UTMD Working Paper
The University of Tokyo
Market Design Center

UTMD-050

Tie-breaking or Not: A Choice Function Approach

Kenzo Imamura
University of Tokyo

Kentaro Tomoeda
University of Technology Sydney

June 4, 2023

Tie-breaking or Not: A Choice Function Approach*

Kenzo Imamura[†] Kentaro Tomoeda[‡]

June 4, 2023

Abstract

This paper considers a new axiom of a choice function called *equal treatment of individuals in an indifference class (ETI)* in the context of matching problems. We show that when a choice function satisfies ETI and two commonly-used axioms, substitutability and size monotonicity, any individual for whom ETI applies must either be always accepted whenever the choice set includes them or be never selected. ETI is also generally incompatible with another axiom, q-acceptance. When ETI, substitutability, and size monotonicity are required, the degree of q-acceptance violation depends on the sum of the sizes of all indifference classes for which ETI applies, but when size monotonicity is replaced by consistency, it is characterized by the size of a particular indifference class. These results clarify the trade-off between ETI and other axioms, which would be helpful in designing a tie-breaking rule.

Keywords: choice function, equal treatment, substitutability, size monotonicity, consistency, acceptance

JEL Classification Numbers: C78, D47, D71

*We would like to thank Keisuke Bando, Daisuke Hirata, Keiichi Kawai, Onur Kesten, Fuhito Kojima, Morimitsu Kurino, Daisuke Oyama, Koji Yokote, and seminar participants at Australasian Economic Theory Workshop 2023, Hitotsubashi University, Keio University, Osaka University, September Workshop on Economics at Otaru 2021, the University of Sydney, the University of Tokyo, and Waseda University for their useful comments and discussions. Imamura gratefully acknowledges financial support by JSPS KAKENHI Grant Number JP23K12443.

[†]Department of Economics, the University of Tokyo, email: imamurak@e.u-tokyo.ac.jp.

[‡]UTS Business School, University of Technology Sydney, email: Kentaro.Tomoeda@uts.edu.au.

1 Introduction

In many choice problems where multiple individuals are chosen, *ex post* fairness among some individuals is often a social desideratum, rather than just *ex ante* fairness achieved by a lottery. In college admissions where the admission criterion is based on scores, a college often treats applicants with the same score equally (i.e., either accepts or rejects all of them) without breaking a tie. Examples include admissions systems in Hungary, Chile, Australia, and China, and a tie-breaking among such applicants is even prohibited by law in Hungary and Chile (Biró and Kiselgof, 2015; Rios et al., 2021). In the distribution of disaster relief, all individuals in the same category (such as children and elderly people) are treated equally. This is because the central authority would want to eliminate any envy among them based on their *ex post* relief allocation and would not prefer a random lottery (Kamada and Kojima, 2023). These examples are in stark contrast to other choice problems such as student selection by primary or secondary schools, where a tie-breaking is used to meet the capacity constraint of schools (Erdil and Ergin, 2008; Abdulkadiroğlu et al., 2009).

While *ex post* equal treatment of certain individuals is a practical requirement, the literature on choice theory and matching theory has not examined "what choice could be made" if we require *ex post* fairness together with other properties of choice rules.[1] It has been known that several axioms on the choice function are important in making two-sided many-to-one matching markets work well. When every choice function is substitutable and consistent, a stable matching is guaranteed to exist in the market and the well-known student-proposing Deferred Acceptance (DA) mechanism finds it (Roth, 1984; Aygün and Sönmez, 2013). With a stronger requirement of substitutability and size monotonicity, DA becomes strategy-proof (Hatfield and Milgrom, 2005). When a tie is not an issue, a responsive choice function would satisfy all these relevant axioms.[2] However, when equal treatment of certain individuals is required, it is not clear what choice functions could meet this fairness while retaining other axioms.

In this paper, we consider *equal treatment of individuals in an indifference class (ETI)* as an axiom of choice functions and analyze the compatibility of this axiom with other ones studied in the literature. We consider a partition of all individuals into indifference classes, and ETI requires any individuals in the same indifference class be treated equally.[3] Our first main theorem shows that if a choice function satisfies ETI, substitutability, and size

[1]See Section 1.1 for related papers on the implication of *ex post* fairness on matching markets.

[2]See Section 2 for the definitions of these axioms and Section 3.1 for the definition of a responsive choice function.

[3]In our model, we allow ETI to be applied to some indifference classes but not necessarily all of them.

monotonicity, then for any given individual for whom ETI applies, it must satisfy either of the following conditions: (i) it always accepts them whenever the choice set includes them; or (ii) it never selects them (Theorem 1). This is a very strong necessary condition because the choice function loses the ability to compare the relevant individual with the others in the choice set. The class of choice functions could be significantly relaxed if each of the three axioms is dropped.

To provide an intuition of this result, let us consider the following two choice functions that are used to satisfy ETI in reality (Biró and Kiselgof, 2015; Rios et al., 2021). A *q-receptive* choice function accepts individuals according to a weak priority ranking, and when it finds multiple indifferent individuals around the "target" capacity q (i.e., the first group of indifferent individuals for whom the size of the chosen set reaches or exceeds q), it accepts up to all such individuals. A *q-unreceptive* choice function accepts individuals in a similar way according to a weak priority ranking, but it rejects all indifferent individuals around the target capacity. Both choice functions share the same spirit as a responsive choice function in the sense that they compare individuals in the choice set. It is easy to verify that they meet ETI and substitutability. Theorem 1 implies that such choice functions must necessarily violate size monotonicity unless they accept or reject all individuals. Size monotonicity requires that the size of the chosen set be weakly increasing when the choice set expands in a set-inclusion sense.

Example 1. Consider three individuals s_1, s_2 and s_3. s_1's priority is higher than s_2 and s_3, and s_2 and s_3 are indifferent. When choice function C is 1-receptive, $C(\{s_1, s_2, s_3\}) = \{s_1\}$ but $C(\{s_2, s_3\}) = \{s_2, s_3\}$. When the choice set is $\{s_2, s_3\}$, both s_2 and s_3 are accepted because they are at the borderline of $q = 1$. When choice function C is 2-unreceptive, $C(\{s_2, s_3\}) = \{s_2, s_3\}$ but $C(\{s_1, s_2, s_3\}) = \{s_1\}$. Under this choice function, s_2 and s_3 are both rejected when $\{s_1, s_2, s_3\}$ applies because they are at the borderline of $q = 2$. Either choice function violates size monotonicity. □

As in this example, it is often the case that colleges have some sense of target capacity and adjust the admission cutoff depending on the applicant pool. In such cases, at least one of ETI, substitutability, and size monotonicity must be compromised. We also formally show that once we replace size monotonicity with consistency, the set of choice functions is significantly broader than Theorem 1, including receptive choice functions (Proposition 2).

ETI is generally incompatible with another important axiom, q-acceptance. q-acceptance requires any individuals to be accepted up to a non-negative integer q. Since the violation of q-acceptance can be interpreted as a departure from the target capacity, it is reasonable to evaluate the violation by its degree. We propose a natural way to measure the degree of q-acceptance violation by considering the worst-case difference in the size of the chosen set, where the worst case is taken out of all possible choice sets.

We characterize the minimum of this degree out of all choice functions that satisfy (i) ETI, substitutability, and size monotonicity, and (ii) ETI, substitutability, and consistency. In case (i), we provide an algorithm to find the minimum and show that the sum of the sizes of all indifference classes for which ETI applies matters (Proposition 3). This is a natural consequence of Theorem 1 that all indifference classes for which ETI applies must always be accepted or rejected by any choice function satisfying these three axioms. The algorithm embeds the *partition problem* in number theory since the degree of q-acceptance violation is minimized by balancing the size of the always-accepted set and the always-rejected set. In case (ii), we introduce a $(q, \bar{q})$*-generalized receptive* choice function that satisfies all three axioms and show that the minimum of q-acceptance violation can be achieved by this choice function by setting the parameter $\bar{q}$ appropriately (Propositions 5 and 6). A $(q, \bar{q})$-generalized receptive choice function has the features of both receptive and unreceptive choice functions. As opposed to case (i), the minimized degree of q-acceptance violation is characterized by the size of a certain indifference class. These highlight the cost of size monotonicity measured by the degree of q-acceptance violation because the minimum in case (i) could be significantly larger than that in case (ii), especially when the size of each indifference class is small. Our results also include cases where choice functions are required to be compatible with a weak priority ranking over individuals.

1.1 Related literature

ETI is motivated by the *ex post* fairness requirement in college admissions with ties. The following papers study real-world matching markets where schools' choices satisfy ETI. Biró and Kiselgof (2015) study ETI in the context of Hungarian college admissions where each college uses an unreceptive choice function.[4] They propose a receptive choice function and examine its welfare and stability consequences to the market. They also find that in markets with receptive or unreceptive choice functions, DA violates strategy-proofness. Rios et al. (2021) independently study ETI and receptive choice functions in the context of Chilean college admissions. Other papers also analyze the implication of ETI for matching markets and mechanisms (Ehlers, 2006; Fleiner and Jankó, 2014; Kamiyama, 2017; Ágoston et al.,

[4] Biró and Kiselgof (2015) do not name the choice rules of schools, but their "H-stable (resp. L-stable) score-limits" correspond to the cutoffs achieved by unreceptive (resp. receptive) choice functions.

2022). Ehlers (2006) considers a strong fairness concept that rules out any envy among tied students and proposes a condition on the priority structure for this fairness to be compatible with efficiency. While these papers study matching markets by focusing on specific stability concepts or choice functions, we investigate the compatibility between ETI and other axioms of choice functions. That is, we characterize the general class of choice functions, not just the ones studied by the papers above, that make matching markets or mechanisms work well (in terms of the existence of stable matchings and the strategy-proofness of stable mechanisms). In this sense, our approach contributes to a better understanding of matching markets where ETI is required.

Our paper contributes to the literature on choice and matching theory.[5] Path independence, which is equivalent to substitutability and consistency, has been studied in the literature of choice theory (Plott, 1973; Aizerman and Malishevski, 1981). Following this, several results have been shown on the choice functions satisfying certain axioms including path independence. For example, Chambers and Yenmez (2017) study path-independent and size monotonic choice functions; Doğan et al. (2021) characterize path-independent and acceptant choice functions. We add to this literature by introducing and examining our new fairness axiom, ETI.

Our results also relate to a broader literature on priority design, which has received considerable attention in recent years. In the context of affirmative action in India, Sönmez and Yenmez (2022) design a choice function that satisfies desirable properties reflecting the laws of India. Echenique and Yenmez (2015) characterize the choice functions used in affirmative action policies such as quotas (Abdulkadiroğlu and Sönmez, 2003) and reserves (Hafalir et al., 2013). Affirmative action with complex constraints has been studied in a variety of settings using the priority design approach (Imamura, 2020; Aygün and Turhan, 2022; Doğan et al., 2022). Other papers examine applications other than affirmative action such as walk zones reserves in school choice (Dur et al., 2018) and medical rationing during a pandemic (Pathak et al., 2021).

Recently, it has been known that achieving size monotonicity is challenging in some real-life applications. One such example is refugee resettlement, where each institution has a capacity and families may have different sizes (Delacrétaz et al., 2020). One of their proposed choice functions is shown to violate size monotonicity.[6] A similar result holds in several applications such as daycare seat allocation and college admissions with budget constraints (Kamada and Kojima, 2023). The violation of size monotonicity in these models

[5] See Alva and Doğan (2021) for a recent survey of this topic.

[6] They also provide another choice function that satisfies size monotonicity.

is due to heterogeneous individual sizes. By contrast, our paper finds that size monotonicity may be violated when ETI is prioritized, which is a different logic from theirs.

2 Model

Consider a choice problem where multiple individuals (e.g., students or applicants) are chosen. Let $\mathcal{S}$ be a finite set of individuals. A choice function $C : 2^{\mathcal{S}} \to 2^{\mathcal{S}}$ is such that $C(S) \subseteq S$ for any $S \subseteq \mathcal{S}$. To introduce our fairness notion, we consider a partition of $\mathcal{S}$, denoted by $\mathscr{I}$. That is, $\mathscr{I}$ is a set of subsets I of $\mathcal{S}$ such that $\cup_{I \in \mathscr{I}} I = \mathcal{S}$ and for any $I, I' \in \mathscr{I}$, $I \neq I'$ imply $I \cap I' = \emptyset$. We call each element I of $\mathscr{I}$ an *indifference class*. A subset of $\mathscr{I}$ is called a *collection of indifference classes*. For any collection of indifference classes $\mathcal{I} \subseteq \mathscr{I}$, define $\mathcal{I}_{\geq 2} := \{I \in \mathcal{I} : |I| \geq 2\}$. In words, $\mathcal{I}_{\geq 2}$ is the set of all indifference classes $I \in \mathcal{I}$ whose size is greater than or equal to two.

Our new axiom of a choice function requires *ex post* equal treatment of individuals in the same indifference class.

Definition 1. Choice function C satisfies *equal treatment of individuals in an indifference class (ETI)* for a collection of indifference classes $\mathcal{I} \subseteq \mathscr{I}$ if for any $S \subseteq \mathcal{S}$, $I \in \mathcal{I}$, and $s, s' \in S \cap I$, $s \in C(S)$ if and only if $s' \in C(S)$.

Note that we allow for any $\mathcal{I} \subseteq \mathscr{I}$ in our analysis. This means that ETI can be applied only for certain indifference classes and not necessarily for all of them. For example, it is possible that children (or elderly people) need to be treated equally while others may not need to be. A college may want to treat students with a certain high score equally but may be willing to break a tie for those with a lower score.

This paper considers the compatibility of ETI with the following three axioms:

1. Substitutability: for any $S, S' \subseteq \mathcal{S}$ with $s \in S \subseteq S'$, $s \in C(S')$ implies $s \in C(S)$.

2. Size monotonicity: for any $S, S' \subseteq \mathcal{S}$ with $S \subseteq S'$, $|C(S)| \leq |C(S')|$.

3. Consistency: for any $S, S' \subseteq \mathcal{S}$ with $C(S') \subseteq S \subseteq S'$, $C(S) = C(S')$.[7]

These axioms are known to be the key to guaranteeing and implementing a stable matching in markets. In two-sided many-to-one matching markets (with contracts), a stable matching exists and can be found by the student-proposing DA mechanism if every choice function

[7]Consistency is also known as "irrelevance of rejected contracts" in the literature of matching with contracts (Aygün and Sönmez, 2013).

is substitutable and consistent (Roth, 1984; Aygün and Sönmez, 2013).[8] In addition, DA becomes strategy-proof if every choice function is substitutable and size monotonic (Hatfield and Milgrom, 2005).

Following these arguments, we consider the compatibility of ETI with two sets of axioms: (i) substitutability and size monotonicity, and (ii) substitutability and consistency. The first exercise allows us to understand how restrictive choice functions would be if we require ETI together with substitutability and size monotonicity, which are often assumed in the matching literature. Note that consistency is implied by the combination of substitutability and size monotonicity. Thus, the second exercise further allows us to study how the requirement can be weakened if we replace size monotonicity with consistency.

3 Implications of ETI

In this section, we provide several characterization results of choice functions that satisfy ETI and other axioms.

3.1 Compatibility with substitutability and size monotonicity

We first show that if we require ETI together with substitutability and size monotonicity, the set of choice functions is severely constrained.

