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Kazuo Murota
The Institute of Statistical Mathematics and
Tokyo Metropolitan University

Akihisa Tamura
Keio University

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Kazuo Murota* and Akihisa Tamura†

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Abstract

The Shapley–Folkman theorem is a statement about the Minkowski sum of (non-convex) sets, expressing the closeness of the Minkowski sum to convexity in a quantitative manner. This paper establishes similar theorems for integrally convex sets and $\mathrm{M}^\natural$-convex sets, which are major classes of discrete convex sets in discrete convex analysis.

Keywords: Discrete convex analysis, Integrally convex set, $\mathrm{M}^\natural$-convex set, Minkowski sum, Shapley–Folkman theorem

Contents

*The Institute of Statistical Mathematics, Tokyo 190-8562, Japan; and Faculty of Economics and Business Administration, Tokyo Metropolitan University, Tokyo 192-0397, Japan, murota@tmu.ac.jp

†Department of Mathematics, Keio University, Yokohama 223-8522, Japan, aki-tamura@math.keio.ac.jp

1 Introduction

The Shapley–Folkman theorem is concerned with the Minkowski sum of (non-convex) sets and expresses the closeness of the Minkowski sum to convexity in a quantitative manner. The theorem was first discovered in the literature of economics (Arrow–Hahn [1], Starr [16, 17]) and found applications also in optimization (Aubin–Ekeland [2], Ekeland–Témam [5], Bertsekas [3, 4]) and other fields of mathematics (Fradelizi–Madiman–Marsiglietti–Zvavitch [7]).

To describe the Shapley–Folkman theorem we need to introduce some terminology and notation. The *Minkowski sum* (or *vector sum*) of sets $S_1, S_2 \subseteq \mathbb{R}^n$ means the subset of $\mathbb{R}^n$ defined by

$$S_1 + S_2 = \{x + y \mid x \in S_1,\ y \in S_2\}. \tag{1.1}$$

This operation can natually be extended to the Minkowski sum $\sum_{i=1}^{m} S_i = S_1 + S_2 + \cdots + S_m$ of an arbitrary number of sets $S_i \subseteq \mathbb{R}^n$ $(i = 1, 2, \ldots, m)$. The Minkowski sum of convex sets is again convex. For any subset S of $\mathbb{R}^n$, we denote its *convex hull* by $\overline{S}$, which is, by definition, the smallest convex set containing S. As is well known, $\overline{S}$ coincides with the set of all convex combinations of (finitely many) elements of S. It is known that $\overline{S_1 + S_2 + \cdots + S_m} = \overline{S_1} + \overline{S_2} + \cdots + \overline{S_m}$.

For any set S $(\subseteq \mathbb{R}^n)$, the *radius* $\mathrm{rad}(S)$ and the *inner radius* $r(S)$ are defined by

$$\mathrm{rad}(S) = \inf_{x \in \mathbb{R}^n} \sup_{y \in S} \|x - y\|_2, \tag{1.2}$$

$$r(S) = \sup_{x \in \overline{S}} \inf_{T} \{\mathrm{rad}(T) \mid T \subseteq S, x \in \overline{T}\}. \tag{1.3}$$

The inner radius $r(S)$ expresses the size of holes or dents in S, and we have $r(S) = 0$ for a convex set S.

The following theorem [1, Theorem B.10] expresses the closeness of the Minkowski sum of (non-convex) sets to convexity in a quantitative manner. This theorem is often referred to as the Shapley–Folkman–Starr theorem, as it was derived by Starr [16] from the Shapley–Folkman theorem [1, Theorem B.9] as a (non-trivial) corollary.

Theorem 1.1 (Shapley–Folkman–Starr)**.** *Let S_i $(i = 1, 2, \ldots, m)$ be compact subsets of $\mathbb{R}^n$ such that $r(S_i) \leq L$ for $i = 1, 2, \ldots, m$ for some $L \in \mathbb{R}$. Let $W = S_1 + S_2 + \cdots + S_m$. For any $x \in \overline{W}$, there exists $z \in W$ that satisfies $\|x - z\|_2 \leq L\sqrt{\min(n, m)}$.* ■

A key fact used in the proof of Theorem 1.1 is the following theorem, which formulates a phenomenon in the Minkowski summation that may be compared to Carathéodory's theorem for convex combinations.

Theorem 1.2 (Shapley–Folkman)**.** *Let $S_i \subseteq \mathbb{R}^n$ $(i = 1, 2, \ldots, m)$, and $W = S_1 + S_2 + \cdots + S_m$. For any $x \in \overline{W}$, there exists a subset I of the index set $\{1, 2, \ldots, m\}$ such that $|I| \leq \min(n, m)$ and $x \in \overline{\sum_{i \in I} S_i} + \sum_{j \in J} S_j$, where $J = \{1, 2, \ldots, m\} \setminus I$.* ■

Theorem 1.2 is ascribed to Shapley and Folkman in [1, Theorem B.8], and is often referred to as the Shapley–Folkman lemma. Although the statement of [1, Theorem B.8] involves an assumption of compactness of each S_i, it is possible to avoid this assumption by using an algebraic proof based on Carathéodory's theorem (Bertsekas [3, Proposition 5.7.1], Fradelizi–Madiman–Marsiglietti–Zvavitch [7, Lemma 2.3]). Alternative proofs of Theorem 1.2 can be found in Ekeland–Témam [5, Appendix I] (without the compactness assumption) and Howe [9] (under the compactness assumption).

