t)\} \in R^{s(\omega)}$, contradicting Condition (iii) of Definition 4.2.

By Steps 1 and 2, we have the equivalence between QCCE outcomes and expectational equilibrium outcomes.

We then show that there is an equivalence between QCCE outcomes, stable outcomes, and core outcomes. Herings and Zhou (2022) show the equivalence between these notions in an economy with hard liquidity constraints, i.e., $\mathcal{E} = (B,S,\Omega,u,M,r)$ such that for each $b \in B$, $r^b = 0$ or $+\infty$. We claim that such an equivalence result, with the same transformation as used in the proof of Proposition 3.3, carries over to economies with $r \in [0,+\infty]^B$. More precisely, we only transform utility functions of buyers who face soft liquidity constraints. An economy with both types of liquidity constraints is transformed into an economy with hard liquidity constraints in such a way that for each buyer $b \in B$ with soft liquidity constraints in the original economy, $\widetilde{u}^b$ is constructed in the same way as in the proof of Proposition 3.3. It is easy to see that the set of QCCEs, stable outcomes and core outcomes in the original economy coincides with the set of QCCEs, stable outcomes and core outcomes in the transformed economy, respectively. Thus, the claim holds.

Existence: Following the same reasoning before, the existence of QCCEs in the economies as studied by Herings and Zhou (2022) implies the existence of QCCEs in the economy of Theorem 4.5.[9] By the equivalence result, we have the existence for all notions defined in this section. **Q.E.D.**

Theorem 4.5 presents a novel equivalence result between QCCE outcomes and expectational equilibrium outcomes. Moreover, it also provides additional insights of how we can use Proposition 3.3 to carry over results for economies with only hard liquidity constraints to economies with both types of liquidity constraints.

In an economy with soft liquidity constraints, using Theorem 4.5, it is easy to verify that the sets of "strongly stable outcomes,"[10] "strict core outcomes,"[11] stable outcomes, core outcomes, and competitive equilibria coincide.[12] By Proposition 3.3, they are non-empty. Thus, we have equivalence and existence results between all equilibrium and stability notions in an economy with soft liquidity constraints. However, in an economy with hard liquidity constraints, a stable outcome may not be strongly stable, a core outcome may not be strict, and the sets of strongly stable outcomes and strict core outcomes can be empty (Herings and Zhou, 2022). Thus, the notions of equilibrium and stability as stated in Theorem 4.5 are more appealing when we study economies with different types of liquidity constraints.

[9] An alternative way to show existence is as follows. Both the model with soft and the model with hard liquidity constraints satisfy the assumptions of Theorem 5.5 of Herings (2020), so the existence of expectational equilibrium follows from that result. By the equivalence result, we obtain the existence result of the other three concepts.

[10] An outcome is strongly stable if (i) no agent involved in a contract prefers the no-trade option and (ii) there are no two agents who can sign a contract making both agents weakly better off and at least one agent strictly better off.

[11] An outcome is a strict core outcome if there is no outcome and a coalition of agents such that (i) agents outside the coalition obtain $\emptyset$, and (ii) each agent in the coalition is weakly better off, with at least one agent being strictly better off.

[12] In an economy with soft liquidity constraints, we only need to show that (1) a QCCE is a competitive equilibrium with quantity constraints $(1, \ldots, 1)$, (2) a stable outcome is strongly stable, and (3) a core outcome is a strict core outcome. All these three statements can be easily verified. Then by Theorem 4.5, the conclusion follows.

5 Convergence Results

In this section, we show a novel structural property, continuity, for the notions studied in the previous section. More precisely, we consider a sequence of economies with increasingly stringent soft liquidity constraints and a corresponding sequence of equilibria, and study whether the limit of the sequence of equilibria is an equilibrium for the limit economy with hard liquidity constraints.

It follows from Theorem 4.5 that if we show the limit result in terms of outcomes for one of the notions of the previous section, then we obtain the corresponding limit result for any of the other notions for free. Rather than focusing on outcomes only, we start with a limit result for QCCEs in terms of both prices and outcomes in Theorem 5.1. It will bring us additional economic insights as discussed after Theorem 5.1 and in Corollary 5.3 and Example 5.4.

The sequence of economies $(\mathcal{E}^n)_{n\in\mathbb{N}} = (B, S, \Omega, u, M, r^n)_{n\in\mathbb{N}}$ is said to converge to an economy $\overline{\mathcal{E}} = (B, S, \Omega, u, M, \overline{r})$ if $\overline{r} = \lim_{n\to\infty} r^n$. A sequence of outcomes $(A^n)_{n\in\mathbb{N}}$ in $\mathcal{A}$ is said to converge to an outcome $A \in \mathcal{A}$ if there is $n' \in \mathbb{N}$ such that, for every $n \geq n'$, $\omega(A^n) = \omega(A)$ and, for every $(\omega, t) \in A$, for every $c^n \in A^n$ with $\omega(c^n) = \omega$, $\lim_{n\to\infty} t(c^n) = t$. A sequence of competitive equilibria $(p^n, A^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^\Omega \times \mathcal{A}$ is said to converge to the competitive equilibrium $(p, A) \in \mathbb{R}^\Omega \times \mathcal{A}$ if $\lim_{n\to\infty} p^n = p$ and $(A^n)_{n\in\mathbb{N}}$ converges to A. A sequence of QCCEs $(p^n, q^n, A^n)_{n\in\mathbb{N}}$ in $\mathbb{R}^\Omega \times \{0,1\}^\Omega \times \mathcal{A}$ is said to converge to the QCCE $(p, q, A) \in \mathbb{R}^\Omega \times \{0,1\}^\Omega \times \mathcal{A}$ if $\lim_{n\to\infty} p^n = p$, and there is $n' \in \mathbb{N}$ such that, for every $n \geq n'$, $q^n = q$, and $(A^n)_{n\in\mathbb{N}}$ converges to A.

We next show a limit result for QCCEs.