Theorem 1. *Suppose that choice function C satisfies ETI for a collection of indifference classes $\mathcal{I}$, substitutability, and size monotonicity. Then for any $I \in \mathcal{I}_{\geq 2}$ and $s \in I$, either of the following conditions holds:*

1. *$s \in C(S)$ for all $S \subseteq \mathcal{S}$ with $s \in S$, or*
2. *$s \notin C(S)$ for all $S \subseteq \mathcal{S}$.*

In words, the necessary condition means that any individual $s \in I \in \mathcal{I}_{\geq 2}$ must be either accepted or rejected irrespective of the choice set. Thus, when these three axioms are required, C loses the ability to compare s with other individuals in S.

The main proof idea is as follows. It suffices to show that the three axioms imply $s \in C(\{s\}) \Rightarrow s \in C(S)$ for any $s \in I \in \mathcal{I}_{\geq 2}$ and $S \ni s$.[9] When the choice set expands by

[8] It is known that substitutability and consistency are equivalent to another single axiom called *path independence*: Choice function C is path independent if for every S and S', $C(S \cup S') = C(C(S) \cup C(S'))$ (Plott, 1973; Aizerman and Malishevski, 1981).

[9] Note that substitutability implies the other direction.

one, substitutability implies that individuals who are originally rejected cannot be chosen from an expanded choice set, and hence the size of the chosen set increases by at most one. On the other hand, since $s \in I \in \mathcal{I}_{\geq 2}$, there is another individual $s' \in I \setminus \{s\}$ who must be treated equally as s. If we suppose $s \in C(\{s\})$ but s were not chosen from some larger choice set $S \ni s$, we can always find a situation where s is chosen from an original choice set but both s and s' would be rejected when the choice set expands by one. This would be a contradiction to size monotonicity because the size of the chosen set strictly decreases.

Since ETI does not require anything regarding those who are in a singleton indifference class, i.e., individuals s such that $\{s\} \in \mathcal{I}$, Theorem 1's condition is not sufficient for the three axioms. The next corollary confirms that when $|I| \geq 2$ for any $I \in \mathscr{I}$ and we consider ETI for $\mathscr{I}$, the same condition becomes sufficient.

Corollary 1. *Suppose $|I| \geq 2$ for any $I \in \mathscr{I}$. Then choice function C satisfies ETI for $\mathscr{I}$, substitutability, and size monotonicity if and only if for any $I \in \mathscr{I}$, either of the following conditions hold:*

1. *$I \cap S \subseteq C(S)$ for all $S \subseteq \mathcal{S}$, or*

2. *$I \cap C(S) = \emptyset$ for all $S \subseteq \mathcal{S}$.*

Proof. The "only if" direction follows from Theorem 1. We show the "if" direction. Clearly, C satisfies ETI for $\mathscr{I}$. Let $\mathscr{I}^a := \{I \in \mathscr{I} : I \cap S \subseteq C(S) \text{ for all } S \subseteq \mathcal{S}\}$. Then we have $C(S) = (\cup_{I \in \mathscr{I}^a} I) \cap S$ for any $S \subseteq \mathcal{S}$. C is substitutable because for any $S, S' \subseteq \mathcal{S}$ with $s \in S \subseteq S'$, $s \in (\cup_{I \in \mathscr{I}^a} I) \cap S'$ implies $s \in (\cup_{I \in \mathscr{I}^a} I) \cap S$. C satisfies size monotonicity because for any $S, S' \subseteq \mathcal{S}$ with $S \subseteq S'$, $|I \cap S| \leq |I \cap S'|$ for all $I \in \mathscr{I}^a$ leads to $|(\cup_{I \in \mathscr{I}^a} I) \cap S| \leq |(\cup_{I \in \mathscr{I}^a} I) \cap S'|$. □

In Theorem 1, the three axioms are independent. Indeed, each of them plays a crucial role and once we drop any of them, the necessary condition would be significantly weakened.

First, it is well-known that a responsive choice function with a strict priority, which does not satisfy ETI in general, satisfies substitutability and size monotonicity. Formally, let *strict priority ranking* $\trianglerighteq$ be a complete, transitive, and antisymmetric binary relationship on $\mathcal{S}$.[10] Given a strict priority ranking $\trianglerighteq$, choice function C is *q-responsive* if

$$C(S) = \begin{cases} S & \text{if } |S| \leq q, \\ \{s \in S : s \trianglerighteq s^*(S)\} & \text{otherwise,} \end{cases}$$

[10]For all $s, s' \in \mathcal{S}$, we write $s \triangleright s'$ to mean $s \trianglerighteq s'$ and $s' \not\trianglerighteq s$.

where $s^*(S) \in S$ is the unique individual who satisfies $|\{s \in S : s \trianglerighteq s^*(S)\}| = q$ when $|S| > q$. Choice function C is *responsive* if it is q-responsive for some integer q. Note that a responsive choice function may not be well-defined with a "weak" priority ranking (that will be defined below) because $s^*(S)$ may not exist.

Second, there are many choice functions that satisfy ETI and substitutability but do not meet the condition in Theorem 1. As discussed in Introduction, receptive and unreceptive choice functions serve as examples. Let *weak priority ranking* $\succeq$ be a complete and transitive binary relationship on $\mathcal{S}$ such that $s \sim s'$ for any $s, s' \in I \in \mathscr{I}$.[11] Let $s^{**}(S)$ be one of the highest priority individuals who satisfy $|\{s \in S : s \succeq s^{**}(S)\}| \geq q$ when $|S| > q$. In other words, $s^{**}(S)$ is one of the individuals for whom the size of the chosen set reaches or exceeds q for the first time once C accepts individuals according to $\succeq$ and treats all indifferent individuals (according to $\succeq$) equally.

Definition 2. Given a weak priority ranking $\succeq$, choice function C is *q-receptive* if

$$C(S) = \begin{cases} S & \text{if } |S| \leq q, \\ \{s \in S : s \succeq s^{**}(S)\} & \text{otherwise.} \end{cases}$$

Choice function C is *receptive* if it is q-receptive for some integer q.

Given a weak priority ranking $\succeq$, choice function C is *q-unreceptive* if

$$C(S) = \begin{cases} S & \text{if } |S| \leq q, \\ \{s \in S : s \succ s^{**}(S)\} & \text{otherwise.} \end{cases}$$

Choice function C is *unreceptive* if it is q-unreceptive for some integer q.

Both receptive and unreceptive choice functions can be seen as a generalization of a responsive choice function to the case with a weak priority ranking. The difference stems from the acceptance of individuals at the "borderline," i.e., individuals s such that $s \sim s^{**}(S)$: A q-receptive choice function accepts all such individuals, while a q-unreceptive choice function rejects all of them. As a result, when $|S| \geq q$, we always have $|C(S)| \geq q$ (resp. $|C(S)| \leq q$) if C is q-receptive (resp. q-unreceptive). It is easy to verify that both satisfy ETI and substitutability. They are more flexible than the necessary condition of Theorem 1 in the sense that the selection from choice set S can depend on S. Then, as implied by Theorem 1, these choice functions violate size monotonicity (see Example 1). Section 3.2 further provides a characterization result of choice functions with ETI when substitutability and consistency are required.

[11] For all $s, s' \in \mathcal{S}$, we write $s \sim s'$ to mean $s \succeq s'$ and $s' \succeq s$, and write $s \succ s'$ to mean $s \succeq s'$ and $s' \not\succeq s$.

Third, when only ETI and size monotonicity are required, the following proposition characterizes the set of choice functions. Here, for a concise characterization, assume $|I| \geq 2$ for every $I \in \mathscr{I}$. A profile of sets of indifference classes $(\tilde{\mathscr{I}}(S))_{S \in 2^{\mathcal{S}}} \subseteq \mathscr{I}^{2^{|\mathcal{S}|}}$ is said to be *increasing* if

$$|S \cap (\cup_{I \in \tilde{\mathscr{I}}(S)} I)| \leq |S' \cap (\cup_{I \in \tilde{\mathscr{I}}(S')} I)|$$

for any $S, S' \in 2^{\mathcal{S}}$ with $S \subseteq S'$.

Proposition 1. *Choice function C satisfies ETI for $\mathscr{I}$ and size monotonicity if and only if $C(S) = S \cap (\cup_{I \in \tilde{\mathscr{I}}(S)} I)$ where $(\tilde{\mathscr{I}}(S))_{S \in 2^{\mathcal{S}}}$ is an increasing profile of sets of indifference classes.*

Theorem 1 implies that with all the three axioms, essentially $C(S) = S \cap (\cup_{I \in \tilde{\mathscr{I}}(S)} I)$ where $\tilde{\mathscr{I}}(S) = \tilde{\mathscr{I}}(S')$ for any $S, S' \in 2^{\mathcal{S}}$ would need to hold.[12] By contrast, the combination of ETI and size monotonicity is more flexible than that because $\tilde{\mathscr{I}}(S) \neq \tilde{\mathscr{I}}(S')$ is possible. For example, consider the following choice function C:

$$C(S) = \begin{cases} S \cap I & \text{if } |S \cap I| \geq a, \\ \emptyset & \text{otherwise.} \end{cases}$$

for some $I \in \mathscr{I}$ and an integer $a \in \{2, \ldots, |I|\}$. This C satisfies the condition of Proposition 1 because we can take $\tilde{\mathscr{I}}(S) = \emptyset$ for any $S \subseteq \mathcal{S}$ with $|S \cap I| < a$ and $\tilde{\mathscr{I}}(S) = \{I\}$ otherwise. Hence, it satisfies both ETI for $\mathscr{I}$ and size monotonicity. This choice function C exhibits complementarity between individuals in I because $C(\{i\}) = \emptyset$ and $C(I') = I'$ for any $i \in I' \subseteq I$ with $|I'| \geq a$.

3.2 Compatibility with substitutability and consistency

Next, we consider consistency instead of size monotonicity. Note that we impose consistency here because substitutability (of every choice function) alone is not sufficient for the existence of a stable matching in matching markets (Aygün and Sönmez, 2013). But since consistency is weaker than the combination of substitutability and size monotonicity, we indeed show that the set of choice functions is significantly larger than the one in Theorem 1. For example, it is easy to verify that a receptive choice function satisfies consistency in addition to ETI and substitutability. Our next result generalizes this observation.

[12] Technically, $\tilde{\mathscr{I}}(S) = \tilde{\mathscr{I}}(S')$ may not need be satisfied for all $S, S' \in 2^{\mathcal{S}}$, but whether or not I is in $\tilde{\mathscr{I}}(S)$ needs to be consistent across all $S \subseteq \mathcal{S}$ such that $S \cap I \neq \emptyset$.

Proposition 2. *Suppose that choice function C satisfies substitutability and consistency. Then C satisfies ETI for a collection of indifference classes $\mathcal{I}$ if and only if for all $I \in \mathcal{I}$, $s \in I$, and $S \subseteq \mathcal{S}$ with $s \in S$,*

$$s \in C(S) \Longleftrightarrow I \subseteq C(S \cup I). \tag{1}$$

To make the difference more salient, we can rewrite Theorem 1 in the following way:

Theorem 1. (Arranged.) *Suppose that choice function C satisfies substitutability and size monotonicity. Then C satisfies ETI for a collection of indifference classes $\mathcal{I}$ if and only if for all $I \in \mathcal{I}_{\geq 2}$, $s \in I$, and $S \subseteq \mathcal{S}$ with $s \in S$,*

$$s \in C(S) \Longleftrightarrow I \subseteq C(\mathcal{S}). \tag{2}$$

This illustrates how flexible choice functions can be by replacing size monotonicity with consistency. Equation (1) requires the selection of an individual s from a choice set S be the same as when $S \cup I$ would apply. By contrast, equation (2) requires the selection of an individual s from a choice set S be the same as when all individuals (i.e., $\mathcal{S}$) would apply. The former allows the selection of s from S to depend on the choice set S while the latter does not, highlighting the flexibility of choice functions when size monotonicity is dropped.

4 Degree of q-acceptance violation

4.1 Measuring the degree of q-acceptance violation

In this section, we consider another important axiom called *acceptance*, which is often used in real-world choice problems. Choice function C is *q-acceptant* if $|C(S)| = \min\{q, |S|\}$ for any $S \subseteq \mathcal{S}$. Choice function C is *acceptant* if it is q-acceptant for some integer q. The idea of acceptance is that there is some rigid capacity q, and any individuals can be accepted up to the capacity q.

Clearly, ETI has a potential conflict with acceptance. For example, if there is an indifference class $I \in \mathcal{I}$ with $|I| > q$, then ETI for $\mathcal{I}$ and q-acceptance are not compatible. Note that there is a reasonable class of choice functions that satisfy acceptance and all axioms considered in this paper except ETI: a responsive choice function with any strict priority is substitutable, size monotonic, consistent, and acceptant. The violation of acceptance can be interpreted as a departure from the target capacity. In many applications such as college admissions, some small departure from the target capacity may be tolerable but a too large fluctuation in the number of accepted individuals would be less desirable. Therefore, it is reasonable to evaluate the violation of acceptance by its degree, and the trade-off between the axioms of choice functions and the violation of acceptance should be discussed when we consider ETI.

To quantify the degree of violation of q-acceptance, we consider the following measure, $\alpha(q,C)$, which is *the worst-case difference in the size of the chosen set*:

$$\alpha(q,C) = \max_{S:S\subseteq\mathcal{S}} \Big| |C(S)| - \min\{q,|S|\} \Big|.$$

This is the maximum difference in the size of the chosen set between a given choice function C and a q-acceptant choice function when all choice sets $S \subseteq \mathcal{S}$ are considered.

The question of this section is as follows: given a set of axioms including ETI, what is the minimum of $\alpha(q,C)$ among all choice functions C that satisfy those axioms? The solution can be interpreted as the necessary flexibility of the size of the chosen set when we require a certain set of axioms. Following Section 3, we consider the following two sets of axioms: (i) ETI, substitutability, and size monotonicity, and (ii) ETI, substitutability, and consistency. In each case of (i) and (ii), we also consider when choice functions are compatible with a weak priority ranking over individuals, which is a natural requirement in many applications. Formally, given a weak priority ranking $\succeq$, choice function C is $\succeq$-*compatible* if for any $S \subseteq \mathcal{S}$ and $s,s' \in S$, $s \in C(S)$ and $s' \notin C(S)$ imply $s \succeq s'$. Note that our analysis in this section covers the case without $\succeq$-compatibility as a special case because $\succeq$-compatibility does not have any restriction when $\succeq$ is such that $s \sim s'$ for any individuals $s,s' \in \mathcal{S}$.

Since any two individuals in the same indifference class are assumed to be indifferent in a weak priority ranking, for the sake of notational simplicity, we write $I \succeq I'$, $I \succeq s'$, or $s \succeq I'$ if $s \succeq s'$ for some $s \in I$ and $s' \in I'$.

4.2 When ETI, substitutability, and size monotonicity are required

Let $\mathcal{C}_1^{\succeq}(\mathcal{I})$ be the set of all choice functions that satisfy ETI for $\mathcal{I}$, substitutability, size monotonicity, and $\succeq$-compatibility. The goal of this section is to provide an algorithm that finds the minimum of $\alpha(q,C)$ among all choice functions C in $\mathcal{C}_1^{\succeq}(\mathcal{I})$ for any inputs $(q,\mathcal{I},\succeq)$.[13] By exploiting Theorem 1, we first derive key necessary conditions for a choice function to be in $\mathcal{C}_1^{\succeq}(\mathcal{I})$.

[13] The environment $(\mathcal{S},\mathscr{I})$ is also the inputs of the algorithm, but we omit it because it is fixed throughout the paper.

