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Endowments-swapping-proof house allocation with feasibility constraints

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Abstract

This paper studies housing markets in the presence of constraints on the number of agents involved in exchanges. We search for mechanisms satisfying *effective endowments-swapping-proofness*, which requires that no pair of agents can gain by "individually rational" swapping their endowments before the mechanism is applied. Our first main result is that when preferences are strict and feasibility constraints are imposed, no mechanism satisfies both *individual rationality* and *effective endowments-swapping-proofness*. To avoid this negative result, we consider two well-known restricted domains: common ranking preferences and single-dipped preferences. When each agent has common ranking preferences, there exists a pairwise exchange mechanism that satisfies *individual rationality* and *effective endowments-swapping-proofness* in the three-agent case; however, in the case with four or more agents, we again obtain a negative result. We further establish that the top trading cycles mechanism is the only pairwise exchange mechanism satisfying *individual rationality* and *effective endowments-swapping-proofness* when preferences are single-dipped.

Keywords: endowments-swapping-proofness; common ranking preferences; single-dipped preferences; top trading cycles; housing markets; kidney exchange.

JEL codes: C71; C78; D47; D71.

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1 Introduction

1.1 Motivation and outline

We study the Shapley and Scarf (1974) housing exchange economy, where each agent is endowed with a heterogeneous indivisible object (house) and has strict preferences over a set of objects. A "mechanism" reallocates the objects under the condition that each agent consumes one and only one object. Applications of this model are diverse: kidney exchange (Roth, Sönmez, and Ünver (2004)), on-campus housing (Abdulkadiroğlu and Sömmez (1999)), school choice (Abdulkadiroğlu and Sömmez (2003)), and airport landing slot assignments (Schummer and Vohra (2013)).

It is well-known that the top trading cycles mechanism (TTC) selects the unique core allocation via the famous Gale's TTC algorithm (Roth and Postlewaite (1977)).[1] Ma (1994) shows that TTC is the only mechanism that is *efficient* (a chosen assignment cannot be changed in a manner that no agent is worse off, and some agent is better off), *individually rational* (no agent is worse off after the reallocation), and *strategy-proof* (no agent ever benefits from misrepresenting his preferences). Following Ma's study, TTC has been widely characterized by other axioms: "Maskin monotonicity" (Takamiya (2001)), "anonymity" (Miyagawa (2002)), "no-envy" (Hashimoto and Saito (2015)), a weak form of *efficiency* (Ekici (2021)), and so forth.[2]

Fujinaka and Wakayama (2018) study this problem from another perspective. They propose a new form of manipulation via endowments, *endowments-swapping-proofness*. This axiom requires that no pair of agents can both strictly benefit from exchanging their endowments before entering the mechanism.[3] Such manipulation by swapping their endowments is theoretically interesting and is

[1]See Section 3 for a definition of the TTC algorithm.

[2]Some studies have recently generalized TTC to the mentioned various popular applications. These generalized versions of TTC play a central role in these applications and have been characterized in several studies (e.g., Pápai (2000); Svensson and Larsson (2005); Sönmez and Ünver (2010); Dur (2013); Ekici (2013); Morrill (2013); Bade (2014); Tang and Zhang (2016); Pycia and Ünver (2017)).

[3]*Endowments-swapping-proofness* applies only to two-agent coalitions. Postlewaite (1979) and Moulin (1995) have already considered the version of *endowments-swapping-proofness* that involves all subsets of agents. However, the mechanism designer can ignore manipulations by large coalitions because such strategic cooperation is difficult for large coalitions. Thus, this coalitional version of *endowments-swapping-proofness* is too strong a requirement. Conversely, collusion by two agents is relatively easy, and thus *endowments-swapping-proofness* is appealing if any pairs can form.

also interesting in real life. For example, in the context of kidney exchange, two patients may have an incentive to swap their donors using legal loopholes (i.e., fake marriages and fake adoptions) to obtain higher-quality kidneys. Fujinaka and Wakayama (2018) provide an alternative characterization of TTC in terms of *endowments-swapping-proofness*: TTC is the only mechanism that satisfies *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness*.

Endowments-swapping-proofness does not require that the swapping before implementing the mechanism is "individually rational." That is, one agent might temporarily receive an object that is strictly worse than his endowment. If so, he may be reluctant to swap his endowment with that of another agent before participating in the mechanism. This motivates us to weaken *endowments-swapping-proofness* to require only that individually rational pre-swapping is not beneficial. We call this weaker, natural axiom *effective endowments-swapping-proofness*.[4] Interestingly, Fujinaka and Wakayama's (2018) characterization still holds even if *endowments-swapping-proofness* is weakened to *effective endowments-swapping-proofness*.

As mentioned, many studies have examined desirable mechanisms in the standard Shapley and Scarf model. However, this model ignores some important aspects of reality, which prevent direct application of the results of this model to real-life problems. One aspect is the constraint on the size of exchanges among agents. For example, in the context of kidney exchange, it is well-known that exchanges involving many donor and patient pairs are infeasible due to the presence of logistic constraints (e.g., the limited number of doctors and rooms in which kidney transplants are performed).[5] Therefore, this paper seeks to find *effectively endowments-swapping-proof* mechanisms in the Shapley and Scarf model with feasibility constraints.

We first establish an impossibility result on the domain of strict preferences: the presence of feasible constraints makes it impossible to construct a mechanism satisfying *individual rationality* and *effective endowments-swapping-proofness* (Theorem 3).

We subsequently examine whether this negative result can be avoided on smaller domains. To analyze this issue, we restrict attention to pairwise exchanges and consider "common ranking" preferences, which is first proposed

[4] Fujinaka and Wakayama (2018) call this axiom "weak endowments-swapping-proofness."

[5] As another example, Nicolò and Rodoríguez-Álvarez (2013a) notice that, in the case of holiday house swaps, legal constraints may prevent exchanges of a larger size than pairwise exchanges.

by Nicolò and Rodoríguez-Álvarez (2017). An agent has common ranking preferences if his ranking of "acceptable" objects coincides with the predetermined common ranking of the objects. In the context of kidney exchange, this domain of preferences is considered natural if each patient prefers kidneys from compatible younger donors to those from older donors. We then show that, on the domain of common ranking preferences, the natural priority mechanism is the only pairwise exchange mechanism that satisfies both *individual rationality* and *effective endowments-swapping-proofness* when there are three agents (Theorem 4).[6] However, in general, the above mentioned impossibility result persists; that is, no pairwise exchange mechanism is *individually rational* and *effectively endowments-swapping-proof* when there are at least four agents (Theorem 5).

We also consider another well-known restricted domain, called "single-dipped" preferences. An agent has single-dipped preferences (with respect to a fixed order of objects) if he has a unique worst object and on each side of this object according to the order, his welfare is strictly increasing away from this object. Interestingly, even when feasible exchanges are restricted to pairwise exchanges, TTC is well-defined on that domain because the size of each cycle formed via the TTC algorithm is either one or two (Proposition 2). This is primarily because there are only two types of agents' best objects when preferences are single-dipped. Consequently, unlike the domain of common ranking preferences, we obtain a possibility result on the domain of single-dipped preferences: TTC is the only pairwise exchange mechanism that satisfies *individual rationality* and *effective endowments-swapping-proofness* (Theorem 7).[7]

1.2 Related literature

Feasibility constraints A typical real-life example where the presence of feasibility constraints becomes a serious concern in our model is living-donor kidney transplantation. Roth et al. (2005) are the first to address the issue of fea-

[6] The natural priority mechanism allocates objects via an algorithm that prioritizes agents that own objects with lower index numbers. In the algorithm, we start with a set of *individually rational* pairwise assignments, and each agent sequentially refines the set of assignments to his best assignments according to priority ordering. See Section 5.1 for a formal definition of this mechanism.

[7] This possibility result no longer holds when each agent has more general single-dipped preferences, called "single-dipped preferences on a tree." In fact, *individual rationality* and *effective endowments-swapping-proofness* are incompatible on this extended domain when only pairwise exchanges are allowed. For a more detailed discussion of the mechanisms on the domain of single-dipped preferences on a tree, see Appendix A.

sibility constraints in the context of kidney exchange and propose *efficient* and *strategy-proof* pairwise exchange mechanisms. Unlike our study, they consider dichotomous preferences, where all compatible kidneys (i.e., acceptable objects) are homogenous for each patient. Some follow-up papers (e.g., Hatfield (2005), Ünver (2010), Yılmaz (2011)) also share this view.

A disadvantage of the assumption of dichotomous preferences is that it does not reflect recent medical findings that some factors, such as the age and health status of the donor, body size, or kidney weight, affect the expected survival of the graft (e.g., Øien et al. (2007), Giral et al. (2010)). Based on these medical findings, Nicolò and Rodoríguez-Álvarez (2012; 2013a) and Balbuzanov (2020) consider another model in which feasibility constraints are imposed. However, compatible kidneys are heterogeneous and each agent has strict preferences.[8] This paper follows their approach. Nicolò and Rodoríguez-Álvarez (2012) provide an impossibility result in that setting: no mechanism satisfies *individual rationality*, *efficiency*, and *strategy-proofness*. Nicolò and Rodoríguez-Álvarez (2013a) show that one cannot escape from this impossibility result by weakening *strategy-proofness* to "ordinal Bayesian incentive compatibility." Balbuzanov (2020) produces another impossibility result, showing the incompatibility between *efficiency* and a fairness property, "anonymity." Our Theorem 3 is considered an *effective endowments-swapping-proofness* counterpart of these results.

Restricted domains Nicolò and Rodoríguez-Álvarez (2017) show that, on the domain of common ranking preferences, the natural priority mechanism is the only pairwise exchange mechanism that satisfies *efficiency*, *individual rationality*, and *strategy-proofness*.[9] Our result indicates that this characterization theorem can no longer hold, except in the three-agent case, when *efficiency* and *strategy-proofness* are replaced by *effective endowments-swapping-proofness*.

Tamura (2023) shows that the characterizations of TTC proposed by Ma (1994) and Fujinaka and Wakayama (2018) persist even if preferences are restricted to being single-dipped. However, Tamura does not consider feasibility constraints on the size of exchanges. From our results, we propose that *strategy-proofness* can be dropped from Tamura's *endowments-swapping-proofness* characterization of TTC on the domain of single-dipped preferences if we focus on pairwise exchanges.

[8] Roth et al. (2004), which is the earliest study on kidney exchange, also consider strict preferences, but ignore limitations on the size of exchanges.

[9] Rodoríguez-Álvarez (2021) specifies the extent to which the domain of common ranking preferences can be enlarged to permit the existence of mechanisms that satisfies the three axioms.

1.3 Organization

The remainder of the paper is organized as follows. Section 2 introduces the model and our axioms. Section 3 reviews the related results and provides new insights under no feasibility constraints. Section 4 states our impossibility result for the model with feasibility constraints. Section 5 considers two restricted domains of preferences and provides our results on these domains. Section 6 concludes. Appendix A discusses the existence of *effective endowments-swapping-proof* mechanisms on the domain of single-dipped preferences on a tree, instead of a line. Appendix B contains the proofs that are omitted from the main text.

2 Preliminaries

2.1 Model

Let $N = \{1, 2, \ldots, n\}$ and $H = \{h_1, h_2, \ldots, h_n\}$ be a finite set of agents and a finite set of objects, respectively. Throughout this paper, we assume that $n \geq 3$. An **assignment** is a bijection $x\colon N \to H$. For convenience, we write x_i for $x(i)$. Here, x_i represents the object agent i receives at x. Let X be the set of assignments. An **endowment** is denoted by $\omega = (\omega_i)_{i \in N} \in X$, where ω_i represents the object owned by agent i.

Given a pair of an assignment and an endowment $(x, \omega) \in X \times X$, we call a sequence $(i_1(= i_{S+1}), \ldots, i_S)$ of agents a **trading cycle** at (x, ω) if for each $\{s, s'\} \subset \{1, \ldots, S\}$ with $s \neq s'$, $i_s \neq i_{s'}$ and for each $s \in \{1, \ldots, S\}$, $x_{i_s} = \omega_{i_{s+1}}$. Given an endowment $\omega \in X$ and an integer $\ell \in \{1, \ldots, n\}$, we say that an assignment $x \in X$ is **ℓ-feasible with respect to ω** if for each trading cycle $(i_1, \ldots, i_S)$ at (x, ω), $|\{i_1, \ldots, i_S\}| \leq \ell$.[10] Denote by $X_\ell(\omega)$ the set of ℓ-feasible assignments with respect to ω.[11]

We assume that agent $i \in N$ has a strict preference relation $\succ_i$ over H. Let $\mathscr{P}$ be the set of strict preferences over H. For each $\succ_0 \in \mathscr{P}$, $\succsim_0$ represents the induced weak preference relation from $\succ_0$; that is, for each $\{h, h'\} \subset H$, $h \succsim_0 h'$ if and only if either $h \succ_0 h'$ or $h = h'$. Let $\mathscr{P}^N$ be the set of strict preference profiles $\succ = (\succ_i)_{i \in N}$ such that for each $i \in N$, $\succ_i \in \mathscr{P}$. We often represent $\succ_i$ as an

[10]Given a set Z, $|Z|$ denotes the cardinality of Z.

