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Abstract

This study investigates efficient and strategy-proof mechanisms for allocating indivisible goods under constraints. First, we examine a scenario with endowments. We find that the *generalized matroid* is a necessary and sufficient condition on the constraint structure for the existence of a mechanism that is Pareto-efficient (PE), individually rational (IR), and strategy-proof (SP). We also demonstrate that a top trading cycles mechanism satisfies PE, IR, and group strategy-proofness (GSP) under any generalized matroid constraint. Second, we examine a scenario without endowments. In this scenario, we demonstrate that the serial dictatorship mechanism satisfies PE, IR, and GSP if the constraint satisfies a condition of *ordered accessibility*. Furthermore, we prove that *accessibility* is a necessary condition for the existence of PE, IR, and GSP mechanisms. Finally, we observe that any two out of the three properties—PE, IR, and GSP—could be achieved under general constraints.

1 Introduction

We study indivisible goods allocation problems, including real-life applications such as student placement in public schools [2], refugee resettlement [8], and student-project assignment [3]. Such applications are often subject to constraints. For instance, school districts may have specific requirements regarding the diversity of the student body in each school, including *type-specific quotas* and *proportional constraints*. Additionally, schools may have *minimal quotas* to determine the minimum number of students required for their operations. In the case of refugee resettlement, the central authority must consider factors such as heterogeneous family sizes and other requirements—for example, job training and language classes—resulting in *multidimensional knapsack constraints*. In student-project assignment problems, in which an instructor can offer multiple projects, certain subsets of projects may have *common quotas* because both projects and instructors have capacity constraints. By framing the allocation problem as a matching problem between students and schools, this study aims to identify the constraints under which a desirable mechanism can be designed.

Regarding the desirable properties of the mechanisms, we focus on Pareto efficiency for students (PE), individual rationality (IR), and strategy-proofness (SP). PE is a natural efficiency requirement because schools are indivisible goods to be allocated in our environment [2]. IR provides students with incentives to participate in the market. SP is a central concept of incentive compatibility in the market design literature. SP mechanisms are strategically simple, as students do not need to consider how other students will report because truthful reporting is a dominant strategy. Furthermore, owing to this simplicity, SP mechanisms are fair, as less informed or less sophisticated students are not disadvantaged relative to their more sophisticated peers. Indeed, SP has played a crucial role in designing reforms in practice, including school admission reforms in several cities [29, 30]. We also examine group strategy

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proofness (GSP), which entails a stronger requirement than SP. GSP mechanisms are robust to student group manipulation. In the context of school choice, instances of coordinated reporting to manipulate mechanisms have been documented in practice [29].

In this study, we consider two scenarios: one with and one without endowments. The applicability of either scenario in real-life applications depends on the specific circumstances. For instance, Delacrétaz et al. [8] suggested that refugee resettlement fits into either scenario. If resettlement agencies already operate established matching procedures, it is effective to improve the obtained baseline matching by utilizing preference information. In this context, the scenario with endowments is applicable. Conversely, if these procedures are absent or if resettlement agencies prefer to establish matching procedures from scratch, the scenario without endowments is suitable.

We first investigate the scenario with endowments in which IR requires that each student be assigned to a school that is at least as good as her endowment. In the standard model of unit capacity constraints, the top trading cycle (TTC) mechanism [35] satisfies PE, IR, and GSP. Suzuki et al. [38] extended this result to the cases where the distributional constraint forms an M-convex set. However, no mechanism can simultaneously achieve PE, IR, and SP under a general constraints. Specifically, Delacrétaz et al. [8] demonstrated that no mechanism satisfies PE, IR, and SP under multidimensional knapsack constraints. The following example, a simplified version of one found in [8], illustrates this fact.

Example 1. Suppose that there are three students $1, 2, 3$ and three schools s_1, s_2, s_3. The preference $\succ_i$ of each student i is given as follows:

$$\succ_1 = (s_3\, s_1\, s_2\, \varnothing), \quad \succ_2 = (s_3\, s_1\, s_2\, \varnothing), \quad \succ_3 = (s_2\, s_3\, \varnothing\, s_1).$$

Here, $\varnothing$ represents the outside option of being unmatched. For this preference, student 1 prefers school s_3 the most and least prefers the outside option $\varnothing$. The constraint $\mathcal{F}_s$ of each school s is given as follows:

$$\mathcal{F}_{s_1} = \big\{\emptyset, \{1\}, \{2\}, \{3\}\big\}, \quad \mathcal{F}_{s_2} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}\big\}, \quad \mathcal{F}_{s_3} = \big\{\emptyset, \{1\}, \{2\}, \{3\}\big\}.$$

Here, $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_3}$ are (unit) capacity constraints, whereas $\mathcal{F}_{s_2}$ is not. Indeed, $\{1, 2\}, \{3\} \in \mathcal{F}_{s_2}$ but $\{1, 3\}, \{2, 3\} \notin \mathcal{F}_{s_2}$. Constraints such as $\mathcal{F}_{s_2}$ appear as budget constraints (e.g., student 3 requires more scholarship money). The endowments of students 1 and 2 are s_2, and the endowment of student 3 is s_3.

It is not difficult to see that there exist only two PE and IR matchings: $\mu_1 = \{(1, s_3), (2, s_1), (3, s_2)\}$ and $\mu_2 = \{(1, s_1), (2, s_3), (3, s_2)\}$. Here, if student 1 misreports her preference as $\succ'_1 = (s_3\, s_2\, \varnothing\, s_1)$ whereas the other students report their true preferences, then μ_1 is a unique PE and IR matching. Similarly, if student 2 misreports her preference as $\succ'_2 = (s_3\, s_2\, \varnothing\, s_1)$ whereas the other students report their true preferences, then μ_2 is a unique PE and IR matching. Hence, in any PE and IR mechanism, either student 1 or 2 can be better off by misreporting their preference, depending on whether the outcome for true reporting is μ_1 or μ_2.

This example raises the question of which constraint structure is crucial for the existence of PE, IR, and SP mechanisms? We identify that *generalized matroid (g-matroid)* is a "necessary and sufficient" condition of constraints to guarantee existence. To establish the sufficiency of the g-matroid constraint, we modify the TTC-M mechanism introduced by Suzuki et al. [37]. Through this modification, we successfully demonstrate that the resulting mechanism possesses the desired properties under g-matroid constraints. This holds true even in the case of a distributional g-matroid constraint. Although our algorithm is based on the TTC-M, it has the advantage of a more general approach to model constraints. Our model handles not only the constraints previously explored by [37] but also a wider range of more complex constraints, as detailed in Section 1.1. To establish the necessity of the g-matroid condition, we demonstrate that if a school has a constraint that is not g-matroid (like $\mathcal{F}_{s_2}$ in Example 1), this implies the existence of a market without PE, IR, and SP mechanisms, even when the constraints of the other schools are limited to capacity constraints. Therefore, g-matroid is a necessary condition within a certain "maximal domain" sense. Note that the investigation of such a "maximal domain" has been employed in related studies such as [16, 19, 20].

Second, we examine the scenario without endowments, where IR requires that each student be assigned to a school that is at least as good as her outside option or remains unassigned. Two points should be noted. First, in this scenario, we assume that the empty matching (every student is unmatched) is feasible. No IR mechanism exists if the empty matching is infeasible. This is because no feasible matching is IR if the outside option is the best for every student. In Section 5, we demonstrate that a PE, IR, and

GSP mechanism always exists even when the empty matching is allowed to be infeasible, as long as the outside option is assumed to be the worst for every student. Second, the above necessary and sufficient condition of g-matroid holds only when arbitrary endowments are allowed. Hence, although the scenario without endowments can be viewed as a special case in which the endowments of all students are outside options, g-matroid is only a sufficient condition for the existence of a mechanism that is PE, IR, and SP. For the market in Example 1 without endowments, the Serial Dictatorship (SD) mechanism satisfies PE, IR, and GSP. Additionally, a modified version of the TTC mechanism under multidimensional knapsack constraints introduced by Delacrétaz et al. [8] satisfies PE, IR, and GSP if there are no endowments. More generally, $\mathcal{F}_{s_2}$ in Example 1, matroid constraints (i.e., g-matroid containing the empty set), and multidimensional knapsack constraints are classified within the general upper bound [20]. This class, also referred to as hereditary or downward-closed, yields the existence of PE, IR, and GSP mechanisms. However, no mechanism can simultaneously achieve PE, IR, and GSP under general constraints. The following example demonstrates this.

Example 2. Suppose there are two students $1, 2$ and two schools s_1, s_2. The constraint $\mathcal{F}_s$ on each school s is defined as follows:

$$\mathcal{F}_{s_1} = \{\emptyset, \{1,2\}\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{\emptyset, \{1\}, \{2\}\}.$$

The constraint $\mathcal{F}_{s_1}$ could appear as a proportional constraint; for example, that the number of male and female students who match with a particular school must be equal. This market does not admit a mechanism that simultaneously satisfies PE, IR, and SP.