Lemma 1. *For a given choice function C, let $A(C) = \{s \in \mathcal{S} : s \in C(S)$ for any $S \subseteq \mathcal{S}$ with $s \in S\}$ and $R(C) = \{s \in \mathcal{S} : s \notin C(S)$ for any $S \subseteq \mathcal{S}\}$. If C is in $\mathcal{C}_1^{\succeq}(\mathcal{I})$, it satisfies the following conditions:*

(i) *$I \subseteq A(C)$ or $I \subseteq R(C)$ for any $I \in \mathcal{I}_{\geq 2}$, and $s \succeq s'$ for any $s \in A(C)$ and $s' \in R(C)$;*

(ii) *for any $I' \in \mathscr{I} \setminus \mathcal{I}_{\geq 2}$, $I' \subseteq A(C)$ if there exists $I \in \mathcal{I}_{\geq 2}$ such that $I \subseteq A(C)$ and $I' \succ I$; and*

(iii) *for any $I' \in \mathscr{I} \setminus \mathcal{I}_{\geq 2}$, $I' \subseteq R(C)$ if there exists $I \in \mathcal{I}_{\geq 2}$ such that $I \subseteq R(C)$ and $I \succ I'$.*

The implication of Lemma 1 is that when we search for the minimum of $\alpha(q, C)$ in $\mathcal{C}_1^{\succeq}(\mathcal{I})$, we can restrict our attention to the choice functions satisfying conditions (i)–(iii). Of theoretical interest, it is worth noting that $\succeq$-compatibility is not enough to derive conditions (ii) and (iii) and we use substitutability and size monotonicity as well.

The next lemma finds the minimum of $\alpha(q, C)$ by fixing the sets of always-accepted and always-rejected individuals. For given disjoint sets $A, R \subseteq \mathcal{S}$, let $\mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$ be the set of all choice functions $C \in \mathcal{C}_1^{\succeq}(\mathcal{I})$ such that $A \cap S \subseteq C(S)$ and $R \cap S \subseteq S \setminus C(S)$ for all $S \subseteq \mathcal{S}$.

Lemma 2. *For disjoint sets $A, R \subseteq \mathcal{S}$ and an integer q, suppose that $\mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$ is nonempty. Then, the following holds:*

$$\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)} \alpha(q, C) = \max\big\{|A| - q, \min\{|R|, q\}\big\}.$$

Lemma 2 shows that the minimum of $\alpha(q, C)$ in $\mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$ is determined by the sizes of A and R. Given Lemmas 1 and 2, our question reduces to how to minimize $\max\big\{|A| - q, \min\{|R|, q\}\big\}$ among all pairs (A, R) that satisfy the conditions (i)–(iii) in Lemma 1. The next example illustrates how we can do this.

Example 2. Consider four indifference classes I_1, I_2, I_3, I_4 with $|I_1| = 15$, $|I_2| = |I_4| = 2$, $|I_3| = 4$, a weak priority ranking $I_1 \sim I_2 \succ I_3 \succ I_4$, $q = 10$, and a collection of indifference classes $\mathcal{I} = \{I_1, I_2, I_4\}$. To find $\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$, it suffices to find $(A, R) \subseteq \mathcal{S}^2$ that minimizes $\max\big\{|A| - 10, \min\{|R|, 10\}\big\}$ by satisfying conditions (i)–(iii) of Lemma 1. Condition (i) implies that I_1, I_2, and I_4 must be included in either A or R. If $I_1 \subseteq R$, we immediately have $\max\big\{|A| - 10, \min\{|R|, 10\}\big\} = 10$. Then, consider $I_1 \subseteq A$ and see if we can achieve a smaller value than 10. There are two possibilities, $I_2 \subseteq A$ or $I_2 \subseteq R$, because $I_1 \sim I_2$ and either case is $\succeq$-compatible. If $I_2 \subseteq A$, we can achieve $\max\big\{|A| - 10, \min\{|R|, 10\}\big\} = \max\big\{17 - 10, \min\{2, 10\}\big\} = 7$ by $I_3 \subseteq \mathcal{S} \setminus (A \cup R)$ and $I_4 = R$. If $I_2 \subseteq R$, condition (iii) requires $I_3 \cup I_4 \subseteq R$ and we achieve $\max\big\{|A| - 10, \min\{|R|, 10\}\big\} = \max\big\{15 - 10, \min\{8, 10\}\big\} = 8$. Thus, $\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C) = 7$ in this example. $\square$

There are two lessons from this example. First, when more than one indifference classes in $\mathcal{I}_{\geq 2}$ are not strictly ranked, we need to consider all possibilities of assigning them to A or R. That is, we need to find the optimal way to partition I_1 and I_2 into A and R. Second, by conditions (ii) and (iii), whether or not an indifference class in $\mathscr{I} \setminus \mathcal{I}_{\geq 2}$ needs to be in A or R depends on the assignment of indifference classes in $\mathcal{I}_{\geq 2}$. When $I_1 \cup I_2 = A$, I_3 does not need to be included in either of A or R. But when $I_2 \subseteq R$, condition (iii) requires $I_3 \subseteq R$ as well.

To generalize the exercise in Example 2, we define an algorithm to find the optimal pair $(A, R) \subseteq \mathcal{S}^2$, which achieves $\max\big\{|A| - q, \min\{|R|, q\}\big\} = \min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$. To do so, the following mathematical objects need to be defined. Given an integer $q \geq 0$, for any $(A, R) \subseteq \mathcal{S}^2$ and any $\chi_1, \chi_2 \subseteq 2^{\mathcal{S}}$, define

$$f(\chi_1, \chi_2, A, R) := \max\Big\{\big(\sum_{Y \in \chi_1} |Y|\big) + |A| - q, \min\big\{\big(\sum_{Y \in \chi_2} |Y|\big) + |R|, q\big\}\Big\}.$$

For $\chi \subseteq 2^{\mathcal{S}}$, let $D(\chi) := \{(\chi_1, \chi_2) \subseteq \chi^2 : \chi_1 \cup \chi_2 = \chi, \chi_1 \cap \chi_2 = \emptyset\}$. Given q, for any $\chi \subseteq 2^{\mathcal{S}}$ and any finite sets A and R, define

$$\pi(\chi, A, R) \in \operatorname*{arg\,min}_{(\chi_1, \chi_2) \in D(\chi)} f(\chi_1, \chi_2, A, R).$$

$\pi_1(\chi, A, R)$ (resp. $\pi_2(\chi, A, R)$) denotes the first (resp. second) element of $\pi(\chi, A, R)$.

Let $\mathscr{I}_1(\succeq) := \{I \in \mathscr{I} : \nexists I' \in \mathscr{I} \text{ such that } I' \succ I\}$. Recursively, $\mathscr{I}_{k+1}(\succeq)$ is defined as $\mathscr{I}_{k+1}(\succeq) := \{I \in \mathscr{I} \setminus (\cup_{l=1}^{k} \mathscr{I}_l(\succeq)) : \nexists I' \in \mathscr{I} \setminus (\cup_{l=1}^{k} \mathscr{I}_l(\succeq)) \text{ such that } I' \succ I\}$.

For $k \in \{2, \ldots, K\}$, when $(\cup_{l=1}^{k-1} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$ is nonempty, let $I_k^a \in (\cup_{l=1}^{k-1} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$ be an indifference class that satisfies $I \succeq I_k^a$ for all $I \in (\cup_{l=1}^{k-1} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$. Define $A(1) := \emptyset$ and $A(k) := \cup_{I \in \cup_{l=1}^{k-1} \mathscr{I}_l(\succeq)} I$ for $k \in \{2, \ldots, K\}$. Define $A^T(1) := \emptyset$ and

$$A^T(k) := \begin{cases} \cup_{I \in \mathscr{I} : I \succeq I_k^a} I & \text{if } (\cup_{l=1}^{k-1} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

for $k \in \{2, \ldots, K\}$.

For $k \in \{1, \ldots, K-1\}$, when $(\cup_{l=k+1}^{K} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$ is nonempty, let $I_k^r \in (\cup_{l=k+1}^{K} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$ be an indifference class that satisfies $I_k^r \succeq I$ for all $I \in (\cup_{l=k+1}^{K} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2}$. Define $R(k) := \cup_{I \in \cup_{l=k+1}^{K} \mathscr{I}_l(\succeq)} I$ for $k \in \{1, \ldots, K-1\}$, and $R(K) := \emptyset$. Define

$$R^B(k) := \begin{cases} \cup_{I \in \mathscr{I} : I_k^r \succeq I} I & \text{if } (\cup_{l=k+1}^{K} \mathscr{I}_l(\succeq)) \cap \mathcal{I}_{\geq 2} \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

for $k \in \{1, \ldots, K-1\}$, and $R^B(K) := \emptyset$.

Define the following three potential solutions, the *interior solution* $\sigma^I : \{1, \ldots, K\} \to \mathbb{N}_0$, the *top-corner solution* $\sigma^T : \{1, \ldots, K\} \to \mathbb{N}_0$, and the *bottom-corner solution* $\sigma^B : \{1, \ldots, K\} \to \mathbb{N}_0$.[14] For each $k \in \{1, \ldots, K\}$,

- $\sigma^I(k) := f\big(\pi_1(\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, A(k), R(k)), \pi_2(\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, A(k), R(k)), A(k), R(k)\big)$,
- $\sigma^T(k) := f(\emptyset, \mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, A^T(k), R(k))$, and
- $\sigma^B(k) := f(\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, \emptyset, A(k), R^B(k))$.

The following *Algorithm* outputs a value $\sigma^* \in \mathbb{N}_0$ from the inputs $(q, \mathcal{I}, \succeq)$.

Step 1. If $\mathscr{I}_1(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ and $\sigma^B(1) > \sigma^I(1)$ hold, terminate the algorithm and define $\sigma^* := \sigma^I(1)$. Otherwise, proceed to Step 2.

Step $k \in \{2, \ldots, K-1\}$. If $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ and $\sigma^B(k) > \min\{\sigma^I(k), \sigma^T(k)\}$ hold, terminate the algorithm and define $\sigma^* := \min\{\sigma^I(k), \sigma^T(k)\}$. Otherwise, proceed to Step $k+1$.

Step K. If $\mathscr{I}_K(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ holds, define $\sigma^* = \min\{\sigma^I(K), \sigma^T(K)\}$. Otherwise, define $\sigma^* := \sigma^B(k^*)$ where k^* is the largest integer with $\mathscr{I}_{k^*}(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$.

This *Algorithm* examines the candidates of $(A, R) \subseteq \mathcal{S}^2$ for minimizing $\max\big\{|A| - q, \min\{|R|, q\}\big\}$ among those satisfying the conditions (i)–(iii) in Lemma 1. The basic idea is that the *Algorithm* considers each $\mathscr{I}_k(\succeq)$ sequentially from high to low priority, and it solves the partition of $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ into either A or R, by fixing the assignment of other indifference classes as given. The problem in each step is a slight modification of the *partition problem* in number theory, which is to find the optimal partition of positive integers into two sets in a way that the difference in the sum of the integers is minimized.[15]

There is a subtle variation in how the assignment of other indifference classes are fixed depending on whether all indifference classes in $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ are to be in A or R. When at least one indifference class in $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ is to be included in each of A and R, conditions (ii) and (iii) of Lemma 1 imply that all individuals ranked strictly higher than $\mathscr{I}_k(\succeq)$ (namely,

[14] $\mathbb{N}_0$ denotes the set of all non-negative integers.

[15] This optimization problem is known to be NP-complete, but there are many algorithms that can solve it in practice. See Korf (1998) for a review of algorithms for the partition problem including approximate algorithms (such as the greedy heuristic and the Karmarkar-Karp algorithm) and exact ones (such as the complete greedy algorithm and the Complete Karmarkar-Karp algorithm).

$A(k)$) must be in A and all individuals ranked strictly lower than $\mathscr{I}_k(\succeq)$ (namely, $R(k)$) must be in R. The interior solution $\sigma^I(k)$ of Step k represents the optimized value of the partition problem with this supposition. On the other hand, when all indifference classes in $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ are to be in R, indifference classes that are not in $\mathcal{I}_{\geq 2}$ and ranked just above $\mathscr{I}_k(\succeq)$ (if such ones exist) do not need to be included in A. $A^T(k)$ reflects this fact, and the top-corner solution $\sigma^T(k)$ of Step k represents the value of f when all indifference classes in $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ are to be in R. Similarly, the bottom-corner solution $\sigma^B(k)$ of Step k represents the value of f when all indifference classes in $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}$ are to be in A. In each step, the *Algorithm* seeks a (potentially) lower value in the next step if $\sigma^B(k)$ is lower than $\sigma^I(k)$ and $\sigma^T(k)$, but terminates otherwise.

Intuitively, the *Algorithm* finds the optimal solution because the size of A monotonically increases while that of R decreases as it proceeds through the steps. This guarantees that $\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\}$ weakly decreases to a certain step and then starts to weakly increase. Lemmas 3 and 4 formalize this intuition and show that the output σ^* of the *Algorithm* is indeed the minimum of all potential solutions examined in the *Algorithm*, i.e, σ^* is equal to $\min_{k \in \{1,\dots,K\}} \big\{ \min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\} \big\}$.

Lemma 3. *Suppose that at Step $k \in \{1, \dots, K-1\}$, $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ holds and the Algorithm proceeds to Step $k+1$. Then, we have $\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\} \geq \sigma^*$.*

Lemma 4. *Suppose that the Algorithm terminates at Step $k \in \{1, \dots, K-1\}$. Then, we have $\sigma^* < \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}$ for all $l \in \{k+1, \dots, K\}$.*

Finally, by showing that the optimal pair $(A, R) \subseteq \mathcal{S}^2$ is always included in those examined in the *Algorithm*, we obtain our desired result.

Proposition 3. *The output σ^* of the Algorithm satisfies $\sigma^* = \min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$.*

While this proposition does not provide a closed-form solution of the optimal solution σ^* in terms of the inputs, there are three properties of σ^* that are worth clarifying. First, the sum of the sizes of all indifference classes in $\mathcal{I}_{\geq 2}$ matters for σ^*, not just the size of one particular indifference class in $\mathcal{I}_{\geq 2}$. This is because σ^* is determined by the sizes of the optimal A and R, which together must cover all indifference classes in $\mathcal{I}_{\geq 2}$. Second, for the same reason as above, σ^* increases as ETI applies for a larger collection of indifference classes $\mathcal{I}' \supseteq \mathcal{I}$ conditional on q and $\succeq$. Third, σ^* is always capped by q because $\alpha(q, C) = q$ for choice function C that does not accept anyone is in $\mathcal{C}_1^{\succeq}(\mathcal{I})$.

Next, we focus on the following two extreme cases and illustrate that the problem can be simplified in different ways.

When $I \sim I'$ for any $I, I' \in \mathscr{I}$.

This case is equivalent to not requiring $\succeq$-compatibility. In such a case, the problem is reduced to a one-shot partition problem, and we can represent the solution in a more concise manner.

Proposition 4. *Suppose that $I \sim I'$ for any $I, I' \in \mathscr{I}$. Given a collection of indifference classes $\mathcal{I}$ and an integer q, $\min_{C\in\mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$ is given as follows:*

$$\min_{C\in\mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q,C) = \begin{cases} q & \text{if } e > 3q \\ \min\left\{q, \frac{e-q}{2} + \min_{r\in\mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right|\right\} & \text{if } e \in (2q, 3q] \\ \frac{e-q}{2} + \min_{r\in\mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right| & \text{if } e \in (q, 2q] \\ 0 & \text{if } e \le q \end{cases}$$

where $e := \sum_{I\in\mathcal{I}_{\ge 2}} |I|$ and $\mathcal{R}(\mathcal{I}) := \{r \in \mathbb{R} : r = |I^{(1)}| + |I^{(2)}| + \cdots + |I^{(l)}|$ for some indifference classes $I^{(1)}, I^{(2)}, \ldots, I^{(l)} \in \mathcal{I}_{\ge 2}\}$.