The objective of this paper is to establish theorems similar to Theorem 1.1 in the context of discrete convex analysis [8, 10, 11, 12]. Section 2 is devoted to the preliminaries from discrete convex analysis, and the main results are described in Section 3. Theorems 3.1 and 3.2 give two variants of the Shapley–Folkman-type theorem for integrally convex sets, and Theorem 3.4 deals with $\mathrm{M}^\natural$-convex sets. The proofs are given in Section 4, where Theorem 1.2 is used.

2 Preliminaries from Discrete Convex Analysis

2.1 Integrally convex sets

Integral convexity is a fundamental concept in discrete convex analysis, introduced by Favati–Tardella [6] for functions defined on the integer lattice $\mathbb{Z}^n$. In this paper we use the concept of integrally convex sets, as formulated in [11, Section 3.4] as a special case of integrally convex functions. The reader is referred to [14] for a recent comprehensive survey on integral convexity.

For $x \in \mathbb{R}^n$ the *integral neighborhood* of x is defined by

$$N(x) = \{z \in \mathbb{Z}^n \mid |x_i - z_i| < 1 \ (i = 1, 2, \ldots, n)\}. \tag{2.1}$$

It is noted that strict inequality " $<$ " is used in this definition and $N(x)$ admits an alternative expression

$$N(x) = \{z \in \mathbb{Z}^n \mid \lfloor x_i \rfloor \leq z_i \leq \lceil x_i \rceil \ \ (i = 1, 2, \ldots, n)\}, \tag{2.2}$$

where, for $t \in \mathbb{R}$ in general, $\lfloor t \rfloor$ denotes the largest integer not larger than t (rounding-down to the nearest integer) and $\lceil t \rceil$ is the smallest integer not smaller than t (rounding-up to the nearest integer). That is, $N(x)$ consists of all integer vectors z between $\lfloor x \rfloor = (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor)$ and $\lceil x \rceil = (\lceil x_1 \rceil, \lceil x_2 \rceil, \ldots, \lceil x_n \rceil)$.

For a set $S \subseteq \mathbb{Z}^n$ and $x \in \mathbb{R}^n$ we call the convex hull of $S \cap N(x)$ the *local convex hull* of S around x. A nonempty set $S \subseteq \mathbb{Z}^n$ is said to be *integrally convex* if the union of the local convex hulls $\overline{S \cap N(x)}$ over $x \in \mathbb{R}^n$ is convex. In other words, a set $S \subseteq \mathbb{Z}^n$ is called integrally convex if

$$\overline{S} = \bigcup_{x \in \mathbb{R}^n} \overline{S \cap N(x)}. \tag{2.3}$$

This condition is equivalent to saying that every point x in the convex hull of S is contained in the convex hull of $S \cap N(x)$, i.e.,

$$x \in \overline{S} \implies x \in \overline{S \cap N(x)}. \tag{2.4}$$

Obviously, every subset of $\{0, 1\}^n$ is integrally convex.

We say that a set $S \subseteq \mathbb{Z}^n$ is *hole-free* if

$$S = \overline{S} \cap \mathbb{Z}^n. \tag{2.5}$$

It is known that an integrally convex set is hole-free; see [14, Proposition 2.2] for a formal proof.

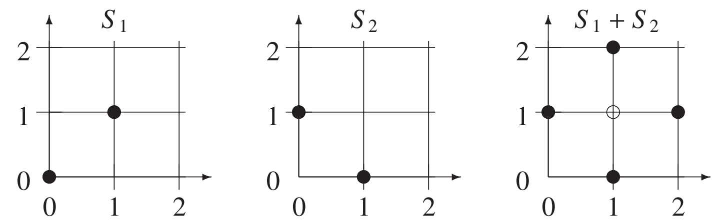

Figure 1: Minkowski sum of discrete sets

2.2 Minkowski sum in discrete convex analysis

Minkowski summation is an intriguing operation in discrete setting. The naive looking relation

$$S_1 + S_2 = \overline{(S_1 + S_2)} \cap \mathbb{Z}^n \tag{2.6}$$

is not always true, as Example 2.1 below shows. It may be said that if (2.6) is true for some class of discrete convex sets, this equality captures a certain essence of the discrete convexity in question.

Example 2.1 ([11, Example 3.15]). The Minkowski sum of $S_1 = \{(0,0),(1,1)\}$ and $S_2 = \{(1,0),(0,1)\}$ is equal to $S_1 + S_2 = \{(1,0),(0,1),(2,1),(1,2)\}$, for which $(1,1) \in \overline{(S_1 + S_2)} \setminus (S_1 + S_2)$. That is, the Minkowski sum $S_1 + S_2$ has a 'hole' at $(1,1)$. See Figure 1. ■

In Example 2.1 above, both S_1 and S_2 are integrally convex. This shows that (2.6) is not guaranteed for integrally convex sets and that the Minkowski sum of integrally convex sets is not necessarily integrally convex.