To obtain a contradiction, suppose that ψ is a mechanism that satisfies PE, IR, and SP. We define 8 preference profiles as follows:

- $P^{(1)} = (s_2 \varnothing s_1,\ s_1 \varnothing s_2)$
- $P^{(2)} = (s_2\, s_1 \varnothing,\ s_1 \varnothing s_2)$
- $P^{(3)} = (s_2\, s_1 \varnothing,\ s_1\, s_2 \varnothing)$
- $P^{(4)} = (s_2 \varnothing s_1,\ s_1\, s_2 \varnothing)$
- $Q^{(1)} = (s_1 \varnothing s_2,\ s_2 \varnothing s_1)$
- $Q^{(2)} = (s_1 \varnothing s_2,\ s_2\, s_1 \varnothing)$
- $Q^{(3)} = (s_1\, s_2 \varnothing,\ s_2\, s_1 \varnothing)$
- $Q^{(4)} = (s_2 \varnothing s_1,\ s_2\, s_1 \varnothing)$

In profile $P^{(1)}$, student 1 prefers s_2, followed by $\varnothing$, and then s_1. Student 2 prefers s_1, followed by $\varnothing$, and then s_2. Based on PE and IR, we derive that $\psi[P^{(1)}] = \{(1, s_2)\}$. For profile $P^{(2)}$, the outcome $\psi[P^{(2)}]$ must be $\{(1, s_1), (2, s_1)\}$ or $\{(1, s_2)\}$ by PE and IR. However, $\psi[P^{(2)}] = \{(1, s_1), (2, s_1)\}$ is impossible by SP because it incentivizes student 1 to misreport so that the preference profile becomes $P^{(1)}$. Hence, we obtain $\psi[P^{(2)}] = \{(1, s_2)\}$. Similarly, we have $\psi[P^{(3)}] = \{(1, s_2)\}$ by PE, IR, and SP. This implies $\psi[P^{(4)}] = \{(1, s_2)\}$ by PE, IR, and SP.

Applying similar reasoning, we can determine that $\psi[Q^{(1)}] = \psi[Q^{(2)}] = \psi[Q^{(3)}] = \{(2, s_2)\}$. In addition, we can conclude that $\psi[Q^{(4)}] = \{(2, s_2)\}$ by PE, IR, and SP. However, this contradicts SP because it incentivizes student 2 to misreport at $P^{(4)}$.

Therefore, we investigate the structural requirements of the constraints for the existence of PE, IR, and SP mechanisms. The SD mechanism satisfies IR and GSP but does not always produce a PE matching (see Section 4.1). To ensure that the SD mechanism always yields a PE matching, the constraints must satisfy the *accessibility* condition, which states that if a school s can accept a nonempty subset of students X, then there must exist a student $i \in X$ such that the school s can accept $X \setminus \{i\}$. Note that the constraint $\mathcal{F}_{s_1}$ in Example 2 is not accessible. Furthermore, we prove that accessibility is a necessary condition to guarantee the existence of a mechanism that satisfies PE, IR, and GSP (Theorem 4). We provide a sufficient condition to ensure the existence of such mechanisms. Even when the constraints are accessible, PE, IR, and SP mechanisms may still not exist, as we will see in Example 4. This indicates that accessibility is an inadequate structure. To address this inadequacy, we propose a stronger form of accessibility, termed *σ-accessibility*, where σ is an order of students. The SD mechanism based on this order σ satisfies PE, IR, and GSP if the constraints are σ-accessible (Theorem 3). An example of a σ-accessible constraint occurs when a school requires that the number of minority students matched to it must not be less than half the number of majority students matched to it. This constraint is σ-accessible for an order σ in which the minority students are ahead of the majority students. Furthermore, the concept of a σ-accessible constraint is relevant to school choice in China [14]. Each district in China contains multiple schools. Although students can apply to schools in other districts, the government imposes limits on the proportion of cross-district students in schools. These constraints are not in the general upper bound class. In contrast, every general upper bound constraint is σ-accessible for any σ.

Subsequently, as a by-product of our results in the two scenarios, we provide implications for the desired mechanisms in refugee resettlement. As noted above, Delacrétaz et al. [8] studied PE, IR, and SP mechanisms under multidimensional knapsack constraints. They established the nonexistence of such mechanisms when endowments are present, but demonstrated their existence in a scenario without endowments. These results regarding the existence of PE, IR, and SP mechanisms can be obtained from our results. Although the multidimensional knapsack constraint may not be a g-matroid, it must be a general upper bound.

1.1 Related work

Our study is closely related to the three papers [13, 37, 38]. These studies explored scenarios with endowments and a generalized TTC, where the distributional constraint is represented by an M-convex set on the vector of the number of students assigned to each school.[1] Suzuki et al. [37, 38] proposed the TTC-M mechanism and proved that it is PE, IR, and GSP. This study makes two major contributions to the literature.

First, we identify that g-matroid is a necessary condition of the constraint structure for the existence of mechanisms that satisfy the three desirable properties. This finding partially addresses the open question posed by Suzuki et al. [38].

Second, our model generalizes theirs because constraints are imposed on the matched student-school pairs. An example of such constraints can be found in academic hiring, where each student (or applicant) has multiple labels based on their expertise, and each school (or university) provides an upper and lower quota on each label [11, 15, 40]. Suppose that there are five students $I = \{1, 2, 3, 4, 5\}$ and two types t_1, t_2. Student 1 has types t_1, t_2, students $2, 3$ have type t_1, student 4 has type t_2, and student 5 does not have any type. If school s has the capacity to accept a maximum of two students but must select at least one student of each type t_1 and t_2, the corresponding constraint can be represented as $\mathcal{F}_s = \big\{\{1\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{3, 4\}\big\}$. Another example is a model in which a student has multiple types but is allocated as one of her types [23]. This model includes important real-life applications, such as affirmative action in India [36] and Brazil [4]. If school s' can select one student of each type t_1, t_2 from the above I, the constraint becomes $\mathcal{F}_{s'} = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\big\}$. Another difference from the model proposed by [37, 38] is that our model includes outside options and allows for unmatched agents. Therefore, our model is flexible enough to include house allocation with existing tenants [1] and kidney exchanges [34] as special cases. In addition, our TTC generalizes the You Request My House—I Get Your Turn (YRMH-IGYT) mechanism [1] and the Top Trading Cycles and Chains (TTCC) mechanism with the SP and PE chain rule [34].

Kamiyama [21] explored the case where the outside option is assumed to be worst for every student (every school is acceptable to any student). He showed that a mechanism, called the Generalized Serial Dictatorship with Project Closures (GSDPC), satisfies PE and SP for general constraints. The GSDPC sequentially assigns each student to her best school to the extent that the remaining students can be feasibly assigned. It is not difficult to see that the GSDPC satisfies GSP. Furthermore, in the scenario without endowments, any mechanism is IR; hence, the GSDPC satisfies PE, IR, and GSP.

Imamura and Kawase [16] studied PE under a general constraint and, in particular, provided a method for checking whether a given matching is Pareto efficient. They identified that matroid is a necessary and sufficient condition for the constraint to characterize the set of PE matchings by the SD. They also introduced the Constrained Serial Dictatorship (CSD) to check PE under general constraints. The CSD is almost the same as the GSDPC; however, it also considers IR. Hence, the CSD can be viewed as a PE and IR mechanism, but it is not SP.

The field of matching under constraints has grown rapidly [2, 7, 10, 12, 17, 18, 22] with a primary focus on stability or fairness. However, our study emphasizes the importance of PE. Several studies

[1] Even though Hafalir et al. [13] investigated how to improve a distributional objective based on student types and did not explicitly analyze constraints, it parallels the analysis of the TTC under constraints.

examined PE mechanisms under constraints [8, 32, 41]. In particular, Delacrétaz et al. [8] studied PE, IR, and SP mechanisms under multidimensional knapsack constraints. As previously highlighted, they established that desired mechanisms do not exist when endowments are present and do exist when they are not. These findings can be derived from our results.

2 Preliminaries

2.1 Model

A market is a tuple $(I, S, (\succ_i)_{i\in I}, (\mathcal{F}_s)_{s\in S}, \omega)$. $I = \{1, 2, \dots, n\}$ is a finite set of students, and S is a finite set of schools. Each student i has a strict preference $\succ_i$ over $S \cup \{\varnothing\}$, where $\varnothing$ means being unmatched (or an outside option). We write $x \succeq_i x'$ if either $x \succ_i x'$ or $x = x'$ holds. $\mathcal{F}_s$ is the family of subsets of students that school s can accept; $\omega\colon I \to S \cup \{\varnothing\}$ is an endowment function, where $\omega(i) = s$ denotes that the endowment of i is $s \in S \cup \{\varnothing\}$. In a scenario without endowments, we assume that $\omega(i) = \varnothing$ for all $i \in I$.

A *matching* μ is a subset of $I \times S$ such that each student i appears at most in one pair of μ; that is, $|\mu \cap \{(i, s) : s \in S\}| \le 1$ for all $i \in I$. For each $i \in I$, we write $\mu(i)$ to denote the school to which i is assigned at μ, that is, $\mu(i) = s$ if $(i, s) \in \mu$ and $\mu(i) = \varnothing$ if $(i, s) \notin \mu$ for all $s \in S$. Similarly, for each $s \in S$, we write $\mu(s)$ to denote the set of students assigned to s at μ, that is, $\mu(s) = \{i \in I : (i, s) \in \mu\}$. A matching is called *feasible* if $\mu(s) \in \mathcal{F}_s$ for all $s \in S$. For notational simplicity, we sometimes add unmatched pairs $(i, \varnothing)$ to a matching, but we ignore such pairs.

Let μ_0 denote the endowment matching, that is, $\mu_0(i) = \omega(i)$ for all $i \in I$. We assume that the endowment matching is feasible, that is, $\mu_0 \in \mathcal{F}$.

2.2 Constraints

The aggregated constraint is sometimes represented by $\mathcal{F} = \big\{X \subseteq I \times S : X(s) \in \mathcal{F}_s\ (\forall s \in S)\big\}$, where $X(s) = \{i \in I : (i, s) \in X\}$.[2] Using this notation, a matching μ is feasible if and only if $\mu \in \mathcal{F}$. In addition, we will also consider a distributional constraint $\mathcal{F} \subseteq I \times S$ that may not be expressible through individual constraints $(\mathcal{F}_s)_{s\in S}$.