This formula allows us to confirm the three general properties of σ^* explained above.

When $I \succ I'$ or $I' \succ I$ holds for any $I, I' \in \mathscr{I}$.

In the other extreme case, all indifference classes are strictly ranked with each other, and hence the partition problem in each step becomes trivial. In such a case, the *Algorithm* is simplified as follows and the solution can be found without involving algorithms for the partition problem. Let I_k be the k-th highest ranked indifference class according to $\succeq$ among those in $\mathcal{I}_{\ge 2}$. Let $A_k := \{s \in \mathcal{S} : s \succeq I_k\}$ for each $k \in \{1, \ldots, K\}$ and $R_k := \{s \in \mathcal{S} : I_{k+1} \succeq s\}$ for each $k \in \{0, \ldots, K-1\}$.

Step 1. If $|A_1| - q \ge \min\{|R_1|, q\}$, terminate the algorithm. Define $\sigma^* := \min\big\{\min\{|R_0|, q\}, |A_1| - q\big\}$. Otherwise, proceed to Step 2.

Step $k \in \{2, \ldots, K-1\}$. If $|A_k| - q \ge \min\{|R_k|, q\}$, terminate the algorithm. Define $\sigma^* := \min\big\{\min\{|R_{k-1}|, q\}, |A_k| - q\big\}$. Otherwise, proceed to Step $k+1$.

Step K. Define $\sigma^* := \min\big\{\min\{|R_{K-1}|, q\}, |A_K| - q\big\}$.

4.3 When ETI, substitutability, and consistency are required

Let $\mathcal{C}_2^{\succeq}(\mathcal{I})$ be the set of all choice functions that satisfy ETI for $\mathcal{I}$, substitutability, consistency, and $\succeq$-compatibility. We first introduce a new class of choice functions, $(q, \bar{q})$-*generalized receptive* choice functions. Then, we show that $\min_{C\in\mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C)$ is achieved by this choice function by setting an appropriate parameter for $\bar{q}$.

To give an intuition for why we propose a new choice function, consider receptive and unreceptive choice functions, which both meet ETI and substitutability. The next example illustrates that a receptive choice function may accept too many individuals, an unreceptive choice function may reject too many, and an alternative does better than both of them with regard to the minimization of $\alpha(q, C)$.

Example 3. Consider five individuals s_1, s_2, s_3, s_4, s_5, target capacity $q = 2$, and a weak priority ranking such that $s_1 \succ s_2 \sim s_3 \sim s_4 \succ s_5$.[16] C and C' denote 2-receptive and 2-unreceptive choice functions, respectively. Then, $C(\{s_1, s_2, s_3, s_4\}) = \{s_1, s_2, s_3, s_4\}$ and $C'(\{s_2, s_3, s_4\}) = \emptyset$ hold, and we can see $\alpha(2, C) = \alpha(2, C') = 2$. Consider another choice function C'' such that

$$C''(S) = \begin{cases} \{s_1\} & \text{if } s_1 \in S, \\ C(S) & \text{otherwise.} \end{cases}$$

Then, it is easy to see that $\alpha(2, C'') = 1$. □

This example suggests that a hybrid C'' of C and C' would achieve a smaller $\alpha(q, C)$ than these two, by either being receptive or unreceptive around the borderline depending on the choice set. Moreover, this C'' satisfies consistency although the 2-unreceptive choice function C' does not.[17] The main innovation of this hybrid function C'' is to carefully choose when it mimics a receptive and an unreceptive choice function. This C'' is indeed one of the $(q, \bar{q})$-generalized receptive choice functions, which we propose. To define them formally, we need the following new class of priorities.

Definition 3. $\succeq$ is a *weak priority ranking with a tie-breaking* if $\succeq$ is a complete and transitive binary relationship on $\mathcal{S}$ such that $s \sim s'$ for any $s, s' \in I \in \mathcal{I}$ and satisfies the following conditions.[18]

Condition T1. For any $I, I' \in \mathscr{I}$ with $I \neq I'$, either $[s \succ s'$ for all $s \in I$ and $s' \in I']$ or $[s' \succ s$ for all $s \in I$ and $s' \in I']$ holds.

Condition T2. For any $I \notin \mathcal{I}$ and $s, s' \in I$ with $s \neq s'$, either $s \succ s'$ or $s' \succ s$ holds.

Compared to a weak priority ranking we use throughout the paper, a weak priority ranking with a tie-breaking is required to be strict between all indifference classes and also all individuals in any $I \notin \mathcal{I}$. Given weak priority ranking with a tie-breaking $\succeq$, we write

[16] We assume that $\mathscr{I} = \{\{s_1\}, \{s_2, s_3, s_4\}, \{s_5\}\}$.

[17] $C'(\{s_1, s_2\}) = \{s_1, s_2\}$ but $C'(\{s_1, s_2, s_3\}) = \{s_1\}$.

[18] For all $s, s' \in \mathcal{S}$, we write $s \sim s'$ to mean $s \succeq s'$ and $s' \succeq s$, and write $s \succ s'$ to mean $s \succeq s'$ and $s' \not\succeq s$.

$I \succ I'$ if $s \succ s'$ for all $(s, s') \in I \times I'$, write $I \succ s'$ if $s \succ s'$ for all $s \in I$, and write $s \succ I'$ if $s \succ s'$ for all $s' \in I'$.

Definition 4. Given a collection of indifference classes $\mathcal{I}$, weak priority ranking with a tie-breaking $\succeq$, and integers $(q, \bar{q})$ with $\bar{q} \geq q$, choice function C is $(q, \bar{q})$-*generalized receptive* if for all $S \subseteq \mathcal{S}$, $C(S)$ is determined by the following algorithm.

> **Step 1.** Let $I^1(S) \in \mathscr{I}$ be an indifference class such that $I^1(S) \cap S \neq \emptyset$ and $I^1(S) \succeq s$ for any $s \in S$. Set $S^1(S) = S \cap I^1(S)$. If $\max_{I \in \mathcal{I}_{\geq 2}} |I| > \bar{q}$, terminate the algorithm and set $C(S) = \emptyset$. If $\max_{I \in \mathcal{I}_{\geq 2}} |I| \leq \bar{q}$ and $|S^1(S)| \geq q$, terminate the algorithm and set $C(S)$ as:
>
> $$C(S) = \begin{cases} S^1(S) & \text{if } I^1(S) \in \mathcal{I}, \\ \{s \in S : s \succeq s^*\} & \text{otherwise,} \end{cases}$$
>
> where $s^* \in S^1(S)$ is defined as an individual with $|\{s \in S : s \succeq s^*\}| = q$. Otherwise, proceed to Step 2.
>
> **Step $t \geq 2$.** If $S^{t-1}(S) = S$, terminate the algorithm and set $C(S) = S$. Let $I^t(S) \in \mathscr{I}$ be an indifference class such that $I^t(S) \cap S \neq \emptyset$ and $I^t(S) \succeq s$ for any $s \in S \setminus S^{t-1}(S)$. Set $S^t(S) = S^{t-1}(S) \cup (S \cap I^t(S))$. If $|S^{t-1}(S)| + \max_{I \in \mathcal{I}_{\geq 2} : I^{t-1}(S) \succ I} |I| > \bar{q}$, terminate the algorithm and set $C(S) = S^{t-1}(S)$. If $|S^{t-1}(S)| + \max_{I \in \mathcal{I}_{\geq 2} : I^{t-1}(S) \succ I} |I| \leq \bar{q}$ and $|S^t(S)| \geq q$, terminate the algorithm and set $C(S)$ as:
>
> $$C(S) = \begin{cases} S^t(S) & \text{if } I^t(S) \in \mathcal{I}, \\ \{s \in S : s \succeq s^*\} & \text{otherwise,} \end{cases}$$
>
> where $s^* \in S^t(S)$ is defined as an individual with $|\{s \in S : s \succeq s^*\}| = q$. Otherwise, proceed to Step $t + 1$.

Choice function C is *generalized receptive* if it is $(q, \bar{q})$-generalized receptive for some integers q and $\bar{q}$.[19]

This choice function is based on the q-receptive choice function, but in some cases, it accepts strictly less than q individuals (even when $|S| > q$). Those cases are described as

[19] A $(q, \bar{q})$-generalized receptive choice function is q-receptive when $\bar{q} \geq |\mathcal{S}|$. Conversely, any q-receptive choice function is generalized receptive if a weak priority ranking with a tie-breaking is used to define it (instead of a weak priority ranking).

follows: given the currently accepted subset $S^{t-1}(S)$ with $|S^{t-1}(S)| < q$, if further accepting the largest indifference class below (no matter whether these individuals are in S) would make the size of the chosen set strictly greater than $\bar{q}$, it ends up accepting only $S^{t-1}(S)$. The subtlety of this construction is in the choice of these cases: it rejects any indifference class below $S^{t-1}(S)$ even when further accepting $S \cap I^t(S)$ would not exceed $\bar{q}$.

To see this point in Example 3, consider C'', which corresponds to the $(2,3)$-generalized receptive choice function, and any $S \ni s_1$. In Step 1, by $|S^1(S)| = |\{s_1\}| < 2$, the algorithm proceeds to Step 2. In Step 2, since $|S^1(S)| + \max_{I \in \mathcal{I}_{\geq 2}: I^1(S) \succ I} |I| = |\{s_1\}| + |\{s_2, s_3, s_4\}| > 3$, the algorithm stops and we have $C''(S) = \{s_1\}$. This means that even when the size of S is two (e.g., $S = \{s_1, s_2\}$ or $\{s_1, s_5\}$) and the parameter q is two, C'' only accepts s_1.

We can show that a generalized receptive choice function satisfies the desired three axioms.

Proposition 5. *Given a collection of indifference classes $\mathcal{I}$ and weak priority ranking with a tie-breaking $\succeq$, for any integers $(q, \bar{q})$ with $\bar{q} \geq q$, $(q, \bar{q})$-generalized receptive choice function C satisfies ETI for $\mathcal{I}$, substitutability, and consistency.*

This choice function satisfies ETI by construction, and it is relatively straightforward to show that it is substitutable. To see why this choice function successfully meets consistency, let us see two cases where an unreceptive choice function violates consistency. The first case is where an unreceptive choice function rejects an individual when another individual from the same indifference class is added to the choice set: in Example 3, $C'(\{s_1, s_2\}) = \{s_1, s_2\}$ but $C'(\{s_1, s_2, s_3\}) = \{s_1\}$. On the other hand, the $(2,3)$-generalized receptive choice function C'' satisfies $C''(\{s_1, s_2\}) = C''(\{s_1, s_2, s_3\}) = \{s_1\}$ by rejecting any individuals following s_1. The second case is where an unreceptive choice function rejects an individual when other individuals with higher priority are added to the choice set: $C'(\{s_1, s_5\}) = \{s_1, s_5\}$ but $C'(\{s_1, s_2, s_3, s_4, s_5\}) = \{s_1\}$. C'' does not have this problem either because $C''(\{s_1, s_5\}) = C''(\{s_1, s_2, s_3, s_4, s_5\}) = \{s_1\}$.

Our goal is to find a generalized receptive choice function that is $\succeq$-compatible and minimizes $\alpha(q, C)$. Since a generalized receptive choice function is defined with a weak priority ranking with a tie-breaking, we need to construct it from a weak priority ranking $\succeq$, which does not satisfy Conditions T1 and T2 in general. We say that *weak priority ranking with a tie-breaking $\succeq^*$ is constructed from $\succeq$* if $\succeq^*$ satisfies the following five conditions.

1. For any $I \in \mathscr{I}_k(\succeq)$ and $I' \in \mathscr{I}_{k'}(\succeq)$, $I \succ^* I'$ if $k < k'$.

2. For any $\mathscr{I}_k(\succeq)$ and $I, I' \in \mathscr{I}_k(\succeq)$ with $I \neq I'$, either $I \succ^* I'$ or $I' \succ^* I$.

3. For any $\mathscr{I}_k(\succeq)$, $I \in \mathscr{I}_k(\succeq) \setminus \mathcal{I}_{\geq 2}$, and $s, s' \in I$ with $s \neq s'$, either $s \succ^* s'$ or $s' \succ^* s$.

4. For any $\mathscr{I}_k(\succeq)$ and $I, I' \in \mathcal{I}_{\geq 2} \cap \mathscr{I}_k(\succeq)$, $I \succ^* I'$ if $|I| > |I'|$.

5. For any $\mathscr{I}_k(\succeq)$, $I \in \mathcal{I}_{\geq 2} \cap \mathscr{I}_k(\succeq)$, and $I' \in \mathscr{I}_k(\succeq) \setminus \mathcal{I}_{\geq 2}$, $I \succ^* I'$.

The first three conditions guarantee that this $\succeq^*$ satisfies Conditions T1 and T2. The first condition also guarantees that generalized receptive choice functions with $\succeq^*$ are $\succeq$-compatible. The fourth condition says that any indifference classes in $\mathcal{I}_{\geq 2} \cap \mathscr{I}_k(\succeq)$ should be ordered by their size. The fifth condition requires that in each $\mathscr{I}_k(\succeq)$, indifference classes in $\mathcal{I}_{\geq 2}$ be prioritized over others. The last two conditions are for the minimization of $\alpha(q, C)$ because as we see below, the size of indifference classes in $\mathcal{I}_{\geq 2}$ in a certain low-priority range contributes to $\alpha(q, C)$.

For any indifference class $I \in \mathscr{I}$ and weak priority ranking with a tie-breaking $\succeq^*$, define $\overline{I} := \{s \in \mathcal{S} : s \succ^* s' \ \forall s' \in I\} \cup I$.[20] Let $I^*(\succeq^*)$ be an indifference class such that $|\overline{I^*(\succeq^*)}| > q$ and $|\overline{I^*(\succeq^*)} \setminus I^*(\succeq^*)| \leq q$. In words, $I^*(\succeq^*)$ is an indifference class at the borderline of q when $\mathcal{S}$ is the choice set. To avoid the trivial case, we assume $|I| \leq 2q$ for all $I \in \mathcal{I}$. Note that if there exists $I \in \mathcal{I}$ such that $|I| > 2q$, then $\min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C) = q$.[21]

Proposition 6. *Given a collection of indifference classes $\mathcal{I}$, an integer q, and a weak priority ranking $\succeq$, suppose $|I| \leq 2q$ for all $I \in \mathcal{I}$ and consider weak priority ranking with a tie-breaking $\succeq^*$ constructed from $\succeq$. Define*

$$\alpha^* := \max \left\{ \mathbb{1}_{\{I^*(\succeq^*) \in \mathcal{I}_{\geq 2}\}} \left[\min \left\{ |\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \right\} \right], \max_{I \in \mathcal{I}_{\geq 2} : I^*(\succeq^*) \succ^* I} \left\lfloor \frac{|I|}{2} \right\rfloor \right\}.$$

Then, we have $\min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C) = \alpha^*$. Moreover, $(q, q + \alpha^*)$-generalized receptive choice function C^* with $\succeq^*$ achieves the minimum, i.e., $\alpha(q, C^*) = \alpha^*$.

