Lecture 3:
Axioms and Mechanisms for Responsive Affirmative Action

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October 20, 2023

Introduction

In Lecture 2 we examined a new way of administering an affirmative action policy, using Non-Equitable PRP mechanisms.

In this lecture we first look at two more applications of PRP mechanisms to affirmative action:

1. A "non-explicit" affirmative action policy which treats students asymmetrically based on their test scores.
2. A reserve-based PRP mechanism which we use to demonstrate trade-offs between DA-R and IA-DA-R.

In the second part of this lecture we introduce further axioms for responsive affirmative action and new mechanisms that satisfy stronger responsiveness properties.

Note: This lecture is based on work in progress, so there are many open questions.

Overview: Part 1 - Two More PRP Mechanisms

- Progressive Choice mechanisms
    - Definition: progressive choices
    - Efficiency
    - Manipulability comparisons
    - Minimal responsiveness
- PRP-R mechanisms
    - Definition
    - "Between" DA-R and IA-DA-R: trade-offs
    - Equivalence of PRP-R and IA-DA-R on a Restricted Domain

“Progressive Choices in School Choice Problems”

by Yuxing Liang and Szilvia Pápai

Progressive Choice Mechanisms: Introduction

Based on **Liang and Pápai (2023)** [work in progress]:

"Progressive Choices in School Choice Problems"

The **priorities of schools over students are identical,** e.g., based on a centralized test score or other "objective" criterion.
From now on: test scores.

Progressive Choice Mechanisms: Introduction

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The **priorities of schools over students are identical,** e.g., based on a centralized test score or other "objective" criterion.
From now on: test scores.

- Each student may have a different number of preference ranks in each component of their preference partition: different across students, but the same number in each component for a specific student.
- This means that each student gets a different number of schools as options that they can apply to in each round of the PRP procedure, following their preference ranking.

Progressive Choices According to Test Scores

- The number of school options allowed for each student depends on the test score of the student: **the lower the test score, the more school options are available** (hence the name: *progressive* choices).
- While technically a superset under the restriction of homogeneous priority profiles, these mechanisms can be seen as non-equitable counterparts of the equitable Application-Rejection (or Parallel) mechanisms of Chen and Kesten (2017).
- Without progressive choices, the identical school priorities lead to a Serial Dictatorship.
- **Serial Dictatorship:** Each student is assigned to her next highest-ranked acceptable school with available seats (if there are any) in the order of a fixed permutation of the students.

Progressive Choice Mechanisms: Rationale

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Progressive Choice Mechanisms: Rationale

- Serial Dictatorships would be fair only if test scores were accurate and based on equal opportunities.
- General rationale for progressive choices: test scores are not necessarily accurate measures of performance.
- Rationale for progressive choices in an affirmative action context: test scores are not a fair measure due to unequal opportunities.
- Progressive choices may provide more equal opportunities if an explicit affirmative action policy is banned (e.g., US) or not politically viable.
- More explicit variations may be based on the identity of students (e.g., minority vs majority or women vs men) independently of the test scores.
- *Note:* none of the results depend on the relationship between the test score and the coarseness of the preference rank partition.

Progressive Choice Mechanisms: Definition

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Choice function:

Step 1: Skip.

Step 2: Select students from the applicant pool based on preference rank classes that are heterogeneous and **progressive in test scores** across students (definition follows).

Step 3: If selection is not resolved then go to the strict priority tie-breaker.

Progressive Choices Mechanisms	**Priority Partition (Step 1)**	**Preference Partition (Step 2)**
	Coarsest	Heterogeneous: progressive in test scores

Progressive Choices in Test Scores: Definition

Identical priorities for each school: $1, \ldots, |S|$
(students are ordered in descending order of their test scores)

For each student $s \in S$, let n_s denote the **number of preference ranks in each component** of the preference rank partition.

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Then each Progressive Choice mechanism is given by a list of progressive choices for students: $n_S = (n_1, \ldots, n_{|S|})$.

Progressive Choice Mechanisms: Example

Example

$C = \{c_1, \dots, c_5\}$ with capacities $q = (1, \dots, 1)$

Progressive choices: $n_1 = n_2 = 1$, $n_3 = 2$, $n_4 = n_5 = 3$

P_1	P_2	P_3	P_4	P_5	$\succ$
$\enclose{circle}{c_2}$	c_2	c_2	c_2	c_3	1
c_1	c_3	$\enclose{circle}{c_3}$	$\enclose{circle}{c_5}$	c_5	2
c_3	$\enclose{circle}{c_1}$	c_1	c_3	c_2	3
c_5	c_4	0	c_1	$\enclose{circle}{c_4}$	4
0	c_5	-	c_4	c_1	5

Round 1: $1 - c_2$, $3 - c_3$, $4 - c_5$

Round 2: $5 - c_4$

Round 3: $2 - c_1$

Note: Student 2 is not assigned to c_3 due to the progressive choices.

Extreme Progressive Choice Mechanisms

Both extreme members are Equitable PRP mechanisms (with homogeneous priorities):

If $n_1 = \ldots = n_{|S|} = 1$: **Immediate Acceptance mechanism**

If $n_1 = \ldots = n_{|S|} = |C|$: **Serial Dictatorship**

Restricted to homogeneous priority profiles, the class of Progressive Choice mechanisms also contains

- all the Application-Rejection mechanisms, for which $n_1 = \ldots = n_{|S|}$.
- the Favored Minority Student mechanisms, for which $n_a = 1$ and $n_i = |C|$, where a denotes majority students and i denotes minority students.

Efficiency of Progressive Choice Mechanisms

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Progressive Choice mechanisms are Pareto-efficient.