Let E be a ground set. A family of subsets $\mathcal{F} \subseteq 2^E$ is a *matroid* if it satisfies the following three properties: (i) $\emptyset \in \mathcal{F}$; (ii) if $X \in \mathcal{F}$ and $X' \subseteq X$, then $X' \in \mathcal{F}$; and (iii) if $X, Y \in \mathcal{F}$ and $|X| < |Y|$, then $y \in Y \setminus X$ exists such that $X \cup \{y\} \in \mathcal{F}$. If $\mathcal{F}_s$ is a matroid for every $s \in S$, then the aggregated constraint $\mathcal{F}$ is also a matroid. Given a matroid $\mathcal{F}$, an element $B \in \mathcal{F}$ is called a *base* if B is an inclusion-wise maximal subset of E in $\mathcal{F}$. According to property (iii), all the bases of a given matroid have the same cardinality. The collection of all the bases is called the *matroid base family*. The matroid base family can be characterized as a nonempty family of subsets $\mathcal{B} \subseteq 2^E$ that satisfies the following property: for any $B, B' \in \mathcal{B}$ and $b \in B \setminus B'$, there exists $b' \in B' \setminus B$ such that $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$.

Matroid constraints include many real-life examples of constraints. Abdulkadiroğlu and Sönmez [2] formally studied type-specific quotas to address student diversity requirements within schools. Kamada and Kojima [17] studied the regional maximum quotas in the context of medical residency matching in Japan. Under the regional maximum quotas, each school belongs to a region, and there is an upper bound on the number of students that can be matched in each region. These constraints are special cases of matroid.

A nonempty family of subsets $\mathcal{F} \subseteq 2^E$ is a g-matroid if, for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, it holds that

(i) $X \setminus \{e\}$ and $Y \cup \{e\} \in \mathcal{F}$, or

(ii) there is $e' \in Y \setminus X$ such that $(X \setminus \{e\}) \cup \{e'\}$ and $(Y \cup \{e\}) \setminus \{e'\}$ are in $\mathcal{F}$.

Alternatively, g-matroid can be characterized by another property [27, 39]: for any $X, Y \in \mathcal{F}$ and $e \in X \setminus Y$, it holds that

(i) $X \setminus \{e\} \in \mathcal{F}$ or $(X \setminus \{e\}) \cup \{e'\} \in \mathcal{F}$ for some $e' \in Y \setminus X$, and

[2] Note that $X \in \mathcal{F}$ may not be a matching because some students may appear multiple times.

(ii) $Y \cup \{e\} \in \mathcal{F}$ or $(Y \cup \{e\}) \setminus \{e'\} \in \mathcal{F}$ for some $e' \in Y \setminus X$.

A g-matroid is also called an M$^\natural$-convex family because the corresponding set of 0–1 vectors is an M$^\natural$-convex set as a subset of $\mathbb{Z}^E$ [26]. It is known that the upper and lower quotas for each class of a laminar[3] classification is a g-matroid. Hence, the upper and lower quotas on the number of students with each label is g-matroid if the labels form a laminar. In addition, if each student has multiple types but is allocated as one of her types, an upper and lower quota on the number of students with each label is g-matroid (even without the laminar structure). In Section 1.1, we provided such constraints. It is not difficult to see that g-matroid is a class that includes both matroid and matroid base family. Moreover, a nonempty family of subsets $\mathcal{F} \subseteq 2^E$ is a g-matroid if and only if there exists a matroid base family $\mathcal{B} \subseteq 2^{E'}$ with $E \subseteq E'$ such that $\mathcal{F} = \{B \cap E : B \in \mathcal{B}\}$ [39].

A family of subsets $\mathcal{F} \subseteq 2^E$ belongs to the class of *general upper bound* (or independence system) if $X \subseteq Y \in \mathcal{F}$ implies $X \in \mathcal{F}$. A family of subsets $\mathcal{F} \subseteq 2^E$ is called *accessible* if for any $X \in \mathcal{F} \setminus \{\emptyset\}$, there exists $e \in X$ such that $X \setminus \{e\} \in \mathcal{F}$. By definition, any nonempty accessible set system must contain the empty set. For an order σ of E, a family of subsets $\mathcal{F} \subseteq 2^E$ is called *σ-accessible* if for any $X \in \mathcal{F} \setminus \{\emptyset\}$, we have $X \setminus \{e\} \in \mathcal{F}$ for $e \in \arg\max\{\sigma^{-1}(e) : e \in X\}$. By definition, every general upper bound is σ-accessible for any σ, and every σ-accessible set system (for some σ) is accessible. In addition, these classes are distinct as $\{\emptyset, \{1\}, \{1, 2\}\}$ is σ-accessible for $\sigma = (1, 2)$ but not general upper bound, and $\{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \{4\}, \{3, 4\}, \{2, 3, 4\}\}$ is accessible but not σ-accessible for any σ.

An example of a σ-accessible constraint is the proportionality ceiling constraint that arises from school choice in a Chinese district. In this context, the government has imposed a proportionality ceiling that states the number of students from outside a district assigned to a school should not exceed a certain fraction of the total number of students assigned [14]. Let us consider a scenario in which there are two students from within the district, denoted as i_1, i_2, and two students from outside the district, denoted as u_1, u_2. If the proportion to be guaranteed is half, the constraint for a school s with capacity two is $\mathcal{F}_s = \{\emptyset, \{i_1\}, \{i_2\}, \{i_1, i_2\}, \{i_1, u_1\}, \{i_1, u_2\}, \{i_2, u_1\}, \{i_2, u_2\}\}$, which is σ-accessible with respect to $\sigma = (i_1, i_2, u_1, u_2)$. In general, the proportionality ceiling constraint is σ-accessible, where σ is an order in which students from within the district are listed before those from outside the district.

Moreover, σ-accessible constraints also appear in various other applications. For instance, Huang [14] introduced a choice function that incorporates a proportionality ceiling constraint. Dur et al. [9] studied a similar choice function in resource allocation during a pandemic. Additionally, Bando and Kawasaki [6] introduced a broader class of choice functions and studied dynamic matching. The constraints induced by these choice functions are σ-accessible. These details are discussed at the end of Section 4.2.

Figure 1 illustrates the relationship among classes of constraints.

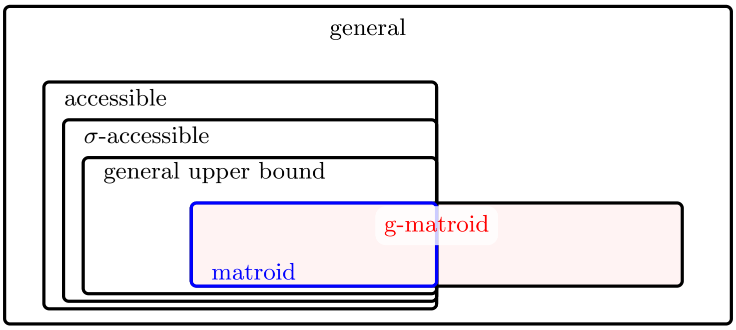

Figure 1: Classes of constraints we deal with in this study.

2.3 Properties

A matching μ is said to *Pareto dominate* μ' if $\mu(i) \succeq_i \mu'(i)$ for all $i \in I$ and $\mu(i) \succ_i \mu'(i)$ for some $i \in I$. A feasible matching μ is called *Pareto efficient (PE)* if there is no feasible matching μ' that Pareto dominates μ. Additionally, a feasible matching μ is called *individually rational (IR)* if $\mu(i) \succeq_i \mu_0(i)$ for all $i \in I$.

[3] A family of sets is called a laminar family if each pair of sets are either disjoint or related by containment.

A mechanism ψ is a map from a preference profile to a feasible matching. A mechanism is PE and IR if it always produces a feasible matching that fulfills the conditions of PE and IR, respectively.

A mechanism ψ is *strategy-proof (SP)* if for every preference profile $\succ_I$, there is no $i \in I$ and her preference $\succ'_i$ such that $\psi[\succ'_i, \succ_{-i}](i) \succ_i \psi[\succ_I](i)$, where $\succ_I = (\succ_j)_{j\in I}$ and $\succ_{-i} = (\succ_j)_{j\in I\setminus\{i\}}$. Intuitively, SP requires that no student can be assigned to a strictly preferred school by misreporting her preference. Similarly, the mechanism ψ is *group strategy-proof (GSP)* if, for every preference profile $\succ_I$, there is no $I' \in 2^I \setminus \{\emptyset\}$ and their preference profile $\succ_{I'}$ such that $\psi[\succ'_{I'}, \succ_{-I'}](i) \succeq_i \psi[\succ_I](i)$ for all $i \in I'$ and $\psi[\succ'_{I'}, \succ_{-I'}](i) \succ_i \psi[\succ_I](i)$ for some $i \in I'$, where $\succ'_{I'} = (\succ'_j)_{j\in I'}$ and $\succ_{-I'} = (\succ_j)_{j\in I\setminus I'}$. In other words, GSP requires that no group of students can make each member weakly better off and that at least one student in the group is strictly better off by jointly misreporting their preferences. Clearly, GSP is a stronger property than SP.

A mechanism is *nonbossy* if no student can influence the assignment of others without changing her own assignment by misreporting her preference. Formally, for every preference profile $\succ_I$, $i \in I$, and her preference $\succ'_i$, $\psi[\succ_I](i) = \psi[\succ'_i, \succ_{-i}](i)$ implies $\psi[\succ_I] = \psi[\succ'_i, \succ_{-i}]$. It is known that a mechanism is GSP if and only if it is SP and nonbossy [28].