Theorem 5.1: Let $(\mathcal{E}^n)_{n\in\mathbb{N}}$ be a sequence of economies that converges to an economy $\overline{\mathcal{E}}$. For every $n \in \mathbb{N}$, let (p^n, q^n, A^n) be a QCCE of the economy $\mathcal{E}^n$.
(i) The sequence $(p^n, A^n)_{n\in\mathbb{N}}$ has a convergent subsequence.
(ii) Let $(\overline{p}, \overline{A})$ be the limit of a convergent subsequence of $(p^n, A^n)_{n\in\mathbb{N}}$. Then there is $\overline{q} \in \{0,1\}^\Omega$ such that $(\overline{p}, \overline{q}, \overline{A})$ is a QCCE of $\overline{\mathcal{E}}$.

Proof:
Part (i): Since the set of trades is finite, there is a subsequence $(A^{n_m})_{m\in\mathbb{N}}$ of $(A^n)_{n\in\mathbb{N}}$ such that the set of trades $\omega(A^{n_m})$ does not depend on n_m. Define $\overline{\Omega} = \omega(A^{n_m})$.
Step 1: For every $\omega \in \overline{\Omega}$, the sequence $(p^{n_m}_\omega)_{m\in\mathbb{N}}$ is bounded.

Fix some $m \in \mathbb{N}$. Let $\omega \in \overline{\Omega}$, $b = b(\omega)$, and $s = s(\omega)$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $U^s(A^{n_m}) = u^s(\omega, p^{n_m}_\omega) \geq u^s(o^s) = u^s(\omega, \underline{t}^s_\omega)$. It follows that $\underline{t}^s_\omega \leq p^{n_m}_\omega$.

There is $t^{n_m}_\omega \geq p^{n_m}_\omega$ such that $u^b(\omega, t^{n_m}_\omega) = U^b(\{(\omega, p^{n_m}_\omega)\}; r^{n_m})$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $u^b(\omega, t^{n_m}_\omega) = U^b(\{(\omega, p^{n_m}_\omega)\}; r^{n_m}) \geq u^b(o^b) = u^b(\omega, \overline{t}^b_\omega)$. It follows that $p^{n_m}_\omega \leq t^{n_m}_\omega \leq \overline{t}^b_\omega$.

We conclude that, for every $m \in \mathbb{N}$, $\underline{t}^s_\omega \leq p^{n_m}_\omega \leq \overline{t}^b_\omega$.

Step 2: For every $\omega \in \Omega \backslash \overline{\Omega}$, the sequence $(p^{n_m}_\omega)_{m \in \mathbb{N}}$ is bounded.

Fix some $m \in \mathbb{N}$. Let $\omega \in \Omega \backslash \overline{\Omega}$, $b = b(\omega)$, and $s = s(\omega)$.

Assume $(A^{n_m})^b = \emptyset$. Consider the case where $p^{n_m}_\omega < M^b$, so $q^{n_m}_\omega = 1$. Since $\min\{0, \overline{t}^b_\omega\} \leq M^b$ and $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$\begin{aligned} U^b(\{\omega, \min\{0, \overline{t}^b_\omega\}\}; r^{n_m}) &= u^b(\omega, \min\{0, \overline{t}^b_\omega\}) \geq u^b(\omega, \overline{t}^b_\omega) = u^b(o^b) = U^b(A^{n_m}; r^{n_m}) \\ &\geq U^b(\{\omega, p^{n_m}_\omega\}; r^{n_m}), \end{aligned}$$

and so $p^{n_m}_\omega \geq \min\{0, \overline{t}^b_\omega\}$. In case $p^{n_m}_\omega \geq M^b \geq 0$, it also holds that $p^{n_m}_\omega \geq \min\{0, \overline{t}^b_\omega\}$.

Assume $(A^{n_m})^b \neq \emptyset$. Consider the case where $p^{n_m}_\omega < M^b$, so $q^{n_m}_\omega = 1$. Let $(A^{n_m})^b = \{(\omega', p^{n_m}_{\omega'})\}$ and $s(\omega') = s'$. Since $\omega' \in \overline{\Omega}$, by Step 1, it holds that $\underline{t}^{s'}_{\omega'} \leq p^{n_m}_{\omega'} \leq \overline{t}^b_{\omega'}$. Let $t^b_\omega \in \mathbb{R}$ be such that $u^b(\omega', \underline{t}^{s'}_{\omega'}) = u^b(\omega, t^b_\omega)$. Since $\underline{t}^{s'}_{\omega'} \leq p^{n_m}_{\omega'}$, $\min\{0, t^b_\omega\} \leq M^b$, and $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$\begin{aligned} U^b(\{\omega, \min\{0, t^b_\omega\}\}; r^{n_m}) &= u^b(\omega, \min\{0, t^b_\omega\}) \geq u^b(\omega', \underline{t}^{s'}_{\omega'}) \geq u^b(\omega', p^{n_m}_{\omega'}) \\ &\geq U^b(A^{n_m}; r^{n_m}) \geq U^b(\{\omega, p^{n_m}_\omega\}; r^{n_m}), \end{aligned}$$

and so $p^{n_m}_\omega \geq \min\{0, t^b_\omega\}$. In case $p^{n_m}_\omega \geq M^b \geq 0$, it also holds that $p^{n_m}_\omega \geq \min\{0, t^b_\omega\}$.

Assume $(A^{n_m})^s = \emptyset$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that $U^s(A^{n_m}) = u^s(o^s) = u^s(\omega, \underline{t}^s_\omega) \geq u^s(\omega, p^{n_m}_\omega)$. It follows that $p^{n_m}_\omega \leq \underline{t}^s_\omega$.

Assume $(A^{n_m})^s \neq \emptyset$. Let $(A^{n_m})^s = \{(\omega', p^{n_m}_{\omega'})\}$ and $b(\omega') = b'$. Since $\omega' \in \overline{\Omega}$, by Step 1, it holds that $\underline{t}^s_{\omega'} \leq p^{n_m}_{\omega'} \leq \overline{t}^{b'}_{\omega'}$. By (s-ii), there is $t^s_\omega \in \mathbb{R}$ such that $u^s(\omega', \overline{t}^{b'}_{\omega'}) = u^s(\omega, t^s_\omega)$. Since $(p^{n_m}, q^{n_m}, A^{n_m})$ is a QCCE of $\mathcal{E}^{n_m}$, it holds that

$$U^s(\{\omega, t^s_\omega\}) = U^s(\{\omega', \overline{t}^{b'}_{\omega'}\}) \geq U^s(\{\omega', p^{n_m}_{\omega'}\}) = U^s(A^{n_m}) \geq U^s(\{\omega, p^{n_m}_\omega\}),$$

and so $p^{n_m}_\omega \leq t^s_\omega$. Consequently, Step 2 holds.