Roughly speaking, $\min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C)$ is determined by the size of an indifference class $I \in \mathcal{I}_{\geq 2}$, which is ranked weakly lower than $I^*(\succeq^*)$ according to $\succeq^*$. The size of an indifference class I with $I \succ^* I^*(\succeq^*)$ does not matter because such an indifference class could always be accepted (for example, consider $\{s_1\}$ in Example 3). Any indifference class $I \in \mathcal{I}_{\geq 2}$ with $I^*(\succeq^*) \succ^* I$ has the possibility that it is at the borderline. Since ETI applies to

[20] The same definition applies to a weak priority ranking. $\overline{I}$ depends on $\succeq^*$ but we omit the dependence since the weak priority ranking with a tie-breaking is fixed in the relevant discussion.

[21] If there exists $I \in \mathcal{I}$ such that $|I| > 2q$, then $C(I) = I$ or $C(I) = \emptyset$ holds for any choice function C satisfying ETI. Then, $\alpha(q, C) \geq ||C(I)| - q| \geq q$ holds. A choice function C' with $C'(S) = \emptyset$ for any $S \subseteq \mathcal{S}$ achieves $\alpha(q, C') = q$.

$I \in \mathcal{I}_{\geq 2}$, the worst case in which I is at the borderline and the chosen set size departs from q most is when the choice set $S \subseteq \mathcal{S}$ is such that $S \subseteq \bar{I}$, $I \subseteq S$ and $|S| = q + \left\lfloor \frac{|I|}{2} \right\rfloor$. For $I^*(\succeq^*)$, since $|I^*(\succeq^*)|$ may only exceed q by a small amount, the worst-case violation in which $I^*(\succeq^*)$ is at the borderline and the chosen set size departs from q most is given by $\min\left\{|\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor\right\}$. α^* is found by the largest out of all such values. Note that the fourth and fifth conditions for the construction of $\succeq^*$ are important because within each $\mathscr{I}_k(\succeq)$, prioritizing any indifference classes in $\mathcal{I}_{\geq 2}$ over others and prioritizing larger indifference classes over smaller ones in $\mathcal{I}_{\geq 2}$ minimize the value of $\alpha(q, C)$.

It is worth noting that substitutability plays a crucial role in the proof of this proposition. Although the minimum is found by a generalized receptive choice function, other choice functions in $\mathcal{C}_2^{\succeq}(\mathcal{I})$ are not necessarily compatible with any weak priority ranking with a tie-breaking. Proposition 6 even covers the case where $\succeq$-compatibility is not required at all because the weak priority ranking $\succeq$ can be such that $I \sim I'$ for any $I, I' \in \mathscr{I}$. When we compare an arbitrary choice function $C \in \mathcal{C}_2^{\succeq}(\mathcal{I})$ and the $(q, q + \alpha^*)$-generalized receptive choice function C^*, we exploit the substitutability of the former and show that $\alpha(q, C)$ is weakly higher than $\alpha(q, C^*)$.

5 Discussions

5.1 When only ETI and substitutability are required

Our analysis in the main sections required consistency, in addition to ETI and substitutability. However, in some applications such as college admissions in Hungary, unreceptive choice functions, which are not consistent, are used. Although a stable matching (in a standard sense) is not guaranteed to exist under these choice functions, Biró and Kiselgof (2015) show that they lead to matchings that satisfy a certain generalization of stability.[22] Then, if we compromise consistency, could we further minimize the violation of q-acceptance? That is, would the minimum of $\alpha(q, C)$ be further lowered than $\min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C)$ if we drop consistency from Proposition 6?

The answer is indeed no. This immediately follows from the proof of Proposition 6. When we show that α^* is the lower bound of all choice functions in $\mathcal{C}_2^{\succeq}(\mathcal{I})$, consistency is not used. Then, the same minimum α^* applies to the class of choice functions with ETI,

[22]Fleiner and Jankó (2014) also study various stability concepts when a choice function is not path independent, including the one proposed by Fleiner (2003). Their focus is on the existence and properties of stable matchings.

substitutability, and $\succeq$-compatibility, and it is achieved by the same generalized receptive choice function. However, note that for some choice set, it is possible that another hybrid of receptive and unreceptive choice functions (which is not consistent) achieves a smaller violation than our generalized receptive choice function. Our measure of violation $\alpha(q, C)$ does not change because this does not happen for the worse case out of all possible choice sets $S \subseteq \mathcal{S}$. Thus, if we take another measure of q-acceptance violation, such as the mean across all possible choice sets, it could be affected by whether consistency is required. We leave the investigation of other approaches to measuring q-acceptance violation for future research.

5.2 Cost of size monotonicity in terms of $\alpha(q, C)$

Our results in Section 4 imply that given ETI and substitutability, there is a clear trade-off between size monotonicity and q-acceptance. That is, the requirement of size monotonicity in addition to ETI, substitutability, and consistency would increase the degree of q-acceptance violation by $\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C) - \min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C)$.[23] Following the discussion in Section 5.1, this increment would be the same when we consider only ETI and substitutability. Since the main benefit of size monotonicity is the strategy-proofness of the DA mechanism in matching markets, our approach provides one way to measure the "cost" of size monotonicity in terms of the magnitude of q-acceptance violation.

To give a sense of how this depends on the structure of indifference classes, consider the following example.

Example 4. Suppose $\mathcal{I} = \mathscr{I}$, $|\mathcal{S}| > 3q$, and $|I| = |I'| \geq 2$ for any $I, I' \in \mathscr{I}$. Then,

$$\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C) - \min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C) = \max\left\{q - \left\lfloor \frac{|I|}{2} \right\rfloor, 0\right\}.$$

$\square$

Here, the cost of size monotonicity increases as the size of each indifference class decreases. When size monotonicity is required, the *Algorithm* in Proposition 3 finds that $\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$ hits the upper bound q. Note that this does not depend on the size of each indifference class. By contrast, we can see that in Proposition 6, $\min_{C \in \mathcal{C}_2^{\succeq}(\mathcal{I})} \alpha(q, C)$ is equal to $\left\lfloor \frac{|I|}{2} \right\rfloor$ in this case. Therefore, size monotonicity would be seen as more costly when each indifference class is smaller. This message can be generalized to a more complex environment.

[23] Note that this is always non-negative by $\mathcal{C}_1^{\succeq}(\mathcal{I}) \subseteq \mathcal{C}_2^{\succeq}(\mathcal{I})$.

Our measure of the cost of size monotonicity helps us understand the implications of policies that determine the structure of indifference classes. For example, in admissions markets where ETI is applied to every priority category, the central authority could make the priority categories finer by using more detailed information.[24] This has indeed happened in Hungary in 2007, where ties became less common and the size of each tie was reduced due to the change in how final scores are calculated (Biró and Kiselgof, 2015). Such policies would weakly reduce both $\min_{C \in \mathcal{C}_1^{\succsim}(\mathcal{I})} \alpha(q, C)$ and $\min_{C \in \mathcal{C}_2^{\succsim}(\mathcal{I})} \alpha(q, C)$ in general, but Example 4 implies that $\min_{C \in \mathcal{C}_1^{\succsim}(\mathcal{I})} \alpha(q, C) - \min_{C \in \mathcal{C}_2^{\succsim}(\mathcal{I})} \alpha(q, C)$ would increase unless the size of each indifference class reduces to one. Therefore, whether or not a size monotonic choice function is used should be discussed together with the partition structure of indifference classes.

6 Concluding remarks

In this paper, we showed that the requirement of ETI can restrict possible choice functions given other standard axioms. Our Theorem 1 can be seen as either a possibility or impossibility result depending on the context. When the capacity is inflexible and $\min_{C \in \mathcal{C}_1^{\succsim}(\mathcal{I})} \alpha(q, C)$ we characterized is seen as too large, the policymaker would need to compromise one of the three axioms. As discussed in Section 3.1, the set of choice functions would be significantly enlarged in either of the three different directions. On the other hand, if the q-acceptance violation is not a problem, Theorem 1 implies that the three axioms are compatible. In this case, the school would need to commit to which indifference classes to accept irrespective of the choice sets. One such example is a choice based on the absolute performance measure rather than the relative performance of the applicants.

Our model focused on the choice problem of one school, and thus the interaction with the matching market with multiple such schools has not been examined. For example, we characterized the q-acceptance violation $\alpha(q, C)$ by taking the worst case out of all possible choice sets, but a potential choice set to each school can be determined by the matching market. Then, possible violation realized in a matching market may not be as large as $\alpha(q, C)$. Further, we consider the standard axioms such as substitutability, size monotonicity, and consistency to make the matching market work well. It is not clear yet how large the implications of the lack of these axioms as well as ETI are to the incentives and welfare of the agents. Therefore, it would be a fruitful direction to examine the impacts of these axioms on the agents' welfare at the matching market level.

[24] Technically, this is seen as a general version of tie-breaking that still leaves some indifference classes.

References

Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2009): "Strategy-Proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match," *American Economic Review*, 99, 1954–78.

Abdulkadiroğlu, A. and T. Sönmez (2003): "School Choice: A Mechanism Design Approach," *American Economic Review*, 93, 729–747.

Ágoston, K. C., P. Biró, E. Kováts, and Z. Jankó (2022): "College admissions with ties and common quotas: Integer programming approach," *European Journal of Operational Research*, 299, 722–734.

Aizerman, M. and A. Malishevski (1981): "General theory of best variants choice: Some aspects," *IEEE Transactions on Automatic Control*, 26, 1030–1040.

Alva, S. and B. Doğan (2021): "Choice and Market Design," *arXiv preprint arXiv:2110.15446*.

Aygün, O. and T. Sönmez (2013): "Matching with Contracts: Comment," *American Economic Review*, 103, 2050–51.

Aygün, O. and B. Turhan (2022): "How to De-reserve Reserves: Admissions to Technical Colleges in India," in *Proceedings of the 23rd ACM Conference on Economics and Computation*, 956–957.

Biró, P. and S. Kiselgof (2015): "College admissions with stable score-limits," *Central European Journal of Operations Research*, 23, 727–741.

Chambers, C. P. and M. B. Yenmez (2017): "Choice and Matching," *American Economic Journal: Microeconomics*, 9, 126–47.

Delacrétaz, D., S. D. Kominers, and A. Teytelboym (2020): "Matching Mechanisms for Refugee Resettlement," Unpublished manuscript.

Doğan, B., S. Doğan, and K. Yildiz (2021): "On Capacity-Filling and Substitutable Choice Rules," *Mathematics of Operations Research*, 46, 856–868.

Doğan, B., K. Imamura, and M. B. Yenmez (2022): "Market Design with Deferred Acceptance: A Recipe for Policymaking," *arXiv preprint arXiv:2209.06777*.

Dur, U., S. D. Kominers, P. A. Pathak, and T. Sönmez (2018): "Reserve Design: Unintended Consequences and the Demise of Boston's Walk Zones," *Journal of Political Economy*, 126, 2457–2479.

Echenique, F. and M. B. Yenmez (2015): "How to Control Controlled School Choice," *American Economic Review*, 105, 2679–94.

Ehlers, L. (2006): "Respecting Priorities when Assigning Students to Schools," Unpublished manuscript.

Erdil, A. and H. Ergin (2008): "What's the Matter with Tie-Breaking? Improving Efficiency in School Choice," *American Economic Review*, 98, 669–89.

Fleiner, T. (2003): "A Fixed-Point Approach to Stable Matchings and Some Applications," *Mathematics of Operations Research*, 28, 103–126.

Fleiner, T. and Z. Jankó (2014): "Choice Function-Based Two-Sided Markets: Stability, Lattice Property, Path Independence and Algorithms," *Algorithms*, 7, 32–59.

Hafalir, I. E., M. B. Yenmez, and M. A. Yildirim (2013): "Effective affirmative action in school choice," *Theoretical Economics*, 8, 325–363.

Hatfield, J. W. and P. R. Milgrom (2005): "Matching with Contracts," *American Economic Review*, 95, 913–935.

Imamura, K. (2020): "Meritocracy versus Diversity," Unpublished manuscript.

Kamada, Y. and F. Kojima (2023): "Fair Matching under Constraints: Theory and Applications," *The Review of Economic Studies*, rdad046.

Kamiyama, N. (2017): "Strategic issues in college admissions with score-limits," *Operations Research Letters*, 45, 105–108.

Korf, R. E. (1998): "A complete anytime algorithm for number partitioning," *Artificial Intelligence*, 106, 181–203.

Pathak, P. A., T. Sönmez, M. U. Ünver, and M. B. Yenmez (2021): "Fair Allocation of Vaccines, Ventilators and Antiviral Treatments: Leaving No Ethical Value Behind in Health Care Rationing," in *Proceedings of the 22nd ACM Conference on Economics and Computation*, 785–786.

Plott, C. R. (1973): "Path Independence, Rationality, and Social Choice," *Econometrica*, 41, 1075–1091.

Rios, I., T. Larroucau, G. Parra, and R. Cominetti (2021): "Improving the Chilean College Admissions System," *Operations Research*, 69, 1186–1205.

Roth, A. E. (1984): "Stability and Polarization of Interests in Job Matching," *Econometrica*, 52, 47–57.

Sönmez, T. and M. B. Yenmez (2022): "Affirmative Action in India via Vertical, Horizontal, and Overlapping Reservations," *Econometrica*, 90, 1143–1176.

Appendix A Omitted proofs

Appendix A.1 Proof of Theorem 1

It suffices to show that for all $I \in \mathcal{I}_{\geq 2}$, $s \in I$, and S with $s \in S$,

$$s \in C(\{s\}) \Longleftrightarrow s \in C(S).$$

($\Leftarrow$) This follows from the substitutability of C.

($\Rightarrow$) Toward a contradiction, suppose that there exist $I \in \mathcal{I}_{\geq 2}$, $s \in I$, and $S \subseteq \mathcal{S}$ with $s \in S$ such that $s \in C(\{s\})$ and $s \notin C(S)$. Then there must exist $S' \subset S$ and $t \in S \setminus S'$ such that $s \in C(S')$ and $s \notin C(S' \cup \{t\})$. Let s' be an arbitrary individual in $I \setminus \{s\}$.

Note that the substitutability of C implies

$$C(X \cup \{y\}) \setminus C(X) \subseteq \{y\} \tag{3}$$

for any arbitrary $X \subseteq \mathcal{S}$ and $y \in \mathcal{S}$.

Case 1. When $s' \in S'$.

ETI for $\mathcal{I}$ implies $\{s, s'\} \subseteq C(S')$ and $\{s, s'\} \cap C(S' \cup \{t\}) = \emptyset$. Together with equation (3) where $X = S'$ and $y = t$, we have $|C(S' \cup \{t\})| < |C(S')|$, which is a contradiction to the size monotonicity of C.

Case 2. When $s' \notin S'$ and $s' \in C(S' \cup \{s'\})$.

First, $s' \neq t$ should hold because ETI for $\mathcal{I}$ implies $s \in C(S' \cup \{s'\})$ but we have $s \notin C(S' \cup \{t\})$. Since C is substitutable, $s \notin C(S' \cup \{t\})$ implies $s \notin C(S' \cup \{s', t\})$. By ETI for $\mathcal{I}$, we have $\{s, s'\} \subseteq C(S' \cup \{s'\})$ and $\{s, s'\} \cap C(S' \cup \{s', t\}) = \emptyset$. Together with equation (3) where $X = S' \cup \{s'\}$ and $y = t$, we have $|C(S' \cup \{s', t\})| < |C(S' \cup \{s'\})|$, which is a contradiction to the size monotonicity of C.

Case 3. When $s' \notin S'$ and $s' \notin C(S' \cup \{s'\})$.[25]

We have $s \in C(S')$, and ETI for $\mathcal{I}$ implies $\{s, s'\} \cap C(S' \cup \{s'\}) = \emptyset$. Together with equation (3) where $X = S'$ and $y = s'$, we have $|C(S' \cup \{s'\})| < |C(S')|$, which is a contradiction to the size monotonicity of C.