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- However, the result is not trivial since the modified priority profiles of these PRP mechanisms are not homogeneous priority profiles (except for the Serial Dictatorship).
- The theorem does not depend on the assumption of progressive choices, that is, on the relationship between test scores and the number of options allowed.

Efficiency of Progressive Choice Mechanisms

Intuition for Theorem 1:

- Due to the identical school priorities, each Progressive Choice mechanism can be seen as an *iterative constrained* Serial Dictatorship, where the constraints are given iteratively by the consecutive preference rank classes based on progressive choices.
- This leads to a permutation of the students such that each student is assigned to her favorite school which still has unassigned seats, in the order of this permutation, subject to the constraints of the current preference rank class, and iterated the same way for each preference rank class.
- Unlike for a Serial Dictatorship, the permutation of students is a function of the preference profile and is not necessarily the same as the homogeneous priority ordering $\succ$.

Efficiency of Progressive Choice Mechanisms: Example

Example (continued)

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0	c_5	-	c_4	c_1	5

A possible permutation of students: $(1, 3, 4, 5, 2)$

Manipulability of Progressive Choice Mechanisms

Definitions of manipulability

A **student** s can **manipulate** profile $(P, \succ)$ for mechanism φ if there exists an alternative preference ordering P'_s such that

$$\varphi_s((P'_s, P_{-s}), \succ) \; P_s \; \varphi_s(P, \succ).$$

A **profile** $(P, \succ)$ is **manipulable** for mechanism φ if there exists a student who can manipulate this profile for φ.

A **mechanism** φ is **manipulable** if there exists at least one manipulable profile for φ. A mechanism is **strategyproof** if it is not manipulable.

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Definition: Manipulability comparisons (Pathak and Sönmez, 2013)

A mechanism φ is **weakly less manipulable** than a mechanism ϕ if all manipulable profiles for φ are also manipulable for ϕ.

Manipulability of Progressive Choice Mechanisms

Definition: Relative permissibility of Progressive Choice mechanisms

A Progressive Choice mechanism φ is **more permissible** than another Progressive Choice mechanism ϕ if, for all $s \in S$,

$$n_s(\varphi) \geq n_s(\phi)$$

with at least one strict inequality.

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If a Progressive Choice mechanism φ is more permissible than another Progressive Choice mechanism ϕ, then φ is weakly less manipulable than ϕ.

Key to the proof: A student s can manipulate profile $(P, \succ)$ for a Progressive Choice mechanism φ if and only if s envies another student s' at this profile (i.e., $\varphi_{s'}(P, \succ)\ P_s\ \varphi_s(P, \succ)$) such that either $s \succ s'$ or the rank of $\varphi_{s'}(P, \succ)$ in $P_{s'}$ is lower than $n_{s'}$.

Manipulability of Progressive Choice Mechanisms

Theorem 2 is similar to the manipulability comparison of Chen and Kesten (2017) but logically independent:

- It is more general since preference rank classes may be heterogeneous (Progressive Choice mechanisms include Non-Equitable PRP mechanisms that treat students asymmetrically).
- It is less general since school priorities are identical.

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- **Conjecture:** the result extends to general heterogeneous priorities.

The theorem indicates a trade-off between incentives and effective affirmative action:

- The most permissible Progressive Choice mechanism is the Serial Dictatorship; it is strategyproof but has no affirmative action impact.
- In the case of only two student groups, incentive considerations suggest that the group to be prioritized should have the coarsest preference partition (as in the Favored Students mechanism).
- However, Theorem 2 provides guidance only regarding incentives. The overall policy should be chosen according to multiple criteria.

Minimal Responsiveness of Progressive Choice Mechanisms

We can define a minimal responsiveness axiom with respect to specific groups to be prioritized (in the case of more than two groups we have to specify the prioritized group).

Let S^a and S^i partition S such that S^i is the prioritized group.

We assume that this partition is consistent with the Progressive Choice mechanism in the sense that for all pairs (a, i) such that $a \in S^a$ and $i \in S^i$, $n_a \leq n_i$.

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Weakly stronger affirmative action policy

A Progressive Choice mechanism φ with $n_S(\varphi)$ has a weakly stronger affirmative action policy than another Progressive Choice mechanism ϕ with $n_S(\phi)$ if, for all $i \in S^i$, $n_i(\varphi) \geq n_i(\phi)$ and, for all $a \in S^a$, $n_a(\varphi) \leq n_a(\phi)$.

Minimal Responsiveness of Progressive Choice Mechanisms

Theorem 3

Progressive Choice mechanisms are minimally responsive.

Note: The theorem depends on the assumption of homogeneous priorities and does not extend to heterogeneous priorities, as the next example demonstrates.

Minimal Responsiveness Counterexample for the General Case

Example (Progressive Choice Mechanisms are not Minimally Responsive With Heterogeneous Priorities)

$C = \{c_1, \ldots, c_4\}$ with capacities $q = (1, \ldots, 1)$.

Progressive choices for ϕ: $n_{a_1} = 1$, $n_{a_2} = n_{a_3} = 2$, $n_{i_1} = n_{i_2} = 3$

Progressive choices for φ: $n_{a_1} = 1$, $n_{a_2} = n_{a_3} = 2$, $n_{i_1} = n_{i_2} = 4$; a stronger affirmative action policy, where $S^i = \{i_1, i_2\}$ is the prioritized group.