[11]Note that $X_n(\omega) = X$ for each $\omega \in X$. However, if $\ell \neq n$, there is $\{\omega', \omega''\} \subset X$ such that $X_\ell(\omega') \neq X_\ell(\omega'')$. For example, consider the case where $n = 3$ and $\ell = 2$. Let $\omega' = (h_1, h_2, h_3)$ and $\omega'' = (h_2, h_3, h_1)$. Then, $X_\ell(\omega') \neq X_\ell(\omega'')$, as $(h_2, h_3, h_1) \in X_\ell(\omega'')$ but $(h_2, h_3, h_1) \notin X_\ell(\omega')$.

ordered list of objects as follows:

$$\begin{array}{c} \succ_i \\ \hline h \\ h' \\ h'' \\ \vdots \end{array}$$

This means that agent i prefers object h the most; further, agent i prefers h to h', h' to h'', and so on. For each $i \in N$ and each $(\succ_i, \omega_i) \in \mathscr{P} \times H$, let $A(\succ_i, \omega_i) = \{h \in H \setminus \{\omega_i\} : h \succ_i \omega_i\}$ be the set of **acceptable** objects for i at $(\succ_i, \omega_i)$.

An **economy** is a pair of a preference profile and an endowment $e = (\succ, \omega) \in \mathscr{P}^N \times X$. Let $\mathscr{E} \subseteq \mathscr{P}^N \times X$ be a set of admissible economies, which we call a **domain**. Denote by $\mathscr{E}^{\text{st}} = \mathscr{P}^N \times X$ the **strict domain**.

Given a domain $\mathscr{E} \subseteq \mathscr{E}^{\text{st}}$, a **mechanism** on $\mathscr{E}$ is a function $f : \mathscr{E} \to X$ that maps each economy $e = (\succ, \omega) \in \mathscr{E}$ to an assignment $f(e) \in X$. Given an integer $\ell \in \{1, \ldots, n\}$, we say that a mechanism f on $\mathscr{E}$ is **ℓ-feasible** if for each $e = (\succ, \omega) \in \mathscr{E}$, $f(e) \in X_\ell(\omega)$. In particular, we say that a mechanism f on $\mathscr{E}$ is a **pairwise exchange mechanism** if it is 2-feasible.

2.2 Axioms

We introduce desirable properties of mechanisms. To explain our main axiom, we begin by introducing the following strategic property: no pair of agents can both strictly benefit from swapping their endowments before they enter the mechanism. To define this property formally, we require additional notation. Given an economy $e = (\succ, \omega) \in \mathscr{E}$ and a pair $\{i, j\} \subset N$, let $e^{i,j} = (\succ, \omega^{i,j}) \in \mathscr{P}^N \times X$ be such that $\omega_i^{i,j} = \omega_j$, $\omega_j^{i,j} = \omega_i$, and for each $k \in N \setminus \{i, j\}$, $\omega_k^{i,j} = \omega_k$.

***Endowments-swapping-proofness*:** There are no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset N$ such that

(i) $e^{i,j} \in \mathscr{E}$, and

(ii) $f_i(e^{i,j}) \succ_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

It should be emphasized that in the definition of *endowments-swapping-proofness*, the pre-swapping is not required to be individually rational; that is, one agent might temporarily receive an object that is strictly worse than his endowment.

Then, one can consider a weaker and more natural version of *endowments-swapping-proofness*, which only requires that individually rational pre-swapping is not profitable.[12]

***Effective endowments-swapping-proofness*:** There are no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset N$ such that

(i) $e^{i,j} \in \mathscr{E}$,

(ii) $\omega_j \in A(\succ_i, \omega_i)$ and $\omega_i \in A(\succ_j, \omega_j)$, and

(iii) $f_i(e^{i,j}) \succ_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

Remark 1. Fujinaka and Wakayama (2018), who first propose both *endowments-swapping-proofness* and *effective endowments-swapping-proofness*, do not include Condition (i), $e^{i,j} \in \mathscr{E}$, in their definitions. This is because they only consider the strict domain and that domain clearly includes any "swapping economy" in which a pair of agents swaps their endowments. Unlike Fujinaka and Wakayama (2018), we consider not only the strict domain but also its restricted domains. There is no guarantee that such restricted domains necessarily include any swapping economy. This makes it necessary for us to require Condition (i) in the definitions of *endowments-swapping-proofness* and *effective endowments-swapping-proofness*. $\diamondsuit$

We also impose the following allocative property, which states that no one is made worse off by participating in a mechanism.

***Individual rationality*:** For each $e = (\succ, \omega) \in \mathscr{E}$ and each $i \in N$,

$$f_i(e) \succsim_i \omega_i.$$

3 Endowments-swapping-proof mechanisms without feasibility constraints

A prominent mechanism on the strict domain is the so-called top trading cycles mechanism. The **top trading cycles mechanism**, or TTC for short, is the mechanism $TTC\colon \mathscr{E}^{\text{st}} \to X$ that selects for each $e \in \mathscr{E}^{\text{st}}$, the assignment $TTC(e)$ obtained via the following algorithm, known as the TTC algorithm:

[12] This notion itself is not new and has been already presented in Fujinaka and Wakayama (2018), who call it "weak endowments-swapping-proofness."

- **Step 1.** Each agent points to the agent who owns his best object. Then, there is at least one trading cycle as there is a finite number of agents. Each agent involved in a cycle is assigned the object along the cycle and removed. If an agent remains, the procedure continues to the next step, and it terminates otherwise.

- **Step $t \geq 2$.** Each remaining agent points to the agent who owns his best object among those remaining. Then, at least one trading cycle exists. Each agent involved in a cycle is assigned the object along the cycle and removed. If an agent remains, the procedure continues to the next step, and it terminates otherwise.

For each $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$ and each $t \in \mathbb{N}$, let $\mathbb{S}_t(e) \subset 2^N$ be the set of groups of agents that form cycles at Step t of TTC, and

$$N_t(e) = \bigcup_{S \in \mathbb{S}_t(e)} \{S\};$$
$$H_t(e) = \{h \in H \colon \omega_i = h \text{ for some } i \in N_t(e)\}.$$

That is, $S = \{i_1(= i_{K+1}), \ldots, i_K\} \in \mathbb{S}_t(e)$ means that for each $k \in \{1, \ldots, K\}$, $i_k \in N \setminus \bigcup_{j=1}^{t-1} N_j(e)$ and $\omega_{i_k} \in H \setminus \bigcup_{j=1}^{t-1} H_j(e)$, and for each $h \in H \setminus (\bigcup_{j=1}^{t-1} H_j(e) \cup \{\omega_{i_{k+1}}\})$, $\omega_{i_{k+1}} \succ_{i_k} h$.

An axiomatic characterization of TTC on the strict domain in terms of *effective endowments-swapping-proofness* has been already presented in Theorem 4 of Fujinaka and Wakayama (2018).

Theorem 1. *A mechanism on $\mathscr{E}^{\text{st}}$ is individually rational, strategy-proof, and effectively endowments-swapping-proof if and only if it is TTC.*[13]

As Example 1 below shows, TTC violates the following strict version of *effective endowments-swapping-proofness*.

***Strict effective endowments-swapping-proofness*:** There are no $e = (\succ, \omega) \in \mathscr{E}$ and $\{i, j\} \subset N$ such that

(i) $e^{i,j} \in \mathscr{E}$,

[13]The notion of *strategy-proofness* requires that no agent should ever be made better off than by telling the truth. This notion is formally stated as follows: For each $e = (\succ, \omega) \in \mathscr{E}$, each $i \in N$, and each $e' = ((\succ'_i, \succ_{-i}), \omega) \in \mathscr{E}$, $f_i(e) \succsim_i f_i(e')$.

(ii) $\omega_j \in A(\succ_i, \omega_i)$ and $\omega_i \in A(\succ_j, \omega_j)$, and

(iii) $f_i(e^{i,j}) \succsim_i f_i(e)$ and $f_j(e^{i,j}) \succ_j f_j(e)$.

Example 1. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_{j\geq 4}$
h_3	h_3	h_1	h_j
h_2	$\vdots$	h_2	$\vdots$
h_1		h_3	
$\vdots$		$\vdots$	

and for each $i \in N$, $\omega_i = h_i$. Then, $TTC(e) = (h_3, h_2, h_1, h_4, \dots, h_n)$ and $TTC(e^{2,3}) = (h_2, h_3, h_1, h_4, \dots, h_n)$. It thus follows that $\omega_3 = h_3 \in A(\succ_2, \omega_2)$ and $\omega_2 = h_2 \in A(\succ_3, \omega_3)$, and

$$TTC_2(e^{2,3}) = h_3 \succ_2 h_2 = TTC_2(e);$$
$$TTC_3(e^{2,3}) = h_1 = TTC_3(e).$$

This implies that TTC violates *strict effective endowments-swapping-proofness*. ■

The next result indicates that on the strict domain, not only TTC, but also all other *individually rational* mechanisms violate *strict effective endowments-swapping-proofness*.

Theorem 2. *No mechanism on $\mathscr{E}^{\text{st}}$ satisfies individual rationality and strict effective endowments-swapping-proofness.*

Proof. See Appendix B. □

4 Endowments-swapping-proof mechanisms with feasibility constraints

As shown above, in the setting without feasibility constraints on the size of trading cycles, TTC is the unique mechanism that satisfies *individual rationality*, *strategy-proofness*, and *effective endowments-swapping-proofness*. A natural question is whether an *effectively endowments-swapping-proof* mechanism satisfying other desirable properties can be found when we impose feasibility constraints on the size of trading cycles. Unfortunately, the next result indicates that as soon as feasibility constraints are imposed, *individual rationality* and *effectively endowments-swapping-proofness* are incompatible.

Theorem 3. *Let $\ell \in \{1, \ldots, n-1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^{\mathrm{st}}$ satisfies individual rationality and effective endowments-swapping-proofness.*

Proof. Suppose, by contradiction, that there is an ℓ-feasible mechanism f on $\mathscr{E}^{\mathrm{st}}$ satisfying the two axioms. Let $\succ \in \mathscr{P}^N$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\ldots$	$\succ_k$	$\ldots$	$\succ_{n-1}$	$\succ_n$
h_2	h_3	h_4	$\ldots$	h_{k+1}	$\ldots$	h_n	h_1
h_3	h_4	h_5	$\ldots$	h_{k+2}	$\ldots$	h_1	h_2
h_4	h_5	h_6	$\ldots$	h_{k+3}	$\ldots$	h_2	h_3
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
h_{n-1}	h_n	h_1	$\ldots$	h_{k-2}	$\ldots$	h_{n-3}	h_{n-2}
h_n	h_1	h_2	$\ldots$	h_{k-1}	$\ldots$	h_{n-2}	h_{n-1}
h_1	h_2	h_3	$\ldots$	h_k	$\ldots$	h_{n-1}	h_n

Since we fix the preference profile $\succ$ in this proof, we write ω for $e = (\succ, \omega)$. Let $\bar{\omega} = (h_1, h_2, \ldots, h_n)$. For each $m \in N (= \{1, 2, \ldots, n\})$ and each integer $m' (\geq n+1)$, let $h_{m'} = h_m$ if $m' \equiv m \pmod{n}$.

Step 1: $f_1(\bar{\omega}) = h_2$. Let $\Omega_1^1 = \{\omega \in X \colon \omega_1 = h_2\}$ and for each $k \in N \setminus \{1\}$,

$$\Omega_k^1 = \{\omega \in X \colon \omega_1 = h_{k+1} \text{ and for each } i \in \{2, \ldots, k\}, \omega_i = h_i\}.$$

Note that $\Omega_n^1 = \{\bar{\omega}\}$. We prove by induction that for each $k \in N$ and each $\omega \in \Omega_k^1$, $f_1(\omega) = h_2$.

Base step. Let $k = 1$ and $\omega \in \Omega_1^1$. By *individual rationality*, $f_1(\omega) = h_2$.

Induction hypothesis. Let $k \in \{2, 3, \ldots, n\}$. For each $k' \in \{1, 2, \ldots, k-1\}$ and each $\omega \in \Omega_{k'}^1$, $f_1(\omega) = h_2$.

Induction step. Let $k \in \{2, 3, \ldots, n\}$ and $\omega \in \Omega_k^1$. Then, ω is represented as follows:

$\succ_1$	$\succ_2$	$\succ_3$	$\cdots$	$\succ_{k-1}$	$\succ_k$
h_2	h_3	h_4	$\cdots$	h_k	h_{k+1}
h_3	h_4	h_5	$\cdots$	h_{k+1}	h_{k+2}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_k	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\boxed{h_{k+1}}$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_1	$\boxed{h_2}$	$\boxed{h_3}$	$\cdots$	$\boxed{h_{k-1}}$	$\boxed{h_k}$

In the above preference table, the boxes indicate the agents' endowments. Suppose, by contradiction, that $f_1(\omega) \neq h_2$. By *individual rationality*, there is $q \in \{3, 4, \dots, k+1\}$ such that $f_1(\omega) = h_q$. Then, we can show the following claim by using the induction hypothesis.

Claim 1. *For each* $i \in \{q-1, q, \dots, k\}$, $f_i(\omega) = h_{i+1}$.

The proof of Claim 1 is in Appendix B. By Claim 1, we have $f_{q-1}(\omega) = h_q$, which contradicts $f_1(\omega) = h_q$.