Appendix A.2 Proof of Proposition 1

If part. Since $C(S)$ is written as $S \cap (\cup_{I \in \tilde{\mathscr{I}}(S)} I)$ where $\tilde{\mathscr{I}}(S)$ is a set of indifference classes for each $S \in 2^{\mathcal{S}}$, for any $I \in \mathscr{I}$, $s \in C(S)$ if and only if $s' \in C(S)$ for any $s, s' \in S \cap I$, which

[25] In this case, $s' = t$ is possible.

implies ETI for $\mathscr{I}$. Further, since $(\tilde{\mathscr{I}}(S))_{S\in 2^{\mathcal{S}}}$ is increasing and $C(S) = S \cap (\cup_{I\in\tilde{\mathscr{I}}(S)} I)$, $|C(S)| \leq |C(S')|$ for any $S, S' \in 2^{\mathcal{S}}$ with $S \subseteq S'$, which implies size monotonicity.

Only if part. Since C satisfies ETI for all indifference classes in $\mathscr{I}$, for every $S \in 2^{\mathcal{S}}$, there must be a set of indifference classes $\tilde{\mathscr{I}}(S) \subseteq \mathscr{I}$ such that $C(S)$ is written as $S \cap (\cup_{I\in\tilde{\mathscr{I}}(S)} I)$. Size monotonicity of C implies $|C(S)| \leq |C(S')|$ for any $S, S' \in 2^{\mathcal{S}}$ with $S \subseteq S'$. Then, we have $|S \cap (\cup_{I\in\tilde{\mathscr{I}}(S)} I)| \leq |S' \cap (\cup_{I\in\tilde{\mathscr{I}}(S')} I)|$ for such S and S', implying that $(\tilde{\mathscr{I}}(S))_{S\in 2^{\mathcal{S}}}$ is increasing.

Appendix A.3 Proof of Proposition 2

If part. For any $S \subseteq \mathcal{S}$, $I \in \mathcal{I}$, and $s, s' \in S \cap I$, $s \in C(S)$ implies $s' \in I \subseteq C(S \cup I)$. Since C is substitutable, $s' \in C(S \cup I)$ implies $s' \in C(S)$. Thus, C satisfies ETI for $\mathcal{I}$.

Only if part. Take arbitrary $I \in \mathcal{I}_{\geq 2}$, $s \in I$, and $S \subseteq \mathcal{S}$ with $s \in S$.

($\Leftarrow$) Since C is substitutable, $s \in I \subseteq C(S \cup I)$ implies $s \in C(S)$.

($\Rightarrow$) When $I \subseteq S$, $s \in C(S)$ and ETI for $\mathcal{I}$ imply $I \subseteq C(S) = C(S \cup I)$, and the proof is done. Suppose $I \nsubseteq S$. Toward a contradiction, suppose that $s \in C(S)$ and $I \nsubseteq C(S \cup I)$. $I \nsubseteq C(S \cup I)$ and ETI for $\mathcal{I}$ imply $I \subseteq (S \cup I) \setminus C(S \cup I)$. Then, we have $s \in C(S)$ and $s \notin C(S \cup I)$ at the same time. However, this implies $C(S \cup I) \neq C(S)$ while $C(S \cup I) \subseteq S$, which is a contradiction to the consistency of C.

Appendix A.4 Proof of Lemma 1

Consider an arbitrary choice function $C \in \mathcal{C}_1^{\succeq}(\mathcal{I})$. Theorem 1 implies the first part of condition (i), and $\succeq$-compatibility implies the second part of condition (i). Take any indifference class $I' \in \mathscr{I}$ such that there exists $I \in \mathcal{I}_{\geq 2}$ with $I \subseteq A(C)$ and $I' \succ I$. Since $I' \subseteq C(\mathcal{S})$ holds by $I \subseteq A(C)$ and $\succeq$-compatibility, $I' \cap S \subseteq C(S)$ must also hold for any $S \subseteq \mathcal{S}$ by the substitutability of C. Then, condition (ii) holds. Take any indifference class $I' \in \mathscr{I}$ such that there exists $I \in \mathcal{I}_{\geq 2}$ with $I \subseteq R(C)$ and $I \succ I'$. Take any $s \in I$ and $s' \in I'$. By $I \subseteq R(C)$ and $\succeq$-compatibility, $C(\{s, s'\}) = \emptyset$. Then, size monotonicity implies $C(\{s'\}) = \emptyset$. By the substitutability of C, $s' \notin C(S)$ for any $S \subseteq \mathcal{S}$, which implies $s' \in R(C)$. Thus, condition (iii) also holds.

Appendix A.5 Proof of Lemma 2

To find the minimum of $\alpha(q, C)$ in $\mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$, we define the following choice function.

Definition 5. Choice function C is *quasi-q-acceptant* if $\mathcal{S}$ is partitioned into three sets A, R, and T, and $C(S)$ satisfies the following three conditions for any $S \subseteq \mathcal{S}$:

1. $s \in C(S)$ for any $s \in A \cap S$;

2. $s \notin C(S)$ for any $s \in R \cap S$; and

3. $|T \cap C(S)| = \max\big\{0, \min\{q - |A \cap S|, |T \cap S|\}\big\}$.

For given $(A, R) \subseteq \mathcal{S}^2$, consider a quasi-$q$-acceptant choice function $C^* \in \mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$ where T is defined as $\mathcal{S} \backslash (A \cup R)$. Take any $S \subseteq \mathcal{S}$. When $|C^*(S)|$ strictly exceeds $\min\{q, |S|\}$, $\min\{q, |S|\} = q$ must happen because otherwise $|C^*(S)| > |S|$ would be a contradiction. Then, we have

$$\big||C^*(S)| - \min\{q, |S|\}\big| = |C^*(S)| - q = |A \cap S| - q$$

because in this case $C^*(S) = A \cap S$ holds by the definition of a quasi-q-acceptant choice function. When $\min\{q, |S|\}$ weakly exceeds $|C^*(S)|$, we have

$$\big||C^*(S)| - \min\{q, |S|\}\big| = \min\{q, |S|\} - |C^*(S)| = \min\{q - |C^*(S)|, |R \cap S|\}.$$

To see the second part of this equation, note that when $|S| \leq q$, $|T \cap S| \leq q - |A \cap S|$ holds, and thus a quasi-q-acceptant choice function should satisfy $C^*(S) = (A \cup T) \cap S$. Then, we obtain

$$\alpha(q, C^*) = \max_{S \subseteq \mathcal{S}} \big||C^*(S)| - \min\{q, |S|\}\big| = \max\big\{|A| - q, \min\{|R|, q\}\big\},$$

since the maximum of $\big||C^*(S)| - \min\{q, |S|\}\big|$ across all $S \subseteq \mathcal{S}$ is achieved when $S = A$ or $S = R$. Given that A and R are taken as fixed, no choice function C in $\mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)$ can achieve a strictly lower value of $\alpha(q, C)$ than $\alpha(q, C^*)$. Thus, $\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I}, A, R)} \alpha(q, C)$ is given by $\max\big\{|A| - q, \min\{|R|, q\}\big\}$.

Appendix A.6 Proof of Lemma 3

Since $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ holds and the *Algorithm* proceeds to Step $k+1$, we must have $\min\{\sigma^I(k), \sigma^T(k)\} \geq \sigma^B(k)$. If $\mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2} = \emptyset$ holds for all $l \in \{k+1, \ldots, K\}$, the proof is done because of $\sigma^B(k) = \sigma^*$. Then, suppose that $\mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ for some $l \in \{k+1, \ldots, K\}$ and let k_1 be the smallest such integer. By definitions, we have $\sigma^T(k_1) = \sigma^B(k)$. Then, $\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\} = \sigma^B(k) = \sigma^T(k_1) \geq \min\{\sigma^I(k_1), \sigma^T(k_1), \sigma^B(k_1)\}$ holds.

If the *Algorithm* terminates at Step k_1, $\min\{\sigma^I(k_1), \sigma^T(k_1), \sigma^B(k_1)\} = \sigma^*$ holds. Otherwise, by the same logic as above, there exists $k_2 \in \{k_1+1, \ldots, K\}$ such that $\min\{\sigma^I(k_1), \sigma^T(k_1), \sigma^B(k_1)\} \geq \min\{\sigma^I(k_2), \sigma^T(k_2), \sigma^B(k_2)\}$. By repeating this argument, we obtain $\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\} \geq \sigma^*$ since there are finite steps in the *Algorithm.*

Appendix A.7 Proof of Lemma 4

Since the *Algorithm* terminates at Step $k \in \{1, \ldots, K-1\}$, $\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2} \neq \emptyset$ and $\sigma^* = \min\{\sigma^I(k), \sigma^T(k)\}$ hold. Define (A_1, R_1) as

$$A_1 := \begin{cases} A(k) \cup \big(\cup_{I \in \pi_1(\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, A(k), R(k))} I \big) & \text{if } \sigma^* = \sigma^I(k), \\ A^T(k) & \text{if } \sigma^* = \sigma^T(k), \end{cases}$$

$$R_1 := \begin{cases} R(k) \cup \big(\cup_{I \in \pi_2(\mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}, A(k), R(k))} I \big) & \text{if } \sigma^* = \sigma^I(k), \\ R(k) \cup \big(\cup_{I \in \mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}} I \big) & \text{if } \sigma^* = \sigma^T(k). \end{cases}$$

Define (A_2, R_2) as

$$A_2 := A(k) \cup \big(\cup_{I \in \mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}} I \big),$$
$$R_2 := R^B(k).$$

By definitions, (A_1, R_1) achieves $\max\big\{|A_1| - q, \min\{|R_1|, q\}\big\} = \min\{\sigma^I(k), \sigma^T(k)\}$, and (A_2, R_2) achieves $\max\big\{|A_2| - q, \min\{|R_2|, q\}\big\} = \sigma^B(k)$. We also have $A_1 \subseteq A_2$ and $R_1 \supseteq R_2$.

Since the *Algorithm* terminates in this step, we should have

$$\max\big\{|A_1| - q, \min\{|R_1|, q\}\big\} < \max\big\{|A_2| - q, \min\{|R_2|, q\}\big\}.$$

We have $R_1 \neq R_2$ because the only possibility with $R_1 = R_2$ would be when $\min\{\sigma^I(k), \sigma^T(k)\} = \sigma^I(k) = \sigma^B(k)$ holds, but this is a contradiction to the supposition that the *Algorithm* terminated at Step k. Thus, $R_1 \supsetneq R_2$ holds. If $|A_2| - q \leq \min\{|R_2|, q\}$ holds, it would lead to $\min\{|R_1|, q\} \leq \max\big\{|A_1| - q, \min\{|R_1|, q\}\big\} < \max\big\{|A_2| - q, \min\{|R_2|, q\}\big\} = \min\{|R_2|, q\}$, which is a contradiction to $|R_1| > |R_2|$. Thus, we should have $\max\big\{|A_2| - q, \min\{|R_2|, q\}\big\} = |A_2| - q$.

For an arbitrary step $l \in \{k+1, \ldots, K\}$, define (A_3, R_3) as

$$A_3 := \begin{cases} A(l) \cup \big(\cup_{I \in \pi_1(\mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2}, A(l), R(l))} I \big) & \text{if } \sigma^I(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}, \\ A^T(l) & \text{if } \sigma^T(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}, \\ A(l) \cup \big(\cup_{I \in \mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2}} I \big) & \text{if } \sigma^B(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}, \end{cases}$$

$$R_3 := \begin{cases} R(l) \cup \big(\cup_{I \in \pi_2(\mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2}, A(l), R(l))} I \big) & \text{if } \sigma^I(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}, \\ R(l) \cup \big(\cup_{I \in \mathscr{I}_l(\succeq) \cap \mathcal{I}_{\geq 2}} I \big) & \text{if } \sigma^T(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}, \\ R^B(l) & \text{if } \sigma^B(l) = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}. \end{cases}$$

(A_3, R_3) achieves $\max\big\{|A_3| - q, \min\{|R_3|, q\}\big\} = \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}$, and $A_2 \subseteq A_3$ and $R_2 \supseteq R_3$ hold by definitions. These lead to $\max\big\{|A_3| - q, \min\{|R_3|, q\}\big\} \geq |A_3| - q \geq |A_2| - q$.

Therefore, we obtain

$$\sigma^* < \sigma^B(k) \leq \min\{\sigma^I(l), \sigma^T(l), \sigma^B(l)\}$$

for all $l \in \{k+1, \ldots, K\}$.

Appendix A.8 Proof of Proposition 3

Let $(\mathcal{A}, \mathcal{R})$ be the set of all pairs of disjoint sets $(A, R) \subseteq \mathcal{S}^2$ that satisfy the conditions (i)-(iii) in Lemma 1. Then by Lemmas 1 and 2, we have

$$\min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C) = \min_{(A,R) \in (\mathcal{A},\mathcal{R})} \min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I},A,R)} \alpha(q, C) = \min_{(A,R) \in (\mathcal{A},\mathcal{R})} \max\big\{|A| - q, \min\{|R|, q\}\big\}. \quad (4)$$

Let $(\mathcal{A}', \mathcal{R}')$ be the set of all pairs of disjoint sets $(A, R) \subseteq \mathcal{S}^2$ that satisfy the conditions (i)–(iii) in Lemma 1 and the following condition (iv): for any $I' \in \mathscr{I} \setminus \mathcal{I}_{\geq 2}$, $I' \subseteq \mathcal{S} \setminus (A \cup R)$ if there is no $I \in \mathcal{I}_{\geq 2}$ such that $[I \subseteq A$ and $I' \succ I]$ or $[I \subseteq R$ and $I \succ I']$. For any $(A, R) \in (\mathcal{A}, \mathcal{R})$ and $T := \mathcal{S} \setminus (A \cup R)$, consider $T' := T \cup \{s \in \mathcal{S} : s \in I' \cap (A \cup R)$ for some $I' \in \mathscr{I} \setminus \mathcal{I}_{\geq 2}$ such that there is no $I \in \mathcal{I}_{\geq 2}$ with $[I \subseteq A$ and $I' \succ I]$ or $[I \subseteq R$ and $I \succ I']\}$, $A' := A \setminus T'$, and $R' := R \setminus T'$. Then, we have $(A', R') \in (\mathcal{A}', \mathcal{R}')$, $A' \subseteq A$ and $R' \subseteq R$, which implies $\max\big\{|A'| - q, \min\{|R'|, q\}\big\} \leq \max\big\{|A| - q, \min\{|R|, q\}\big\}$. Then, we have

$$\min_{(A,R) \in (\mathcal{A},\mathcal{R})} \max\big\{|A| - q, \min\{|R|, q\}\big\} = \min_{(A,R) \in (\mathcal{A}',\mathcal{R}')} \max\big\{|A| - q, \min\{|R|, q\}\big\}. \quad (5)$$

Further, since (A', R') satisfies conditions (i)–(iv), there must exist $k \in \{1, \ldots, K\}$ such that (A', R') is equal to either $(A^T(k), R(k) \cup (\cup_{I \in \mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}} I))$, $(A(k) \cup (\cup_{I \in \mathscr{I}_k(\succeq) \cap \mathcal{I}_{\geq 2}} I), R^B(k))$, or $(A(k) \cup (\cup_{I \in \mathscr{I}_k^1(\succeq) \cap \mathcal{I}_{\geq 2}} I), R(k) \cup (\cup_{I \in \mathscr{I}_k^2(\succeq) \cap \mathcal{I}_{\geq 2}} I))$, where $\mathscr{I}_k^1(\succeq)$ is some subset of $\mathscr{I}_k(\succeq)$ and $\mathscr{I}_k^2(\succeq) = \mathscr{I}_k(\succeq) \setminus \mathscr{I}_k^1(\succeq)$. In other words, every $(A', R') \in (\mathcal{A}', \mathcal{R}')$ is examined in the minimization of the interior solution, the top-corner solution, or the bottom-corner solution of some step of the *Algorithm*. This implies

$$\min_{(A,R) \in (\mathcal{A}',\mathcal{R}')} \max\big\{|A| - q, \min\{|R|, q\}\big\} = \min_{k \in \{1,\ldots,K\}} \big\{\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\}\big\}. \quad (6)$$

Finally, Lemmas 3 and 4 show that $\min_{k \in \{1,\ldots,K\}} \big\{\min\{\sigma^I(k), \sigma^T(k), \sigma^B(k)\}\big\} = \sigma^*$. Combining this with equations (4), (5), and (6), we obtain $\sigma^* = \min_{C \in \mathcal{C}_1^{\succeq}(\mathcal{I})} \alpha(q, C)$.