By Steps 1 and 2, it follows that the sequence $(p^n, A^n)_{n \in \mathbb{N}}$ has a convergent subsequence.

Part (ii): Without loss of generality, assume the sequence $(p^n, A^n)_{n \in \mathbb{N}}$ is convergent and has limit $(\overline{p}, \overline{A})$. Define $\overline{\Omega} = \omega(\overline{A})$.

Let $\overline{q} \in \{0,1\}^\Omega$ be such that (i) for every $\omega \in \overline{\Omega}$, $\overline{q}_\omega = 1$, and (ii), for every $\omega \in \Omega \backslash \overline{\Omega}$, if $r^{b(\omega)} = +\infty$ and $\overline{p}_\omega \geq M^{b(\omega)}$, $\overline{q}_\omega = 0$, and if $r^{b(\omega)} < +\infty$ or $\overline{p}_\omega < M^{b(\omega)}$, $\overline{q}_\omega = 1$. Since for every $n \in \mathbb{N}$, A^n is a QCCE outcome and the sequence $(A^n)_{n \in \mathbb{N}}$ converges to $\overline{A}$, it is easily seen that $\overline{A}$ is an outcome. We show next that $(\overline{p}, \overline{q}, \overline{A})$ is a QCCE of the limit economy $\overline{\mathcal{E}}$.

Step 1: Condition (i) of Definition 4.1 holds.

By contradiction, suppose that there is a buyer $b \in B$ such that $\overline{A}^b \notin \delta^b(\overline{p}, \overline{q}; \overline{r})$.

Case 1: $\overline{A}^b = \emptyset$.

Since $\overline{A}^b = \emptyset \notin \delta^b(\overline{p}, \overline{q}; \overline{r})$, there is $\omega' \in \Omega^b$ such that $\{(\omega', \overline{p}_{\omega'})\} \in \delta^b(\overline{p}, \overline{q}; \overline{r})$, $U^b(\emptyset; \overline{r}) < U^b(\{(\omega', \overline{p}_{\omega'})\}; \overline{r})$, and $\overline{q}_{\omega'} = 1$. If $\overline{r}^b = +\infty$, it holds by the construction of $\overline{q}$ that

$\overline{p}_{\omega'} < M^b$. For sufficiently large n, we have that $p^n_{\omega'} < M^b$, $q^n_{\omega'} = 1$, and $U^b(\emptyset; r^n) = u^b(o^b) < u^b(\omega', p^n_{\omega'}) = U^b(\{(\omega', p^n_{\omega'})\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\overline{r}^b < +\infty$, we have for sufficiently large n that $U^b(\emptyset; r^n) = u^b(o^b) < U^b(\{(\omega', p^n_{\omega'})\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Case 2: $\overline{A}^b \neq \emptyset$.

Let ω be the unique element of $\omega(\overline{A}^b)$. First, we show that if $\overline{r}^b = +\infty$, then $\overline{p}_\omega \leq M^b$. By contradiction, suppose that $\overline{p}_\omega > M^b$. There is $\varepsilon > 0$ such that $\overline{p}_\omega > M^b + \varepsilon$. For sufficiently large n, we have that $p^n_\omega \geq M^b + \varepsilon$ and $r^n \varepsilon > \overline{t}^b_\omega$. This implies that

$$U^b(A^n; r^n) = u^b(\omega, r^n(p^n_\omega - M^b) + p^n_\omega) \leq u^b(\omega, r^n\varepsilon) < u^b(\omega, \overline{t}^b_\omega) = u^b(o^b) = U^b(\emptyset; r^n),$$

contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. Consequently, we have that $\overline{p}_\omega \leq M^b$.

Since $\overline{A}^b \notin \delta^b(\overline{p}, \overline{q}; \overline{r})$, there is $\tilde{A}^b \in \delta^b(\overline{p}, \overline{q}; \overline{r})$ such that $U^b(\overline{A}^b; \overline{r}) < U^b(\widetilde{A}^b; \overline{r})$.

Case 2-1: $\overline{r}^b = +\infty$.

By the analysis above, it holds that $\overline{p}_\omega \leq M^b$. For every $n \in \mathbb{N}$, we have that

$$U^b(\{(\omega, p^n_\omega)\}; r^n) = u^b(\omega, p^n_\omega + r^{n,b} \max\{0, p^n_\omega - M^b\}) \leq u^b(\omega, p^n_\omega).$$

Assume $\widetilde{A}^b = \emptyset$. Since $\lim_{n\to\infty} u^b(\omega, p^n_\omega) = u^b(\omega, \overline{p}_\omega) = U^b(\overline{A}^b; \overline{r})$, for sufficiently large n, we have that $U^b(\{(\omega, p^n_\omega)\}; r^n) < U^b(\emptyset; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Assume $\widetilde{A}^b \neq \emptyset$. Let $\widetilde{\omega}$ be the unique element of $\omega(\widetilde{A}^b)$. It holds that $\overline{q}_{\widetilde{\omega}} = 1$. Since $\overline{r}^b = +\infty$, by the construction of $\overline{q}$, it holds that $\overline{p}_{\widetilde{\omega}} < M^b$. Since $\lim_{n\to\infty} u^b(\omega, p^n_\omega) = u^b(\omega, \overline{p}_\omega) = U^b(\overline{A}^b; \overline{r})$, it follows that $U^b(\{(\omega, p^n_\omega)\}; r^n) < U^b(\{(\widetilde{\omega}, p^n_{\widetilde{\omega}})\}; r^n)$ for sufficiently large n, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Case 2-2: $\overline{r}^b < +\infty$.