P_{a_1}	P_{a_2}	P_{a_3}	P_{i_1}	P_{i_2}
$\boxed{\underline{c_4}}$	$\underline{c_3}$	$\underline{c_1}$	c_2	$\underline{c_2}$
	$\boxed{c_1}$	$\boxed{c_2}$	c_4	$\boxed{c_3}$
			c_3	c_1
			c_1	c_4
			$\boxed{\underline{0}}$	

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$
a_2	a_3	i_2	a_1
i_1	i_2	a_2	i_1
a_3	a_1	i_1	

Under the stronger affirmative action policy φ (matching in squares) student i_1 is indifferent and student i_2 is worse off than under ϕ (underlined matching).

Further Research Questions

1. Stability comparison of Progressive Choice mechanisms?
2. Which results generalize to heterogeneous (general) school priorities?

Empirical Paper on Progressive Choice Mechanisms

“Towards a Better Major Assignment: An Empirical Study with Students in China” by **Xintong Han, Yuxing Liang, and Yan Zeng (working paper)**

- Empirical study using a data set of college major choices at a Chinese university.
- Demonstrates the potential of the Progressive Choice mechanisms to improve the welfare of under-represented (low test-score) students.
- At the same time, the welfare loss for students who are not prioritized is minimal.
- A counterfactual analysis is also carried out with two different numbers of school options based on the students’ gender (not on their test score).
- This provides more evidence that such asymmetric treatment of students may be helpful in the affirmative action context.

PRP-R Mechanisms

PRP-R Mechanisms: Introduction

- New results, not in any paper yet.
- Another PRP mechanism with an affirmative action policy.
- This mechanism is based on minority reserve seats.
- Thus, it provides an explicit affirmative action policy, combining the PRP approach with reserve-based affirmative action.
- Similar to the IA-DA-R mechanism (note: the IA-DA-R mechanism cannot be defined as a PRP mechanism).
- Demonstrates trade-offs between DA-R and IA-DA-R.

PRP-R Mechanisms: Definition

- Given a reserve-based affirmative action policy r, split each school $c \in C$ into a "reserve" school c_r and a "regular" school c_g with the following schools capacities:

  Reserve school c_r: $q_{c_r} = r_c$
  the number of reserved seats at school c

  Regular school c_g: $q_{c_g} = q_c - r_c$
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- Since these are PRP mechanisms, they can be defined by the three lexicographic steps of the school choice function:
  1) priority classes 2) preference classes 3) strict-priority tie-breaking.

- Steps 1) and 2) differ depending on the type of the school.

PRP-R Mechanisms: Definition

Preferences for PRP-R mechanisms:

- For all students $s \in S$, the ordering of the split-off schools are consistent with P_s.
- The reserve and regular schools corresponding to the original school c are ranked adjacently corresponding to the original position of school c in the preference order P_s.

For minority students: c_r first, then c_g

reserve school first - essential

For majority students: c_g first, then c_r
regular school first - not essential

PRP-R Mechanisms: Definition

PRP-R Mechanisms	**Priority Partition (Step 1)**	**Preference Partition (Step 2)**
Reserve schools c_r:	Minority students first in one top-priority class; Majority students next: finest, according to $\succ_c$	Minority students: finest Majority students: coarsest
Regular schools c_g:	Including all students: finest, according to $\succ_c$	All students: coarsest

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- Minority students have the finest preference partition for reserve schools resembling the immediate acceptances of minority students to minority reserve seats by IA-DA-R.
- Regular (non-reserved) seats are treated as in the DA: the finest priority partition and the coarsest preference partition for each student.
- Majority students have tentative acceptances, as in the DA, for reserved seats that are not (yet) assigned to minority students.

PRP-R vs DA-R: A Comparison in the PRP Framework

PRP-R Mechanisms	**Priority Partition (Step 1)**	**Preference Partition (Step 2)**
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DA-R Mechanisms	**Priority Partition (Step 1)**	**Preference Partition (Step 2)**
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  Respects the affirmative action policy: reserve schools prioritize minority students through the top priority class which consists of all minority students.

- PRP-R mechanisms are not minimally responsive.
- It is a future research question whether PRP-R mechanisms have better responsiveness properties according to some other criterion than DA-R (see the next example and the equivalence result with IA-DA-R on a restricted domain).

Example

Example (Comparison of DA-R to PRP-R and IA-DA-R)

Let $S^M = \{a_1, a_2\}$ and $S^m = \{i_1, i_2\}$ be the sets of majority and minority students. Let $C = \{c_1, c_2, c_3\}$ with capacities $q = (1, 1, 1)$.

P_{a_1}	P_{a_2}	P_{i_1}	P_{i_2}
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If the affirmative action policy is $r = (0, 1, 0)$:

the DA-R matching (in squares) is not minimally responsive;

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If there is no affirmative action policy ($r = (0, 0, 0)$), the outcome of all three mechanisms is the DA (underlined).

If the affirmative action policy is $r = (0, 1, 0)$:

the DA-R matching (in squares) is not minimally responsive;

the PRP-R and IA-DA-R matchings are the same (circled), and makes minority student i_2 better off than in the DA.

PRP-R Mechanisms: Fairness Properties

PRP-R mechanisms satisfy Minority Fairness:

- No priority violation is caused by majority students:
  majority students face the finest priority partition according to $\succ_c$ at both reserve and regular schools.

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- Minority students cause at most r_c priority violations at school c:
  minority students have the top priority only at reserve schools, and face the finest priority partition according to $\succ_c$ at regular schools.

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- Minority students cause at most r_c priority violations at school c: minority students have the top priority only at reserve schools, and face the finest priority partition according to $\succ_c$ at regular schools.

Priority violations within the set of minority students:

- Priority violations may occur, but PP-stability is satisfied within the minority group.
- Either the original priorities $\succ_c$ or the relative preference ranks determine the relative selection of minority students by each school (whether reserve or regular) in a PRP-R mechanism.
- Note that the same does not hold for IA-DA-R.