Step 2: For each $i \in N \setminus \{1\}$, $f_i(\bar{\omega}) = h_{i+1}$. Let $i \in N \setminus \{1\}$. Let $\Omega^i_1 = \{\omega \in X : \omega_i = h_{i+1}\}$ and for each $k \in N \setminus \{1\}$,

$$\Omega^i_k = \{\omega \in X \colon \omega_i = h_{i+k} \text{ and for each } j \in \{i+1, \dots, i+k-1\}, \omega_j = h_j\}.$$

Note that $\Omega^i_n = \{\bar{\omega}\}$. By argument similar to Step 1, we can show that for each $k \in N$ and each $\omega \in \Omega^i_k$, $f_i(\omega) = h_{i+1}$.

Step 3: Conclusion. By Steps 1 and 2, we have that for each $i \in N$, $f_i(\bar{\omega}) = h_{i+1}$. This means $f(\bar{\omega}) \notin X_\ell(\bar{\omega})$, which is a contradiction. □

5 Pairwise exchanges on restricted domains

We have so far observed that *individual rationality* and *effective endowments-swapping-proofness* are incompatible on the strict domain when the size of trading cycles is limited. However, these two axioms might be compatible if one restricts the domain of strict preferences to a special class of preferences. This section examines whether the two axioms are compatible on a restricted domain when we focus on pairwise exchanges. Here we consider two well-known restricted domains:

common ranking preferences (Nicolò and Rodríguez-Álvarez (2017); Rodríguez-Álvarez (2021)) and single-dipped preferences.

5.1 Common ranking preferences

We begin our discussion by providing a formal definition of common ranking preferences. An agent who has a common ranking preference orders acceptable objects according to a predetermined ranking of objects that is common to all agents. Here we consider the common ranking in which objects are naturally ordered; that is, for each $\{j,k\} \in N$ with $j < k$, h_j is ranked higher than h_k. For each $i \in N$ and $\omega_i \in H$, we say that agent i's preference relation $\succ_i \in \mathscr{P}$ is a **common ranking preference with respect to $\boldsymbol{\omega_i}$** if for each $\{h_j, h_k\} \subseteq A(\succ_i, \omega_i)$,

$$h_j \succ_i h_k \iff j < k.$$

Let $\mathscr{P}_{\omega_i} \subset \mathscr{P}$ be the set of common ranking preferences with respect to ω_i. Given $\omega \in X$, let $\mathscr{P}_\omega = \prod_{i=1}^n \mathscr{P}_{\omega_i}$. Denote by $\mathscr{E}^{\text{cm}} = \bigcup_{\omega \in X} \{\mathscr{P}_\omega \times \{\omega\}\}$ the **common ranking domain**.

Given an endowment $\omega \in X$, a **priority ordering** at ω, $\sigma[\![\omega]\!]\colon N \to N$, is a permutation such that the k-th agent in the permutation is the agent with the k-th priority. Let $\sigma = \{\sigma[\![\omega]\!]\colon \omega \in X\}$ be a **priority ordering**. The **natural priority ordering** is the priority ordering σ^* such that for each $\omega \in X$ and each $i \in N$, if $\omega_i = h_k$, then $\sigma^*[\![\omega]\!](i) = k$.

We introduce a pairwise exchange mechanism that selects the assignment obtained the following algorithm:

$\boldsymbol{\sigma}$-priority algorithm. Pick any priority ordering σ and any economy $e = (\succ, \omega) \in \mathscr{E}$:

- $\mathbb{X}_0^\sigma(e) = \mathcal{I}(e) = \{x \in X_2(\omega)\colon \text{for each } i \in N, x_i \succsim_i \omega_i\}$.
- For each $t \in N$, let $\mathbb{X}_t^\sigma(e) \subseteq \mathbb{X}_{t-1}^\sigma(e)$ be such that:

$$\mathbb{X}_t^\sigma(e) = \left\{ x \in \mathbb{X}_{t-1}^\sigma(e)\colon \begin{array}{l} \text{there is no } y \in \mathbb{X}_{t-1}^\sigma(e) \text{ such that} \\ y_{(\sigma[\![\omega]\!])^{-1}(t)} \succ_{(\sigma[\![\omega]\!])^{-1}(t)} x_{(\sigma[\![\omega]\!])^{-1}(t)} \end{array} \right\}.$$

The σ-priority algorithm works as follows. Given an economy $e = (\succ, \omega) \in \mathscr{E}$, we start with the set $\mathcal{I}(e)$, which denotes the set of *individually rational* pairwise assignments for e. At Step 1, the first agent in the priority ordering $\sigma[\![\omega]\!]$ selects his best pairwise assignments from the set of individually rational pairwise assignments. The selection proceeds iteratively. In general, at Step $t \geq 2$, the t-th agent in the priority ordering $\sigma[\![\omega]\!]$ selects his best pairwise assignments from those that have survived in the previous steps. The **natural priority mechanism** is the pairwise exchange mechanism $P\colon \mathscr{E}^{\mathrm{cm}} \to X$ such that for each $e \in \mathscr{E}^{\mathrm{cm}}$, $P(e) \in \mathbb{X}_n^{\sigma^*}(e)$.

Remark 2. Note that for each $e \in \mathscr{E}^{\mathrm{cm}}$, $\mathbb{X}_n^{\sigma}(e) \neq \emptyset$ and $|\mathbb{X}_n^{\sigma}(e)| = 1$. Accordingly, we confirm that the natural priority mechanism is well-defined. ◊

Remark 3. There is a priority ordering $\bar{\sigma}(\neq \sigma^*)$ such that for each $e \in \mathscr{E}^{\mathrm{cm}}$, $\mathbb{X}_n^{\sigma^*}(e) = \mathbb{X}_n^{\bar{\sigma}}(e)$. Specifically, it satisfies the following: for each $\omega \in X$ and each $i \in N$, if $\omega_i = h_k$ and $k \leq n-2$, then $\bar{\sigma}[\![\omega]\!](i) = k$. Then, we also refer to the mechanism that selects the assignment via the $\bar{\sigma}$-priority algorithm as the natural priority mechanism, although $\bar{\sigma}$-priority algorithm is not based on the natural priority ordering. ◊

The next result indicates that on the common ranking domain, no pairwise exchange mechanism meets both *individual rationality* and *effective endowments-swapping-proofness* except for the natural priority mechanism.

Proposition 1. *If a pairwise exchange mechanism on $\mathscr{E}^{\mathrm{cm}}$ is individually rational and effectively endowments-swapping-proof, then it is the natural priority mechanism.*

Proof. See Appendix B. □

Note that Proposition 1 says nothing about whether the natural priority mechanism on the common ranking domain satisfies *effective endowments-swapping-proofness*. In fact, when there are three agents, the natural priority mechanism on the common ranking domain is the only pairwise exchange mechanism satisfying both *individual rationality* and *effective endowments-swapping-proofness*.

Theorem 4. *Suppose $n = 3$. A pairwise exchange mechanism on $\mathscr{E}^{\mathrm{cm}}$ is individually rational and effectively endowments-swapping-proof if and only if it is the natural priority mechanism.*

Proof. See Appendix B. □

Unfortunately, the natural priority mechanism on the common ranking domain violates *effective endowments-swapping-proofness* when there are at least four agents, as illustrated by the four-agent example below (the example can easily be adapted to $n > 4$).

Example 2. Suppose $N = \{1,2,3,4\}$. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$
h_2	h_3	h_1	h_1
h_3	h_4	h_4	h_2
h_1	h_2	h_3	h_4
h_4	h_1	h_2	h_3

and $\omega = (h_1, h_2, h_3, h_4)$. Then, for each $i \in N$, $\sigma^*[\![\omega]\!](i) = i$, and hence $P(e) = (h_3, h_4, h_1, h_2)$. Consider $e^{2,4}$. By $\omega^{2,4} = (h_1, h_4, h_3, h_2)$, $\sigma^*[\![\omega^{2,4}]\!](1) = 1$, $\sigma^*[\![\omega^{2,4}]\!](2) = 4$, $\sigma^*[\![\omega^{2,4}]\!](3) = 3$, and $\sigma^*[\![\omega^{2,4}]\!](4) = 2$. Thus, $P(e^{2,4}) = (h_2, h_3, h_4, h_1)$. Then, $e^{2,4} \in \mathscr{E}^{\text{cm}}$, $\omega_4 = h_4 \in A(\succ_2, \omega_2)$ and $\omega_2 = h_2 \in A(\succ_4, \omega_4)$, and

$$P_2(e^{2,4}) = h_3 \succ_2 h_4 = P_2(e);$$
$$P_4(e^{2,4}) = h_1 \succ_4 h_2 = P_4(e),$$

which implies that P violates *effective endowments-swapping-proofness*. ■

The observation in Example 2, together with Proposition 1, yields the following impossibility result.

Theorem 5. *Suppose $n \geq 4$. No pairwise exchange mechanism on $\mathscr{E}^{\text{cm}}$ satisfies individual rationality and effective endowments-swapping-proofness.*

Theorem 5 is in sharp contrast with Nicolò and Rodríguez-Álvarez's (2017) possibility result that the natural priority mechanism on the common ranking domain is the only pairwise exchange mechanism satisfying *individual rationality*, *efficiency*, and *strategy-proofness*.

Furthermore, by strengthening *effective endowments-swapping-proofness* to *strict effective endowments-swapping-proofness*, we face a similar impossibility result even in the three-agent case.

Theorem 6. *No pairwise exchange mechanism on $\mathscr{E}^{\text{cm}}$ satisfies individual rationality and strict effective endowments-swapping-proofness.*

Proof. From Theorem 4 and Theorem 5, it suffices to show that the natural priority mechanism on $\mathscr{E}^{\text{cm}}$ violates *strict effective endowments-swapping-proofness* in the three-agent case. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$
h_2	h_3	h_1
h_3	h_2	h_2
h_1	h_1	h_3

and $\omega = (h_1, h_2, h_3)$. Then, for each $i \in N$, $\sigma^*[\![\omega]\!](i) = i$, and hence $P(e) = (h_3, h_2, h_1)$. Consider $e^{2,3}$. By $\omega^{2,3} = (h_1, h_3, h_2)$, $\sigma^*[\![\omega^{2,3}]\!](1) = 1$, $\sigma^*[\![\omega^{2,3}]\!](2) = 3$, and $\sigma^*[\![\omega^{2,3}]\!](3) = 2$. Thus, $P(e^{2,3}) = (h_2, h_3, h_1)$. Then, $e^{2,3} \in \mathscr{E}^{\text{cm}}$, $\omega_3 = h_3 \in A(\succ_2, \omega_2)$ and $\omega_2 = h_2 \in A(\succ_3, \omega_3)$, and

$$P_2(e^{2,3}) = h_3 \succ_2 h_2 = P_2(e);$$
$$P_3(e^{2,3}) = h_1 = P_3(e),$$

which implies that P violates *strict effective endowments-swapping-proofness*. □

5.2 Single-dipped preferences

This section considers another restricted domain of preferences, called "single-dipped" preferences. We first describe a formal definition of single-dipped preferences. To do this, we consider a linear order $<$ on H. Without loss of generality, we fix a linear order $<$ on H as:

$$h_1 < h_2 < \cdots < h_n.$$

Given $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped** with respect to $<$ if there is an object, $d(\succ_i) \in H$, such that

(i) for each $h \in H \setminus \{d(\succ_i)\}$, $h \succ_i d(\succ_i)$;

(ii) for each $\{h, h'\} \subset H \setminus \{d(\succ_i)\}$, if either $h' < h < d(\succ_i)$ or $d(\succ_i) < h < h'$, then $h' \succ_i h$.

We denote by $\mathscr{S}_\vee \subset \mathscr{P}$ the set of single-dipped preference relations. Let $\mathscr{E}^\vee = \mathscr{S}_\vee^N \times X$ be the **single-dipped domain**.

Interestingly, TTC on the single-dipped domain is a pairwise exchange mechanism because the size of each trading cycle generated by TTC is either one or two. Moreover, on this domain, TTC emerges as the unique mechanism satisfying *individual rationality* and *effective endowments-swapping-proofness*.

Proposition 2. *TTC on $\mathscr{E}^\vee$ is a pairwise exchange mechanism.*

Proof. Let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$. Without loss of generality, for each $i \in N$, $\omega_i = h_i$. We denote $\underline{i}(t)$ (resp. $\bar{i}(t)$) the lowest (resp. highest) index among the set of remaining agents at Step t. Note that $\underline{i}(1) = 1$ and $\bar{i}(1) = n$.

We first consider Step 1 of TTC. Since preferences are single-dipped, then for each $i \in N$ and each $h \in H \setminus \{\omega_{\underline{i}(1)}, \omega_{\bar{i}(1)}\}$, we have either

(a) $\omega_{\underline{i}(1)} = \omega_1 \succ_i h$ and $\omega_{\underline{i}(1)} \succ_i \omega_{\bar{i}(1)}$, or

(b) $\omega_{\bar{i}(1)} = \omega_n \succ_i h$ and $\omega_{\bar{i}(1)} \succ_i \omega_{\underline{i}(1)}$.