3 Scenario with Endowments

In this section, we establish that g-matroid is a maximal domain for the existence of PE, IR, and SP mechanisms in a scenario with endowments. To demonstrate this, we first prove that a TTC mechanism satisfies PE, IR, and GSP if the constraints are g-matroid. Subsequently, we construct a market that permits no PE, IR, and SP mechanisms for each constraint $\mathcal{F}_{s^*}$ that is not g-matroid.

3.1 Mechanism for g-matroid constraints

We provide a TTC mechanism that satisfies PE, IR, and GSP when the constraints are g-matroid. We derive this mechanism by utilizing the TTC-M mechanism introduced by Suzuki et al. [37, 38]. The TTC-M mechanism maintains PE, IR, and GSP for any distributional constraint that can be represented by an M-convex set on the vector of the number of students assigned to each school. Let $\chi_e \in \{0,1\}^E$ be the eth unit vector. A set of integer vectors $\mathcal{V} \subseteq \mathbb{Z}^E_{\geq 0}$ is an M-convex set if for all $v, v' \in \mathcal{V}$ and all $e \in E$ with $v_e > v'_e$, there exists $f \in E$ with $v_f < v'_f$ such that $v - \chi_e + \chi_f \in \mathcal{V}$ and $v' + \chi_e - \chi_f \in \mathcal{V}$ [25].

Note that the TTC-M mechanism cannot be directly applied to our setting. The primary reason for this is that in our setting, the constraints are not imposed on the number of students assigned to each school but rather on the matched student-school pairs. In addition, our setting allows students to be unmatched whereas their model does not.

To utilize the TTC-M mechanism, we construct a virtual market $(I, \tilde{S}, (\tilde{\succ}_i)_{i\in I}, \tilde{\mathcal{F}}, \tilde{\omega})$ from the given market $(I, S, (\succ_i)_{i\in I}, \mathcal{F}, \omega)$. The set of schools in the virtual market is defined as the set of student-school pairs $\tilde{S} \coloneqq \big\{(i,s) \,:\, i \in I,\ s \in S \cup \{\varnothing\}\big\}$. Each student $i \in I$ has a strict preference $\tilde{\succ}_i$ over $\tilde{S}$ such that for any $(i_1, s_1), (i_2, s_2) \in \tilde{S}$, we have

(i) $(i_1, s_1) \tilde{\succ}_i (i_2, s_2) \iff s_1 \succ_i s_2$ if $i_1 = i_2 = i$, and

(ii) $(i_1, s_1) \tilde{\succ}_i (i_2, s_2)$ if $i_1 = i$ and $i_2 \neq i$.

The distributional constraint $\tilde{\mathcal{F}} \subseteq \mathbb{Z}^{\tilde{S}}_{\geq 0}$ is defined as follows:

$$\tilde{\mathcal{F}} \coloneqq \big\{\nu \in \{0,1\}^{\tilde{S}} \,:\, \textstyle\sum_{(i,s)\in\tilde{S}} \nu_{(i,s)} = |I| \quad \text{and} \quad \{(i,s) \in I \times S \,:\, \nu_{(i,s)} = 1\} \in \mathcal{F}\big\}.$$

The endowment function satisfies $\tilde{\omega}(i) = (i, \omega(i))$ for each $i \in I$. Note that if $\mathcal{F}$ is a g-matroid, then $\tilde{\mathcal{F}}$ is an M-convex set. Indeed, $\mathcal{F}' = \big\{\nu \subseteq \tilde{S} \,:\, \nu \cap (I \times S) \in \mathcal{F}\big\}$ is also a g-matroid by definition. Further, $\tilde{\mathcal{F}}$ can be obtained from $\mathcal{F}'$ by truncating it with cardinality $|I|$ (i.e., $\tilde{\mathcal{F}} = \{\nu \in \mathcal{F}' \,:\, |\nu| = |I|\}$), and such a truncation induces a matroid base family [39]. As the class of matroid base families is a subclass of M-convex sets, $\mathcal{F}$ is an M-convex set.

The TTC-M mechanism runs as follows. Let $\rhd$ be a common priority order over the students I. Without loss of generality, we may assume that $1 \rhd 2 \rhd \cdots \rhd n$. In each round, every (virtual) school $(i,s) \in \tilde{S}$ selects a student. If (i,s) belongs to the endowment matching, then it selects i. Otherwise,

(i,s) selects the highest priority student among the students i' for which (i,s) can be added to the current matching by removing $(i',\omega(i'))$ without violating feasibility. This mechanism gives the selected student the right to obtain a seat. Each student selects the right to obtain her top applicable school seat. Subsequently, students with such rights can trade seats among themselves by constructing trading cycles. Implement the trade indicated by this cycle, and all the involved students are removed from the market. If any students remain, the procedure continues.

Note that a trading cycle can be interpreted as an alternating cycle in the *exchange graph* of a g-matroid intersection. This correspondence can be established by constructing an instance of the g-matroid intersection problem whose common ground set is the set of student-school pairs $\tilde{S}$. One g-matroid is the distributional constraint $\tilde{\mathcal{F}}$, and the other is a partition matroid $\mathcal{M}$ that ensures each student appears at most once. In other words, $X \in \mathcal{M}$ if $|X \cap \{(i,s) \in \tilde{S} : s \in S \cup \{\varnothing\}\}| \le 1$ for all $i \in I$. For a feasible matching μ, the exchange graph is a directed bipartite graph with bipartition μ and $\tilde{S} \setminus \mu$. A pair $(y,x) \in \mu \times (\tilde{S} \setminus \mu)$ is an arc if $(\mu \setminus \{y\}) \cup \{x\} \in \tilde{\mathcal{F}}$ and $(x,y) \in (\tilde{S} \setminus \mu) \times \mu$ is an arc if $(\mu \setminus \{y\}) \cup \{x\} \in \mathcal{M}$. To preserve the feasibility of matching after trading, it is sufficient to select a cycle in the exchange graph that does not contain shortcuts [24]. A standard method for selecting such a cycle is to select a shortest cycle. However, such a selection rule does not satisfy strategy-proofness [16]. The TTC-M mechanism instead selects cycles without shortcuts by utilizing the priority order.

Formally, our TTC mechanism is described in Algorithm 1. At the beginning of round k, the set of remaining students is $I^{(k-1)}$, and each student $i \in I^{(k-1)}$ is matched with $\mu^{(k-1)}(i) = (i,\omega(i))$. Each student $i \in I \setminus I^{(k-1)}$ exits the market matched with $\tilde{\mu}^{(k-1)}(i)$. The set of schools to which student $i \in I^{(k-1)}$ has a chance of being matched with is represented as $S_i^{(k)}$. Then, each student $i \in I^{(k-1)}$ points to $(i,p_i^{(k)})$ where $p_i^{(k)}$ is the most preferred school in $S_i^{(k)}$. Each virtual school (i,s) points to the most prioritized student i' whom (i,s) can add by removing $(i',\omega(i'))$.

Algorithm 1: Generalized TTC

input : a market $(I,S,(\succ_i)_{i\in I},\mathcal{F},\omega)$
output: a matching $\tilde{\mu}$

Let $\mu^{(0)} \leftarrow \{(i,\omega(i)) : i \in I\}$, $\tilde{\mu}^{(0)} \leftarrow \emptyset$, and $I^{(0)} \leftarrow I$;
for $k \leftarrow 1,2,\dots$ **do**
  if $I^{(k-1)} = \emptyset$ **then return** $\tilde{\mu}^{(k-1)}$;
  foreach $i \in I^{(k-1)}$ **do**
    Let $S_i^{(k)} \leftarrow \{s \in S \cup \{\varnothing\} : (\mu^{(k-1)} \setminus \{(i',\omega(i'))\}) \cup \tilde{\mu}^{(k-1)} \cup \{(i,s)\} \in \mathcal{F}\ (\exists i' \in I^{(k-1)})\}$;
    Let $p_i^{(k)}$ be the most preferred school in $S_i^{(k)}$ for i;
    i points to $(i,p_i^{(k)})$;
  foreach $(i,s) \in \{(i,p_i^{(k)}) : i \in I^{(k-1)}\}$ **do**
    if $(i,s) \in \mu^{(k-1)}$ **then** (i,s) points to i;
    else
      Let $I_{(i,s)}^{(k)} \leftarrow \{i' \in I^{(k-1)} : (\mu^{(k-1)} \setminus \{(i',\omega(i'))\}) \cup \tilde{\mu}^{(k-1)} \cup \{(i,s)\} \in \mathcal{F}\}$;
      (i,s) points to the most prioritized (smallest index) student in $I_{(i,s)}^{(k)}$;
  Identify a cycle $(i_1,(i_1,p_{i_1}^{(k)}),i_2,(i_2,p_{i_2}^{(k)}),\dots,i_r,(i_r,p_{i_r}^{(k)}))$;
  $\mu^{(k)} \leftarrow \mu^{(k-1)} \setminus \{(i_1,\omega(i_1)),\dots,(i_r,\omega(i_r))\}$;
  $\tilde{\mu}^{(k)} \leftarrow \tilde{\mu}^{(k-1)} \cup \{(i_1,p_{i_1}^{(k)}),\dots,(i_r,p_{i_r}^{(k)})\}$;
  $I^{(k)} \leftarrow I^{(k-1)} \setminus \{i_1,\dots,i_r\}$;

For clarity, we provide an example of how our TTC mechanism works.