PRP-R Mechanisms: Incentive Properties

- The PRP-R mechanism is manipulable.

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PRP-R Mechanisms: Incentive Properties

- The PRP-R mechanism is manipulable.
- Minority students can manipulate to obtain a reserve school due to the finest preference partition at reserve schools, similarly to IA manipulations.
- Thus, a PRP-R mechanism is obviously manipulable by minority students.
- **A PRP-R mechanism cannot be manipulated by majority students.**
- This follows from the PRP Manipulation Theorem, since majority students have the coarsest preference partition at both reserve schools and regular schools.

Summary Table: Trade-Offs

	NW	AA	Responsive	MF	Stable S^m	SP
DA-R	✓	✓	Not Minimally Responsive	✓	Stable	SP Min SP Maj
PRP-R	✓	✓	**Not Minimally Responsive**	✓	PP-stable	— Min SP Maj
IA-DA-R	✓	✓	Minimally Responsive	✓	Not PP-stable	— Min — Maj

Note: All three mechanisms become the DA when there is no affirmative action policy (i.e., $r = (0, \ldots, 0)$).

NW: Non-wasteful

AA: Respects the affirmative action policy

MF: Minority Fair

Stable S^m : Stability within the set of minority students

SP: Strategyproofness

A Restricted Domain

A specific preference domain:

RR-domain (Reserve-Restricted domain)

- At each preference profile P each school c with a minority reserve r_c is ranked first by at least r_c minority students.
- This domain restriction is likely to be satisfied if
  - there are few minority reserve seats compared to the number of minority students;
  - highly popular schools have more minority reserve seats than less popular schools.

PRP-R and IA-DA-R Equivalence on the RR-Domain

Recall the following impossibility theorem from Lecture 1:

Impossibility on the general domain

There is no mechanism with minority allotments which is non-wasteful, respects the affirmative action policy, is minimally responsive, minority fair, and strategyproof for majority students.

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Possibility on the RR-domain

On the RR-domain there exists a mechanism with minority allotments which is non-wasteful, respects the affirmative action policy, is minimally responsive, minority fair and strategyproof for majority students.

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On the RR-domain there exists a mechanism with minority allotments which is non-wasteful, respects the affirmative action policy, is minimally responsive, minority fair and strategyproof for majority students.

The PRP-R and IA-DA-R mechanisms are equivalent on the RR-domain and have all of the above properties.

Further Research Questions

- What are the responsiveness properties of PRP-R compared to DA-R?
- Maximal domain to reconcile non-wastefulness, respecting the affirmative action policy, minimal responsiveness, minority fairness and strategyproofness for majority students?
- A more systematic way of tracing out the trade-offs among responsiveness, fairness within the minority student group and incentives?

“Responsive Affirmative Action: Axioms and Policies”

by Muntasir Chaudhury and Szilvia Pápai

Overview: Part 2

Based on **Chaudhury and Pápai (2023b)** [work in progress]:

"Responsive Affirmative Action: Axioms and Policies"

We study **four responsiveness axioms** (including minimal responsiveness), all of which concern the welfare improvement of minority students when the strength of the affirmative action policy changes.

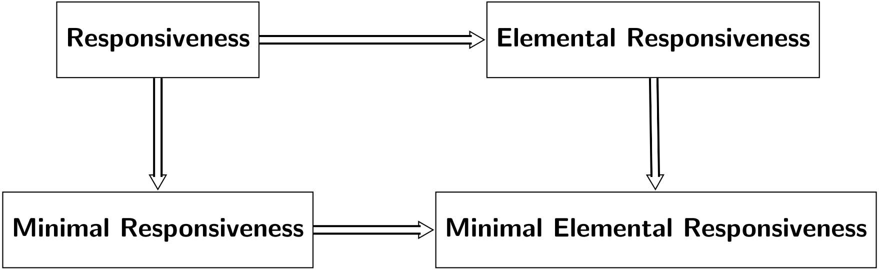
	Strong welfare requirement	**Weak welfare requirement**
Compares all AA to stronger AA policy	Responsive	Minimally Responsive
Compares no AA to any AA policy	Elementally Responsive	Minimally Elementally Responsive

Overview: Part 2 (continued)

- We show first that previously studied minimally responsive reserve-based or quota-based mechanisms are not elementally responsive and, therefore, not responsive.
- Then we introduce two mechanisms that are more responsive than MDA, IA-R and IA-DA-R:
    1. **Divided DA (DDA)** mechanism: responsive
    2. **Guaranteed DA (GDA)** mechanism: elementally responsive and minimally responsive
- With the aid of these two mechanisms we further explore the compatibility and incompatibility of welfare and fairness axioms.

Four Responsiveness Axioms

Logical Relationships Among Responsiveness Axioms



Two axioms compare an affirmative action policy to **no affirmative action policy:** elemental responsiveness and minimal elemental responsiveness.

Two axioms compare **any two affirmative action policies** where one is weakly stronger than the other: responsiveness and minimal responsiveness.

- **Elemental:** refers to comparing to no affirmative action policy.
- **Minimal:** refers to the weaker of the two welfare requirements for minority students.

Axioms of Responsiveness

- Given minority reserves, recall that r' represents a **weakly stronger affirmative action policy** than r if for all $c \in C$, $r'_c \geq r_c$, i.e., $r' \geq r$.
- **No affirmative action policy:** $r = \mathbf{0}$.