A subclass of integrally convex sets, called M$^\natural$-convex sets, is well-behaved with respect to Minkowski summation. A set $S \subseteq \mathbb{Z}^n$ is called *M$^\natural$-convex* if it enjoys the following exchange property:

> For any $x, y \in S$ and $i \in \{1,2,\ldots,n\}$ with $x_i > y_i$, we have
> (i) $x - \mathbf{1}^i \in S$, $y + \mathbf{1}^i \in S$ or
> (ii) there exists some $j \in \{1,2,\ldots,n\}$ such that $x_j < y_j$, $x - \mathbf{1}^i + \mathbf{1}^j \in S$, and $y + \mathbf{1}^i - \mathbf{1}^j \in S$,

where $\mathbf{1}^i$ denotes the ith unit vector for $i = 1,2,\ldots,n$. It is known that the Minkowski sum of M$^\natural$-convex sets is M$^\natural$-convex ([11, Section 4.6], [11, Theorem 6.15], [13, Theorem 3.13]). The following theorem states this fact.

Theorem 2.1. *The Minkowski sum $W = S_1 + S_2 + \cdots + S_m$ of M$^\natural$-convex sets $S_i \subseteq \mathbb{Z}^n$ $(i = 1,2,\ldots,m)$ is an M$^\natural$-convex set.* ■

Corollary 2.2. *For the Minkowski sum $W = S_1 + S_2 + \cdots + S_m$ of M$^\natural$-convex sets $S_i \subseteq \mathbb{Z}^n$ $(i = 1,2,\ldots,m)$, we have $\overline{W} \cap \mathbb{Z}^n = W$.*

Proof. W is an M$^\natural$-convex set by Theorem 2.1. Any M$^\natural$-convex set is an integrally convex set, for which (2.5) holds. □

See [13, Section 3.5] for the Minkowski sum operation for other kinds of discrete convex sets (such as L$^\natural$-convex sets, multimodular sets, and discrete midpoint convex sets). In particular, we mention that the Minkowski sum of two L$^\natural$-convex sets is not necessarily L$^\natural$-convex but it is integrally convex. Hence (2.6) is true for two L$^\natural$-convex sets. It is also noted that the Minkowski sum of three L$^\natural$-convex sets is no longer integrally convex.

3 Results

In this section we present our main results, the Shapley–Folkman-type theorems for integrally convex sets and for M$^\natural$-convex sets. To state the theorems we need to define functions

$$\alpha(n,m) = (1 - \frac{1}{n}) \min(n,m), \qquad \beta(n,m) = \frac{1}{2}\sqrt{n \cdot \min(n,m)}, \tag{3.1}$$

where n is the dimension of the space and m is the number of Minkowski summands. The proofs are given in Section 4.

Theorem 3.1. *Let $S_i \subseteq \mathbb{Z}^n$ $(i = 1, 2, \ldots, m)$ be integrally convex sets and $W = S_1 + S_2 + \cdots + S_m$, where $n \geq 2$. For any $x \in \overline{W}$, there exists $z \in W$ that satisfies $\|x - z\|_\infty \leq \alpha(n,m)$. If $x \in \overline{W} \cap \mathbb{Z}^n$, in particular, then $\|x - z\|_\infty \leq \lfloor \alpha(n,m) \rfloor = \min(n,m) - 1$.* ■

Theorem 3.2. *Let $S_i \subseteq \mathbb{Z}^n$ $(i = 1, 2, \ldots, m)$ be integrally convex sets and $W = S_1 + S_2 + \cdots + S_m$. For any $x \in \overline{W}$, there exists $z \in W$ that satisfies $\|x - z\|_2 \leq \beta(n,m)$ (and hence $\|x - z\|_\infty \leq \beta(n,m)$). If $x \in \overline{W} \cap \mathbb{Z}^n$, in particular, then $\|x - z\|_\infty \leq \lfloor \beta(n,m) \rfloor$.* ■

Example 3.1. In Figure 1 (Example 2.1), we have $n = 2$, $m = 2$, $\alpha(n,m) = \beta(n,m) = 1$. For $x = (1,1) \in \overline{S_1 + S_2}$, which is a 'hole,' we can take $z = (1,0) \in S_1 + S_2$ satisfying $\|x - z\|_\infty \leq 1$. ■

A combination of Theorems 3.1 and 3.2 implies that, for any $x \in \overline{W}$, there exists $z \in W$ that satisfies

$$\|x - z\|_\infty \leq \min\{\alpha(n,m), \beta(n,m)\} \qquad (n \geq 2, m \geq 1); \tag{3.2}$$

if $x \in \overline{W} \cap \mathbb{Z}^n$, in particular, then

$$\|x - z\|_\infty \leq \min\{\lfloor \alpha(n,m) \rfloor, \lfloor \beta(n,m) \rfloor\} \qquad (n \geq 2, m \geq 1). \tag{3.3}$$

The following proposition determines which is smaller between $\alpha(n,m)$ and $\beta(n,m)$ depending on (n,m). The proof is given in Section 4.3. Roughly speaking, $\alpha(n,m)$ is smaller when m is small, and $\beta(n,m)$ is smaller when m is large.

Proposition 3.3.
(1) *Case of $n = 2$: $\alpha(2,m) = \beta(2,m) = 1$ for all $m \geq 2$.*
(2) *Case of $m = 1$: $\alpha(n,1) < \beta(n,1)$ for all $n \geq 2$.*
(3) *Case of $m \geq 2$: $\alpha(n,m) > \beta(n,m)$ if $3 \leq n \leq 4m - 3$, and $\alpha(n,m) < \beta(n,m)$ if $n \geq 4m - 2$.*
■

The values of $\lfloor \alpha(n,m) \rfloor$ and $\lfloor \beta(n,m) \rfloor$ used in (3.3) for an integral point x are shown below. For each (n,m), the smaller of the two is in boldface.