Thus,

$$\mathbb{S}_1(e) \in \Big\{ \{\{\underline{i}(1), \bar{i}(1)\}\}, \{\{\underline{i}(1)\}, \{\bar{i}(1)\}\}, \{\{\underline{i}(1)\}\}, \{\{\bar{i}(1)\}\} \Big\},$$

which implies that $N_1(e) \subseteq \{\underline{i}(1), \bar{i}(1)\} = \{1, n\}$. We next consider Step $t \geq 2$. Since preferences are single-dipped, then for each $i \in N \setminus \bigcup_{j=1}^{t-1} N_j(e)$ and each $h \in H \setminus \left(\bigcup_{j=1}^{t-1} H_j(e) \cup \{\omega_{\underline{i}(t)}, \omega_{\bar{i}(t)}\} \right)$, we have either

(a) $\omega_{\underline{i}(t)} \succ_i h$ and $\omega_{\underline{i}(t)} \succ_i \omega_{\bar{i}(t)}$, or

(b) $\omega_{\bar{i}(t)} \succ_i h$ and $\omega_{\bar{i}(t)} \succ_i \omega_{\underline{i}(t)}$.

Thus,

$$\mathbb{S}_t(e) \in \Big\{ \{\{\underline{i}(t), \bar{i}(t)\}\}, \{\{\underline{i}(t)\}, \{\bar{i}(t)\}\}, \{\{\underline{i}(t)\}\}, \{\{\bar{i}(t)\}\} \Big\},$$

which implies that $N_t(e) \subseteq \{\underline{i}(t), \bar{i}(t)\}$. Hence, we observe that the size of each cycle formed in each step is either one or two. This implies that *TTC* on $\mathscr{E}^{\vee}$ is a pairwise exchange mechanism. □

Theorem 7. *A pairwise exchange mechanism on $\mathscr{E}^{\vee}$ is individually rational and effectively endowments-swapping-proof if and only if it is TTC.*

Proof. See Appendix B. □

Recall that *effective endowments-swapping-proofness* is weaker than *endowments-swapping-proofness*. Moreover, it is easy to see that TTC on $\mathscr{E}^{\vee}$ satisfies *endowments-swapping-proofness*. Using these facts together with Theorem 7, we obtain the following corollary.

Corollary 1. *A pairwise exchange mechanism on $\mathscr{E}^{\vee}$ is individually rational and endowments-swapping-proof if and only if it is TTC.*

Remark 4. Both Theorem 7 and Corollary 1 no longer hold if we consider the size of exchanges larger than pairwise exchanges. That is, we can construct a non-TTC mechanism that satisfies *individual rationality* and (*effective*) *endowments-swapping-proofness*. To demonstrate this, consider $n = 3$ and the following 3-feasible mechanism: for each $e \in \mathscr{E}^{\vee}$,

$$f^{\vee}(e) = \begin{cases} (h_2, h_3, h_1) & \text{if } e = (\succ', \omega') \\ TTC(e) & \text{otherwise,} \end{cases}$$

where

$\succ'_1$	$\succ'_2$	$\succ'_3$
h_3	h_3	h_1
h_2	h_2	h_2
h_1	h_1	h_3

and $\omega' = (h_1, h_2, h_3)$. It is clear that this mechanism is *individually rational*. Additionally, it satisfies *effective endowments-swapping-proofness*.[14] Indeed, Tamura (2023) has shown that *strategy-proofness* is indispensable for the characterization of TTC using (*effective*) *endowments-swapping-proofness* on the singe-dipped domain without feasibility constraints. That is, our results indicate that one can drop *strategy-proofness* in Tamura's characterization of TTC if pairwise exchanges only are allowed. ◊

Remark 5. The domain of single-dipped preferences considered in this section can be referred to as the domain of single-dipped preferences on a line. We can consider a generalization of this domain, called the domain of single-dipped preferences on a "tree." Without feasibility constraints on the size of exchanges, the characterization of TTC holds on the domain of single-dipped preferences on a tree (Tamura (2023)). However, our characterizations of TTC (Theorem 7 and Corollary 1) no longer hold on the domain of single-dipped preferences on a tree. We discuss it in detail in Appendix A. ◊

[14] For each $e = (\succ, \omega) \in \mathscr{E}^{\vee}$, if $\succ \neq \succ'$, $f^{\vee}(e) = TTC(e)$. Since TTC is *effectively endowments-swapping-proof*, no pair of agents has an incentive to swap their endowments at $e = (\succ, \omega)$ with $\succ \neq \succ'$. Therefore, we now consider $e = (\succ, \omega)$ with $\succ = \succ'$. If $\omega = \omega'$, agents 2 and 3 have no incentive to swap their endowments with another agent because they have received their best objects; that is, no pair of agents gains by swapping their endowments at $(\succ', \omega')$. Thus, we consider the case $\omega \neq \omega'$. Then, $f^{\vee}(\succ', \omega) = TTC(\succ', \omega)$. Since TTC is *efficient* at $(\succ', \omega)$, $TTC(\succ', \omega) \in \{(h_3, h_2, h_1), (h_2, h_3, h_1)\}$. In both cases, two of the three agents receive their best objects. Hence, no pair of agents has an incentive to swap their endowments. Hence we conclude that $f^{\vee}$ is *effectively endowments-swapping-proof*.

As the following example shows, TTC violates *strict effective endowments-swapping-proofness* even on the single-dipped domain.

Example 3. Let $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ be such that

$\succ_1$	$\succ_{i\geq 2}$
h_n	h_1
h_{n-1}	h_2
$\vdots$	$\vdots$
h_2	h_{n-1}
h_1	h_n

and $\omega = (h_1, h_2, \ldots, h_n)$. Then, $TTC(e) = (h_n, h_2, h_3, \ldots, h_{n-1}, h_1)$ and $TTC(e^{1,2}) = (h_n, h_1, h_3, \ldots, h_{n-1}, h_2)$. Hence, $e^{1,2} \in \mathscr{E}^{\vee}$, $\omega_2 = h_2 \in A(\succ_1, \omega_1)$ and $\omega_1 = h_1 \in A(\succ_2, \omega_2)$, and

$$TTC_1(e^{1,2}) = h_n = TTC_1(e);$$
$$TTC_2(e^{1,2}) = h_1 \succ_2 h_2 = TTC_2(e),$$

in violation of *strict effective endowments-swapping-proofness*. ■

By Theorem 7 and Example 3, we have the following corollary.

Corollary 2. *No pairwise exchange mechanism on $\mathscr{E}^{\vee}$ satisfies individual rationality and strict effective endowments-swapping-proofness.*

Before completing this section, we check the independence of axioms in Theorem 7.

Example 4 (Dropping *effective endowments-swapping-proofness*). Consider the following pairwise exchange mechanism, NT: for each $e = (\succ, \omega) \in \mathscr{E}^{\vee}$, $NT(e) = \omega$. This mechanism is *individually rational*, but not *effectively endowments-swapping-proof*. ■

Example 5 (Dropping *individual rationality*). Consider the following pairwise exchange mechanism, $f^{\leftrightarrow}$: for each $e = (\succ, \omega) \in \mathscr{E}^{\vee}$,

- if n is even, then for each $i \in \{1, \ldots, \frac{n}{2}\}$, $f^{\leftrightarrow}_{2i-1}(e) = \omega_{2i}$ and $f^{\leftrightarrow}_{2i}(e) = \omega_{2i-1}$;
- if n is odd, then $f^{\leftrightarrow}_n(e) = \omega_n$ and for each $i \in \left\{1, \ldots, \frac{n-1}{2}\right\}$, $f^{\leftrightarrow}_{2i-1}(e) = \omega_{2i}$ and $f^{\leftrightarrow}_{2i}(e) = \omega_{2i-1}$.

It is easy to see that this mechanism violates *individual rationality*. We show below that $f^{\leftrightarrow}$ is *effectively endowments-swapping-proof*. Suppose, by contradiction, that there are $e = (\succ, \omega) \in \mathscr{E}^{\vee}$ and $\{i, j\} \subset N$ such that

(i) $e^{i,j} \in \mathscr{E}^{\vee}$,

(ii) $\omega_j \in A(\succ_i, \omega_i)$ and $\omega_i \in A(\succ_j, \omega_j)$, and

(iii) $f_i^{\leftrightarrow}(e^{i,j}) \succ_i f_i^{\leftrightarrow}(e)$ and $f_j^{\leftrightarrow}(e^{i,j}) \succ_j f_j^{\leftrightarrow}(e)$.

Let

$$\mathcal{N}^{\leftrightarrow} = \begin{cases} \left\{\{2i-1, 2i\} \subset N \colon \exists\, i \in \left\{1, \ldots, \frac{n}{2}\right\}\right\} & \text{if } n \text{ is even} \\ \left\{\{2i-1, 2i\} \subset N \colon \exists\, i \in \left\{1, \ldots, \frac{n-1}{2}\right\}\right\} & \text{if } n \text{ is odd.} \end{cases}$$

There are two cases.

• **Case 1: $\{i, j\} \in \mathcal{N}^{\leftrightarrow}$.** Without loss of generality, we assume $\{i, j\} = \{1, 2\}$. By (ii), $\omega_2 \succ_1 \omega_1$. Then, by the definition of $f^{\leftrightarrow}$, $f_1^{\leftrightarrow}(e) = \omega_2$ and $f_1^{\leftrightarrow}(e^{1,2}) = \omega_2^{1,2} = \omega_1$. By (iii), $f_1^{\leftrightarrow}(e^{1,2}) = \omega_1 \succ_1 \omega_2 = f_1^{\leftrightarrow}(e)$, which contradicts $\omega_2 \succ_1 \omega_1$.

• **Case 2: $\{i, j\} \notin \mathcal{N}^{\leftrightarrow}$.** Without loss of generality, we assume $i = 1$. By $\{i = 1, j\} \notin \mathcal{N}^{\leftrightarrow}$, $j \neq 2$. Then, by the definition of $f^{\leftrightarrow}$, $f_1^{\leftrightarrow}(e^{1,j}) = \omega_2^{1,j} = \omega_2 = f_1^{\leftrightarrow}(e)$, which contradicts (iii). ■

6 Conclusion

This paper searched for *effectively endowments-swapping-proof* mechanisms in the presence of feasibility constraints on trading cycles. We found that on the strict domain, *individual rationality* and *effective endowments-swapping-proofness* are incompatible under the feasibility constraints. To escape from this negative result, we considered two well-known domains of preferences: common ranking preferences and single-dipped preferences. First, we showed that if there are three agents, then the natural priority mechanism is the only pairwise exchange mechanism on the common ranking domain that satisfies *individual rationality* and *effective endowments-swapping-proofness*; otherwise, the two axioms are incompatible on the common ranking domain with pairwise exchanges. Second, we established that on the single-dipped domain, TTC is the only pairwise exchange mechanism that satisfies *individual rationality* and *effective endowments-swapping-proofness*.

We close our discussion by mentioning four possible extensions of the model.

Other restricted domains Since we considered single-dipped preferences, we can also consider its dual version,"single-peaked" preferences. Recently interest has been growing in mechanisms on the single-peaked domain. (Bade (2019); Liu (2022); Tamura (2022); Tamura and Hosseini (2022); Fujinaka and Wakayama (2023)). According to these studies, there are many non-TTC mechanisms on the single-peaked domain that satisfy certain desirable properties if there are no feasibility constraints. Thus, it remains open to clarify the structure of *effectively endowments-swapping-proof* mechanisms on the single-peaked domain both with and without feasibility constraints.

Beyond pairwise exchanges We only considered pairwise exchanges on our two restricted domains. It would be interesting to study exchanges that involve more than two agents on those domains. Nicolò and Rodríguez-Álvarez (2017) show that on the common ranking domain, no ℓ-feasible (where $3 \leq \ell < n$) mechanism satisfies *individual rationality*, (*constrained*) *efficiency*, and *strategy-proofness*. It remains open as to whether a similar negative result holds when using *effective endowments-swapping-proofness* instead of *efficiency* and *strategy-proofness*. Tamura's (2023) characterization of TTC on the single-dipped domain implies that TTC is the unique ℓ-feasible (where $3 \leq \ell < n$) mechanism satisfying *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness*. Remark 4 stated that the characterization no longer holds without *strategy-proofness*. Finding non-*strategy-proof* ℓ-feasible mechanisms satisfying *individual rationality* and *effective endowments-swapping-proofness* is an interesting future research topic.

Weak preferences Our setting does not allow the preferences of agents to exhibit indifferences. Nicolò and Rodríguez-Álvarez (2017) and Rodríguez-Álvarez (2021) extend the common ranking domain to domains where the preferences of the agents might be weak. They call these "age based domains" and propose a pairwise exchange mechanism on those domains that satisfies *individual rationality*, (*constrained*) *efficiency*, and *strategy-proofness*. It is an open question as to whether there is a pairwise exchange mechanism on age based domains that satisfies *individual rationality* and *effective endowments-swapping-proofness*.

Probabilistic mechanisms Balbuzanov (2020) succeeds in finding an *efficient* and "anonymous" pairwise exchange mechanism on the strict domain by allowing randomness, whereas no deterministic mechanism satisfies the two properties. However, he shows that under certain mild conditions, no pairwise exchange mechanism on the strict domain satisfies *individual rationality*, *efficiency*, and *strategy-proofness* even if randomness is admitted. Thus, it is an open question as to whether there is a probabilistic pairwise exchange mechanism on the strict domain satisfying *individual rationality* and *effective endowments-swapping-proofness*.

A Appendix: Single-dipped preferences on a tree

Section 5 considers the domain of single-dipped preferences on a line. This preference domain can be extended to the domain of single-dipped preferences on a "tree." Tamura (2023) has characterized TTC as the only mechanism that satisfies *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness* on this extended domain without restrictions on the size of possible exchanges. Here, we discuss whether Tamura's characterization holds even when there is a restriction on the size of possible exchanges, and we search for *effective endowments-swapping-proof* mechanisms in this setting.