Axioms of Responsiveness

- Given minority reserves, recall that r' represents a **weakly stronger affirmative action policy** than r if for all $c \in C$, $r'_c \geq r_c$, i.e., $r' \geq r$.
- **No affirmative action policy:** $r = \mathbf{0}$.

A mechanism φ is **responsive** if a weakly stronger affirmative action policy weakly Pareto-dominates the outcome of the initial affirmative action policy for minority students.

Definition: Responsiveness

A mechanism φ is responsive if for all r, r' such that $r' \geq r$, all $(P, \succ)$, and all minority students $i \in S^m$, $\varphi_i(r', P, \succ)\ R_i\ \varphi_i(r, P, \succ)$.

This is the strongest responsiveness axiom.

Axioms of Responsiveness

A mechanism φ is **elementally responsive** if an affirmative action policy weakly Pareto-dominates the outcome for minority students when there is no affirmative action policy.

Definition: Elemental responsiveness

A mechanism φ is elementally responsive if for all affirmative action policies r, all $(P, \succ)$, and all minority students $i \in S^m$, $\varphi_i(r, P, \succ)\ R_i\ \varphi_i(\mathbf{0}, P, \succ)$.

Even if a stronger affirmative action policy may not necessarily result in a (weak) Pareto-improvement for the minority, the introduction of an affirmative action policy is expected to do so.

Axioms of Responsiveness

A mechanism φ is **minimally responsive** if a weakly stronger affirmative action policy never results in a Pareto-inferior outcome for minority students.

Definition: Minimal responsiveness

A mechanism φ is minimally responsive if for all r, r' such that $r' \geq r$, all $(P, \succ)$, if $\varphi_{S^m}(r, P, \succ) \neq \varphi_{S^m}(r', P, \succ)$ then there exists $i \in S^m$ such that $\varphi_i(r', P, \succ)\ P_i\ \varphi_i(r, P, \succ)$.

This concept is the same as *respecting the spirit of quota-based affirmative action* introduced by Kojima (2012) and the *minimal responsiveness* axiom of Doğan (2016); the weak responsiveness axiom studied so far in these lectures.

Axioms of Responsiveness

A mechanism φ is **minimally elementally responsive** if an affirmative action policy never results in a Pareto-inferior outcome for minority students when compared to no affirmative action.

Definition: Minimal elemental responsiveness

A mechanism φ is elementally responsive if for all affirmative action policies r and all $(P, \succ)$, if $\varphi_{S^m}(r, P, \succ) \neq \varphi_{S^m}(\mathbf{0}, P, \succ)$ then there exists $i \in S^m$ such that $\varphi_i(r, P, \succ)\ P_i\ \varphi_i(\mathbf{0}, P, \succ)$.

This is the weakest responsivess requirement, which only guarantees that at least one minority student gains when an affirmative action policy is introduced, given that the affirmative action policy has any impact.

Responsiveness Properties of Previous Mechanisms

Responsiveness Properties of Previous Mechanisms

Not minimally elementally responsive mechanisms

- DA-Q (Abdulkadiroğlu and Sönmez, 2003; Kojima, 2012)
- DA-R (Hafalir et al., 2013)
- PRP-R

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Minimally responsive, but not elementally responsive mechanisms

- MDA (Doğan, 2016)

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- IA-R (Afacan and Salman, 2016; Doğan and Klaus, 2018)

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- PRP-R

Minimally responsive, but not elementally responsive mechanisms

- MDA (Doğan, 2016)
- IA-R (Afacan and Salman, 2016; Doğan and Klaus, 2018)
- IA-DA-R (Chaudhury and Pápai, 2023a)

DA-Q, DA-R and PRP-R Are Not Minimally Elementally Responsive

Example

Let $S^M = \{a_1, a_2\}$ and $S^m = \{i_1, i_2\}$ be the sets of majority and minority students. Let $C = \{c_1, c_2, c_3\}$ with capacities $q = (1, 1, 1)$.

P_{a_1}	P_{a_2}	P_{i_1}	P_{i_2}
$\underline{c_2}$	$\boxed{\underline{c_3}}$	$\underline{c_1}$	c_1
$\boxed{c_1}$	c_1	$\boxed{c_2}$	c_2
0	0	c_3	c_3
		0	$\boxed{\underline{0}}$

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
a_1	a_1	a_1
a_2	i_1	i_1
i_1	i_2	a_2
i_2	a_2	i_2

If there is no affirmative action policy ($v = (0, 0, 0)$), the outcome of all three mechanisms is the DA (underlined).
If the affirmative action policy is $v = (0, 1, 0)$, the DA-Q, DA-R and PRP-R matchings are the same (in squares).
Minority student i_1 is worse off, and minority student i_2 is indifferent compared to the DA matching.

MDA, IA-R and IA-DA-R Are Not Elementally Responsive

Example

Let $S^M = \{a_1\}$ and $S^m = \{i_1, i_2\}$ be the sets of majority and minority students. Let $C = \{c_1, c_2, c_3\}$ with capacities $q = (1, 1, 1, 1)$.

P_{a_1}	P_{i_1}	P_{i_2}
$\underline{c_1}$	$\boxed{c_1}$	c_1
$\boxed{c_3}$	$\underline{c_2}$	$\underline{c_3}$
0	0	$\boxed{c_2}$

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
a_1	i_1	a_1
i_1	a_1	i_2
i_2	i_2	i_1

If there is no affirmative action policy ($r = (0, 0, 0)$), the outcome of all three mechanisms is the underlined matching.

If the affirmative action policy is $r = (0, 1, 0)$, the MDA, IA-R and IA-DA-R matchings are the same (in squares).

Minority student 3 is worse off with the stronger minority reserve policy r' compared to $r = (0, 0, 0)$.