	$m = 1$		$m = 2$		$m = 3$		$m = 4$		$m = 5$	
	$\lfloor \alpha \rfloor$	$\lfloor \beta \rfloor$	$\lfloor \alpha \rfloor$	$\lfloor \beta \rfloor$	$\lfloor \alpha \rfloor$	$\lfloor \beta \rfloor$	$\lfloor \alpha \rfloor$	$\lfloor \beta \rfloor$	$\lfloor \alpha \rfloor$	$\lfloor \beta \rfloor$
$n = 2$	0	0	1	1	1	1	1	1	1	1
$n = 3$	0	0	1	1	2	**1**	2	**1**	2	**1**
$n = 4$	**0**	1	1	1	2	**1**	3	**2**	3	**2**
$n = 8$	**0**	1	**1**	2	2	2	3	**2**	4	**3**
$n = 12$	**0**	1	**1**	2	**2**	3	3	3	4	**3**
$n = 16$	**0**	2	**1**	2	**2**	3	**3**	4	4	4

The particular case of Theorem 3.1 for $m = 1$ is worthy of attention. For $m = 1$, we have $\alpha(n, 1) = 1 - 1/n$ for $n \geq 2$, and hence $\lfloor \alpha(n, 1) \rfloor = 0$ for all $n \geq 2$. The latter (i.e., $\lfloor \alpha(n, 1) \rfloor = 0$) corresponds to the fact that $S = \overline{S} \cap \mathbb{Z}^n$ for an integrally convex set S. A combination of the former (i.e., $\alpha(n, 1) = 1 - 1/n$) with Theorem 2.1 results in a sharp bound for the case of M$^\natural$-convex summands S_i.

Theorem 3.4. *Let $S_i \subseteq \mathbb{Z}^n$ $(i = 1, 2, \ldots, m)$ be M$^\natural$-convex sets and $W = S_1 + S_2 + \cdots + S_m$, where $n \geq 2$. For any $x \in \overline{W}$, there exists $z \in W$ that satisfies $\|x - z\|_\infty \leq 1 - 1/n$.*

Proof. Since the Minkowski sum of M$^\natural$-convex sets remains to be M$^\natural$-convex (Theorem 2.1), W is an M$^\natural$-convex set, and hence it is an integrally convex set. By Theorem 3.1 with $m = 1$, there exists $z \in W$ that satisfies $\|x - z\|_\infty \leq \alpha(n, 1) = 1 - 1/n$. □

Remark 3.1. A recent paper by Nguyen–Vohra [15] gives an interesting variant of the Shapley–Folkman theorem. A polytope P with vertices in $\{0, 1\}^n$ is called Δ-*uniform* if each edge of P, say, $v - u$ with $v, u \in \{0, 1\}^n$, has ℓ_1-norm at most Δ. The theorem of Nguyen and Vohra (to be called "Theorem NV" here) states the following: Let S_i $(i = 1, 2, \ldots, m)$ be subsets of $\{0, 1\}^n$ such that each $\overline{S_i}$ is Δ-uniform, and let $W = S_1 + S_2 + \cdots + S_m$. Then, for any $x \in \overline{W} \cap \mathbb{Z}^n$, there exists $z \in W$ that satisfies $\|x - z\|_\infty \leq \Delta - 1$. The following comparisons may be made between Theorem NV and our result (Theorem 3.1).

- Theorem NV deals exclusively with integral vectors x in $\overline{W}$, while Theorem 3.1 can cope with real vectors x as well.
- For any summand sets $S_i \subseteq \{0, 1\}^n$ $(i = 1, 2, \ldots, m)$, we can take $\Delta = n$ and Theorem NV affords a bound $\|x - z\|_\infty \leq n - 1$, while Theorem 3.1 gives $\|x - z\|_\infty \leq \lfloor \alpha(n, m) \rfloor = \min(n, m) - 1$. When $n \leq m$, the two bounds coincide, whereas Theorem 3.1 gives a better bound if $n > m$.
- Theorem NV captures a property of summand sets S_i in terms of edge vectors, while Theorem 3.1 exploits no specific properties. Recall that any subset of $\{0, 1\}^n$ is integrally convex.
- When each summand S_i is an M$^\natural$-convex set contained in $\{0, 1\}^n$ (e.g., arising from the independent sets of a matroid), we have $\Delta = 2$ and Theorem NV gives a bound $\|x - z\|_\infty \leq 1$. However, we can actually assert that $\|x - z\|_\infty = 0$, since every $x \in \overline{W} \cap \mathbb{Z}^n$ belongs to W in this case (see Corollary 2.2).
- When each summand S_i arises from a delta-matroid (see [8, Section 3.5(b)] for the definition), we have $\Delta = 2$ and Theorem NV gives a bound $\|x - z\|_\infty \leq 1$, which is new (to the best knowledge of the authors). ■

4 Proofs

4.1 Proofs of Theorems 3.1 and 3.2

In this section we prove the main theorems (Theorems 3.1 and 3.2) of this paper. For the proof Theorem 3.1, we need the following lemma concerning a subset of $\{0, 1\}^n$ in general, which may be useful in some other contexts.