A.1 Definitions and preliminary results

To formally define single-dipped preferences on a tree, we now introduce some graph theoretical notions. An **(indirected) graph** is a pair $G = (H, E)$, where $E \subset \{\{h', h''\} \subset H \colon h' \neq h''\}$ is the set of **edges**. The **degree** of object $h \in H$ is the number of edges that contain h; that is, $|\{\{h', h''\} \in E \colon h \in \{h', h''\}\}|$. Given an object $h \in H$, we say that h is a **leaf** if the degree of h is one. We denote by $\mathbb{L}$ the set of leaves in G.[15] Given $\{h', h''\} \subset H$ with $h' \neq h''$, a **path from h' to h'' in $G = (H, E)$** is a sequence $(h^1, \ldots, h^K)$ such that $h^1 = h'$, $h^K = h''$, $|\{h^1, \ldots, h^K\}| = K$, and for each $k \in \{1, \ldots, K-1\}$, $\{h^k, h^{k+1}\} \in E$. A graph $G = (H, E)$ is a **tree** if

(i) it is connected (i.e., for each $\{h', h''\} \subset H$ with $h' \neq h''$, there is a path from h' to h'' in G), and

(ii) it has no cycle (i.e., there is no sequence $(h^1, \ldots, h^K)$ such that $K \geq 3$, $h^1 = h^K$, for each $k \in \{1, \ldots, K-1\}$, $\{h^k, h^{k+1}\} \in E$, and for each $\{k', k''\} \subset \{1, \ldots, K\}$ such that $k' \neq k''$ and $\{k', k''\} \neq \{1, K\}$, $h^{k'} \neq h^{k''}$).

It is well-known that if a graph G is a tree, then, for each $\{h', h''\} \subset H$ with $h' \neq h''$, there is a unique path from h' to h'' in G (See, for example, Theorem 2.1.4 in West (2001)). We often denote the path from h' to h'' by $[h', h'']$. For each $\{h, h', h''\} \subset H$, we write $h \in [h', h'']$ if h is on the path from h' to h''.

Given a tree $G = (H, E)$ and an agent $i \in N$, we say that i's preference relation $\succ_i \in \mathscr{P}$ is **single-dipped on the tree G** if there is an object, $d(\succ_i) \in H$, such that

[15] Formally, it should be $\mathbb{L}(G)$, but unless otherwise specified, we omit G for simplicity.

(i) for each $h \in H \setminus \{d(\succ_i)\}$, $h \succ_i d(\succ_i)$;

(ii) for each $h, h' \in H \setminus \{d(\succ_i)\}$, if $h \in [d(\succ_i), h']$, then $h' \succ_i h$.

Given a tree G, we denote the set of single-dipped preferences on the tree G by $\mathscr{P}_G \subset \mathscr{P}$. Let $\mathscr{E}^G = \mathscr{P}_G^N \times X$.

Remark 6. Note that, for each $i \in N$ and each $\succ_i \in \mathscr{P}_G$, i's best object at $\succ_i$ among H is a leaf. To observe this, let $h \in H \setminus \mathbb{L}$. We only consider the case where $h \neq d(\succ_i)$; if $h = d(\succ_i)$, it is evident that h is not his best object at $\succ_i$. By $h \neq d(\succ_i)$, there is the unique path from $d(\succ_i)$ to h in the tree G, $[d(\succ_i), h] = (h^1 = d(\succ_i), \ldots, h^K = h)$. By $h \notin \mathbb{L}$, the degree of h is greater than 1. Thus, there is $h' \in H$ such that $h' \neq h^{K-1}$ and $\{h, h'\} \in E$. Since G has no cycle, for each $k \in \{1, \ldots, K\}$, $h' \neq h^k$. Hence, $[d(\succ_i), h'] = (h^1 = d(\succ_i), \ldots, h^K = h, h')$. Since $h \in [d(\succ_i), h']$ and $\succ_i$ is single-dipped on G, $h' \succ_i h$, which implies that h is not i's best object at $\succ_i$. Hence, i's best object at $\succ_i$ must be in $\mathbb{L}$. $\Diamond$

It is worthwhile to mention that TTC on the domain of single-dipped preferences on a tree is a $|\mathbb{L}|$-feasible exchange mechanism. In addition, we observe that the maximal size of possible exchanges under TTC is $|\mathbb{L}|$.

Proposition 3. *Suppose that G is a tree. Then, TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$-feasible.*

Proof. See Appendix B. $\square$

Proposition 4. *Suppose that G is a tree. Then, the maximal size of possible trading cycles under TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$.*

Proof. Without loss of generality, assume $\mathbb{L} = \left\{h_1, h_2, \ldots, h_{|\mathbb{L}|}\right\}$. Let $e = (\succ, \omega) \in \mathscr{E}^G$ be such that

$\succ_1$	$\succ_2$	$\cdots$	$\succ_k$	$\cdots$	$\succ_{\vert\mathbb{L}\vert-1}$	$\succ_{\vert\mathbb{L}\vert}$
h_2	h_3	$\cdots$	h_{k+1}	$\cdots$	$h_{\vert\mathbb{L}\vert}$	h_1
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$

for each $i \in N$, $\omega_i = h_i$. Then, for each $i \in \{1, 2, \ldots, |\mathbb{L}| - 1\}$, $TTC_i(e) = h_{i+1}$ and $TTC_{|\mathbb{L}|}(e) = h_1$, that is, the size of this trading cycle is $|\mathbb{L}|$. By Proposition 3, since for each $e \in \mathscr{E}^G$, each $t \in \mathbb{N}$, and each $S \in \mathbb{S}_t(e)$, $|S| \leq |\mathbb{L}|$, the maximal size of possible trading cycles under TTC is $|\mathbb{L}|$. $\square$

A.2 Severe feasibility constraints

Theorem 7 and Corollary 1 characterized TTC as the only *individually rational* and (*effectively*) *endowments-swapping-proof* pairwise exchange mechanism on the domain of single-dipped preferences on a line. However, since the maximal size of possible exchanges under TTC is $|\mathbb{L}|$ (Proposition 4), we cannot extend this characterization of TTC to the domain of single-dipped preferences on a tree when there are three or more leaves and possible exchanges restrict attention to pairwise ones. Furthermore, we can show that when there are three or more leaves, no pairwise exchange mechanism satisfies *individual ratinality* and *effective endowmens-swapping-proofness*. More generally, as shown below, this negative result holds as long as the possible exchanges are less than the number of leaves. Since *effective endowments-swapping-proofness* is much weaker than *endowments-swapping-proofness*, our negative result implies that Tamura's (2023) characterization no longer holds under such a "severe" constraint on the size of possible exchanges.

Theorem 8. *Suppose that G is a tree. Let $\ell \in \{1, 2, \dots, |\mathbb{L}| - 1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^G$ satisfies individual rationality and effective endowments-swapping-proofness.*

Proof. Without loss of generality, assume $\mathbb{L} = \left\{h_1, h_2, \dots, h_{|\mathbb{L}|}\right\}$. Suppose, by contradiction, that there is an ℓ-feasible mechanism f on $\mathscr{E}^G$ satisfying the two axioms. Let $\bar{e} = (\succ, \bar{\omega}) \in \mathscr{E}^G$ be such that

$\succ_1$	$\succ_2$	$\succ_3$	$\cdots$	$\succ_k$	$\cdots$	$\succ_{|\mathbb{L}|-1}$	$\succ_{|\mathbb{L}|}$
h_2	h_3	h_4	$\cdots$	h_{k+1}	$\cdots$	$h_{|\mathbb{L}|}$	h_1
h_3	h_4	h_5	$\cdots$	h_{k+2}	$\cdots$	h_1	h_2
h_4	h_5	h_6	$\cdots$	h_{k+3}	$\cdots$	h_2	h_3
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$h_{|\mathbb{L}|-1}$	$h_{|\mathbb{L}|}$	h_1	$\cdots$	h_{k-2}	$\cdots$	$h_{|\mathbb{L}|-3}$	$h_{|\mathbb{L}|-2}$
$h_{|\mathbb{L}|}$	h_1	h_2	$\cdots$	h_{k-1}	$\cdots$	$h_{|\mathbb{L}|-2}$	$h_{|\mathbb{L}|-1}$
h_1	h_2	h_3	$\cdots$	h_k	$\cdots$	$h_{|\mathbb{L}|-1}$	$h_{|\mathbb{L}|}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$

and for each $i \in N$, $\bar{\omega}_i = h_i$. By the argument similar to that in Theorem 3, we have that for each $i \in \{1, 2, \dots, |\mathbb{L}| - 1\}$, $f_i(\bar{e}) = h_{i+1}$ and $f_{|\mathbb{L}|}(\bar{e}) = h_1$. However, by $\ell < |\mathbb{L}|$, $f(\bar{e}) \notin X_\ell(\bar{\omega})$, which is a contradiction. $\square$

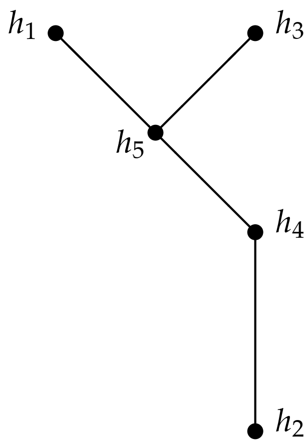

Figure 1: Tree in Example 6

Corollary 3. *Suppose that G is a tree and $|\mathbb{L}| \geq 3$. Then, no pairwise exchange mechanism on $\mathscr{E}^G$ satisfies individual rationality and (effective) endowments-swapping-proofness.*

Corollary 4. *Suppose that G is a tree. Let $\ell \in \{1, 2, \ldots, |\mathbb{L}| - 1\}$. Then, no ℓ-feasible mechanism on $\mathscr{E}^G$ satisfies individual rationality and endowments-swapping-proofness.*

A.3 Mild feasibility constraints

Based on Proposition 3, one might think that on the domain of single-dipped preferences on a tree, TTC can be characterized by means of *individual rationality* and *effective endowments-swapping-proofness* if $|\mathbb{L}|$-feasible exchanges are allowed. However, this conjecture is not true whenever $|\mathbb{L}| \geq 3$. In fact, if $|\mathbb{L}| \geq 3$, we can construct a non-TTC mechanism that is $|\mathbb{L}|$-feasible, *individually rational*, and *effectively endowments-swapping-proof*. We blow provide an example of such a mechanism.

Example 6. Consider $N = \{1, 2, 3, 4, 5\}$ and $H = \{h_1, h_2, h_3, h_4, h_5\}$. Suppose that a tree G is represented as Figure 1. Then, $\mathbb{L} = \{h_1, h_2, h_3\}$. Let $\check{e} = (\check{\succ}, \check{\omega}) \in \mathscr{E}^G$ be such that

$\check{\succ}_1$	$\check{\succ}_2$	$\check{\succ}_3$	$\check{\succ}_4$	$\check{\succ}_5$
h_2	h_1	h_3	h_2	h_1
h_4	h_2	h_1	h_4	h_2
h_1	h_3	h_2	h_1	h_3
h_3	h_4	h_4	h_3	h_4
h_5	h_5	h_5	h_5	h_5

and for each $i \in N$, $\check{\omega}_i = h_i$. Note that $TTC(\check{e}) = (h_2, h_1, h_3, h_4, h_5)$ and by Proposition 3, TTC is a 3-feasible mechanism. Let $f^{\nabla} : \mathscr{E}^G \to X$ be a 3-feasible mechanism such that for each $e \in \mathscr{E}^G$,

$$f^{\nabla}(e) = \begin{cases} (h_4, h_1, h_3, h_2, h_5) & \text{if } e = \check{e}, \\ TTC(e) & \text{otherwise.} \end{cases}$$

It is obvious that this mechanism is *individually rational*. Moreover, f^{∇} satisfies (*effective*) *endowments-swapping-proofness*.[16] ■

Note that mechanism f^{∇} defined in Example 6 violates *strategy-proofness*. To see this, let $\check{\succ}'_1 \in \mathscr{P}_G$ be such that

$\check{\succ}'_1$
h_2
h_1
h_4
h_3
h_5

Then,

$$f_1^{\nabla}((\check{\succ}'_1, \check{\succ}_{-1}), \check{\omega}) = TTC_1((\check{\succ}'_1, \check{\succ}_{-1}), \check{\omega}) = h_2 \check{\succ}_1 h_4 = f_1^{\nabla}(\check{e}).$$

Thus, agent 1 with preferences $\check{\succ}_1$ can benefit from announcing false preferences $\check{\succ}'_1$. This suggests that, by adding *strategy-proofness*, one could obtain a characterization of TTC. Recall here that when there is no restriction on the size of possible exchanges, Tamura (2023) proposes a characterization of TTC by means of *individual rationality*, *strategy-proofness*, and *endowments-swapping-proofness*. In fact, Tamura's characterization holds true even when the size of possible exchanges is greater than or equal to the number of leaves. This is simply because the $|\mathbb{L}|$-feasibility of TTC (Proposition 3) makes it possible for TTC to satisfy such a "mild" feasibility constraint. It is also noteworthy that Tamura's characterization still holds when *endowments-swapping-proofness* is weakened to *effective endowments-swapping-proofness*.[17] Since, as mentioned above, TTC satisfies the

[16]See Appendix B for the proof of this fact.