Two New Mechanisms Which Satisfy Stronger Responsiveness Axioms

Two New Mechanisms Which Satisfy Stronger Responsiveness Axioms

We introduce two new mechanisms, the **Divided DA (DDA)** mechanism and the **Guaranteed DA (GDA)** mechanism.

- These are the first two mechanisms that are proposed in the literature which satisfy more than minimal responsiveness.
- DDA is responsive, and GDA is elementally responsive in addition to satisfying minimal responsiveness.
- Both mechanisms are based on what we call **minority seats** (to be defined).
- Intuitively, minority seats are the seats that minority students are "qualified for" either due to the minority reserves or due to the assignments of minority students in the DA in the absence of affirmative action.

Minority Seats: Formal Definition

Definition: Minority seats

Fix a minority reserve policy r and a profile $(P, \succ)$.
Let $m_c^{DA}(P, \succ)$ denote the number of minority students assigned to school c in the DA matching at $(P, \succ)$.

The number of **minority seats** at each school $c \in C$ is

$$\hat{m}_c(r_c, P, \succ) = \max\left(r_c,\ m_c^{DA}(P, \succ)\right).$$

- At each school the number of minority seats is either the number of minority reserve seats or the number of seats assigned at this school to minority students in the DA matching, whichever is larger.
- Note that the number of minority seats depends on the profile $(P, \succ)$.

DDA Mechanism

Divided DA Mechanism (DDA): Introduction

- The DDA mechanism divides the matching procedure into two steps that relies on the number of minority seats $\hat{m}_c$ at each school $c \in C$.
- In the first step, the assignments of minority students is determined based on the **minority seats**, and in the second step the majority students receive their assignments.

DDA Mechanism: Definition

Fix a minority reserve policy r and a profile $(P, \succ)$.

Step 1 - Minority matching

Run the DA mechanism with minority students S^m only, while restricting the number of seats at each school $c \in C$ to the number of minority seats $\hat{m}_c(r_c, P, \succ)$.
For all $c \in C$, let μ_c^m be the set of minority students matched to c in this step.

Step 2 - Majority matching

Run the DA mechanism with majority students S^M for the remaining seats (seats not assigned in Step 1) at each school: the number of seats available to majority students is $\bar{q}_c = q_c - |\mu_c^m|$ for all $c \in C$.
For all $c \in C$, let μ_c^M be the set of majority students matched to c in this step.

In both steps the matching is final, and the set of students matched to school c is $\mu_c = \mu_c^m \cup \mu_c^M$, for all $c \in C$.

DDA Mechanism: Illustrative Example

Example

Let $S^M = \{1, 2\}$ and $S^m = \{3, 4, 5, 6\}$, and let $C = \{c_1, c_2, c_3, c_4\}$ with capacities $q = (2, 2, 1, 1)$. Let the minority reserves be $r = (1, 0, 0, 0)$.

P_1	P_2	P_3	P_4	P_5	P_6
$\boxed{c_1}$	c_1	$\boxed{c_1}$	c_1	$\boxed{c_3}$	$\boxed{c_4}$
c_3	c_4	c_2	$\boxed{c_2}$	c_1	c_1
c_2	c_3	c_3	c_4	c_4	c_3
c_4	$\boxed{c_2}$	c_4	c_3	c_2	c_2

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$
1	5	1	1
2	6	2	2
3	1	5	6
6	2	6	5
5	3	3	3
4	4	4	4

The DDA matching is given by the assignments in squares.

DDA Mechanism: Illustrative Example

Example (continued)

- The DA matching is $\mu^{\mathrm{DA}} = (c_1, c_1, c_2, c_2, c_3, c_4)$. Thus, the DA list for minority students is $m^{DA}(P, \succ) = (0, 2, 1, 1)$ and the minority seat list is
$$\hat{m}_c(r, P, \succ) = (1, 2, 1, 1).$$
- **Step 1:** Run the DA mechanism with the set of minority students $\{3, 4, 5, 6\}$ only, using $\hat{m}_c(r, P, \succ)$ as the capacity list. This yields the minority matching $\mu_3^m = \{c_1\}$, $\mu_4^m = \{c_2\}$, $\mu_5^m = \{c_3\}$, and $\mu_6^m = \{c_4\}$.
- **Step 2:** Run the DA mechanism on the remaining capacity of the schools, $\bar{q}_c = (1, 1, 0, 0)$, with the set of majority students $\{1, 2\}$ only. This step yields the majority matching $\mu_1^M = \{c_1\}$, $\mu_2^M = \{c_2\}$.
- The **final matching** of the DDA mechanism is $\mu_c = \mu_c^m \cup \mu_c^M$ for all $c \in C$, which gives the following matching: $(c_1, c_2, c_1, c_2, c_3, c_4)$.

Responsiveness of the DDA Mechanism

Theorem 4

The DDA mechanism is responsive.

Responsiveness of the DDA Mechanism

Theorem 4

The DDA mechanism is responsive.

Intuition for the proof:

If $r' \geq r$ then $\hat{m}_c\left(r'_c, P, \succ\right) \geq \hat{m}_c\left(r_c, P, \succ\right)$, which means that minority students have at least as many seats at each school at the weakly stronger affirmative action policy r' as at r.

Then the *resource monotonicity* of of the DA (Chambers and Yenmez, 2017; Ehlers and Klaus, 2016) implies the result.

The DDA Mechanism is Wasteful

Example

Let $S^M = \{1,4\}$ and $S^m = \{2,3\}$, and let $C = \{c_1, c_2, c_3, c_4, c_5\}$ with capacities $q = (1,1,1,1,1)$. Let the minority reserve policy be $r = (1,0,0,0,0)$.