Lemma 4.1. *Let $S \subseteq \{0,1\}^n$, where $n \geq 2$. For any $x \in \overline{S}$, there exists $v^* \in S$ that satisfies*

$$\|x - v^*\|_\infty \leq 1 - \frac{1}{n}. \tag{4.1}$$

Proof. The proof is given in Section 4.2. □

Remark 4.1. The bound $\|x - v^*\|_\infty \leq 1 - 1/n$ in Lemma 4.1 is tight. For example, for $S = \{\mathbf{1}^i \mid i = 1,2,\ldots,n\} = \{(1,0,0,\ldots,0,0),(0,1,0,\ldots,0,0),\ldots,(0,0,0,\ldots,0,1)\}$ and $x = (1/n, 1/n, \ldots, 1/n) \in \overline{S}$, we have $\|x - v\|_\infty = 1 - 1/n$ for all $v \in S$. ■

We can prove Theorem 3.1 as follows. Since

$$x \in \overline{S_1 + S_2 + \cdots + S_m} = \overline{S_1} + \overline{S_2} + \cdots + \overline{S_m},$$

the vector x can be represented as a convex combination of some elements of $\overline{S_1}, \overline{S_2}, \ldots, \overline{S_m}$. That is,

$$x = \sum_{i=1}^{m} y^i \tag{4.2}$$

for some $y^i \in \overline{S_i}$ $(i = 1,2,\ldots,m)$. Let

$$T_i = S_i \cap N(y^i) \tag{4.3}$$

for $i = 1,2,\ldots,m$, where $N(y^i)$ is the integral neighborhood of y^i defined in (2.1). Since each S_i is integrally convex, we may use (2.4) to obtain $y^i \in \overline{S_i \cap N(y^i)} = \overline{T_i}$. Then (4.2) shows $x \in \overline{T_1 + T_2 + \cdots + T_m}$.

By Theorem 1.2 (Shapley–Folkman's lemma) there exists $I \subseteq \{1,2,\ldots,m\}$ such that $|I| \leq \min(n,m)$ and $x \in \sum_{i \in I} \overline{T_i} + \sum_{j \in J} T_j$, where $J = \{1,2,\ldots,m\} \setminus I$. Therefore,

$$x = \sum_{i \in I} x^i + \sum_{j \in J} z^j$$

for some $x^i \in \overline{T_i}$ $(i \in I)$ and $z^j \in T_j$ $(j \in J)$. Lemma 4.1 implies that, for each $i \in I$, there exists $v^i \in T_i$ satisfying $\|x^i - v^i\|_\infty \leq 1 - 1/n$. Define

$$z = \sum_{i \in I} v^i + \sum_{j \in J} z^j,$$

which belongs to $T_1 + T_2 + \cdots + T_m$ $(\subseteq S_1 + S_2 + \cdots + S_m = W)$. We then have

$$\|x - z\|_\infty = \| \sum_{i \in I} (x^i - v^i) \|_\infty \leq \sum_{i \in I} \|x^i - v^i\|_\infty \leq \left(1 - \frac{1}{n}\right) |I| \leq \alpha(n,m).$$

Finally, if $x \in \overline{W} \cap \mathbb{Z}^n$, we have $\mathbb{Z} \ni \|x - z\|_\infty \leq \alpha(n,m)$, whereas $\lfloor \alpha(n,m) \rfloor = \min(n,m) - 1$. This completes the proof of Theorem 3.1.

The proof of Theorem 3.2 is as follows. Each T_i in (4.3) is contained in a translated unit cube, that is, $T_i \subseteq a^i + \{0,1\}^n$ for some $a^i \in \mathbb{Z}^n$, from which follows that $r(T_i) = \mathrm{rad}(T_i) \leq \sqrt{n}/2$ for $i = 1,2,\ldots,m$. Hence we can take $L = \sqrt{n}/2$ in Theorem 1.1 (Shapley–Folkman–Starr theorem), to obtain

$$\|x - z\|_2 \leq L \sqrt{\min(n,m)} = (\sqrt{n}/2) \sqrt{\min(n,m)} = \beta(n,m).$$

Finally, if $x \in \overline{W} \cap \mathbb{Z}^n$, we have $\mathbb{Z} \ni \|x - z\|_\infty \leq \|x - z\|_2 \leq \beta(n,m)$, from which $\|x - z\|_\infty \leq \lfloor \beta(n,m) \rfloor$. Thus Theorem 3.2 is proved.

4.2 Proof of Lemma 4.1

In this section we prove Lemma 4.1, which states that for any $x \in \overline{S}$, there exists $v^* \in S$ satisfying $\|x - v^*\|_\infty \leq 1 - 1/n$ in (4.1). Let $N = \{1, 2, \ldots, n\}$. Without loss of generality, we may assume that $x_i \geq 1/2$ for all $i \in N$. (If $I = \{i \in N \mid x_i < 1/2\}$ is nonempty, change x_i to $1 - x_i$ for all $i \in I$, and change S similarly.) Represent x as a convex combination of the points of S as $x = \sum_{u \in S} \lambda_u u$, where $\sum_{u \in S} \lambda_u = 1$ and $\lambda_u \geq 0$ $(u \in S)$. We first note the following fact.