If $\widetilde{A}^b = \emptyset$, we have for n sufficiently large that $U^b(\{(\omega, p^n_\omega)\}; r^n) < U^b(\emptyset; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\widetilde{A}^b \neq \emptyset$, let $\widetilde{\omega}$ be the unique element of $\omega(\widetilde{A}^b)$. Since $\overline{r}^b < +\infty$, it holds that $\overline{q}_{\widetilde{\omega}} = 1$. For n sufficiently large, we have that $U^b(\{(\omega, p^n_\omega)\}; r^n) < U^b(\{(\widetilde{\omega}, p^n_{\widetilde{\omega}})\}; r^n)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Thus the statement of Step 1 holds.

Step 2: Condition (ii) of Definition 4.1 holds.

By contradiction, suppose that there is $s \in S$ such that $\overline{A}^s \notin \delta^s(\overline{p})$.

In case $\overline{A}^s = \emptyset$, there is $\omega' \in \Omega^s$ such that $U^s(\overline{A}^s) < U^s(\{(\omega', \overline{p}_{\omega'})\})$. For n sufficiently large, we have that $U^s(\emptyset) = U^s(A^n) < U^s(\{(\omega', p^n_{\omega'})\})$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

In case $\overline{A}^s \neq \emptyset$, let ω be the unique element of $\omega(\overline{A}^s)$. Then there is $\widetilde{A}^s \in \delta^s(\overline{p})$ such that $U^s(\overline{A}^s) < U^s(\widetilde{A}^s)$. If $\widetilde{A}^s = \emptyset$, we have for n sufficiently large that $U^s(\{\omega, p^n_\omega\}) = U^s(A^n) < U^s(\emptyset)$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$. If $\widetilde{A}^s \neq \emptyset$, let $\widetilde{\omega}$ be the unique element of $\omega(\widetilde{A}^s)$. For n sufficiently large, we have that $U^s(\{\omega, p^n_\omega\}) = U^s(A^n) < U^s(\{(\widetilde{\omega}, p^n_{\widetilde{\omega}})\})$, contradicting that (p^n, q^n, A^n) is a QCCE of $\mathcal{E}^n$.

Step 3: Condition (iii) of Definition 4.1 holds.

This follows immediately from the construction of $\overline{q}$.

Consequently, $(\overline{p}, \overline{q}, \overline{A})$ is a QCCE of $\overline{\mathcal{E}}$. **Q.E.D.**

Theorem 5.1 considers sequences $(p^n, A^n)_{n \in \mathbb{N}}$ of prices and outcomes and *leaves out* the quantity constraints q^n. Indeed, the limit of any convergent sequence of QCCEs of $\mathcal{E}^n$ may not be a QCCE of the limit economy. For instance, we can take a sequence of competitive equilibria of a sequence of economies with soft liquidity constraints, whereas the limit economy with hard liquidity constraints does not have a competitive equilibrium. In addition, QCCE trades in the limit economy may not be equal to the trades compatible with the limit of competitive equilibrium allocations in economies with increasingly stringent liquidity constraints. In the last paragraph in Appendix A.4, we illustrate these points via numerical examples.

Combining Theorem 4.5 and Theorem 5.1, we obtain the following limit result.

Proposition 5.2: Let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence of economies that converges to an economy $\overline{\mathcal{E}}$. For every $n \in \mathbb{N}$, let A^n be an expectational equilibrium outcome (respectively stable outcome or core outcome) of $\mathcal{E}^n$.
(i) The sequence $(A^n)_{n \in \mathbb{N}}$ has a convergent subsequence.
(ii) Let $\overline{A}$ be the limit of a convergent subsequence of $(A^n)_{n \in \mathbb{N}}$. Then $\overline{A}$ is an expectational equilibrium outcome (respectively stable outcome or core outcome) of $\overline{\mathcal{E}}$.

Since the sets of strongly stable outcomes and strict core outcomes can be empty in economies with hard liquidity constraints, we conclude that a convergent sequence of strongly stable outcomes or strict core outcomes in economies with increasingly stringent soft liquidity constraints may not be a strongly stable outcome or strict core outcome at the limit economy with hard liquidity constraints.

A particularly interesting special case of Theorem 5.1 connects competitive equilibria in economies with increasingly stringent soft liquidity constraints to a QCCE at the limit economy with hard liquidity constraints. The formal statement is as follows.[13]

Corollary 5.3: Let $(\mathcal{E}^n)_{n \in \mathbb{N}}$ be a sequence of economies with soft liquidity constraints that converges to an economy $\overline{\mathcal{E}}$ where at least one buyer is subject to a hard liquidity constraint. For every $n \in \mathbb{N}$, let $(p^n, A^n)_{n \in \mathbb{N}}$ be a competitive equilibrium of the economy $\mathcal{E}^n$. Then $(p^n, A^n)_{n \in \mathbb{N}}$ contains a convergent subsequence whose limit $(\overline{p}, \overline{A})$ is compatible with QCCE prices and outcome in the limit economy $\overline{\mathcal{E}}$.

[13] We illustrate this statement numerically in the last paragraph in Appendix A.4.

Finally, we remark that even if a competitive equilibrium exists in an economy with hard liquidity constraints and $(p^n, A^n)_{n\in\mathbb{N}}$ is a convergent sequence of competitive equilibria in this economy, then its limit $(\overline{p}, \overline{A})$ may not be a competitive equilibrium in the same economy. Example 5.4 illustrates this point.

Example 5.4: Take $B = \{b_1, b_2\}$, $S = \{s\}$, and $\Omega = \{\omega_1, \omega_2\}$, where buyer b_1 can trade ω_1 with s, and buyer b_2 can trade ω_2 with s. For seller s, for every $t \in \mathbb{R}$, $u^s(\omega_1, t) = u^s(\omega_2, t) = t$ and $u^s(o^s) = 0$. For buyer b_i, $u^{b_i}(o^{b_i}) = 0$ and, for every $(\omega_i, t) \in \Omega^{b_i} \times \mathbb{R}$, $u^{b_i}(\omega_i, t) = V^{b_i}(\omega_i) - t$.[14] Let $M^{b_1} = 2$, $M^{b_2} = 10$, $V^{b_1}(\omega_1) = 5$, $V^{b_2}(\omega_2) = 4$, $r^{b_1} = +\infty$, and $r^{b_2} = 0$.