P_1	P_2	P_3	P_4
$\underline{c_2}$	$\underline{c_1}$	c_3	c_1
c_3	c_2	$\underline{c_4}$	c_4
c_1	c_3	c_1	$\underline{c_5}$
c_4	c_4	c_2	c_3
c_5	c_5	c_5	c_2

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$	$\succ_{c_5}$
4	2	1	3	4
2	1	3	4	3
1	3	2	1	2
3	4	4	2	1

In the DDA matching (underlined) minority student 3 is assigned to c_4. However, $c_3 \; P_3 \; c_4$ and c_3 has an empty seat, so DDA is wasteful.

Note: In general, only minority students may experience wastefulness in the DDA mechanism. This contrasts with quota-based mechanisms, where only majority students are affected by wastefulness.

GDA Mechanism

Guaranteed DA Mechanism (GDA): Introduction

- Since the Divided DA mechanism is wasteful, we introduce another mechanism which is not wasteful, the Guaranteed DA (GDA) mechanism, which also has nice responsiveness properties, although it does not satisfy the axiom of responsiveness itself.
- The GDA mechanism is also based on minority seats, but instead of allocating minority seats in the first step to minority students only, the GDA mechanism guarantees that each minority student gets at least as good an assignment as they would get in the DDA.
- These minimal assignments are guaranteed by placing each minority student at the top of the priority ordering for their guaranteed school, and then the DA is run with this modified priority profile.
- This one-step DA procedure prevents wastefulness.

GDA Mechanism: Definition

Fix a minority reserve policy r and a profile $(P, \succ)$.

Step 1 - Determining the guarantees

Find the minority seats $\hat{m}$ and run Step 1 (minority matching) of the Divided DA algorithm. Let μ^m denote the matching obtained for minority students in this step. The assignments made by μ^m are the guaranteed minimum assignments to minority students in the final matching.

Step 2 - DA matching with guarantees

Modify the priorities based on the matching for minority students μ^m that was obtained in Step 1 as a function of $(r, P, \succ)$: for each school c, let the set of students in μ^m_c be ranked at the top of the priority ordering of school c, so each minority student assigned to c in Step 1 is among the top q_c-ranked students in the priority ordering of school c. Let all other priorities remain the same. Denote this priority profile by $\tilde{\succ}(r, P, \succ)$.
Run the DA with priority profile $\tilde{\succ}(r, P, \succ)$ to find the final matching.

GDA Mechanism: Illustrative Example

Example

Let $S^M = \{1, 4, 5\}$ and $S^m = \{2, 3\}$, and let $C = \{c_1, \ldots, c_6\}$ with capacities $q = (1, \ldots, 1)$. Let the minority reserves be $r = (1, 0, 0, 0, 0, 0)$.

P_1	P_2	P_3	P_4	P_5
$\boxed{c_2}$	c_6	$\boxed{c_3}$	c_1	$\boxed{c_6}$
c_3	$\boxed{c_1}$	c_4	$\boxed{c_4}$	c_3
c_1	c_2	c_1	c_5	c_1

$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\succ_{c_4}$	$\succ_{c_5}$	$\succ_{c_6}$
4	2	5	3	4	5
2	1	1	5	2	2
3	3	3	4	1	1
1	4	2	2	3	3
5	5	4	1	5	4

- The minority seat list is $\hat{m}_c(r, P, \succ) = (1, 1, 0, 1, 0, 0)$.
- **Step 1:** run the DA mechanism using the capacity list $\hat{m}_c(r, P, \succ)$ with the set of minority students $\{2, 3\}$ to find the guaranteed seats for minority students.

GDA Mechanism: Illustrative Example

Example (continued)

- This yields the minority matching $\mu_2^m = \{c_1\}$ and $\mu_3^m = \{c_4\}$. These are the guaranteed seats.
- **Step 2:** the priorities for schools c_1 and c_4 are updated to $\hat{\succ}_{c_1}$ and $\hat{\succ}_{c_4}$ (see below).

$\hat{\succ}_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$	$\hat{\succ}_{c_4}$	$\succ_{c_5}$	$\succ_{c_6}$
2	2	5	**3**	4	5
4	1	1	5	2	2
3	3	3	4	1	1
1	4	2	2	3	3
5	5	4	1	5	4

- Run the DA using the above updated priority profile.
- The **final matching** is $(c_2, c_1, c_3, c_4, c_6)$ (in squares).

Responsiveness Properties of the GDA Mechanism

Theorem 5

The GDA mechanism is elementally responsive and minimally responsive.

Responsiveness Properties of the GDA Mechanism

Theorem 5

The GDA mechanism is elementally responsive and minimally responsive.

- **Elemental responsiveness** holds since the DDA is responsive, and thus the guarantee is weakly preferred by each minority student under any minority reserve policy compared to no minority reserves. Then the fact that with no affirmative action policy the GDA and DDA matchings are the same implies the result.

Responsiveness Properties of the GDA Mechanism

Theorem 5

The GDA mechanism is elementally responsive and minimally responsive.

- **Elemental responsiveness** holds since the DDA is responsive, and thus the guarantee is weakly preferred by each minority student under any minority reserve policy compared to no minority reserves. Then the fact that with no affirmative action policy the GDA and DDA matchings are the same implies the result.
- **Minimal responsiveness** also follows from the responsiveness of DDA and from the fact that a minority student cannot be rejected by the school at which she has a guaranteed seat. Thus, the first minority student who receives a different assignment under a stronger minority reserve policy due to the guarantees is guaranteed to be better off.