Claim 1: If $\lambda_v \geq 1/n$ for some $v \in S$, then $\|x - v\|_\infty \leq 1 - 1/n$ for such v.
(Proof of Claim 1) Since

$$x - v = \sum_{u \in S} \lambda_u (u - v) = \sum_{u \neq v} \lambda_u (u - v),$$

we obtain

$$\begin{aligned} \|x - v\|_\infty &= \max_{i \in N} \{ \Big| \sum_{u \neq v} \lambda_u (u_i - v_i) \Big| \} \leq \max_{i \in N} \{ \sum_{u \neq v} \lambda_u |u_i - v_i| \} \\ &\leq \sum_{u \neq v} \lambda_u = 1 - \lambda_v \leq 1 - \frac{1}{n}. \end{aligned}$$

(End of proof of Claim 1)

To prove (4.1) by contradiction, we assume

$$\|x - v\|_\infty > 1 - \frac{1}{n} \quad \text{for all } v \in S. \tag{4.4}$$

We shall derive a contradiction as follows. We first define a partition of S into two subsets, $S = S_1^0 \cup S_1^1$, where S_1^1 is nonempty under (4.4). Then S_1^1 is partitioned into S_2^0 and S_2^1, where S_2^1 is nonempty under (4.4). Continuing this way, we obtain partitions of S of the form

$$\begin{aligned} S &= S_1^0 \cup S_1^1 = S_1^0 \cup (S_2^0 \cup S_2^1) \\ &= S_1^0 \cup S_2^0 \cup (S_3^0 \cup S_3^1) = \cdots = \left(\bigcup_{j=1}^{n-1} S_j^0 \right) \cup S_{n-1}^1, \end{aligned}$$

where $S_{j-1}^1 = S_j^0 \cup S_j^1$ and $S_j^1 \neq \emptyset$ for each $j = 1, 2, \ldots, n-1$ (with the convention of $S_0^1 = S$). At the final stage, we show that $S_{n-1}^1 \neq \emptyset$ leads to a contradiction to (4.4).

The first partition $S = S_1^0 \cup S_1^1$ is defined as follows. By (4.4) there exists $i_1 \in N$ and $u \in S$ satisfying $|x_{i_1} - u_{i_1}| > 1 - 1/n$, where $u_{i_1} = 0$ since $x_{i_1} \geq 1/2$ by our assumption. Thus we have

$$x_{i_1} > 1 - \frac{1}{n}. \tag{4.5}$$

With reference to the component i_1, we classify the vectors in S into two subsets:

$$S_1^0 = \{v \in S \mid v_{i_1} = 0\}, \quad S_1^1 = \{v \in S \mid v_{i_1} = 1\}. \tag{4.6}$$

Since $x_{i_1} = \sum_{v \in S_1^1} \lambda_v$, it follows from (4.5) that

$$\sum_{v \in S_1^1} \lambda_v > 1 - \frac{1}{n}, \qquad \sum_{v \in S_1^0} \lambda_v < \frac{1}{n}. \tag{4.7}$$

In particular, $S^1_1 \neq \emptyset$. It also follows from (4.5) that

$$\text{For every } v \in S^1_1: \quad |x_{i_1} - v_{i_1}| = 1 - x_{i_1} < \frac{1}{n} \leq 1 - \frac{1}{n}, \tag{4.8}$$

where $n \geq 2$ is used. Let $S^1_0 = S$.

Claim 2: For $j = 1, 2, \ldots, n-1$, we can choose an index $i_j \in N \setminus \{i_1, i_2, \ldots, i_{j-1}\}$ which defines a partition of S^1_{j-1} into two parts

$$S^0_j = \{v \in S^1_{j-1} \mid v_{i_j} = 0\}, \quad S^1_j = \{v \in S^1_{j-1} \mid v_{i_j} = 1\} \tag{4.9}$$

such that

$$x_{i_j} > 1 - \frac{1}{n}, \tag{4.10}$$

$$\text{For every } v \in S^1_j: \quad |x_{i_j} - v_{i_j}| = 1 - x_{i_j} \leq 1 - \frac{1}{n}, \tag{4.11}$$

$$\sum_{v \in S^1_j} \lambda_v > 1 - \frac{j}{n}, \qquad \sum_{v \in S^0_j} \lambda_v < \frac{1}{n}. \tag{4.12}$$

(Proof of Claim 2) For $j = 1$ we have (4.9)–(4.12) from (4.5)–(4.8). Assuming we have chosen $i_1, i_2, \ldots, i_j$ (where $j < n-1$) satisfying (4.9)–(4.12), we choose the next index i_{j+1} as follows. For each $v \in S^1_j$ we have $|x_{i_k} - v_{i_k}| \leq 1 - 1/n$ for $k = 1, 2, \ldots, j$ by (4.11) while $\|x - v\|_\infty > 1 - 1/n$ by (4.4). Hence there exists $i_{j+1} \in N \setminus \{i_1, i_2, \ldots, i_j\}$ and $u \in S^1_j$ satisfying $|x_{i_{j+1}} - u_{i_{j+1}}| > 1 - 1/n$, where $u_{i_{j+1}} = 0$ since $x_{i_{j+1}} \geq 1/2$ by our assumption. Thus we obtain

$$x_{i_{j+1}} > 1 - \frac{1}{n}, \tag{4.13}$$

which is (4.10) for $j+1$. With the use of this i_{j+1} we define a partition $S^1_j = S^0_{j+1} \cup S^1_{j+1}$ by (4.9) for $j+1$. Then $S = (S^0_1 \cup \cdots \cup S^0_j) \cup (S^0_{j+1} \cup S^1_{j+1})$ and

$$\begin{aligned} 1 - \frac{1}{n} < x_{i_{j+1}} &= \sum_{v \in S^1_{j+1}} \lambda_v + \sum_{k=1}^{j} \sum_{v \in S^0_k} \lambda_v v_{i_{j+1}} \\ &\leq \sum_{v \in S^1_{j+1}} \lambda_v + \sum_{k=1}^{j} \sum_{v \in S^0_k} \lambda_v \qquad (4.14) \\ &= 1 - \sum_{v \in S^0_{j+1}} \lambda_v. \qquad (4.15) \end{aligned}$$