For every $n \in \mathbb{N}$, let $(p^n, A^n) \in \mathbb{R}^\Omega \times \mathcal{A}$ be such that $\omega(A^n) = \{\omega_2\}$ and

$$p^n_{\omega_1} = p^n_{\omega_2} = 4 - \frac{2n}{1+n}.$$

It is easily seen that, for every $n \in \mathbb{N}$, (p^n, A^n) is a competitive equilibrium in this economy. At the limit when $n \to +\infty$, we have that $\overline{p} = (\overline{p}_{\omega_1}, \overline{p}_{\omega_2}) = (2, 2)$ and $\overline{A} = \{(\omega_2, \overline{p}_{\omega_2})\}$. However, $(\overline{p}, \overline{A})$ is not a competitive equilibrium in the limit economy as the no-trade option does not belong to the demand set of agent 1 when prices are equal to $\overline{p}$. △

In contrast, in an economy with soft liquidity constraints, the limit of a convergent sequence of competitive equilibria is a competitive equilibrium as well.

6 Discussion

In this section, we consider the case where liquidity constraints correspond to more general forms of borrowing costs and show the robustness of our results.

For every buyer $b \in B$, let $g^b : [0, +\infty) \to [0, +\infty]$ be a *general borrowing cost function* that takes one of the following three forms:

- (No borrowing constraints) For every $t \in [0, +\infty)$, $g^b(t) = 0$.
- (Hard borrowing constraints) (i) $g^b(0) = 0$ and (ii) for every $t \in (0, +\infty)$, $g^b(t) = +\infty$,
- (General soft borrowing constraints) (i) $g^b(0) = 0$, (ii) g^b is continuous and strictly increasing on $[0, +\infty)$, and (iii) the range of g^b is $[0, +\infty)$.

The explanation of the first two cases is straightforward. In the third case of general soft borrowing constraints, Condition (i) says that the cost of borrowing an amount of zero is equal to zero. Condition (ii) requires that the borrowing cost is strictly increasing in the

[14] The derivation of the induced utility functions over outcomes can be found in Appendix A.1.

amount of borrowed money. Condition (iii) states that borrowing costs increase without limit. Let $g = (g^b)_{b \in B}$ be the profile of general borrowing cost functions.

In the previous sections, the soft liquidity constraint is represented by a constant interest rate. It corresponds to the special case of a general borrowing cost function where, for every $t \geq 0$, $g^b(t) = r^b \cdot t$. On the other hand, the general borrowing cost function allows interest rates to vary with respect to the amount of borrowed money. Consider the general borrowing cost function defined by, for every $t \in [0, +\infty)$, $g^b(t) = r^b e^t \cdot t$ where $r^b > 0$. Here $r^b e^t$ is the interest rate that has to be paid when the borrowed amount equals t. It is strictly increasing with respect to the amount of borrowed money t.

For every $b \in B$, for every contract $c \in \Omega^b \times \mathbb{R}$, the transfer plus borrowing cost for contract c is now given by

$$t^+(c; g^b) = \begin{cases} t(c), & \text{if } t(c) \leq M^b, \\ t(c) + g^b(t(c) - M^b), & \text{if } t(c) > M^b. \end{cases}$$

The primitives of an economy with general borrowing constraints are summarized by $\mathcal{E}^* = (B, S, \Omega, u, M, g)$. In such an economy, the profile of interest rates r is replaced by the profile of general borrowing cost functions g.

Now we discuss how our results extend to an economy with general borrowing constraints. We argue that such an economy can be transformed to an economy where buyers face either no or hard liquidity constraints. Let some economy with general borrowing constraints $\mathcal{E}^* = (B, S, \Omega, u, M, g)$ be given. For every buyer $b \in B$ with no borrowing constraint, set $\widetilde{r}^b = 0$ and $\widetilde{u}^b = u^b$. For every buyer $b \in B$ facing a general soft borrowing constraint, set $\widetilde{r}^b = 0$ and construct $\widetilde{u}^b$ by following the construction in the proof of Proposition 3.3.[15] For every buyer $b \in B$ subject to a hard borrowing constraint, set $\widetilde{r}^b = +\infty$ and $\widetilde{u}^b = u^b$. Since there is no change on the sellers side, we get a transformed economy $\widetilde{\mathcal{E}} = (B, S, \Omega, ((\widetilde{u}^b)_{b \in B}, (u^s)_{s \in S}), M, \widetilde{r})$. Using the construction of $\widetilde{\mathcal{E}}$, it is easily verified that the sets of equilibria in $\mathcal{E}^*$ and $\widetilde{\mathcal{E}}$ coincide. Since all the results obtained in the previous sections hold for economies without liquidity constraints or with hard liquidity constraints, they can be extended to economies with general borrowing constraints.

7 Concluding remarks

We study a matching with contracts model where buyers may face soft or hard liquidity constraints or both. The economy with hard constraints can also be obtained as the limit of a sequence of economies with increasingly stringent soft constraints. Competitive

[15] To be precise, define $\widetilde{u}^b : (\Omega^b \times \mathbb{R}) \cup \{o^b\} \to \mathbb{R}$ by $\widetilde{u}^b(o^b) = u^b(o^b)$ and, for every $c \in \Omega^b \times \mathbb{R}$, $\widetilde{u}^b(c) = u^b(\omega(c), t^+(c; g^b))$. It follows from the properties of g^b that $\widetilde{u}^b$ satisfies (b-i) and (b-ii).

equilibrium always exists with soft liquidity constraints, but may fail to exist with hard liquidity constraints. In particular, the limit of a sequence of competitive equilibria for economies with increasingly stringent soft liquidity constraints may fail to be a competitive equilibrium of the limit economy with hard liquidity constraints.

We argue that two alternative notions of competitive equilibrium, quantity constrained competitive equilibrium (QCCE) and expectational equilibrium, as well as solution concepts like stable outcomes and core outcomes, do not suffer from such deficiencies. We show the equivalence, existence, and limit results of all these notions. Our results provide new insights for the continuity properties of equilibria in matching models with different types of liquidity constraints.

Appendix

In this appendix, we provide several numerical examples that illustrate ideas and equilibrium notions in the main text.