[17]Given the feature mentioned in Remark 6, we can show this by using arguments similar to the proof of Theorem 4 in Fujinaka and Wakayama (2018) (or Theorem 1 in this paper). The proof is available upon request.

mild feasibility constraint on the size of possible exchange, we obtain the following result:

Theorem 9. *Suppose that G is a tree. Let $\ell \geq |\mathbb{L}|$. Then, an ℓ-feasible mechanism on $\mathscr{E}^G$ satisfies individual rationality, strategy-proofness, and effective endowments-swapping-proofness if and only if it is TTC.*

B Appendix: Omitted proofs

B.1 Proof of Theorem 2

Without loss of generality, we assume $n = 3$. Suppose, by contradiction, that a mechanism f satisfies the two axioms. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{st}}$ be such that

$\succ_1$	$\succ_2$	$\succ_3$
h_3	h_3	h_1
h_2	h_1	h_2
h_1	h_2	h_3

and $\omega = (h_1, h_2, h_3)$. We proceed in three steps.

Step 1: $f(e) = (h_3, h_2, h_1)$. It suffices to show $(f_1(e), f_3(e)) = (h_3, h_1)$, as this immediately implies $f(e) = (h_3, h_2, h_1)$. Suppose, by contradiction, that $f_1(e) \neq h_3$. Consider $e^{1,3}$. Then, $e^{1,3} \in \mathscr{E}^{\text{st}}$, $\omega_3 = h_3 \in A(\succ_1, \omega_1)$ and $\omega_1 = h_1 \in A(\succ_3, \omega_3)$, and by *individual rationality*,

$$f_1(e^{1,3}) = h_3 \succ_1 f_1(e);$$
$$f_3(e^{1,3}) = h_1 \succsim_3 f_3(e),$$

in violation of *strict effective endowments-swapping-proofness*. Hence, $f_1(e) = h_3$. A similar argument leads to $f_3(e) = h_1$.

Step 2: $f(e^{2,3}) = (h_2, h_3, h_1)$. Let $\bar{e} = e^{2,3}$. By *individual rationality*, $f_2(\bar{e}) = h_3$. Suppose, by contradiction, that $(f_1(\bar{e}), f_3(\bar{e})) = (h_1, h_2)$. Consider $\bar{e}^{1,3}$. Then, $\bar{e}^{1,3} \in \mathscr{E}^{\text{st}}$, $\omega_3^{2,3} = h_2 \in A(\succ_1, \omega_1^{2,3} = h_1)$ and $\omega_1^{2,3} = h_1 \in A(\succ_3, \omega_3^{2,3} = h_2)$, and by *individual rationality*,

$$f_1(\bar{e}^{1,3}) \succsim_1 h_2 \succ_1 h_1 = f_1(\bar{e});$$
$$f_3(\bar{e}^{1,3}) = h_1 \succ_3 h_2 = f_3(\bar{e}),$$

in violation of *strict effective endowments-swapping-proofness*.

Step 3: Conclusion. By Steps 1 and 2, it holds that $e^{2,3} \in \mathscr{E}^{\text{st}}$, $\omega_3 = h_3 \in A(\succ_2, \omega_2)$ and $\omega_2 = h_2 \in A(\succ_3, \omega_3)$, and

$$f_2(e^{2,3}) = h_3 \succ_2 h_2 = f_2(e);$$
$$f_3(e^{2,3}) = h_1 = f_3(e),$$

in violation of *strict effective endowments-swapping-proofness*. □

B.2 Proof of Claim 1

We prove this claim by induction.

Base step. Suppose, by contradiction, that $f_k(\omega) \neq h_{k+1}$. Consider $\omega^{1,k}$. Then, $\omega^{1,k}$ is represented as

$\succ_1$	$\succ_2$	$\succ_3$	$\cdots$	$\succ_{k-1}$	$\succ_k$
h_2	h_3	h_4	$\cdots$	h_k	$\boxed{h_{k+1}}$
h_3	h_4	h_5	$\cdots$	h_{k+1}	h_{k+2}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\boxed{h_k}$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_{k+1}	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_1	$\boxed{h_2}$	$\boxed{h_3}$	$\cdots$	$\boxed{h_{k-1}}$	h_k

That is, $\omega^{1,k} \in \Omega^1_{k-1}$. By the induction hypothesis of Theorem 3, $f_1(\omega^{1,k}) = h_2$. By *individual rationality*, $f_k(\omega^{1,k}) = h_{k+1}$. Hence, $(\succ, \omega^{1,k}) \in \mathscr{E}^{\text{st}}$, $\omega_k = h_k \in A(\succ_1, \omega_1 = h_{k+1})$ and $\omega_1 = h_{k+1} \in A(\succ_k, \omega_k = h_k)$, and

$$
\begin{aligned}
f_1(\omega^{1,k}) &= h_2 \succ_1 h_q = f_1(\omega); \\
f_k(\omega^{1,k}) &= h_{k+1} \succ_k f_k(\omega),
\end{aligned}
$$

in violation of *effective endowments-swapping-proofness*.

Induction hypothesis. Let $j \in \{q-1, q, \ldots, k-1\}$. For each $i \in \{j+1, j+2, \ldots, k\}$, $f_i(\omega) = h_{i+1}$.

Induction step. Let $j \in \{q-1, q, \ldots, k-1\}$. Suppose, by contradiction, that $f_j(\omega) \neq h_{j+1}$. By the induction hypothesis of this claim,

$$
f_j(\omega) \notin \{h_{j+2}, h_{j+3}, \ldots, h_{k+1}\}.
$$

Hence, $h_{k+1} \succ_j f_j(\omega)$. Consider $\omega^{1,j}$. Then, $\omega^{1,j}$ is represented as

$\succ_1$	$\succ_2$	$\succ_3$	$\cdots$	$\succ_{j-1}$	$\succ_j$	$\cdots$	$\succ_{k-1}$	$\succ_k$
h_2	h_3	h_4	$\cdots$	h_j	h_{j+1}	$\cdots$	h_k	h_{k+1}
h_3	h_4	h_5	$\cdots$	h_{j+1}	h_{j+2}	$\cdots$	h_{k+1}	h_{k+2}
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\boxed{h_{k+1}}$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\boxed{h_j}$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_{k+1}	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$
h_1	$\boxed{h_2}$	$\boxed{h_3}$	$\cdots$	$\boxed{h_{j-1}}$	h_j	$\cdots$	$\boxed{h_{k-1}}$	$\boxed{h_k}$

That is, $\omega^{1,j} \in \Omega^1_{j-1}$. By the induction hypothesis of Theorem 3, $f_1(\omega^{1,j}) = h_2$. By *individual rationality*, $f_j(\omega^{1,k}) \succsim_j h_{k+1}$. Hence, $(\succ, \omega^{1,j}) \in \mathscr{E}^{\text{st}}$, $\omega_j = h_j \in A(\succ_1, \omega_1 = h_{k+1})$, $\omega_1 = h_{k+1} \in A(\succ_j, \omega_j = h_j)$, and

$$f_1(\omega^{1,j}) = h_2 \succ_1 h_q = f_1(\omega);$$
$$f_j(\omega^{1,j}) \succsim_j h_{k+1} \succ_j f_j(\omega),$$

in violation of *effective endowments-swapping-proofness*. □

B.3 Proof of Proposition 1

Before proving this proposition, we provide additional notions. For each $e = (\succ, \omega) \in \mathscr{E}$, each $\{x, y\} \subset X_2(\omega)$, and each $\{i, j\} \subset N$ with $i \neq j$, **$\{i, j\}$ weakly blocks x at e via y** if

(i) $y_i = \omega_j$ and $y_j = \omega_i$;

(ii) $y_i \succsim_i x_i$ and $y_j \succ_j x_j$.

For each $e = (\succ, \omega) \in \mathscr{E}$, an assignment $x \in X_2(\omega)$ is in the **strict core** for e if $x \in \mathcal{I}(e)$ and there are no pair $\{i, j\} \subset N$ with $i \neq j$ and assignment $y \in X_2(\omega)$ such that $\{i, j\}$ weakly blocks x at e via y. We denote by $\mathcal{C}(e)$ the strict core for e.

We prove Proposition 1 by a series of lemmas. The first lemma (Lemma 1) states that any "individually rational swapping" economy in which a pair of agents swaps their endowments before participating the given mechanism belongs to the common ranking domain. The second lemma (Lemma 2) states that the assignment chosen by a pairwise exchange mechanism satisfying *individual rationality* and *effective endowments-swapping-proofness* is in the strict core. The last lemma (Lemma 3) states that the strict core is a singleton consisting of the assignment chosen by the natural priority mechanism. Note that Lemma 3 has already appeared in Nicolò and Rodríguez-Álvarez (2013b). However, for completeness, we below provide the proof of Lemma 3 by accommodating its proof to our setting.

Lemma 1. *For each $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{cm}}$ and each $\{i, j\} \subset N$ with $i \neq j$, if $\omega_i^{i,j} = \omega_j \in A(\succ_i, \omega_i)$ and $\omega_j^{i,j} = \omega_i \in A(\succ_j, \omega_j)$, then $e^{i,j} \in \mathscr{E}^{\mathrm{cm}}$.*

Proof. Let $\{h_k, h_{k'}\} \subseteq A(\succ_i, \omega_i^{i,j} = \omega_j)$. Since $\omega_i^{i,j} = \omega_j \in A(\succ_i, \omega_i)$, $\{h_k, h_{k'}\} \subseteq A(\succ_i, \omega_i)$. By $\succ_i \in \mathscr{P}_{\omega_i}$, it holds that

$$h_k \succ_i h_{k'} \iff k < k'.$$

Thus, $\succ_i \in \mathscr{P}_{\omega_i^{i,j}}$. Similarly, $\succ_j \in \mathscr{P}_{\omega_j^{i,j}}$. These imply that $e^{i,j} \in \mathscr{E}^{\mathrm{cm}}$. $\square$

Lemma 2. *If a pairwise exchange mechanism f on $\mathscr{E}^{\mathrm{cm}}$ is individually rational and effectively endowments-swapping-proof, then for each $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{cm}}$, $f(e) \in \mathcal{C}(e)$.*

Proof. Suppose, by contradiction, that there is $e = (\succ, \omega) \in \mathscr{E}^{\mathrm{cm}}$ with $f(e) \notin \mathcal{C}(e)$. By $f(e) \in \mathcal{I}(e)$, there are $\{i, j\} \subset N$ with $i \neq j$ and $y \in X_2(\omega)$ such that $\{i, j\}$ weakly blocks $f(e)$ at e via y, that is, (i) $y_i = \omega_j$ and $y_j = \omega_i$ and (ii) $y_i \succsim_i f_i(e)$ and $y_j \succ_j f_j(e)$. If $\omega_i = f_j(e)$,

$$\omega_i = y_j \succ_j f_j(e) = \omega_i,$$

which is a contradiction. Thus, $\omega_i \neq f_j(e)$, which together with $f(e) \in X_2(\omega)$ implies that $\omega_j \neq f_i(e)$. By *individual rationality*,

$$y_i = \omega_j \succ_i f_i(e) \succsim_i \omega_i;$$
$$y_j = \omega_i \succ_j f_j(e) \succsim_i \omega_j.$$

We consider $e^{i,j}$. Then, $\omega_i^{i,j} = \omega_j \in A(\succ_i, \omega_i)$ and $\omega_j^{i,j} = \omega_i \in A(\succ_j, \omega_j)$, and by

Lemma 1, $e^{i,j} \in \mathscr{E}^{\text{cm}}$. Further, by *individual rationality*,

$$f_i(e^{i,j}) \succsim_i \omega_i^{i,j} = \omega_j \succ_i f_i(e);$$
$$f_j(e^{i,j}) \succsim_j \omega_j^{i,j} = \omega_i \succ_j f_j(e),$$

in violation of *effective endowments-swapping-proofness*. □

Lemma 3. *For each $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$, $\mathcal{C}(e) = \{P(e)\}$.*

Proof. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$. We proceed in three steps.

Step 1: For each $y \in X_2(\omega) \setminus \{P(e)\}$, $y \notin \mathcal{C}(e)$. Let $y \in X_2(\omega) \setminus \{P(e)\}$. If $y \notin \mathcal{I}(e)$, then $y \notin \mathcal{C}(e)$. Thus, we assume $y \in \mathcal{I}(e)$. Let $i \in N$ be such that $y_i \neq P_i(e)$ and for each $i' \in N$ with $\sigma^*[\![\omega]\!](i') < \sigma^*[\![\omega]\!](i)$, $y_{i'} = P_{i'}(e)$. Let $j \in N$ be such that $P_i(e) = \omega_j$. Since $y \in \mathcal{I}(e)$ and for each $i' \in N$ with $\sigma^*[\![\omega]\!](i') < \sigma^*[\![\omega]\!](i)$, $y_{i'} = P_{i'}(e)$, it holds that $y \in \mathbb{X}^{\sigma^*}_{\sigma^*[\![\omega]\!](i)-1}(e)$. Since $\succ_i$ is strict, $y_i \neq P_i(e)$, and $y \in \mathcal{I}(e)$, by the definitions of P,

$$P_i(e) = \omega_j \succ_i y_i \succsim_i \omega_i, \tag{1}$$

which implies that $i \neq j$. Further, $\sigma^*[\![\omega]\!](i) < \sigma^*[\![\omega]\!](j)$; otherwise, by the definition of i, $P_i(e) = \omega_j$, and $\{y, P(e)\} \subset X_2(\omega)$, it holds that $y_j = P_j(e) = \omega_i$ and $y_i = P_i(e) = \omega_j$, which is a contradiction. Suppose $\omega_i = h_\ell$. Since $P_j(e) = \omega_i \in A(\succ_j, \omega_j)$, by the definition of $\mathscr{E}^{\text{cm}}$, we have that for each $k \in N$ with $\omega_k \succ_j \omega_i$, $\omega_k = h_{\ell'}$ with $\ell' < \ell$ and thus, $\sigma^*[\![\omega]\!](k) < \sigma^*[\![\omega]\!](i)$ by the definition of σ^*. Recall that for each $k \in N$ with $\sigma^*[\![\omega]\!](k) < \sigma^*[\![\omega]\!](i)$, $P_k(e) = y_k$. These imply that for each $k \in N$ with $\omega_k \succ_j \omega_i$, $y_j \neq \omega_k$; if $y_j = \omega_k$, by $\{y, P(e)\} \subset X_2(\omega)$, it holds that $y_k = P_k(e) = \omega_j$ and $y_j = P_j(e) = \omega_k$, which contradicts $P_j(e) = \omega_i$. In addition, by $y_i \neq P_i(e) = \omega_j$, $y_j \neq P_j(e) = \omega_i$. Hence,

$$P_j(e) = \omega_i \succ_j y_j. \tag{2}$$

Hence, by (1) and (2), $\{i, j\}$ weakly blocks y at $\succ$ via $P(e)$, which implies $y \notin \mathcal{C}(e)$.