The second inequality of (4.12) for $j+1$ follows from (4.15). In (4.14) we have $\sum_{v \in S^0_k} \lambda_v \leq 1/n$ for $k = 1, 2, \ldots, j$ by the second inequality of (4.12), and therefore,

$$1 - \frac{1}{n} < \sum_{v \in S^1_{j+1}} \lambda_v + \frac{j}{n}.$$

Thus we obtain

$$\sum_{v \in S^1_{j+1}} \lambda_v > 1 - \frac{j+1}{n},$$

which is the first inequality of (4.12) for $j+1$. For every $v \in S^1_{j+1}$ we have (4.13) and $v_{i_{j+1}} = 1$, from which we obtain

$$|x_{i_{j+1}} - v_{i_{j+1}}| = 1 - x_{i_{j+1}} < \frac{1}{n} \leq 1 - \frac{1}{n},$$

showing (4.11) for $j + 1$. (End of proof of Claim 2)

By (4.12) for $j = n - 1$, we have $S^1_{n-1} \neq \emptyset$. Since $S^1_{n-1} \subseteq S^1_j$ for all $j \leq n - 1$, any $v \in S^1_{n-1}$ has the property that $v_{i_k} = 1$ for $k = 1, 2, \dots, n - 1$, and $v_{i_n} \in \{0, 1\}$. If S^1_{n-1} contains $v^* = (1, 1, \dots, 1)$, this vector satisfies $\|x - v^*\|_\infty \leq 1 - 1/n$, since

$$|x_{i_j} - v^*_{i_j}| = 1 - x_{i_j} \leq 1 - \frac{1}{n} \qquad (j = 1, 2, \dots, n - 1)$$

by (4.11) and

$$|x_{i_n} - v^*_{i_n}| = 1 - x_{i_n} \leq \frac{1}{2} \leq 1 - \frac{1}{n}.$$

This contradicts (4.4). Otherwise, S^1_{n-1} consists of a unique element u^* with $u^*_{i_n} = 0$ and $u^*_i = 1$ for $i \neq i_n$. By the first inequality of (4.12) for $j = n - 1$ we have $\lambda_{u^*} > 1 - (n - 1)/n = 1/n$, which, by Claim 1, implies $\|x - u^*\|_\infty \leq 1 - 1/n$, which is also a contradiction to (4.4). The proof of Lemma 4.1 is thus completed.

4.3 Proof of Proposition 3.3

In this section we prove Proposition 3.3 to determine which is smaller between $\alpha(n, m)$ and $\beta(n, m)$.

(1) When $n = 2$ and $m \geq 2$, we have

$$\alpha(2, m) = \left(1 - \frac{1}{2}\right) \min(2, m) = 1, \quad \beta(2, m) = \frac{1}{2} \sqrt{2 \cdot \min(2, m)} = 1.$$

(2) When $m = 1$ and $n \geq 2$, we have

$$\alpha(n, 1) = \left(1 - \frac{1}{n}\right) \min(n, 1) = 1 - \frac{1}{n}, \quad \beta(n, 1) = \frac{1}{2} \sqrt{n \cdot \min(n, 1)} = \frac{1}{2} \sqrt{n}.$$

When $n = 2$, we have $\alpha(2, 1) = 1/2$, $\beta(2, 1) = \sqrt{2}/2 = 0.7...$, and hence $\alpha(2, 1) < \beta(2, 1)$. When $n = 3$, we have $\alpha(3, 1) = 2/3$, $\beta(3, 1) = \sqrt{3}/2 = 0.86...$, and hence $\alpha(3, 1) < \beta(3, 1)$. When $n \geq 4$, we have $\alpha(n, 1) < 1$, $\beta(n, 1) = \frac{1}{2}\sqrt{n} \geq 1$, and hence $\alpha(n, 1) < \beta(n, 1)$.

(3) The claim is concerned with the cases with $m \geq 2$ and $n \geq 3$. The combination of Case A and Case B below covers all such cases.

Case A: When $n \geq 3$ and $n \leq m$, we have

$$\alpha(n, m) = \left(1 - \frac{1}{n}\right) n = n - 1, \quad \beta(n, m) = \frac{1}{2} \sqrt{n \cdot n} = \frac{n}{2}.$$

Therefore, $\alpha(n, m) > \beta(n, m)$.