A.1 Integration of matching models with soft and hard liquidity constraints

There is a finite set of buyers B and a finite set of sellers S. Each seller owns one item and is only willing to sell if the price is above a reserve price. Buyers want to acquire one item at most. We take the set of trades equal to $\Omega = B \times S$. For every seller $s \in S$, for every $\omega, \omega' \in \Omega^s$, for every $t \in \mathbb{R}$, $u^s(\omega, t) = u^s(\omega', t) = t$ and $u^s(o^s) \geq 0$ denotes the reserve price for the item in the possession of seller s.

For every buyer $b \in B$, $u^b(o^b) = 0$ and, for every $(\omega, t) \in \Omega^b \times \mathbb{R}$, $u^b(\omega, t) = V^b(\omega) - t$. Buyer $b \in B$ is endowed with an amount of money $M^b \in \mathbb{R}_+$. If the payment exceeds M^b, then buyer b can borrow money against a market interest rate $\rho \in [0, +\infty]$, which is the same for all buyers.

For every $A \in \mathcal{A}$, for every $b \in B$, in case $A^b = \emptyset$, $U^b(A; r) = u^b(o^b) = 0$ and in case $A^b = \{(\omega, t)\}$, if buyer b faces a soft liquidity constraint, i.e., $\rho \in [0, +\infty)$,

$$U^b(A; r) = \begin{cases} V^b(\omega) - t, & \text{if } t \leq M^b, \\ V^b(\omega) - t - \rho(t - M^b), & \text{if } t > M^b, \end{cases}$$

and if buyer b faces a hard liquidity constraint, i.e., $\rho = +\infty$,

$$U^b(A; r) = \begin{cases} V^b(\omega) - t, & \text{if } t \leq M^b, \\ -\infty, & \text{if } t > M^b. \end{cases}$$

A.2 Competitive equilibria in the case of soft liquidity constraints

Consider the economy in Appendix A.1 in the presence of soft liquidity constraints with common interest rate $\rho \in [0, +\infty)$.

Take $B = \{b_1, b_2, b_3\}$, $S = \{s_1, s_2\}$, and $\Omega = \{\omega_{11}, \omega_{22}, \omega_{31}, \omega_{32}\}$, where buyer b_1 trades ω_{11} with s_1, buyer b_2 trades ω_{22} with s_2, and buyer b_3 trades ω_{31} with seller s_1 and ω_{32} with seller s_2. For every agent $i \in B \cup S$, $u^i(o^i) = 0$. Otherwise, utility functions are as described in Appendix A.1. Buyers' monetary endowments are as follows: $M^{b_1} = 2$, $M^{b_2} = 3$, and $M^{b_3} = 10$. Let $V^{b_1}(\omega_{11}) = 5$, $V^{b_2}(\omega_{22}) = 4$, $V^{b_3}(\omega_{31}) = 6$, and $V^{b_3}(\omega_{32}) = 7$.

We define $(p(\rho), A) \in \mathbb{R}^{\Omega} \times \mathcal{A}$ by

$$\begin{aligned} p_{\omega_{11}}(\rho) &= p_{\omega_{31}}(\rho) = \tfrac{3+2\rho}{1+\rho}, \\ p_{\omega_{22}}(\rho) &= p_{\omega_{32}}(\rho) = \tfrac{4+3\rho}{1+\rho}, \\ A &= \{(\omega_{11}, p_{\omega_{11}}(\rho)), (\omega_{32}, p_{\omega_{32}}(\rho))\}. \end{aligned}$$

We show that $(p(\rho), A)$ satisfies Conditions (i) and (ii) of Definition 3.1 and conclude that it is a competitive equilibrium.

First, $(p(\rho), A)$ satisfies Condition (i). For buyer b_1 it holds that $p_{\omega_{11}}(\rho) > 2 = M^{b_1}$. Since

$$t^+((\omega_{11}, p_{\omega_{11}}(\rho)); \rho) = p_{\omega_{11}}(\rho) + \rho(p_{\omega_{11}}(\rho) - M^{b_1}) = \tfrac{3+2\rho}{1+\rho} + \rho(\tfrac{3+2\rho}{1+\rho} - 2) = 3 < 5 = V^{b_1}(\omega_{11}),$$

buyer b_1 strictly prefers buying ω_{11} at price $p_{\omega_{11}}(\rho)$ over the no-trade option. Thus $A^{b_1} = \{(\omega_{11}, p_{\omega_{11}}(\rho))\} \in \delta^{b_1}(p(\rho); \rho) = \{\{(\omega_{11}, p_{\omega_{11}}(\rho))\}\}$.

For buyer b_2 it holds that $p_{\omega_{22}} > 3 = M^{b_2}$. Since

$$t^+((\omega_{32}, p_{\omega_{32}}(\rho)); \rho) = p_{\omega_{22}}(\rho) + \rho(p_{\omega_{22}}(\rho) - M^{b_2}) = \tfrac{4+3\rho}{1+\rho} + \rho(\tfrac{4+3\rho}{1+\rho} - 3) = 4 = V^{b_2}(\omega_{22}),$$

buyer b_2 is indifferent between acquiring ω_{22} against price $p_{\omega_{22}}(\rho)$ and not trading. Thus $A^{b_2} = \emptyset \in \delta^{b_2}(p(\rho); \rho) = \{\emptyset, \{(\omega_{22}, p_{\omega_{22}}(\rho))\}\}$.

For buyer b_3 it holds that $p_{\omega_{31}}(\rho), p_{\omega_{32}}(\rho) < 10 = M^{b_3}$. Since $6 - p_{\omega_{31}}(\rho) = 7 - p_{\omega_{32}}(\rho) > 0$, at prices $p_{\omega_{31}}(\rho)$ and $p_{\omega_{32}}(\rho)$, buyer b_3 is indifferent between ω_{31} and ω_{32} and strictly prefers both trades to the no-trade option. We find that $A^{b_3} = \{(\omega_{32}, p_{\omega_{32}}(\rho))\} \in \delta^{b_3}(p(\rho); \rho) = \{\{(\omega_{31}, p_{\omega_{31}}(\rho))\}, \{(\omega_{32}, p_{\omega_{32}}(\rho))\}\}$.