Step 2: $P(e) \in \mathcal{C}(e)$. Since $P(e) \in \mathcal{I}(e)$, it suffices to show that no pair weakly blocks $P(e)$ at e. Suppose, by contradiction, that there are $\{i, j\} \subset N$ with $i \neq j$ and $y \in X_2(\omega)$ such that $\{i, j\}$ weakly blocks $P(e)$ at e via y. Without loss of generality, $\sigma^*[\![\omega]\!](i) < \sigma^*[\![\omega]\!](j)$. Since preferences are strict, $y \neq P(e)$, $\{y, P(e)\} \subset$

$X_2(\omega)$, and $P(e) \in \mathcal{I}(e)$, we have that

$$y_i = \omega_j \succ_i P_i(e) \succsim_i \omega_i; \tag{3}$$
$$y_j = \omega_i \succ_j P_j(e) \succsim_i \omega_j. \tag{4}$$

Let $k \in N$ be such that $P_i(e) = \omega_k$. There are two cases.

• **Case 1: $\sigma^*[\![\omega]\!](k) < \sigma^*[\![\omega]\!](i)$.** By $k \neq i$ and (3),

$$\omega_j \succ_i P_i(e) = \omega_k \succ_i \omega_i.$$

By this and $e \in \mathscr{E}^{\text{cm}}$, $\sigma^*[\![\omega]\!](j) < \sigma^*[\![\omega]\!](k)$, which contradicts $\sigma^*[\![\omega]\!](k) < \sigma^*[\![\omega]\!](i) < \sigma^*[\![\omega]\!](j)$.

• **Case 2: $\sigma^*[\![\omega]\!](i) \leq \sigma^*[\![\omega]\!](k)$.** By (3) and (4), there is $x \in \mathbb{X}_0^{\sigma^*}(e) = \mathcal{I}(e)$ such that $(x_i, x_j) = (\omega_j, \omega_i)$. However, there is no $x \in \mathbb{X}^{\sigma^*}_{\sigma^*[\![\omega]\!](i)-1}(e)$ such that $(x_i, x_j) = (\omega_j, \omega_i)$; otherwise, by the definition of P, $P_i(e) \succsim_i \omega_j$, which contradicts (3). These imply that there is $i' \in N$ such that $\sigma^*[\![\omega]\!](i') < \sigma^*[\![\omega]\!](i)$ and $(P_{i'}(e), P_j(e)) = (\omega_j, \omega_{i'})$. By $e \in \mathscr{E}^{\text{cm}}$, $\{\omega_{i'}, \omega_i\} \subset A(\succ_j, \omega_j)$, and $\sigma^*[\![\omega]\!](i') < \sigma^*[\![\omega]\!](i)$,

$$P_j(e) = \omega_{i'} \succ_j \omega_i = y_j,$$

which contradicts (4).

Step 3: Conclusion. By Steps 1 and 2, we have $\mathcal{C}(e) = \{P(e)\}$. □

Proof of Proposition 1. Let f be a pairwise exchange mechanism on $\mathscr{E}^{\text{cm}}$ satisfying the two axioms. By Lemma 2 and Lemma 3, we have $f = P$. □

B.4 Proof of Theorem 4

The "only if" part follows from Proposition 1. We next show the "if" part. The definition of P immediately implies *individual rationality* of P. We now prove that if $n = 3$, then P is *effectively endowments-swapping-proof*. Let $e = (\succ, \omega) \in \mathscr{E}^{\text{cm}}$. Without loss of generality, we assume $\omega = (h_1, h_2, h_3)$. Thus, $\sigma^*[\![\omega]\!](1) = 1$, $\sigma^*[\![\omega]\!](2) = 2$, and $\sigma^*[\![\omega]\!](3) = 3$. Suppose, by contradiction, that there is a pair $\{i, j\} \subset N$ such that

(i) $e^{i,j} \in \mathscr{E}^{\text{cm}}$,

(ii) $\omega_j \in A(\succ_i, \omega_i)$ and $\omega_i \in A(\succ_j, \omega_j)$, and

(iii) $P_i(e^{i,j}) \succ_i P_i(e)$ and $P_j(e^{i,j}) \succ_j P_j(e)$.

There are three cases.

• **Case 1: $\{i, j\} = \{1, 2\}$.** By $\omega_2 = h_2 \in A(\succ_1, \omega_1)$, $\omega_1 = h_1 \in A(\succ_2, \omega_2)$, and $e \in \mathscr{E}^{\text{cm}}$, h_2 or h_1 is agent 1's or 2's best object at $\succ_1$ or $\succ_2$, respectively. Hence, by the definition of P, $(P_1(e), P_2(e)) = (h_2, h_1)$, which contradicts (iii).

• **Case 2: $\{i, j\} = \{1, 3\}$.** By $\omega_3 = h_3 \in A(\succ_1, \omega_1)$ and $\omega_1 = h_1 \in A(\succ_3, \omega_3)$, $x = (h_3, h_2, h_1) \in \mathbb{X}_0^{\sigma^*}(e) = \mathcal{I}(e)$. Hence, by the definition of P, $P_1(e) \succsim_1 h_3$. Further, by (ii) and (iii),

$$P_1(e^{1,3}) = h_2 \succ_1 P_1(e) = h_3 \succ_1 \omega_1 = h_1.$$

By $P(e) \in X_2(\omega)$ and $P_1(e) = \omega_3 = h_3$, $P_3(e) = \omega_1 = h_1$. Since $\omega_1 = h_1 \in A(\succ_3, \omega_3)$ and $e \in \mathscr{E}^{\text{cm}}$, h_1 is agent 3's best object at $\succ_3$, which contradicts $P_3(e^{1,3}) \succ_3 P_3(e) = h_1$.

• **Case 3: $\{i, j\} = \{2, 3\}$.** There are two subcases.

∘ *Subcase 3.1*: $P_1(e) \neq \omega_1 = h_1$. By $P(e) \in X_2(\omega)$, there is $k \in \{2, 3\}$ such that $P_k(e) = \omega_1 = h_1$. Further, by $P(e) \in \mathcal{I}(e)$ and $e \in \mathscr{E}^{\text{cm}}$, h_1 is agent k's best object at $\succ_k$. Hence, agent k has no incentive to collude with another agent, which is a contradiction.

∘ *Subcase 3.2*: $P_1(e) = \omega_1 = h_1$. By (ii), $\omega_3 = h_3 \in A(\succ_2, \omega_2 = h_2)$ and $\omega_2 = h_2 \in A(\succ_3, \omega_3 = h_3)$. Since $P_1(e) = \omega_1 = h_1$, $x = (h_1, h_3, h_2) \in \mathbb{X}_2^{\sigma^*}(e)$. By $h_3 \succ_2 h_2$, $(P_2(e), P_3(e)) = (h_3, h_2)$. This together with (ii) and (iii) implies that

$$P_2(e^{2,3}) \succ_2 P_2(e) = h_3 \succ_2 \omega_2 = h_2;$$
$$P_3(e^{2,3}) \succ_3 P_3(e) = h_2 \succ_3 \omega_3 = h_3.$$

It follows from this that $P_2(e^{2,3}) = P_3(e^{2,3}) = h_1$, which is a contradiction. □

B.5 Proof of Theorem 7

The "if" part follows from Tamura (2023) because the size of cycles formed in the TTC algorithm is either one or two even without feasibility constraints. Thus, it suffices to show the "only if" part. The following lemma, which immediately follows from Proposition 2, is useful in this proof.

Lemma 4. *For each* $e = (\succ, \omega) \in \mathscr{E}^\vee$, *each* $t \in \mathbb{N}$, *and each* $S \in \mathbb{S}_t(e)$, *we have either* $|S| = 1$ *or* $|S| = 2$.

We now prove that for each $e = (\succ, \omega) \in \mathscr{E}^\vee$, each $t \in \mathbb{N}$, and each $i \in N_t(e)$, $f_i(e) = TTC_i(e)$. Let $e = (\succ, \omega) \in \mathscr{E}^\vee$. Without loss of generality, suppose $\omega_i = h_i$ for each $i \in N$. We use induction on t.

BASE STEP. $t = 1$. Let $S \in \mathbb{S}_1(e)$. By Lemma 4, there are two cases.

- **Case 1: $|S| = 1$.** Then, $S \in \{\{1\}, \{n\}\}$. Without loss of generality, suppose $S = \{1\}$. Then, ω_1 is agent 1's best object at $\succ_1$. By *individual rationality*, we have $f_1(\succ) = \omega_1 = TTC_1(\succ)$.
- **Case 2: $|S| = 2$.** Then, $S = \{1, n\}$ and

$$\begin{array}{cc} \succ_1 & \succ_n \\ \hline \omega_n & \omega_1 \\ \vdots & \vdots \end{array}$$

Suppose, by contradiction, that $(f_1(\succ), f_n(\succ)) \neq (TTC_1(\succ), TTC_n(\succ)) = (\omega_n, \omega_1)$. Without loss of generality, we assume $f_1(\succ) \neq \omega_n$. Since f is a pairwise exchange mechanism, $f_n(\succ) \neq \omega_1$. Consider $e^{1,n}$. Then, $e^{1,n} \in \mathscr{E}^\vee$, $\omega_1^{1,n} = \omega_n \in A(\succ_1, \omega_1)$ and $\omega_n^{1,n} = \omega_1 \in A(\succ_n, \omega_n)$, and by *individual rationality*,

$$\begin{aligned} f_1(e^{1,n}) &= \omega_1^{1,n} = \omega_n \succ_1 f_1(e) \succsim_1 \omega_1; \\ f_n(e^{1,n}) &= \omega_n^{1,n} = \omega_1 \succ_n f_n(e) \succsim_n \omega_n, \end{aligned}$$

in violation of *effective endowments-swapping-proofness*.

From these two cases, we have that for each $i \in N_1(e)$, $f_i(e) = TTC_i(e)$.

INDUCTION HYPOTHESIS. For each $t \in \{1, 2, \ldots, r-1\}$ and each $i \in N_t(e)$, $f_i(e) = TTC_i(e)$.

INDUCTION STEP. Let $t = r$. By the induction hypothesis,

$$\bigcup_{j=1}^{r-1} H_j(e) = \left\{ h \in H \colon f_i(e) = h \text{ for some } i \in \bigcup_{j=1}^{r-1} N_j(e) \right\}. \tag{5}$$

Consider $S \in \mathbb{S}_r(e)$. By the discussion in Proposition 2, we know that

$$S \in \{\{\underline{i}(r), \bar{i}(r)\}, \{\underline{i}(r)\}, \{\bar{i}(r)\}\}.$$

There are two cases.

- **Case 1: $S \in \{\{\underline{i}(r)\}, \{\bar{i}(r)\}\}$.** Without loss of generality, we assume $S = \{\underline{i}(r)\}$. Then, $\{\underline{i}(r)\}$ forms a cycle at Step r of the TTC algorithm at e. Hence, for each $h \in H \setminus (\bigcup_{j=1}^{r-1} H_j(e) \cup \{\omega_{\underline{i}(r)}\})$, $\omega_{\underline{i}(r)} \succ_{\underline{i}(r)} h$. By (5), $f_{\underline{i}(r)}(e) \in H \setminus \bigcup_{j=1}^{r-1} H_j(e)$. These together with *individual rationality*, $f_{\underline{i}(r)}(e) = \omega_{\underline{i}(r)} = TTC_{\underline{i}(r)}(e)$.