Example 3. Let $I = \{1,2,3,4,5\}$ and $S = \{s_1,s_2\}$. Suppose that students $1,2$ prefer s_2, s_1, $\varnothing$ in this order, and students $3,4,5$ prefer s_1, s_2, $\varnothing$ in this order. The constraints $\mathcal{F}$ is a g-matroid that is defined as the aggregation of

$$\mathcal{F}_{s_1} = \{I' \subseteq I : |I' \cap \{2,3,5\}| \le 1\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{I' \subseteq I : 1 \le |I'| \le 2\}.$$

Let the endowments be $(\omega(1),\omega(2),\omega(3),\omega(4),\omega(5)) = (s_1,s_1,s_2,\varnothing,\varnothing)$, that is, the endowment matching is $\mu^{(0)} = \{(1,s_1),(2,s_1),(3,s_2)\}$.

In round 1 of Algorithm 1, student 1 points to $(1, s_2)$, $(1, s_2)$ points to 1, student 2 points to $(2, s_2)$, $(2, s_2)$ points to 1, and so on (see Figure 2a). Note that $\{(2, s_1), (3, s_2), (2, s_2)\}$ is in $\mathcal{F}$ although it is not a matching. The cycle identified at line 13 is $(1, (1, s_2))$. Hence, we obtain $\mu^{(1)} = \{(2, s_1), (3, s_2)\}$, $\tilde{\mu}^{(1)} = \{(1, s_2)\}$, and $I^{(1)} = \{2, 3, 4, 5\}$.

In round 2, the cycle identified at line 13 is $(2, (2, s_2), 3, (3, s_1))$ (see Figure 2b). Thus, we obtain $\mu^{(2)} = \emptyset$, $\tilde{\mu}^{(2)} = \{(1, s_2), (2, s_2), (3, s_1)\}$, and $I^{(2)} = \{4, 5\}$.

In round 3, there are two cycles $(4, (4, s_1))$ and $(5, (5, \varnothing))$ (see Figure 2c). Note that student 5 cannot point to s_1, as student 3 was matched to s_1 in round 2, and therefore, $s_1 \notin S_5^{(3)}$. The trades indicated by these cycles are implemented in rounds 3 and 4. Consequently, we obtain the matching $\tilde{\mu}^{(4)} = \{(1, s_2), (2, s_2), (3, s_1), (4, s_1)\}$.

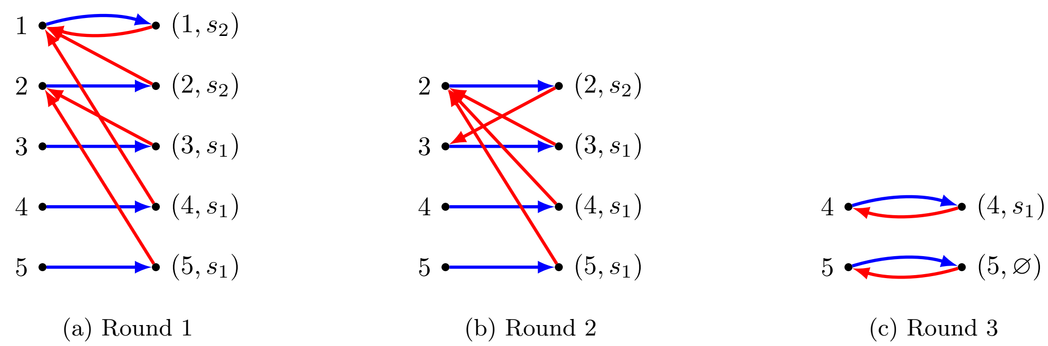

Figure 2: Cycles obtained by the TTC in Example 3. The blue and red arrows represent the relationship to which students and virtual schools are pointing, respectively. Virtual schools that have not been pointed to by any student are omitted.

The following theorem holds because the TTC-M mechanism satisfies PE, IR, and GSP when the distributional constraint is represented by an M-convex set on the vector of the number of students assigned to each school.

Theorem 1. The generalized TTC mechanism (Algorithm 1) satisfies PE, IR, and GSP if the distributional constraints form a g-matroid.

Finally, we briefly discuss the computational complexity of Algorithm 1. As at least one student is fixed in each iteration, the number of iterations is at most $O(|I|)$. The running time of each iteration is $O(|I| \cdot |S|)$ (assuming that the feasibility of matchings can be checked in a constant time). Therefore, the total running time is at most $O(|I|^2 \cdot |S|)$.

3.2 Impossibility for non-g-matroid constraints

Next, we demonstrate that the g-matroid structure is necessary for the existence of a mechanism that satisfies PE, IR, and SP.

Theorem 2. Fix a set of students I and a school s^* with the constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not a g-matroid. Then, there must exist a market $(I, S, (\mathcal{F}_s)_{s \in S}, \omega)$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I : |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism simultaneously satisfies PE, IR, and SP.

Proof. As $\mathcal{F}_{s^*}$ is not a g-matroid, there exist subsets X and Y in $\mathcal{F}_{s^*}$ and a student e in $X \setminus Y$, such that,

(i) $X \setminus \{e\} \notin \mathcal{F}_{s^*}$ and $(X \setminus \{e\}) \cup \{e'\} \notin \mathcal{F}_{s^*}$ for any $e' \in Y \setminus X$, or

(ii) $Y \cup \{e\} \notin \mathcal{F}_{s^*}$ and $(Y \cup \{e\}) \setminus \{e'\} \notin \mathcal{F}_{s^*}$ for any $e' \in Y \setminus X$.

Here, we provide the proof for the case in which (i) holds. We defer the proof for the case when (ii) holds to Appendix A, as it can be demonstrated in a similar manner.

Suppose that there exist $X, Y \in \mathcal{F}_{s^*}$ and $e \in X \setminus Y$ such that $X \setminus \{e\} \notin \mathcal{F}_{s^*}$ and $(X \setminus \{e\}) \cup \{f\} \notin \mathcal{F}_{s^*}$ for any $f \in Y \setminus X$. Let $Z \in \mathcal{F}_{s^*}$ be a set of students such that $(X \cap Y) \subseteq Z \subseteq (X \cup Y) \setminus \{e\}$. Such a set Z must exist because Y satisfies the condition. Among all sets Z that satisfy this condition, we select a set that maximizes $|X \cap Z|$.

We consider the following two cases separately: (a) $|X \setminus Z| = 1$ and (b) $|X \setminus Z| \geq 2$.

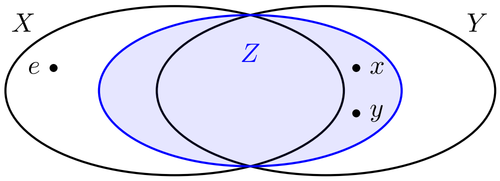

Figure 3: Case a

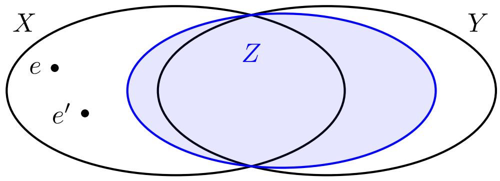

Figure 4: Case b

Case a: $|X \setminus Z| = 1$. In this case, we have $X \cap Z = X \setminus \{e\}$. In addition, we have $|Z \setminus X| \geq 2$ because $(X \setminus \{e\}) \cup J = Z \in \mathcal{F}_{s^*}$ by setting $J = Z \setminus X$. We select two students $x, y \in Z \setminus X$ arbitrarily (see Figure 3). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. Additionally, let the endowments be $\omega(e) = t$, $\omega(i) = s^*$ for each $i \in Z$, and $\omega(i) = \varnothing$ for each $i \notin Z \cup \{e\}$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Z$, $|\mu_0(t)| = 1$, and $|\mu_0(u)| = 0 \leq 1$.

Suppose that the students' preferences of the students are given as follows:

- $\succ_e = (s^* \, t \cdots)$,
- $\succ_i = (s^* \cdots)$ for each $i \in X \setminus \{e\}$,
- $\succ_x = (t \, u \, s^* \cdots)$,
- $\succ_y = (t \, u \, s^* \cdots)$,
- $\succ_i = (\varnothing \, s^* \cdots)$ for each $i \in Z \setminus (X \cup \{x, y\})$,
- $\succ_i = (\varnothing \cdots)$ for each $i \notin X \cup Z$.

Let μ_x be the matching such that x matches to u and every other student matches to her most favorite school (or her outside option). Similarly, let μ_y be the matching such that y matches to u and every other student matches to her most favorite school. Then, μ_x and μ_y are feasible since $\mu_x(s^*) = \mu_y(s^*) = X$. Furthermore, we can observe that only μ_x and μ_y are PE and IR. By symmetry, we can assume, without loss of generality, that a mechanism outputs μ_x. Suppose that x misreports her preference as $t \succ'_x s^* \succ'_x \cdots$. With this misreporting, the unique PE and IR matching is μ_y. Hence, any PE and IR mechanism cannot satisfy SP.

Case b: $|X \setminus Z| \geq 2$. Let e' be an arbitrary student in $X \setminus (Z \cup \{e\})$ (see Figure 4). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. In addition, let the endowments be $\omega(e) = t$, $\omega(e') = u$, $\omega(i) = s^*$ for each $i \in Z$, $\omega(i) = \varnothing$ for each $i \in I \setminus (Z \cup \{e, e'\})$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Z$ and $|\mu_0(t)| = |\mu_0(u)| = 1$.

Suppose that students' preferences of the students are defined as follows:

- $\succ_e = (u \, s^* \, t \cdots)$,
- $\succ_{e'} = (s^* \, t \, u \cdots)$,
- $\succ_i = (s^* \cdots)$ for each $i \in X \cap Z$,
- $\succ_i = (\varnothing \, s^* \cdots)$ for each $i \in Z \setminus X$,
- $\succ_i = (s^* \, \varnothing \cdots)$ for each $i \in X \setminus (Z \cup \{e, e'\})$,
- $\succ_i = (\varnothing \cdots)$ for each $i \notin X \cup Z$.