Responsiveness Properties of the GDA Mechanism

The GDA mechanism does not satisfy responsiveness.

Intuition:

- The GDA mechanism does not preserve the responsiveness of DDA precisely because it eliminates the wastefulness of DDA.
- By doing so, some minority student who was able to get a higher-ranked school than their guaranteed assignment at some profile may not be able to do so at the same profile when a stronger affirmative action policy increases minority reserve seats.

Fairness and Incentive Properties of DDA and GDA

Fairness Properties

The DDA and GDA mechanisms do not satisfy Minority Fairness.

Recall that Minority Fairness requires that at each profile

a) no majority student violates another student's priority

b) at most r_c minority students violate the priority of another student at each school c

Fairness Properties

The DDA and GDA mechanisms do not satisfy Minority Fairness.

Recall that Minority Fairness requires that at each profile

a) no majority student violates another student's priority

b) at most r_c minority students violate the priority of another student at each school c

- Requirement a) is satisfied by both DDA and GDA.
- However, requirement b) may not be satisfied.
- The number of minority seats may exceed the number reserved seats at a school c (when the DA matching assigns more minority students to a school than its reserved seats).
- Then it is possible that all these minority seats at c are assigned to minority students who have a lower priority than majority students at c.

Minority Seat Fairness

Minority Seat Fairness

Given a minority reserve policy r and a profile $(P, \succ) \in \mathcal{P} \times \Pi$, we call a **matching μ minority seat fair** with respect to r and $(P, \succ)$ if, at profile $(P, \succ)$, it satisfies the following conditions:

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1. no majority student violates another student's priority in μ;
2. at most $\hat{m}_c(r_c, P, \succ)$ minority students violate the priority of a majority student at each school c in μ;

Minority Seat Fairness

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Given a minority reserve policy r and a profile $(P, \succ) \in \mathcal{P} \times \Pi$, we call a **matching** μ **minority seat fair** with respect to r and $(P, \succ)$ if, at profile $(P, \succ)$, it satisfies the following conditions:

1. no majority student violates another student's priority in μ;
2. at most $\hat{m}_c(r_c, P, \succ)$ minority students violate the priority of a majority student at each school c in μ;
3. no minority student violates another minority student's priority in μ.

A **mechanism** φ is **minority seat fair** if for all minority reserve policies r and all profiles $(P, \succ) \in \mathcal{P} \times \Pi$, $\varphi(r, P, \succ)$ is minority fair with respect to r and $(P, \succ)$.

- Minority Seat Fairness and Minority Fairness are logically independent of each other.
- Both DDA and GDA satisfy Minority Seat Fairness.

Incentive Properties

- The Divided DA and Guaranteed DA mechanisms are not strategyproof.
- The Divided DA and Guaranteed DA mechanisms can be manipulated by both majority and minority students.
- The intuition underlying both of these negative results is that the minority seats depend on the DA matching, and subsequent steps are based on the DA matching, which makes these mechanisms vulnerable to misrepresenting the preferences to obtain a lower-ranked outcome in the DA (which itself is strategyproof), and benefit from this in a subsequent step in either the DDA or the GDA algorithm.

Impossibility Conjectures and Possibility Results

Impossibility Conjectures: Incompatible Axioms

Impossibility Conjecture 1

- Non-wastefulness
- Respecting the affirmative action policy
- Elemental responsiveness
- Minority fairness

Impossibility Conjectures: Incompatible Axioms

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Implication: IA-DA-R is not elementally responsive.

Impossibility Conjectures: Incompatible Axioms

Impossibility Conjecture 1

- Non-wastefulness
- Respecting the affirmative action policy
- Elemental responsiveness
- Minority fairness

Implication: IA-DA-R is not elementally responsive.

Impossibility Conjecture 2

- Non-wastefulness
- Respecting the affirmative action policy
- Responsiveness
- Minority seat fairness

Impossibility Conjectures: Incompatible Axioms

Impossibility Conjecture 1

- Non-wastefulness
- Respecting the affirmative action policy
- Elemental responsiveness
- Minority fairness

Implication: IA-DA-R is not elementally responsive.

Impossibility Conjecture 2

- Non-wastefulness
- Respecting the affirmative action policy
- Responsiveness
- Minority seat fairness

Implications: DDA is wasteful; GDA is not responsive.

Possibility Results: Compatible Axioms

Possibility Result Based on DDA

- ~~Non-wastefulness~~
- Respecting the affirmative action policy
- Responsiveness
- Minority seat fairness

Satisfied by: DDA

Possibility Results: Compatible Axioms

Possibility Result Based on DDA

- ~~Non-wastefulness~~
- Respecting the affirmative action policy
- Responsiveness
- Minority seat fairness

Satisfied by: DDA

Possibility Result Based on GDA

- Non-wastefulness
- Respecting the affirmative action policy
- ~~Responsiveness~~ Minimal responsiveness and Elemental Responsiveness
- Minority seat fairness

Satisfied by: GDA

Summary

- We investigate mechanisms with a reserve-based affirmative action policy that have stronger responsiveness properties than minimal responsiveness.
- We introduce the DDA mechanism which is responsive but wasteful.
- We also introduce the non-wasteful GDA mechanism that is both minimally responsive and elementally responsive, but not responsive.
- Neither of the two mechanisms are strategyproof for either majority or minority students.
- We explore the compatibility of welfare and fairness axioms in view of the properties of these two mechanisms.

Further Research Questions

- Prove or disprove the impossibility conjectures?
- Possibility and impossibility results using different fairness axioms, such as Minority Fairness or Weak Minority Seat Fairness (drop 3. in Minority Seat Fairness)?
- Possibility and impossibility results involving incentives?

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