Appendix A.9 Proof of Proposition 4

For any choice function C in $\mathcal{C}_1^{\succeq}(\mathcal{I})$, by condition (i) of Lemma 1, any indifference class in $\mathcal{I}_{\geq 2}$ needs to be included in either the always-accepted set $A(C)$ and the always-rejected set

$R(C)$. Then by Lemma 2, the minimum of $\alpha(q, C)$ across all choice functions in $\mathcal{C}_1^{\succ}(\mathcal{I})$ can be written as

$$\min_{C \in \mathcal{C}_1^{\succ}(\mathcal{I})} \alpha(q, C) = \min_{r \in \mathcal{R}(\mathcal{I})} \max\left\{ \min\{r, q\}, \max\{e-r-q, 0\}\right\} = \min_{r \in \mathcal{R}(\mathcal{I})} \max\left\{ \min\{r, q\}, e-r-q\right\},$$

where r represents the cardinality of $R(C)$. Note that the second equality is because of $\min\{r, q\} \geq 0$.

To begin, q is always guaranteed as an upper bound of $\min_{C \in \mathcal{C}(\mathcal{I})} \alpha(q, C)$ since $r = e \in \mathcal{R}(\mathcal{I})$ achieves $\max\left\{ \min\{r, q\}, e - r - q\right\} = \min\{r, q\} \leq q$.

When $e \leq q$, because of $e-q \leq 0$, $\min_{C \in \mathcal{C}(\mathcal{I})} \alpha(q, C) = 0$ can be achieved by $r = 0 \in \mathcal{R}(\mathcal{I})$.

When $e \in (q, 2q]$, $r > q$ would lead to the upper bound of q. Thus, let us consider the case with $r \leq q$. In this case, the optimal r is to minimize $\max\{r, e-r-q\}$, which is to take $r \in \mathcal{R}(\mathcal{I})$ as close to $\frac{e-q}{2}$ as possible. Then, given the constraint that r must be chosen from the set of integers in $\mathcal{R}(\mathcal{I})$, $\min_{r \in \mathcal{R}(\mathcal{I})} \max\{r, e-r-q\}$ is given by $\frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right|$. By $e \in (q, 2q]$, $\frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right| \leq q$ always holds and this is indeed optimal across all $r \in \mathcal{R}(\mathcal{I})$.

When $e \in (2q, 3q]$, $\min_{r \in \mathcal{R}(\mathcal{I})} \max\{r, e-r-q\}$ is given by $\frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right|$ in the same way as above. However, $\frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right|$ may or may not exceed q when $e \in (2q, 3q]$. Since q is guaranteed as an upper bound, the minimum of $\alpha(q, C)$ is given by $\min\left\{q, \frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right|\right\}$.

When $e > 3q$, $\frac{e-q}{2} + \min_{r \in \mathcal{R}(\mathcal{I})} \left|\frac{e-q}{2} - r\right| > q$ always holds, which implies that it is impossible to take r to make $\min_{C \in \mathcal{C}(\mathcal{I})} \alpha(q, C)$ lower than q. Therefore, $\min_{C \in \mathcal{C}(\mathcal{I})} \alpha(q, C)$ is always q.

Appendix A.10 Proof of Proposition 5

C clearly satisfies ETI for $\mathcal{I}$. First, we show that C satisfies consistency. It suffices to show that for any $S \subseteq \mathcal{S}$ and $s \in S \setminus C(S)$, $C(S) = C(S \setminus \{s\})$. Let l be the last step of the algorithm when S is considered.

Case 1. $s \notin S \cap I^l(S)$.

Then $s \in S \setminus S^l(S)$. By the definition of $I^l(S)$, $I^l(S) \succ s$. This implies that $I^t(S) = I^t(S \setminus \{s\})$ and $S^t(S) = S^t(S \setminus \{s\})$ for any t with $1 \leq t \leq l$. Since $S^l(S) = S^l(S \setminus \{s\})$, l is also the last step of the algorithm when $S \setminus \{s\}$ is considered. Thus, $C(S) = C(S \setminus \{s\})$.

Case 2. $s \in S \cap I^l(S)$.

$I^{l-1}(S) \succ s$ implies $I^t(S) = I^t(S \setminus \{s\})$ and $S^t(S) = S^t(S \setminus \{s\})$ for any t with $1 \leq t < l$. Since $S \setminus C(S) \neq \emptyset$, either (i) $|S^{l-1}(S)| + \max_{I \in \mathcal{I}_{\geq 2}: I^{l-1}(S) \succ I} |I| > \bar{q}$ or (ii)

$|S^{l-1}(S)| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S)\succ I}|I| \leq \bar{q}$ and $|S^l(S)| \geq q$ holds. In case (i), $|S^{l-1}(S\setminus\{s\})| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S\setminus\{s\})\succ I}|I| > \bar{q}$ holds since $I^{l-1}(S) = I^{l-1}(S\setminus\{s\})$ and $S^{l-1}(S) = S^{l-1}(S\setminus\{s\})$. Thus, $C(S) = S^{l-1}(S) = S^{l-1}(S\setminus\{s\}) = C(S\setminus\{s\})$. Consider case (ii). $s \in S\setminus C(S)$ implies $I^l(S) \notin \mathcal{I}$. Thus, there exists $s^* \in S$ such that $C(S) = \{s' \in S : s' \succeq s^*\}$ and $|\{s' \in S : s' \succeq s^*\}| = q$. $s \in S\setminus C(S)$ implies $s^* \succ s$, which implies $|\{s' \in S : s' \succeq s^*\}| = |\{s' \in S\setminus\{s\} : s' \succeq s^*\}| = q$. Thus, $C(S) = C(S\setminus\{s\})$.

Next, we show that C satisfies substitutability: for any $S \subseteq \mathcal{S}$ and $s \in S$, $C(S)\setminus\{s\} \subseteq C(S\setminus\{s\})$. If $s \in S\setminus C(S)$, then $C(S)\setminus\{s\} \subseteq C(S\setminus\{s\})$ by consistency. Suppose that $s \in C(S)$ and $s \in I^l(S)$: i.e., s is chosen at Step l when S is considered. Let l^* be the last step of the algorithm when S is considered. It suffices to show $(C(S)\setminus\{s\})\cap I^l(S) \subseteq C(S\setminus\{s\})$ and $C(S)\cap I^t(S) \subseteq C(S\setminus\{s\})$ for any t with $1 \leq t \leq l^*$ and $t \neq l$.

For any t with $t < l$, we have $I^t(S) = I^t(S\setminus\{s\})$ and $S^t(S) = S^t(S\setminus\{s\})$. Thus, $C(S)\cap I^t(S) \subseteq C(S\setminus\{s\})$. For $I^l(S)$, if $C(S)\cap I^l(S) = \{s\}$, then clearly $(C(S)\setminus\{s\})\cap I^l(S) = \emptyset \subseteq C(S\setminus\{s\})$. Suppose $C(S)\cap I^l(S) \supsetneq \{s\}$. By $C(S)\cap I^l(S) \neq \emptyset$, we have $|S^{l-1}(S)| < q$ and $|S^{l-1}(S)| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S)\succ I}|I| \leq \bar{q}$. Since $S^{l-1}(S) = S^{l-1}(S\setminus\{s\})$ and $I^{l-1}(S) = I^{l-1}(S\setminus\{s\})$, we have $|S^{l-1}(S\setminus\{s\})|$ and $|S^{l-1}(S\setminus\{s\})| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S\setminus\{s\})\succ I}|I| \leq \bar{q}$. Thus, $C(S\setminus\{s\})\cap I^l(S\setminus\{s\}) \neq \emptyset$. $C(S)\cap I^l(S) \supsetneq \{s\}$ implies $(S\setminus\{s\})\cap I^l(S) \neq \emptyset$. Thus, $I^l(S) = I^l(S\setminus\{s\})$. If $I^l(S) \in \mathcal{I}$, then $(C(S)\setminus\{s\})\cap I^l(S) \subseteq C(S\setminus\{s\})$. Suppose $I^l(S) \notin \mathcal{I}$. Since $|(S\setminus\{s\})\cap I^l(S\setminus\{s\})| = |(S\setminus\{s\})\cap I^l(S)| < |S\cap I^l(S)|$ and $S^{l-1}(S) = S^{l-1}(S\setminus\{s\})$, we have $(C(S)\setminus\{s\})\cap I^l(S) \subseteq C(S\setminus\{s\})$.

If $S^l(S) = C(S)$, then the proof is done. Suppose $S^l(S) \subsetneq C(S)$. That is, $l < l^*$ and $C(S)\cap I^{l+1}(S) \neq \emptyset$. There are two cases to consider. Case 1: $(S\setminus\{s\})\cap I^l(S) \neq \emptyset$. By $C(S)\cap I^{l+1}(S) \neq \emptyset$, we have $|S^l(S)| < q$ and $|S^l(S)| + \max_{I\in\mathcal{I}_{\geq 2}:I^l(S)\succ I}|I| \leq \bar{q}$. Since $(S\setminus\{s\})\cap I^l(S) \neq \emptyset$, we have $S^l(S) = S^l(S\setminus\{s\})$ and $I^l(S) = I^l(S\setminus\{s\})$. Thus, we have $|S^l(S\setminus\{s\})| < q$ and $|S^l(S\setminus\{s\})| + \max_{I\in\mathcal{I}_{\geq 2}:I^l(S\setminus\{s\})\succ I}|I| \leq \bar{q}$. These imply $C(S\setminus\{s\})\cap I^{l+1}(S) \neq \emptyset$. If $I^{l+1}(S) \in \mathcal{I}$, then $C(S)\cap I^{l+1}(S) \subseteq C(S\setminus\{s\})$. Suppose $I^{l+1}(S) \notin \mathcal{I}$. By $|S^l(S\setminus\{s\})| = |S^l(S)\setminus\{s\}| < |S^l(S)|$, we have $C(S)\cap I^{l+1}(S) \subseteq C(S\setminus\{s\})$. Case 2: $(S\setminus\{s\})\cap I^l(S) = \emptyset$. Since $(S\setminus\{s\})\cap I^l(S) = \emptyset$, we have $I^{l+1}(S) = I^l(S\setminus\{s\})$. Now, we have

$$
\begin{aligned}
&|S^{l-1}(S\setminus\{s\})| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S\setminus\{s\})\succ I}|I| \\
&= |S^{l-1}(S)| + \max_{I\in\mathcal{I}_{\geq 2}:I^{l-1}(S)\succ I}|I| \\
&\leq \bar{q}.
\end{aligned}
$$

The first equality follows from $S^{l-1}(S\setminus\{s\}) = S^{l-1}(S)$ and $I^{l-1}(S\setminus\{s\}) = I^{l-1}(S)$. The second inequality follows from $s \in C(S)$. Note $|S^{l-1}(S \setminus \{s\})| = |S^{l-1}(S)| < q$. These imply $C(S \setminus \{s\}) \cap I^{l+1}(S) \neq \emptyset$. If $I^{l+1}(S) \in \mathcal{I}$, then $C(S) \cap I^{l+1}(S) \subseteq C(S \setminus \{s\})$. Suppose $I^{l+1}(S) \notin \mathcal{I}$. By $|S^{l-1}(S \setminus \{s\})| = |S^{l}(S) \setminus \{s\}| < |S^{l}(S)|$, $C(S) \cap I^{l+1}(S) \subseteq C(S \setminus \{s\})$.

If $S^{l+1}(S) = C(S)$, then the proof is done. Suppose $S^{l+1}(S) \subsetneq C(S)$. That is, $l+1 < l^*$ and $C(S) \cap I^{l+2}(S) \neq \emptyset$. By $C(S) \cap I^{l+2}(S) \neq \emptyset$, we have $|S^{l+1}(S)| < q$ and $|S^{l+1}(S)| + \max_{I \in \mathcal{I}_{\geq 2}: I^{l+1}(S) \succ I} |I| \leq \bar{q}$. The latter implies $|S^{l+1}(S) \setminus \{s\}| + \max_{I \in \mathcal{I}_{\geq 2}: I^{l+1}(S) \succ I} |I| \leq \bar{q}$. Note that either $I^{l+2}(S) = I^{l+2}(S \setminus \{s\})$ or $I^{l+2}(S) = I^{l+1}(S \setminus \{s\})$ holds. If $I^{l+2}(S) = I^{l+2}(S \setminus \{s\})$, then $I^{l+1}(S) = I^{l+1}(S \setminus \{s\})$ and $S^{l+1}(S) = S^{l+1}(S \setminus \{s\})$. Thus, we have $|S^{l+1}(S \setminus \{s\})| < q$ and $|S^{l+1}(S \setminus \{s\})| + \max_{I \in \mathcal{I}_{\geq 2}: I^{l+1}(S \setminus \{s\}) \succ I} |I| \leq \bar{q}$. These imply $C(S \setminus \{s\}) \cap I^{l+2}(S \setminus \{s\}) \neq \emptyset$. If $I^{l+2}(S) = I^{l+1}(S \setminus \{s\})$, then $I^{l+1}(S) = I^{l}(S \setminus \{s\})$ and $S^{l+1}(S) = S^{l}(S \setminus \{s\})$. Thus, we have $|S^{l}(S \setminus \{s\})| < q$ and $|S^{l}(S \setminus \{s\})| + \max_{I \in \mathcal{I}_{\geq 2}: I^{l}(S \setminus \{s\}) \succ I} |I| \leq \bar{q}$. These imply $C(S \setminus \{s\}) \cap I^{l+1}(S \setminus \{s\}) \neq \emptyset$. That is, $C(S \setminus \{s\}) \cap I^{l+2}(S) \neq \emptyset$ in both cases. If $I^{l+2}(S) \in \mathcal{I}$, then $C(S) \cap I^{l+2}(S) \subseteq C(S \setminus \{s\})$. Suppose $I^{l+2}(S) \notin \mathcal{I}$. By $|S^{l+1}(S) \setminus \{s\}| < |S^{l+1}(S)|$, we have $C(S) \cap I^{l+2}(S) \subseteq C(S \setminus \{s\})$. If $l+2 < l^*$, we can apply the same argument for $I^{t}(S)$ for any t with $l+2 < t \leq l^*$ and obtain $C(S) \cap I^{t}(S) \subseteq C(S \setminus \{s\})$.