Second, $(p(\rho), A)$ satisfies Condition (ii). Since $p_{\omega_{11}}(\rho) = p_{\omega_{31}}(\rho) > 0$ and $p_{\omega_{22}}(\rho) = p_{\omega_{32}}(\rho) > 0$, seller s_1 is willing to supply either ω_{11} or ω_{31} and seller s_2 is willing to supply either ω_{22} or ω_{32}. We conclude that

$$\begin{aligned} A^{s_1} &= \{(\omega_{11}, p_{\omega_{11}}(\rho))\} \in \delta^{s_1}(p(\rho)) = \{\{(\omega_{11}, p_{\omega_{11}}(\rho))\}, \{(\omega_{31}, p_{\omega_{31}}(\rho))\}\}, \\ A^{s_2} &= \{(\omega_{32}, p_{\omega_{32}}(\rho))\} \in \delta^{s_2}(p(\rho)) = \{\{(\omega_{22}, p_{\omega_{22}}(\rho))\}, \{(\omega_{32}, p_{\omega_{32}}(\rho))\}\}. \end{aligned}$$

A.3 Non-existence of competitive equilibria

We consider the economy of Appendix A.2 in the presence of hard liquidity constraints, which can be modeled by setting $\rho = +\infty$. In such a case, $p(+\infty)$ is simply written as p. The utility function of buyer b_1 over outcomes exhibits a discontinuity at a price equal to the monetary endowment $M^{b_1} = 2$. For every $t \leq 2$ it holds that

$$U^{b_1}(\{(\omega_{11}, t)\}; \rho) = V^{b_1}(\omega_{11}) - t = 5 - t \geq 3$$

and for every $t > 2$ we have $U^{b_1}(\{(\omega_{11}, t)\}; \rho) = -\infty$. The utility functions of buyers b_2 and b_3 display similar discontinuities at prices equal to their monetary endowments.

We argue that there is no competitive equilibrium. By contradiction, suppose there is a competitive equilibrium (p, A). Since each buyer can trade with at most one seller, there is at least one buyer who does not trade. In case this is buyer b_1, it holds that $p_{\omega_{11}} > 2$. Therefore, instead of choosing the no-trade option, seller s_1 prefers to trade. Since b_1 chooses the no-trade option, s_1 must trade with buyer b_3 at a price $p_{\omega_{31}} \geq p_{\omega_{11}} > 2$. This implies that $p_{\omega_{32}} \geq p_{\omega_{31}} + 1 > 3$, since otherwise buyer b_3 is not willing to trade with s_1. Therefore, rather than choosing the no-trade option, seller s_2 prefers to trade and can trade with buyers b_2 and b_3. Since b_3 trades with s_1, seller s_2 must trade with buyer b_2 at a price $p_{\omega_{22}} \geq p_{\omega_{32}} > 3$. Since $p_{\omega_{22}}$ exceeds the monetary endowment of buyer b_2 and $\rho = +\infty$, this leads to a contradiction.

In case buyer b_2 does not trade or buyer b_3 does not trade, similar contradictions can be derived. Thus, there is no competitive equilibrium.

A.4 Illustration of QCCEs

Consider the economy with hard liquidity constraints in Appendix A.3. As argued before, it has no competitive equilibrium. We illustrate that it has QCCEs.

Let $(\overline{p}, \overline{q}, \overline{A}) \in \mathbb{R}^\Omega \times \{0,1\}^\Omega \times \mathcal{A}$ be defined by

$$\begin{aligned}
\overline{p} &= (\overline{p}_{\omega_{11}}, \overline{p}_{\omega_{22}}, \overline{p}_{\omega_{31}}, \overline{p}_{\omega_{32}}) = (2, 3, 2, 3), \\
\overline{q} &= (\overline{q}_{\omega_{11}}, \overline{q}_{\omega_{22}}, \overline{q}_{\omega_{31}}, \overline{q}_{\omega_{32}}) = (1, 0, 1, 1), \\
\overline{A} &= \{(\omega_{11}, 2), (\omega_{32}, 3)\}.
\end{aligned}$$

We show that $(\overline{p}, \overline{q}, \overline{A})$ is a QCCE by verifying that it satisfies Conditions (i), (ii), and (iii) of Definition 4.1.

First, $(\overline{p}, \overline{q}, \overline{A})$ satisfies Condition (i). For buyer b_1, it holds that $\gamma^{b_1}(\overline{p}, \overline{q}) = \{\emptyset, \{(\omega_{11}, 2)\}\}$ and $\overline{A}^{b_1} = \{(\omega_{11}, 2)\} \in \delta^{b_1}(\overline{p}, \overline{q}; \rho) = \{\{(\omega_{11}, 2)\}\}$. For buyer b_2, we have $\gamma^{b_2}(\overline{p}, \overline{q}) = \{\emptyset\}$ and so $\overline{A}^{b_2} = \emptyset \in \delta^{b_2}(\overline{p}, \overline{q}; \rho) = \{\emptyset\}$. For buyer b_3, it holds that $\gamma^{b_3}(\overline{p}, \overline{q}) = \{\emptyset, \{(\omega_{31}, 2)\}, \{(\omega_{32}, 3)\}\}$ and so $\overline{A}^{b_3} = \{(\omega_{32}, 3)\} \in \delta^{b_3}(\overline{p}, \overline{q}; \rho) = \{\{(\omega_{31}, 2)\}, \{(\omega_{32}, 3)\}\}$.

Second, $(\overline{p}, \overline{q}, \overline{A})$ satisfies Condition (ii). Since all trades involving seller s_1 have the same price, it is easy to see that $\overline{A}^{s_1} = \{(\omega_{11}, 2)\} \in \delta^{s_1}(\overline{p}) = \{\{(\omega_{11}, 2)\}, \{(\omega_{31}, 2)\}\}$. The same argument applies to seller s_2, so $\overline{A}^{s_2} = \{(\omega_{32}, 3)\} \in \delta^{s_2}(\overline{p}) = \{\{(\omega_{22}, 3)\}, \{(\omega_{32}, 3)\}\}$.

A quantity constraint is expected for trade ω_{22} by b_2, who is subject to a hard financial constraint and has a monetary endowment equal to $p_{\omega_{22}}$. It follows that $(\overline{p}, \overline{q}, \overline{A})$ satisfies Condition (iii).