Let μ be the matching produced by a PE, IR, and SP mechanism. By IR, we have $X \cap Z \subseteq \mu(s^*) \subseteq X \cup Z$. If $e \notin \mu(s^*)$, then we must have $\mu(s^*) \subseteq Z$ by the maximality of $|X \cap Z|$. Hence, $\mu(e) \neq s^*$ implies $\mu(e') \neq s^*$. Let us consider three subcases depending on $\mu(e)$: (b1) $\mu(e) = t$, (b2) $\mu(e) = s^*$, and (b3) $\mu(e) = u$.

Case b1: $\mu(e) = t$. In this case, $\mu(e') \neq s^*$ and $\mu(e') = u$. This means that μ is not PE because e and e' can be better off by swapping their allocated schools, which is a contradiction.

Case b2: $\mu(e) = s^*$**.** Suppose that e misreports s^* as being unacceptable (i.e., submitting $\succ'_e = (u\, t \cdots)$). Then, e must be matched with u in any PE and IR matching, which contradicts SP.

Case b3: $\mu(e) = u$**.** In this case, $\mu(e') \neq s^*$ and $\mu(e') = t$. Suppose that e' misreports that t as being unacceptable (i.e., submitting $\succ'_e = (s^*\, u \cdots)$). Then, e' must be matched with s^* because there exists a unique PE and IR matching $\{(i, s^*) : i \in X\}$, which contradicts SP.

□

4 Scenario without Endowments

In this section, we consider a scenario without endowments. We first prove that, for any order σ of students, the SD mechanism with σ satisfies PE, IR, and GSP if the constraints are σ-accessible. We then observe that PE, IR, and SP mechanisms may not exist even when the constraints are accessible. Furthermore, we demonstrate that accessibility is a necessary condition for the existence of PE, IR, and GSP mechanisms.

4.1 SD mechanism for accessible constraints

Let Σ be the set of all permutations of the students. The SD mechanism considers students one by one in a predetermined order $\sigma \in \Sigma$. In each step of the mechanism, the current student is given the opportunity to select her most preferred school from the remaining available schools, subject to the imposed constraint. Once the student makes a choice, the student is fixed on their assignment to the school of her choice. The SD mechanism is described formally in Algorithm 2.

Algorithm 2: Serial Dictatorship (SD)

input : a market $(I, S, (\succ_i)_{i \in I}, \mathcal{F})$ and $\sigma \in \Sigma$
output: a matching
Let $\tilde{\mu} \leftarrow \emptyset$;
for $k \leftarrow 1, 2, \ldots, |I|$ **do**
  $r \leftarrow \arg\max_{\succ_{\sigma(k)}} \left\{ s \in S \cup \{\varnothing\} : \tilde{\mu} \cup \{(\sigma(k), s)\} \in \mathcal{F} \right\}$;
  if $r \in S$ **then** $\mu^{(k)} \leftarrow \mu^{(k-1)} \cup \{(\sigma(k), r)\}$;
return $\mu^{(|I|)}$;

For instance, if we apply the SD mechanism with $\sigma = (i_1, i_2, i_3)$ to the market in Example 1 ignoring endowment, the resulting matching is $\{(i_1, s_3), (i_2, s_1), (i_3, s_2)\}$.

The SD mechanism is IR because each student can at least choose the option of being unmatched. Furthermore, the mechanism is GSP because of the sequential nature of the mechanisms. Indeed, as each student selects her preferred school in her turn, there is no room for a group of students to coordinate and manipulate the outcome strategically.

Unfortunately, the SD mechanism does not satisfy PE under general constraints, even when there is only one school.[4] To observe this, consider a market with $I = \{1, 2\}$, $S = \{s\}$, $\succ_1 = \succ_2 = (s\, \varnothing)$, and $\mathcal{F}_s = \big\{\emptyset, \{1, 2\}\big\}$. In this market, the SD mechanism outputs a matching in which no student matches to s, regardless of the order. However, the unique PE and IR matching is the matching in which both students are matched to school s. The essential reason why the SD mechanism fails to output the matching is that the constraint $\mathcal{F}_s$ is not accessible.

By contrast, the SD mechanism is PE if the individual constraints are σ-accessible for a common $\sigma \in \Sigma$. We prove this fact for a more general case in which the distributional constraint is σ-accessible. A distributional constraint $\mathcal{F}$ is σ-accessible if $\mu \setminus \{(i, \mu(i))\} \in \mathcal{F}$ for any feasible nonempty matching $\mu \in \mathcal{F} \setminus \{\emptyset\}$ and $i \in \arg\max\{\sigma^{-1}(i) : i \in I, \mu(i) \neq \varnothing\}$. Note that, for any σ-accessible individual constraints $(\mathcal{F}_s)_{s \in S}$, the aggregated constraint $\mathcal{F}$ is also σ-accessible. For a market with a σ-accessible distributional constraint, suppose, on the contrary, that the SD mechanism outputs a matching μ that is not PE. Then, there exists a feasible matching μ' $(\neq \mu)$ that Pareto-dominates μ. Let k be the smallest

[4] In contrast, the CSD [16] is PE, IR, and GSP for any market consisting of only one school s. The mechanism is PE and IR in general. In addition, it is GSP since each student can only indicate whether she desires the school s, and misreporting affects the outcome only when it makes the agent worse off.

index such that $\mu(\sigma(k)) \neq \mu'(\sigma(k))$. Then, we have $\mu(\sigma(j)) = \mu'(\sigma(j))$ for $j = 1, 2, \ldots, k-1$ and $\mu'(\sigma(k)) \succ_{\sigma(k)} \mu(\sigma(k))$. This leads to a contradiction because $\sigma(k)$ could have chosen $\mu'(\sigma(k))$ on her turn in the SD mechanism. Hence, we have the following theorem.

Theorem 3. If the distributional constraint is σ-accessible for an order $\sigma \in \Sigma$, the SD mechanism (Algorithm 2) with σ satisfies PE, IR, and GSP.

4.2 Impossibility for non-accessible constraints

If the constraint is not necessarily σ-accessible, there exist markets where it is impossible to satisfy PE, IR, and SP simultaneously, not only for the SD mechanism but also for the other mechanisms.

Example 4. Let $I = \{1, 2\}$ and $S = \{s_1, s_2\}$. The constraint $\mathcal{F}_s$ of each school s is defined as follows:

$$\mathcal{F}_{s_1} = \{\emptyset, \{2\}, \{1,2\}\} \quad \text{and} \quad \mathcal{F}_{s_2} = \{\emptyset, \{1\}, \{1,2\}\}.$$

Note that $\mathcal{F}_{s_1}$ and $\mathcal{F}_{s_2}$ are accessible. Suppose that the true preference $\succ_i$ of each student i is given as follows:

$$\succ_1 = (s_1\, s_2\, \varnothing) \quad \text{and} \quad \succ_2 = (s_2\, s_1\, \varnothing).$$

It is not difficult to see that there exist only two PE and IR matchings for their true preferences: $\mu_1 \coloneqq \{(1, s_1), (2, s_1)\}$ and $\mu_2 \coloneqq \{(1, s_2), (2, s_2)\}$. If student 1 misreports her preference as $\succ_1' = (s_1\, \varnothing\, s_2)$, whereas student 2 reports her true preference $\succ_2$, then μ_1 is the unique PE and IR matching. Conversely, if student 1 reports her true preference $\succ_1$, and student 2 misreports her preference as $\succ_2' = (s_2\, \varnothing\, s_1)$, then μ_2 is the unique PE and IR matching. Therefore, in any PE and IR mechanism, either student 1 or 2 can benefit from misreporting their preferences. This means that no mechanism can simultaneously satisfy PE, IR, and SP for the market.

Consequently, accessibility is insufficient to guarantee the existence of a mechanism that satisfies PE, IR, and SP. Nevertheless, accessibility is a necessary condition for the existence of a mechanism that satisfies PE, IR, and GSP.

Theorem 4. Fix a set of students I and a school s^* with the constraint $\mathcal{F}_{s^*}$. Suppose that $\mathcal{F}_{s^*}$ is not accessible. Then, there must exist a market $(I, S, (\mathcal{F}_s)_{s \in S})$ with $s^* \in S$ and $\mathcal{F}_s = \{X \subseteq I : |X| \leq 1\}$ for all $s \in S \setminus \{s^*\}$ such that no mechanism simultaneously satisfies PE, IR, and GSP.

Proof. We consider a market in which $S = \{s^*, t\}$ and $\mathcal{F}_t = \{X \subseteq I : |X| \leq 1\}$. As $\mathcal{F}_{s^*}$ is not accessible, there exists a nonempty $X^* \in \mathcal{F}_{s^*}$ such that $X^* \setminus \{i\} \notin \mathcal{F}_{s^*}$ for all $i \in X^*$. Note that X^* must contain at least two students because $\emptyset \in \mathcal{F}_{s^*}$ by assumption. Suppose, to the contrary, that there exists a mechanism ψ that satisfies PE, IR, and GSP. Note that ψ is also nonbossy because it is GSP.