Appendix A.11 Proof of Proposition 6

We begin by showing that $\alpha(q, C) = \alpha^*$ is achieved when choice function C is a $(q, \bar{q})$-generalized receptive choice function with $\bar{q} = q + \alpha^*$. First, we show that $\alpha(q, C^*) \leq \alpha^*$. By the definition of a generalized receptive choice function, for any $S \subseteq \mathcal{S}$, $|C^*(S)| \leq \bar{q}$. Thus, $|C^*(S)| - q \leq \alpha^*$. We show that for any $S \subseteq \mathcal{S}$, $q - |C^*(S)| \leq \alpha^*$ If $q - |C^*(S)| \leq 0$, then clearly $q - |C^*(S)| \leq \alpha^*$. Suppose $q - |C^*(S)| > 0$. By the definition of a generalized receptive choice function, there exists t such that $C^*(S) = S^{t-1}(S)$ and $|S^{t-1}(S)| + \max_{I \in \mathcal{I}_{\geq 2}: I^{t-1}(S) \succ^* I} |I| > \bar{q}$. Take any $I' \in \arg\max_{I \in \mathcal{I}_{\geq 2}: I^{t-1}(S) \succ^* I} |I|$. Note $|C^*(S)| + |I'| \geq \bar{q} + 1$. We show that $|I'| \leq 2\alpha^* + 1$. Note that either $I^*(\succeq^*) \succ^* I'$or $I' = I^*(\succeq^*)$ holds. If $I^*(\succeq^*) \succ^* I'$, then $|I'| \leq 2\alpha^* + 1$ since $\max_{I \in \mathcal{I}_{\geq 2}: I^*(\succeq^*) \succ^* I} \left\lfloor \frac{|I|}{2} \right\rfloor \leq \alpha^*$ and $\frac{|I|}{2} - \left\lfloor \frac{|I|}{2} \right\rfloor \in \{0, \frac{1}{2}\}$. Suppose $I' = I^*(\succeq^*)$. If $\min \left\{ |\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \right\} = |\overline{I^*(\succeq^*)}| - q$, then $|\overline{I^*(\succeq^*)}| - q \leq \alpha^*$. Since $\bar{q} = q + \alpha^*$, we have $|\overline{I^*(\succeq^*)}| \leq \bar{q}$. However, $S^{t-1}(S) \cup I^*(\succeq^*) \subseteq \overline{I^*(\succeq^*)}$ implies $|C^*(S)| + |I^*(\succeq^*)| \leq \bar{q}$, which is a contradiction. Thus, $\min \left\{ |\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \right\} = \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor$, which implies $|I^*(\succeq^*)| = |I'| \leq 2\alpha^* + 1$. These facts together imply $q - |C^*(S)| \leq |I'| - (\alpha^* + 1) \leq \alpha^*$. The first inequality follows from $|C^*(S)| + |I'| \geq \bar{q} + 1$, and the second inequality follows from $|I'| \leq 2\alpha^* + 1$.

Next, we show that there exists $S \subseteq \mathcal{S}$ such that $|C^*(S)| - q = \alpha^*$. If $I^*(\succeq^*) \in \mathcal{I}_{\geq 2}$ and $\alpha^* = |\overline{I^*(\succeq^*)}| - q$, then $|C^*(\overline{I^*(\succeq^*)})| - q = \alpha^*$. Suppose $\alpha^* = \left\lfloor \frac{|I'|}{2} \right\rfloor$ for some $I' \in \{I \in$

$\mathcal{I}_{\geq 2} : I^*(\succeq^*) \succ^* I\} \cup \{I^*(\succeq^*)\}$. Then, there exists $S \subseteq \mathcal{S}$ such that $S \subseteq \overline{I'}$, $I' \subseteq S$, and $|S| = q + \left\lfloor \frac{|I'|}{2} \right\rfloor$. Since $q + \left\lfloor \frac{|I'|}{2} \right\rfloor = \bar{q}$, we have $C^*(S) = S$. Thus, $|C^*(S)| - q = \alpha^*$.

Now we show Proposition 6. By the construction of $\succeq^*$, C^* satisfies $\succeq$-compatibility. This fact and Proposition 5 together imply $C^* \in \mathcal{C}_2^{\succeq}(\mathcal{I})$. Thus, by $\alpha(q, C^*) = \alpha^*$, it suffices to show $\alpha(q, C) \geq \alpha(q, C^*)$ for any $C \in \mathcal{C}_2^{\succeq}(\mathcal{I})$. Take any arbitrary $C \in \mathcal{C}_2^{\succeq}(\mathcal{I})$. Since the proof is done if $\big||C(\mathcal{S})| - q\big| \geq \alpha(q, C^*)$, suppose $\big||C(\mathcal{S})| - q\big| < \alpha(q, C^*)$. This implies $\alpha(q, C^*) > 0$, and hence either (i) $I^*(\succeq^*) \in \mathcal{I}_{\geq 2}$ and $\alpha(q, C^*) = \min\left\{|\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor\right\}$, or (ii) $\alpha(q, C^*) = \max\left\{\left\lfloor \frac{|I|}{2} \right\rfloor : I \in \mathcal{I}_{\geq 2}, I^*(\succeq^*) \succ^* I\right\}$ must hold.

<u>[1] When $I^*(\succeq^*) \in \mathcal{I}_{\geq 2}$ and $\alpha(q, C^*) = \min\left\{|\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor\right\}$ holds.</u>

First, we have $|\overline{I^*(\succeq^*)}| \geq q + \alpha(q, C^*)$ because of $\alpha(q, C^*) = \min\left\{|\overline{I^*(\succeq^*)}| - q, \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor\right\}$.

Let $\mathscr{I}_k(\succeq)$ be a class to which $I^*(\succeq^*)$ belongs. There are two cases to consider. Case 1: $\mathscr{I}_k(\succeq) = \{I^*(\succeq^*)\}$. Consider $C(\mathcal{S})$. If $I^*(\succeq^*) \subseteq C(\mathcal{S})$ holds, then $\overline{I^*(\succeq^*)} \subseteq C(\mathcal{S})$ by $\mathscr{I}_k(\succeq) = \{I^*(\succeq^*)\}$ and $\succeq$-compatibility. However, we have $|C(\mathcal{S})| \geq |\overline{I^*(\succeq^*)}| \geq q + \alpha(q, C^*)$, which contradicts with the assumption $\big||C(\mathcal{S})| - q\big| < \alpha(q, C^*)$. Thus, $I^*(\succeq^*) \subseteq \mathcal{S} \setminus C(\mathcal{S})$ holds. Then $I^*(\succeq^*) \cup \{s \in \mathcal{S} : I^*(\succeq^*) \succ^* s\} \subseteq \mathcal{S} \setminus C(\mathcal{S})$ by $\mathscr{I}_k(\succeq) = \{I^*(\succeq^*)\}$ and $\succeq$-compatibility. Thus, we have $\alpha(q, C) \geq q - (|\overline{I^*(\succeq^*)}| - |I^*(\succeq^*)|)$. When $\alpha(q, C^*) = |\overline{I^*(\succeq^*)}| - q$ holds, which implies $|\overline{I^*(\succeq^*)}| - q \leq \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor$, we have

$$\begin{aligned}
\alpha(q, C) &\geq q - (|\overline{I^*(\succeq^*)}| - |I^*(\succeq^*)|) \\
&= |I^*(\succeq^*)| - (|\overline{I^*(\succeq^*)}| - q) \\
&\geq |I^*(\succeq^*)| - \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \\
&\geq \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \\
&\geq \alpha(q, C^*).
\end{aligned}$$

Next, suppose $\alpha(q, C^*) = \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor$ and $|\overline{I^*(\succeq^*)}| - q > \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor$. Then, there exists $S \subseteq \overline{I^*(\succeq^*)} \setminus I^*(\succeq^*)$ such that $|S \cup I^*(\succeq^*)| = q + \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor$. In either case of $I^*(\succeq^*) \subseteq C(S \cup I^*(\succeq^*))$ or $I^*(\succeq^*) \subseteq (S \cup I^*(\succeq^*)) \setminus C(S \cup I^*(\succeq^*))$, we have $\alpha(q, C) \geq \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor = \alpha(q, C^*)$ by $\mathscr{I}_k(\succeq) = \{I^*(\succeq^*)\}$ and $\succeq$-compatibility.

Case 2: $\mathscr{I}_k(\succeq) \neq \{I^*(\succeq^*)\}$. By $\mathcal{S} \setminus \overline{I^*(\succeq^*)} = \{s \in \mathcal{S} : I^*(\succeq^*) \succ^* s\}$, $|\overline{I^*(\succeq^*)}| \geq q + \alpha(q, C^*)$ implies $|\{s \in \mathcal{S} : I^*(\succeq^*) \succ^* s\}| \leq |\mathcal{S}| - q - \alpha(q, C^*)$. Further, by $|C(\mathcal{S})| < q + \alpha(q, C^*)$, $|\{s \in \mathcal{S} : I^*(\succeq^*) \succ^* s\}| < |\mathcal{S} \setminus C(\mathcal{S})|$ holds. Thus, we have $\overline{I^*(\succeq^*)} \cap (\mathcal{S} \setminus C(\mathcal{S})) \neq \emptyset$. We show that there exists $I \subseteq \overline{I^*(\succeq^*)} \cap (\mathcal{S} \setminus C(\mathcal{S}))$ such that $|I| \geq |I^*(\succeq^*)|$.

If $I^*(\succeq^*) \subseteq \mathcal{S} \setminus C(\mathcal{S})$ holds, then we have $I = I^*(\succeq^*)$. Suppose $I^*(\succeq^*) \subseteq C(\mathcal{S})$. Then, by $\overline{I^*(\succeq^*)} \cap (\mathcal{S} \setminus C(\mathcal{S})) \neq \emptyset$ and $\succeq$-compatibility, there exists $I \subseteq \overline{I^*(\succeq^*)} \cap (\mathcal{S} \setminus C(\mathcal{S}))$ such that $I \sim I^*(\succeq^*)$. By the construction of $\succeq^*$, we have $|I| \geq |I^*(\succeq^*)|$ and $I \in \mathcal{I}_{\geq 2}$. By $|C(\mathcal{S})| > q - \alpha(q, C^*) \geq q - \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \geq q - \left\lfloor \frac{|I|}{2} \right\rfloor$, we have $|C(\mathcal{S})| + |I| > q + \left\lfloor \frac{|I|}{2} \right\rfloor$. Then, there exists $S \subseteq \mathcal{S}$ such that $S \subseteq C(\mathcal{S}) \cup I$, $I \subseteq S$, and $|S| = q + \left\lfloor \frac{|I|}{2} \right\rfloor$. By the substitutability of C, $S \cap C(\mathcal{S}) \subseteq C(S)$ holds, and hence by ETI for $\mathcal{I}$, $C(S)$ must be equal to either $S \cap C(\mathcal{S})$ or S itself. Since the size of these two sets are $|S \cap C(\mathcal{S})| = q - \left\lceil \frac{|I|}{2} \right\rceil$ and $|S| = q + \left\lfloor \frac{|I|}{2} \right\rfloor$, in either case, we have $\alpha(q, C) \geq \big||C(S)| - q\big| \geq \left\lfloor \frac{|I|}{2} \right\rfloor \geq \left\lfloor \frac{|I^*(\succeq^*)|}{2} \right\rfloor \geq \alpha(q, C^*)$.[26]

[2] When $\alpha(q, C^*) = \max_{I \in \mathcal{I}_{\geq 2} : I^*(\succeq^*) \succ^* I} \left\lfloor \frac{|I|}{2} \right\rfloor$ holds.

Let $I' \in \mathcal{I}_{\geq 2}$ be such that $\left\lfloor \frac{|I'|}{2} \right\rfloor = \max_{I \in \mathcal{I}_{\geq 2} : I^*(\succeq^*) \succ^* I} \left\lfloor \frac{|I|}{2} \right\rfloor$. There are two cases to consider. Case 1: $I^*(\succeq^*) \succ I'$. There exists $S \subseteq \overline{I^*(\succeq^*)}$ such that $|S \cup I'| = q + \left\lfloor \frac{|I'|}{2} \right\rfloor$. Either $I' \subseteq C(S \cup I')$ or $I' \subseteq (S \cup I') \setminus C(S \cup I')$ holds by ETI for $\mathcal{I}$ and $I' \in \mathcal{I}_{\geq 2}$. Thus, in either case, we have $\alpha(q, C) \geq \left\lfloor \frac{|I'|}{2} \right\rfloor = \alpha(q, C^*)$ by $\succeq$-compatibility. Case 2: $I^*(\succeq^*) \sim I'$. Note that $|I'| > \left\lfloor \frac{|I'|}{2} \right\rfloor = \alpha(q, C^*)$. By $|\overline{I^*(\succeq^*)}| \geq q$ and $I^*(\succeq^*) \succ^* I'$, we have $|\overline{I'}| > q + \alpha(q, C^*)$. By $\mathcal{S} \setminus \overline{I'} = \{s \in \mathcal{S} : I' \succ^* s\}$, $|\overline{I'}| > q + \alpha(q, C^*)$ implies $|\{s \in \mathcal{S} : I' \succ^* s\}| < |\mathcal{S}| - q - \alpha(q, C^*)$. Further, by $|C(\mathcal{S})| < q + \alpha(q, C^*)$, $|\{s \in \mathcal{S} : I' \succ^* s\}| < |\mathcal{S} \setminus C(\mathcal{S})|$ holds. Thus, we have $\overline{I'} \cap (\mathcal{S} \setminus C(\mathcal{S})) \neq \emptyset$. We show that there exists $I \subseteq \overline{I'} \cap (\mathcal{S} \setminus C(\mathcal{S}))$ such that $|I| \geq |I'|$. If $I' \subseteq \mathcal{S} \setminus C(\mathcal{S})$ holds, then we have $I = I'$. Suppose $I' \subseteq C(\mathcal{S})$. Then, by $\overline{I'} \cap (\mathcal{S} \setminus C(\mathcal{S})) \neq \emptyset$ and $\succeq$-compatibility, there exists $I \subseteq \overline{I'} \cap (\mathcal{S} \setminus C(\mathcal{S}))$ such that $I \sim I'$. By the construction of $\succeq^*$, we have $|I| \geq |I'|$ and $I \in \mathcal{I}_{\geq 2}$. By $|C(\mathcal{S})| > q - \alpha(q, C^*) \geq q - \left\lfloor \frac{|I'|}{2} \right\rfloor \geq q - \left\lfloor \frac{|I|}{2} \right\rfloor$, we have $|C(\mathcal{S})| + |I| > q + \left\lfloor \frac{|I|}{2} \right\rfloor$. Then, there exists $S \subseteq \mathcal{S}$ such that $S \subseteq C(\mathcal{S}) \cup I$, $I \subseteq S$, and $|S| = q + \left\lfloor \frac{|I|}{2} \right\rfloor$. By the substitutability of C, $S \cap C(\mathcal{S}) \subseteq C(S)$ holds, and hence by ETI for $\mathcal{I}$, $C(S)$ must be equal to either $S \cap C(\mathcal{S})$ or S itself. Since the size of these two sets are $|S \cap C(\mathcal{S})| = q - \left\lceil \frac{|I|}{2} \right\rceil$ and $|S| = q + \left\lfloor \frac{|I|}{2} \right\rfloor$, in either case, we have $\alpha(q, C) \geq \big||C(S)| - q\big| \geq \left\lfloor \frac{|I|}{2} \right\rfloor \geq \left\lfloor \frac{|I'|}{2} \right\rfloor \geq \alpha(q, C^*)$.

[26] $\lceil x \rceil$ is the smallest integer that is greater than or equal to x.