Case B: When $n \geq 3$, $m \geq 2$, and $m < n$, we have

$$\alpha(n, m) = \left(1 - \frac{1}{n}\right) m, \qquad \beta(n, m) = \frac{1}{2} \sqrt{n \cdot m}.$$

Therefore, we have

$$\alpha < \beta \iff \left(1 - \frac{1}{n}\right)m < \frac{1}{2}\sqrt{n \cdot m} \iff \sqrt{m} < \frac{\sqrt{n}}{2}\frac{1}{1 - 1/n} \iff m < \frac{n^3}{4(n-1)^2}. \tag{4.16}$$

Define

$$\theta(n) = \frac{n^3}{4(n-1)^2}. \tag{4.17}$$

Since $\theta(n)$ is not an integer for any integer $n \geq 3$, we have that $\alpha \neq \beta$ for all (n, m), and that

$$\alpha < \beta \iff m < \theta(n), \qquad \alpha > \beta \iff m > \theta(n). \tag{4.18}$$

Case B-1: When $n = 3$, we have $\theta(3) = 27/16 = 1.6875$, and hence $\alpha(3, 2) > \beta(3, 2)$ by (4.18). Note that $\{m \in \mathbb{Z} \mid m \geq 2, m < n\}$ consists of $m = 2$ only.

Case B-2: When $n = 4$, we have $\theta(4) = 16/9 = 1.77...$, and hence $\alpha(4, m) > \beta(4, m)$ for $m = 2, 3$. Note that $\{m \in \mathbb{Z} \mid m \geq 2, m < n\}$ consists of $m = 2, 3$ only.

Case B-3: When $n \geq 5$, the threshold value $\theta(n)$ can be estimated as

$$\frac{n+2}{4} < \frac{n^3}{4(n-1)^2} < \frac{n+3}{4} \qquad (n \geq 5). \tag{4.19}$$

Indeed, the first inequality of (4.19) holds since

$$\frac{n+2}{4} < \frac{n^3}{4(n-1)^2} \iff (n+2)(n-1)^2 < n^3 \iff 3n > 2,$$

and the second inequality of (4.19) follows from

$$\frac{n^3}{4(n-1)^2} < \frac{n+3}{4} \iff n^3 < (n+3)(n-1)^2 > 0 \iff n^2 - 5n + 3 > 0$$

and $n^2 - 5n + 3 = n(n-5) + 3 > 0$. It follows from (4.18) and (4.19) that

$$\begin{aligned} \alpha < \beta \quad &\text{if} \quad n \geq 5, 2 \leq m \leq (n+2)/4, \\ \alpha > \beta \quad &\text{if} \quad n \geq 5, (n+3)/4 \leq m < n, \end{aligned}$$

or equivalently,

$$\begin{aligned} \alpha < \beta \quad &\text{if} \quad n \geq 5, 2 \leq m, n \geq 4m - 2, \\ \alpha > \beta \quad &\text{if} \quad n \geq 5, 2 \leq m < n \leq 4m - 3. \end{aligned}$$

This completes the proof of Proposition 3.3.

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References

[1] Arrow, K.J., Hahn, F.H.: General Competitive Analysis. Holden–Day, San Francisco (1971)

[2] Aubin, J.P., Ekeland, I.: Estimates of the duality gap in nonconvex optimization. Mathematics of Operations Research **1**, 225–245 (1976)

[3] Bertsekas, D.P.: Convex Optimization Theory. Athena Scientific, Belmont, Mass. (2009)

[4] Bertsekas, D.P.: Nonlinear Programming, 3rd edn. Athena Scientific, Belmont, Mass. (2016)

[5] Ekeland, I., Témam, I.: Convex Analysis and Variational Problems. SIAM, Philadelphia (1999)

[6] Favati, P., Tardella, F.: Convexity in nonlinear integer programming. Ricerca Operativa **53**, 3–44 (1990)

[7] Fradelizi, M., Madiman, M., Marsiglietti, A., Zvavitch, A.: The convexification effect of Minkowski summation. EMS Surveys in Mathematical Sciences **5**, 1–64 (2018)

[8] Fujishige, S.: Submodular Functions and Optimization, 2nd edn. Elsevier, Amsterdam (2005)

[9] Howe, R.E.: On the tendency toward convexity of the vector sum of sets. In: Terlaky, T., Curtis, F.E. (eds.) Modeling and Optimization: Theory and Applications (Springer Proceedings in Mathematics & Statistics 21), pp. 67–76. Springer, New York (2012)

[10] Murota, K.: Discrete convex analysis. Mathematical Programming **83**, 313–371 (1998)

[11] Murota, K.: Discrete Convex Analysis. Society for Industrial and Applied Mathematics, Philadelphia (2003)

[12] Murota, K.: Discrete convex analysis: A tool for economics and game theory. Journal of Mechanism and Institution Design **1**, 151–273 (2016)

[13] Murota, K.: A survey of fundamental operations on discrete convex functions of various kinds. Optimization Methods and Software **36**, 472–518 (2021)

[14] Murota, K., Tamura, A.: Recent progress on integrally convex functions. Japan Journal of Industrial and Applied Mathematics, Published online: April 27, 2023. DOI 10.1007/s13160-023-00589-4 (open access)

[15] Nguyen, T., Vohra, R.: Δ-substitute preferences and equilibria with indivisibilities. Proceedings of the 22nd ACM Conference on Economics and Computation (EC'21), 2021, pp. 735–736. https://doi.org/10.1145/3465456.3467575

[16] Starr, R.M.: Quasi-equilibria in markets with non-convex preferences. Econometrica **17**, 25–38 (1969)

[17] Starr, R. M.: Shapley–Folkman theorem. In: Durlauf, S.N., Blume, L.E. (ed.), The New Palgrave Dictionary of Economics, 2nd ed. Palgrave Macmillan, London (2008) https://doi.org/10.1057/978-1-349-95121-5_1904-2