For each $i \in X^*$, we define $P^{(i)}$ as a preference profile such that $P_i^{(i)} = (t\, \varnothing\, s^*)$, $P_j^{(i)} = (s^*\, \varnothing\, t)$ for each $j \in X^* \setminus \{i\}$, and $P_j^{(i)} = (\varnothing\, s^*\, t)$ for each $j \in I \setminus X^*$. By PE and IR, student i must be matched with school t at $P^{(i)}$ (i.e., $(i, t) \in \psi[P^{(i)}]$). In addition, at least one student $j \in X^* \setminus \{i\}$ is unmatched at $P^{(i)}$ (i.e., $(j, s^*) \notin \psi[P^{(i)}]$) because $X^* \setminus \{i\} \notin \mathcal{F}_{s^*}$. We draw an arrow from each student $i \in X^*$ to an agent $j \in X^* \setminus \{i\}$ who is unmatched at $P^{(i)}$. Then, there must be at least one cycle. Let $(i_1, i_2, \ldots, i_k)$ be such a cycle, where $(i_{\ell+1}, s^*) \notin \psi[P^{(i_\ell)}]$ for $\ell = 1, 2, \ldots, k$ (we use i_{k+1} to represent i_1 for simplicity). Note that $k \geq 2$ because there is no self-loop. For each $j \in \{1, 2, \ldots, k\}$, we define the preference profiles $\hat{P}^{(i_j)}$, $\hat{\hat{P}}^{(i_j)}$, and $Q^{(i_j)}$ as follows:

- $\hat{P}_{i_j}^{(i_j)} = (t\, s^*\, \varnothing)$ and $\hat{P}_i^{(i_j)} = P_i^{(i_j)}$ for each $i \in I \setminus \{i_j\}$;
- $\hat{\hat{P}}_{i_{j+1}}^{(i_j)} = (s^*\, t\, \varnothing)$ and $\hat{\hat{P}}_i^{(i_j)} = \hat{P}_i^{(i_j)}$ for each $i \in I \setminus \{i_{j+1}\}$;
- $Q_{i_{j+1}}^{(i_j)} = (t\, s^*\, \varnothing)$ and $Q_i^{(i_j)} = \hat{\hat{P}}_i^{(i_j)}$ for each $i \in I \setminus \{i_{j+1}\}$.

The preference profiles are summarized in Table 1. By PE and SP, we have $\psi[\hat{P}^{(i_j)}](i_j) = t$ for $j = 1, 2, \ldots, k$. Hence, by nonbossiness, $\psi[\hat{P}^{(i_j)}] = \psi[P^{(i_j)}]$ for $j = 1, 2, \ldots, k$. Moreover, by a similar argument, we also have $\psi[Q^{(i_j)}] = \psi[\hat{\hat{P}}^{(i_j)}] = \psi[\hat{P}^{(i_j)}] = \psi[P^{(i_j)}]$.

	$\{i_1,\ldots,i_k\}\setminus\{i_j,i_{j+1}\}$	i_j	i_{j+1}	$X^*\setminus\{i_1,\ldots,i_k\}$	$I\setminus X^*$
$P^{(i_j)}$	$(s^*\,\varnothing\,t)$	$(t\,\varnothing\,s^*)$	$(s^*\,\varnothing\,t)$	$(s^*\,\varnothing\,t)$	$(\varnothing\,s^*\,t)$
$\hat{P}^{(i_j)}$	$(s^*\,\varnothing\,t)$	$(t\,s^*\,\varnothing)$	$(s^*\,\varnothing\,t)$	$(s^*\,\varnothing\,t)$	$(\varnothing\,s^*\,t)$
$\hat{\hat{P}}^{(i_j)}$	$(s^*\,\varnothing\,t)$	$(t\,s^*\,\varnothing)$	$(s^*\,t\,\varnothing)$	$(s^*\,\varnothing\,t)$	$(\varnothing\,s^*\,t)$
$Q^{(i_j)}$	$(s^*\,\varnothing\,t)$	$(t\,s^*\,\varnothing)$	$(t\,s^*\,\varnothing)$	$(s^*\,\varnothing\,t)$	$(\varnothing\,s^*\,t)$
R	$(t\,s^*\,\varnothing)$	$(t\,s^*\,\varnothing)$	$(t\,s^*\,\varnothing)$	$(s^*\,\varnothing\,t)$	$(\varnothing\,s^*\,t)$

Table 1: Preference profiles in the proof of Theorem 4

Now, let us consider the preference profile R such that $R_i = (t\,s^*\,\varnothing)$ for each $i \in \{i_1,\ldots,i_k\}$, $R_i = (s^*\,\varnothing\,t)$ for each $i \in X^*\setminus\{i_1,\ldots,i_k\}$, and $R_i = (\varnothing\,s^*\,t)$ for each $i \in I\setminus X^*$. Recall that school t has a capacity of one. By symmetry, we may assume, without loss of generality, that no student other than i_k is matched to t in $\psi[R]$ (i.e., $\psi[R](i_k) = t$ or $\psi[R] = \{(i,s^*) : i \in X^*\}$). For each $j \in \{1,2,\ldots,k\}$, let $R^{(j)}$ be the preference profile such that $R^{(j)}_i = (t\,s^*\,\varnothing)$ for each $i \in \{i_j,\ldots,i_k\}$, $R^{(j)}_i = (s^*\,\varnothing\,t)$ for each $i \in X^*\setminus\{i_j,\ldots,i_k\}$, and $R^{(j)}_i = (\varnothing\,s^*\,t)$ for each $i \in I\setminus X^*$. Note that $R^{(1)} = R$ and $R^{(k-1)} = Q^{(i_{k-1})}$. By PE, IR, and GSP, it is not difficult to see that $\psi[R] = \psi[R^{(1)}] = \psi[R^{(2)}] = \cdots = \psi[R^{(k-1)}] = \psi[Q^{(i_{k-1})}]$. This implies that $(i_{k-1},t) \in \psi[Q^{(i_{k-1})}] = \psi[R]$. However, this contradicts the assumption that no student other than i_k is matched to t in $\psi[R]$. Hence, it can be concluded that no mechanism simultaneously satisfies PE, IR, and GSP. $\square$

Finally, we discuss the relationship between the results obtained in this section and the existence of stable matchings. To facilitate this discussion, we introduce a model to allocate indivisible goods with priorities. In this model, each school s is endowed with a priority, which is represented by a choice function over sets of students. Let $\mathrm{Ch}_s\colon 2^I \to 2^I$ be the choice function of $s \in S$, where $\mathrm{Ch}_s(X) \subseteq X$ for all $X \subseteq I$. The choice function Ch_s induces the feasibility constraint $\mathcal{F}_s = \{X \subseteq I : \mathrm{Ch}_s(X) = X\}$. The condition $\mathrm{Ch}_s(X) = X$ is called individual rationality of school s. A matching μ is *stable* if it is individually rational for both sides and there exists no $(i,s) \in I \times S$ such that $s \succ_i \mu(i)$ and $i \in \mathrm{Ch}_s(\mu(s) \cup \{i\})$.

We introduce conditions that impose restrictions on the priorities. A choice function Ch satisfies *path-independence* [31] if for any sets of students X and Y, we have $\mathrm{Ch}(X\cup Y) = \mathrm{Ch}(\mathrm{Ch}(X)\cup\mathrm{Ch}(Y))$. Furthermore, a choice function Ch satisfies *unidirectional substitutes and complements conditions* [9, 14] if there exists an ordered type $t\colon I \to \mathbb{R}$ such that for any $X \subseteq I$ and $i \in \mathrm{Ch}(X)$, the following conditions hold: (a) $\{i' \in \mathrm{Ch}(X)\setminus\{i\} : t(i') = t(i)\} \subseteq \{i' \in \mathrm{Ch}(X\setminus\{i\}) : t(i') = t(i)\}$, and (b) $\{i' \in \mathrm{Ch}(X) : t(i') < t(i)\}\setminus\{i\} = \{i' \in \mathrm{Ch}(X\setminus\{i\}) : t(i') < t(i)\}$.

When every choice function satisfies path-independence, a stable matching exists [5, 33]. Intuitively, a path-independent choice function rules out complementarities, which is associated with the nonexistence of stable matchings. However, Huang [14] demonstrated that a choice function can accommodate a specific type of complementarity. When every choice function satisfies unidirectional substitutes and complements conditions for a common t, a stable matching still exists. Note that a path-independent choice function induces a general upper bound. Moreover, a choice function that satisfies unidirectional substitutes and complements induces a σ-accessible constraint, as discussed in a similar manner to the arguments presented in Section 2.2.

A non-accessible constraint is associated with stronger complementarities. A choice function Ch with the following complementarities leads to a non-accessible constraint: there exists $X \subseteq I$ with $\mathrm{Ch}(X) \neq \emptyset$ such that for any $i \in \mathrm{Ch}(X)$, we have $\mathrm{Ch}(\mathrm{Ch}(X)\setminus\{i\}) \subsetneq \mathrm{Ch}(X)\setminus\{i\}$. The set $\mathrm{Ch}(X)$ with such an X is non-accessible in the feasibility constraint induced by Ch. This type of complementarity is encountered in choice functions under proportional constraints and is also observed in matchings involving couples. The presence of this complementarity leads to the nonexistence of a stable matching. Importantly, this complementarity not only implies the absence of stable matchings but also rules out the existence of mechanisms that satisfy the properties of PE, IR, and GSP, as required by our necessity of accessibility.

5 Discussion and Conclusion

This study investigated the existence of efficient and strategy-proof mechanisms in indivisible goods allocation problems under general constraints.

In a scenario with endowments, we revealed that the g-matroid is a maximal domain under which we can guarantee the existence of a PE, IR, and SP mechanism. The same statement holds true even if we replace SP with GSP. In the scenario without endowments, we demonstrated that the SD mechanism satisfies PE, IR, and GSP if the constraints are σ-accessible for a common σ. We also proved that accessibility is a necessary condition to ensure the existence of PE, IR, and GSP mechanisms. Identifying the most general class of constraints under which PE, IR, and SP mechanisms exist remains open.