A similar argument shows that $(\overline{p}, q', A')$ such that $q' = (q'_{\omega_{11}}, q'_{\omega_{22}}, q'_{\omega_{31}}, q'_{\omega_{32}}) = (0, 1, 1, 1)$ and $A' = \{(\omega_{22}, 3), (\omega_{31}, 2)\}$ is a QCCE.

Note that the limit of any convergent sequence of QCCEs may not be a QCCE of the limit economy. A sequence of competitive equilibria $(p^n(\rho^n), A^n)$ in Appendix A.2 corresponds to a sequence of QCCEs $(p^n(\rho^n), q^n, A^n)$ with $q^n = (1, \ldots, 1)$. However, as shown in Appendix A.3, at the limit economy there is no competitive equilibrium so there is no QCCE that is compatible with quantity constraints $\overline{q} = (1, \ldots, 1)$. That said, the limit of a sequence of competitive equilibria $(p^n(\rho^n), A^n)$ in Appendix A.2 for increasing interest rates ρ^n yields prices $\overline{p}$ and outcome $\overline{A}$ corresponding to the QCCE $(\overline{p}, \overline{q}, \overline{A})$ of the limit economy. Note also that not every set of QCCE trades at the limit economy corresponds to trades compatible with the limit of competitive equilibrium allocations in economies with increasingly stringent soft liquidity constraints. For example, $(\overline{p}, q', A')$ is a QCCE with set of equilibrium trades equal to $\{\omega_{31}, \omega_{22}\}$. However, $\{\omega_{31}, \omega_{22}\}$ is not compatible with any competitive equilibrium in an economy with soft liquidity constraints, see Appendix A.2.

A.5 Illustration of expectational equilibria

Consider the economy with soft liquidity constraints of Appendix A.2. Let $(A, Q, R) \in \mathcal{A} \times 2^{\overline{Y}} \times 2^{\overline{Y}}$ be defined by

$$\begin{aligned}
A &= \{(\omega_{11}, \tfrac{3+2\rho}{1+\rho}), (\omega_{32}, \tfrac{4+3\rho}{1+\rho})\}, \\
Q &= \{(\omega, t) \in \overline{Y} \mid \omega \in \{\omega_{11}, \omega_{31}\},\ t < \tfrac{3+2\rho}{1+\rho}\} \cup \{(\omega, t) \in \overline{Y} \mid \omega \in \{\omega_{22}, \omega_{32}\},\ t < \tfrac{4+3\rho}{1+\rho}\}, \\
R &= \{(\omega, t) \in \overline{Y} \mid \omega \in \{\omega_{11}, \omega_{31}\},\ t > \tfrac{3+2\rho}{1+\rho}\} \cup \{(\omega, t) \in \overline{Y} \mid \omega \in \{\omega_{22}, \omega_{32}\},\ t > \tfrac{4+3\rho}{1+\rho}\}.
\end{aligned}$$

We show that (A, Q, R) is an expectational equilibrium by verifying Conditions (i), (ii), and (iii) of Definition 4.2.

First, (A, Q, R) satisfies Condition (i). For buyer b_1 we have

$$\begin{aligned}
\gamma^{b_1}(Q) &= \{\{(\omega_{11}, t)\} \in X^{b_1} \mid t \geq \tfrac{3+2\rho}{1+\rho}\} \cup \{\emptyset\}, \\
A^{b_1} &= \{(\omega_{11}, \tfrac{3+2\rho}{1+\rho})\} \in \delta^{b_1}(Q; \rho) = \{\{(\omega_{11}, \tfrac{3+2\rho}{1+\rho})\}\}.
\end{aligned}$$

For buyer b_2 it holds that

$$\begin{aligned}
\gamma^{b_2}(Q) &= \{\{(\omega_{22}, t)\} \in X^{b_2} \mid t \geq \tfrac{4+3\rho}{1+\rho}\} \cup \{\emptyset\}, \\
A^{b_2} &= \emptyset \in \delta^{b_2}(Q; \rho) = \{\{(\omega_{22}, \tfrac{4+3\rho}{1+\rho})\}, \emptyset\}.
\end{aligned}$$

For buyer b_3 it holds that

$$\begin{aligned}\gamma^{b_3}(Q) &= \{\{(\omega_{31},t)\}\in X^{b_3}\mid t\geq \tfrac{3+2\rho}{1+\rho}\}\cup\{\{(\omega_{32},t)\}\in X^{b_3}\mid t\geq \tfrac{4+3\rho}{1+\rho}\}\cup\{\emptyset\},\\ A^{b_3} &= \{(\omega_{32},\tfrac{4+3\rho}{1+\rho})\}\in\delta^{b_3}(Q;\rho)=\{\{(\omega_{31},\tfrac{3+2\rho}{1+\rho})\},\{(\omega_{32},\tfrac{4+3\rho}{1+\rho})\}\}.\end{aligned}$$

Second, (A,Q,R) satisfies Condition (ii). For seller s_1 we have

$$\begin{aligned}\gamma^{s_1}(R) &= \{\{(\omega,t)\}\in X^{s_1}\mid \omega\in\{\omega_{11},\omega_{31}\},\ t\leq \tfrac{3+2\rho}{1+\rho}\}\cup\{\emptyset\},\\ A^{s_1} &= \{(\omega_{11},\tfrac{3+2\rho}{1+\rho})\}\in\delta^{s_1}(R)=\{\{(\omega_{11},\tfrac{3+2\rho}{1+\rho})\},\{(\omega_{31},\tfrac{3+2\rho}{1+\rho})\}\}.\end{aligned}$$

For seller s_2 it holds that

$$\begin{aligned}\gamma^{s_2}(R) &= \{\{(\omega,t)\}\in X^{s_2}\mid \omega\in\{\omega_{22},\omega_{32}\},\ t\leq \tfrac{4+3\rho}{1+\rho}\}\cup\{\emptyset\},\\ A^{s_2} &= \{(\omega_{32},\tfrac{4+3\rho}{1+\rho})\}\in\delta^{s_2}(R)=\{\{(\omega_{22},\tfrac{4+3\rho}{1+\rho})\},\{(\omega_{32},\tfrac{4+3\rho}{1+\rho})\}\}.\end{aligned}$$

It is easily verified that Condition (iii) holds as well.

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