In a scenario without endowments, we formulate an integer linear program (ILP) to determine the existence of PE, IR, and SP mechanisms for a given market. In the case where $I = \{1, 2, 3\}$, $S = \{s_1, s_2\}$, $\mathcal{F}_{s_1} = \{X \subseteq I : |X| \neq 2\}$, and $\mathcal{F}_{s_2} = \{X \subseteq I : |X| \leq 1\}$, the Gurobi solver with the ILP revealed that no such mechanism exists. The irreducible inconsistent subsystem obtained for the market contains relationships among 43 preferences, making it challenging to discern its underlying structure. Whether accessibility is necessary for the existence of PE, IR, and SP mechanisms remains for future research.

In a scenario with endowments, Delacrétaz et al. [8] presented stronger nonexistence results under multidimensional knapsack constraints. For example, the desired mechanism does not exist even when PE and IR are replaced by the property that a mechanism Pareto improves upon every Pareto-inefficient endowment. We call this property *Pareto-improving* (PI). Formally, a mechanism φ is PI if, for any preference profile $\succ_I$ at which the endowment matching μ_0 is Pareto inefficient, $\varphi[\succ_I](i) \succeq_i \mu_0(i)$ for all $i \in I$ and $\varphi[\succ_I](i) \succ_i \mu_0(i)$ for some $i \in I$. PI is a weaker requirement than the conjunction of PE and IR. Delacrétaz et al. [8] showed by example that no PI and SP mechanism exists under multidimensional knapsack constraints. In contrast, a PI and SP mechanism exists in Example 1. Thus, we are left with the following question: Which class of constraints is necessary and sufficient for the existence of PI and SP mechanisms?

In both scenarios, with and without endowments, any two of the three properties PE, IR, and GSP can be achieved under general constraints. It is evident that the mechanism which always outputs the endowment matching satisfies both IR and GSP. To satisfy PE and GSP, we can utilize a generalized SD mechanism that sequentially assigns each student to her best school in a predetermined order, ensuring that the remaining students can be feasibly assigned. To observe that the outcome μ of the mechanism is PE, suppose, to the contrary, that there exists a feasible matching μ' that is a Pareto improvement of μ. Let i^* be the first student assigned to a school other than $\mu(i^*)$ in the mechanism. Then, $\mu'(i^*) \succ_{i^*} \mu(i^*)$; however, this contradicts the behavior of the generalized SD mechanism. Additionally, the mechanism is GSP because if a student does not select her preferred school in her turn, she will not receive another chance to do so. This mechanism is equivalent to the GSDPC proposed by Kamiyama [21]. Regarding PE and IR, they can be achieved by using the CSD mechanism [16]. The CSD mechanism sequentially assigns each student to her best school in a predetermined order, while ensuring that the remaining students can be assigned to produce a feasible IR matching. Clearly, this mechanism satisfies IR. The property of PE follows from the fact that a matching is PE if it is PE under the IR constraint. Note that the CSD mechanism is not SP because each student is assigned to a school depending on the preferences of the later students.

Finally, let us discuss the case in which the endowment matching μ_0 is infeasible. In this case, no IR matchings exist, especially when every student prefers her own endowment the most. Therefore, we have no option but to abandon IR. Moreover, abandoning IR is a natural choice when allocating chores in a scenario without endowments. Nevertheless, even without IR, we can still attain PE and GSP by employing the GSDPC mechanism under any constraints, as long as at least one feasible matching exists.

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A Ommited Part in the Proof of Theorem 2

Here, we provide proof of Theorem 2 for the case when (ii) holds.

Suppose that there exist $X, Y \in \mathcal{F}_{s^*}$ and $e \in X \setminus Y$ such that $Y \cup \{e\} \notin \mathcal{F}_{s^*}$ and $(Y \cup \{e\}) \setminus \{f\} \notin \mathcal{F}_{s^*}$ for any $f \in Y \setminus X$. Let $Z \in \mathcal{F}_{s^*}$ be a set of students such that $(X \cap Y) \cup \{e\} \subseteq Z \subseteq X \cup Y$. Such a set Z must exist because X satisfies the condition. Among all sets Z that satisfy this condition, we select a set that minimizes $|X \cap Z|$.

We consider the following two cases separately: (c) $|X \cap Z| = |X \cap Y| + 1$ and (d) $|X \cap Z| \geq |X \cap Y| + 2$.

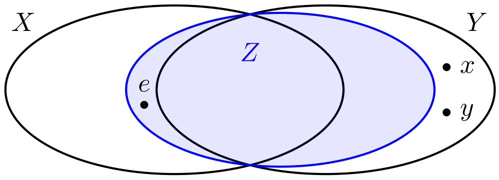

Figure 5: Case c

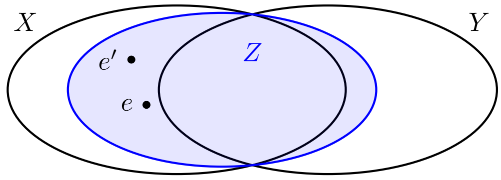

Figure 6: Case d

Case c: $|X \cap Z| = |X \cap Y| + 1$. In this case, we have $X \cap Z = (X \cap Y) \cup \{e\}$. In addition, we have $|Y \setminus Z| \geq 2$ because $(Y \cup \{e\}) \setminus J = Z \in \mathcal{F}_{s^*}$ by setting $J = Y \setminus Z$. We select two students $x, y \in Y \setminus Z$ arbitrarily (see Figure 5). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. Additionally, let the endowments be $\omega(e) = t$, $\omega(i) = s^*$ for each $i \in Y$, and $\omega(i) = \varnothing$ for each $i \notin Y \cup \{e\}$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Y$, $|\mu_0(t)| = 1$, and $|\mu_0(u)| = 0 \leq 1$.

Suppose that the students' preferences of the students are given as follows:

- $\succ_e = (s^* t \cdots)$,
- $\succ_i = (s^* \cdots)$ for each $i \in Z \setminus \{e\}$,
- $\succ_x = (t\, u\, s^* \cdots)$,
- $\succ_y = (t\, u\, s^* \cdots)$,
- $\succ_i = (\varnothing\, s^* \cdots)$ for each $i \in Y \setminus (Z \cup \{x, y\})$,
- $\succ_i = (\varnothing \cdots)$ for each $i \notin Z \cup Y$.

Let μ_x be the matching such that x matches to u and every other student matches to her most favorite school (or her outside option). Similarly, let μ_y be the matching such that y matches to u and every other student matches to her most favorite school. Then, μ_x and μ_y are feasible because $\mu_x(s^*) = \mu_y(s^*) = Z$. Furthermore, we can observe that only μ_x and μ_y are PE and IR. By symmetry, we can assume, without loss of generality, that a mechanism outputs μ_x. Suppose that x misreports her preference as $t \succ'_x s^* \succ'_x \cdots$. With this misreporting, the unique PE and IR matching is μ_y. Hence, any PE and IR mechanism cannot satisfy SP.

Case d: $|X \cap Z| \geq |X \cap Y| + 2$. Let e' be an arbitrary student in $Z \setminus (Y \cup \{e\})$ (see Figure 6). We consider a market in which the set of schools is $S = \{s^*, t, u\}$ and $\mathcal{F}_t = \mathcal{F}_u = \{I' \subseteq I : |I'| \leq 1\}$. In addition, let the endowments be $\omega(e) = u$, $\omega(e') = t$, $\omega(i) = s^*$ for each $i \in Y$, $\omega(i) = \varnothing$ for each $i \in I \setminus (Y \cup \{e, e'\})$. The endowment matching μ_0 for this market is feasible because $\mu_0(s^*) = Y$ and $|\mu_0(t)| = |\mu_0(u)| = 1$.

Suppose that students' preferences of the students are defined as follows:

- $\succ_e = (s^*\, t\, u \cdots)$,
- $\succ_{e'} = (u\, s^*\, t \cdots)$,
- $\succ_i = (s^* \cdots)$ for each $i \in Z \cap Y$,
- $\succ_i = (\varnothing\, s^* \cdots)$ for each $i \in Y \setminus Z$,
- $\succ_i = (s^*\, \varnothing \cdots)$ for each $i \in Z \setminus (Y \cup \{e, e'\})$,
- $\succ_i = (\varnothing \cdots)$ for each $i \notin Z \cup Y$.

Let μ be the matching produced by a PE, IR, and SP mechanism. By IR, we have $Z \cap Y \subseteq \mu(s^*) \subseteq Z \cup Y$. If $e \in \mu(s^*)$, then we must have $X \cap Z \subseteq \mu(s^*)$ by the minimality of $|X \cap Z|$. Hence, $\mu(e') \neq s^*$ implies $\mu(e) \neq s^*$. Let us consider three subcases depending on $\mu(e')$: (d1) $\mu(e') = t$, (d2) $\mu(e') = s^*$, and (d3) $\mu(e') = u$.

Case d1: $\mu(e') = t$. In this case, $\mu(e) \neq s^*$ and $\mu(e) = u$. This means that μ is not PE because e and e' can be better off by swapping their allocated schools, which is a contradiction.

Case d2: $\mu(e') = s^*$. Suppose that e' misreports s^* as being unacceptable (i.e., submitting $\succ'_{e'} = (u\, t \cdots)$). Then, e' must be matched with u in any PE and IR matching, which contradicts SP.

Case d3: $\mu(e') = u$. In this case, $\mu(e) \neq s^*$ and $\mu(e) = t$. Suppose that e misreports t as being unacceptable (i.e., submitting $\succ'_e = (s^*\, u \cdots)$). Then, e must be matched with s^* because there exists a unique PE and IR matching $\{(i, s^*) : i \in Z\}$, which contradicts SP.