- **Case 2: $S = \{\underline{i}(r), \bar{i}(r)\}$.** Then, $\{\underline{i}(r), \bar{i}(r)\}$ forms a cycle at Step r of the TTC algorithm at e. Hence, for each $h \in H \setminus (\bigcup_{j=1}^{r-1} H_j(e) \cup \{\omega_{\underline{i}(r)}, \omega_{\bar{i}(r)}\})$,

$$\omega_{\bar{i}(r)} \succ_{\underline{i}(r)} h \quad \text{and} \quad \omega_{\bar{i}(r)} \succ_{\underline{i}(r)} \omega_{\underline{i}(r)}; \tag{6}$$

$$\omega_{\underline{i}(r)} \succ_{\bar{i}(r)} h \quad \text{and} \quad \omega_{\underline{i}(r)} \succ_{\bar{i}(r)} \omega_{\bar{i}(r)}. \tag{7}$$

Suppose, by contradiction, that $(f_{\underline{i}(r)}(e), f_{\bar{i}(r)}(e)) \neq (TTC_{\underline{i}(r)}(e), TTC_{\bar{i}(r)}(e)) = (\omega_{\bar{i}(r)}, \omega_{\underline{i}(r)})$. Without loss of generality, we assume $f_{\underline{i}(r)}(e) \neq \omega_{\bar{i}(r)}$. Since f is a pairwise exchange mechanism, $f_{\bar{i}(r)}(e) \neq \omega_{\underline{i}(r)}$. By (5), $\{f_{\underline{i}(r)}(e), f_{\bar{i}(r)}(e)\} \subset H \setminus \bigcup_{j=1}^{r-1} H_j(e)$. Thus, by (6) and (7)

$$\omega_{\bar{i}(r)} \succ_{\underline{i}(r)} f_{\underline{i}(r)}(e);$$
$$\omega_{\underline{i}(r)} \succ_{\bar{i}(r)} f_{\bar{i}(r)}(e).$$

Consider $e^{\underline{i}(r),\bar{i}(r)}$. Then, $e^{\underline{i}(r),\bar{i}(r)} \in \mathscr{E}^{\vee}$, $\omega_{\underline{i}(r)}^{\underline{i}(r),\bar{i}(r)} = \omega_{\bar{i}(r)} \in A(\succ_{\underline{i}(r)}, \omega_{\underline{i}(r)})$ and $\omega_{\bar{i}(r)}^{\underline{i}(r),\bar{i}(r)} = \omega_{\underline{i}(r)} \in A(\succ_{\bar{i}(r)}, \omega_{\bar{i}(r)})$, and by *individual rationality*,

$$f_{\underline{i}(r)}(e^{\underline{i}(r),\bar{i}(r)}) \succsim_{\underline{i}(r)} \omega_{\underline{i}(r)}^{\underline{i}(r),\bar{i}(r)} = \omega_{\bar{i}(r)} \succ_{\underline{i}(r)} f_{\underline{i}(r)}(e) \succsim_{\underline{i}(r)} \omega_{\underline{i}(r)};$$
$$f_{\bar{i}(r)}(e^{\underline{i}(r),\bar{i}(r)}) \succsim_{\bar{i}(r)} \omega_{\bar{i}(r)}^{\underline{i}(r),\bar{i}(r)} = \omega_{\underline{i}(r)} \succ_{\bar{i}(r)} f_{\bar{i}(r)}(e) \succsim_{\bar{i}(r)} \omega_{\bar{i}(r)},$$

in violation of *effective endowments-swapping-proofness*. Hence, $(f_{\underline{i}(r)}(e), f_{\bar{i}(r)}(e)) = (TTC_{\underline{i}(r)}(e), TTC_{\bar{i}(r)}(e))$.

From Cases 1 and 2, for each $i \in N_r(e)$, $f_i(e) = TTC_i(e)$. □

B.6 Proof of Proposition 3

Let $e = (\succ, \omega) \in \mathscr{E}^G$. Recall that, for each $t \in \mathbb{N}$, $N_t(e)$ is the set of agents that form cycles at Step t of TTC at e and $H_t(e)$ is the set of objects that are assigned to agents in $N_t(e)$. We now introduce additional notation:

- $N^1 = N$ and for each $t \geq 2$, $N^t = N^{t-1} \setminus N_{t-1}(e)$;
- $G^1 = (H^1, E^1) = (H, E)$ and for each $t \geq 2$, $G^t = (H^t, E^t)$, where $H^t = H^{t-1} \setminus H_{t-1}(e)$ and $E^t = \{\{h', h''\} \in E^{t-1} \colon \{h', h''\} \subset H^t\}$;
- for each $i \in N^1$, $d^1(\succ_i) = d(\succ_i)$ and for each $t \geq 2$ and each $i \in N^t$, $d^t(\succ_i)$ denotes i's worst object at $\succ_i$ among H^t (i.e., $d^t(\succ_i) \in H^t$ and for each $h \in H^t \setminus \{d^t(\succ_i)\}$, $h \succ_i d^t(\succ_i)$).

We will observe below that for each $t \geq 2$, the graph $G^t = (H^t, E^t)$ is a tree. We denote by $\mathbb{L}^t$ the set of leaves in G^t. Note that $\mathbb{L}^1 = \mathbb{L}$. Moreover, for each $t \in \mathbb{N}$ and each $\{h', h''\} \subset H^t$ with $h' \neq h''$, we denote by $[h', h'']^t$ the unique path from h' to h'' in G^t. We now consider each step of TTC.

STEP 1 OF TTC. As stated previously, for each $i \in N^1 = N$, i's best object at $\succ_i$ among $H^1 = H$ is in $\mathbb{L}^1 = \mathbb{L}$. Hence, $N_1(e) \subset \{i \in N^1 \colon \omega_i \in \mathbb{L}^1\}$ and $H_1(e) \subset \mathbb{L}^1$. This implies that the size of each trading cycle formed at Step 1 is less than or equal to $|\mathbb{L}^1|$.

STEP 2 OF TTC. Note that the set of remaining agents (resp. objects) is $N^2 = N^1 \setminus N_1(e)$ (resp. $H^2 = H^1 \setminus H_1(e)$). We present a series of claims before completing the proof.

Claim 2. *G^2 is a tree.*

Proof of Claim 2. Since $H^2 = H^1 \setminus H_1(e)$ and $H_1(e) \subset \mathbb{L}^1$, by Lemma 2.1.3 in West (2001), G^2 is a tree. □

Claim 3. $|\mathbb{L}^2| \leq |\mathbb{L}^1|$.

Proof of Claim 3. Note that, by $H_1(e) \subset \mathbb{L}^1$,

$$|\mathbb{L}^1| = |\mathbb{L}^1 \cap H_1(e)| + |\mathbb{L}^1 \setminus H_1(e)| = |H_1(e)| + |\mathbb{L}^1 \setminus H_1(e)|;$$
$$|\mathbb{L}^2| = |\mathbb{L}^2 \cap \mathbb{L}^1| + |\mathbb{L}^2 \setminus \mathbb{L}^1|.$$

In what follows, we show that (i) $|\mathbb{L}^2 \cap \mathbb{L}^1| \leq |\mathbb{L}^1 \setminus H_1(e)|$ and (ii) $|\mathbb{L}^2 \setminus \mathbb{L}^1| \leq |H_1(e)|$, which together imply that $|\mathbb{L}^2| \leq |\mathbb{L}^1|$.

(i) Let $h \in \mathbb{L}^2 \cap \mathbb{L}^1$. By $h \in \mathbb{L}^2 \subset H^2$, $h \notin H_1(e)$, which implies that $h \in \mathbb{L}^1 \setminus H_1(e)$. Hence, $\mathbb{L}^2 \cap \mathbb{L}^1 \subset \mathbb{L}^1 \setminus H_1(e)$ and $|\mathbb{L}^2 \cap \mathbb{L}^1| \leq |\mathbb{L}^1 \setminus H_1(e)|$.

(ii) Note that the degree of $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$ in G^2 is equal to 1 and that in G^1 is greater than 1. Then, for each $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$, there is $\hat{h} \in H_1(e) (\subset \mathbb{L}^1)$ such that $\{h, \hat{h}\} \in E^1$.[18] Thus, we can construct a mapping $\alpha\colon \mathbb{L}^2 \setminus \mathbb{L}^1 \to H_1(e)$ such that for each $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$, $\alpha(h) \in H_1(e)$ with $\{h, \alpha(h)\} \in E^1$. We now show that α is injective, which immediately implies that $|\mathbb{L}^2 \setminus \mathbb{L}^1| \leq |H_1(e)|$. Suppose, by contradiction, that there is $\{h', h''\} \subset \mathbb{L}^2 \setminus \mathbb{L}^1$ such that $h' \neq h''$ but $\alpha(h') = \alpha(h'')$. Then, by $\{\{h', \alpha(h')\}, \{h'', \alpha(h'') = \alpha(h')\}\} \subset E^1$, the degree of $\alpha(h') = \alpha(h'')$ in G_1 is greater than 1, which is a contradiction to $\alpha(h') = \alpha(h'') \in \mathbb{L}^1$. □

Claim 4. *For each $i \in N^2$, $\succ_i$ is single-dipped on G^2.*

Proof of Claim 4. By the definition of $d^2(\succ_i)$, $d^2(\succ_i) \in H^2$ and for each $h \in H^2 \setminus \{d^2(\succ_i)\}$, $h \succ_i d^2(\succ_i)$. Next, let $\{h', h''\} \subset H^2 \setminus \{d^2(\succ_i)\}$ be such that $h' \in [d^2(\succ_i), h'']^2 = (h^1 = d^2(\succ_i), \dots, h^K = h'')$. Note that, for each $k \in \{1, \dots, K-1\}$, by $\{h^k, h^{k+1}\} \in E^2$, $\{h^k, h^{k+1}\} \in E^1$. Hence, $[d^2(\succ_i), h'']^1 = (h^1 = d^2(\succ_i), \dots, h^K = h'') = [d^2(\succ_i), h'']^2$. There are two cases.

• **Case 1: $\boldsymbol{d^1(\succ_i) \in H^2}$.** It is obvious that $d^2(\succ_i) = d^1(\succ_i)$. Since $\succ_i$ is single-dipped on G^1 and $h' \in [d^2(\succ_i) = d^1(\succ_i), h'']^1$, $h'' \succ_i h'$.

• **Case 2: $\boldsymbol{d^1(\succ_i) \notin H^2}$.** Then, $d^1(\succ_i) \in H_1(e) \subset \mathbb{L}^1$. This implies that the degree of $d^1(\succ_1)$ in G^1 is equal to 1. Let $h^* \in H^1$ be the unique object such that $\{d^1(\succ_i), h^*\} \in E^1$. Then, $h^* \in H^2$.[19] We now show that $h^* = d^2(\succ_i)$; that is, for each $h \in H^2 \setminus \{h^*\}$, $h \succ_i h^*$. Let $h \in H^2 \setminus \{h^*\}$. By $h \in H^1$, we can find $[d^1(\succ_i), h]^1 = (\bar{h}^1 = d^1(\succ_i), \bar{h}^2, \dots, \bar{h}^{\bar{K}} = h)$. Since h^* is the unique object such that $\{d^1(\succ_i), h^*\} \in E^1$, $\bar{h}^2 = h^*$, and thus, $h^* \in [d^1(\succ_i), h]^1$. Since $\succ_i$ is single-dipped on G^1, $h \succ_i h^*$. Additionally, since $[d^2(\succ_i) = h^*, h'']^1 = (h^1 = d^2(\succ_i) = h^*, \dots, h^K = h'')$ and $\{d^1(\succ_i), h^*\} \in E^1$, $[d^1(\succ_i), h'']^1 = (d^1(\succ_i), h^1 = d^2(\succ_i) = h^*, \dots, h^K = h'')$. By $h' \in [d^2(\succ_i), h'']^2 = [d^2(\succ_i), h'']^1$, $h' \in [d^1(\succ_i), h'']^1$. Since $\succ_i$ is single-dipped on G^1, $h'' \succ_i h'$. □

Since G^2 is a tree (Claim 2) and for each $i \in N^2$, $\succ_i$ is single-dipped on G^2 (Claim 4), we have that for each $i \in N^2$, i's best object at $\succ_i$ among H^2 is in $\mathbb{L}^2$ (Remark 6). Hence, $N_2(e) \subset \{i \in N^2 \colon \omega_i \in \mathbb{L}^2\}$ and $H_2(e) \subset \mathbb{L}^2$. This

[18] Otherwise, there is $h \in \mathbb{L}^2 \setminus \mathbb{L}^1$ such that for each $\hat{h} \in H^1$ with $\{h, \hat{h}\} \in E^1$, $\hat{h} \notin H_1(e)$. Then, the degree of h in G^2 is equal to that in G^1, which is a contradiction.

[19] If $h^* \in H_1(e)$, then $h^* \in \mathbb{L}^1$ and the degree of h^* in G^1 is equal to 1. This implies that $H^1 = \{d^1(\succ_i), h^*\}$, $E^1 = \{\{d^1(\succ_i), h^*\}\}$, and $H^2 = H^1 \setminus H_1(e) = \emptyset$; that is, the TTC algorithm terminates at Step 1, a contradiction.

together with Claim 3 implies that the size of each trading cycle formed at Step 2 is less than or equal to $|\mathbb{L}^1|$.

By repeating this argument, we observe that the size of each trading cycle formed in each step of TTC is less than or equal to $|\mathbb{L}^1|$. This implies that TTC on $\mathscr{E}^G$ is $|\mathbb{L}|$-feasible. □

B.7 Omitted proof in Example 6

Here, we show that $f^{\triangledown}$ satisfies (*effective*) *endowments-swapping-proofness*. To observe this, consider $e = (\succ, \omega) \in \mathscr{E}^G$ and $\{i, j\} \subset N$ with $i \neq j$. If $\{e, e^{i,j}\} \subset \mathscr{E}^G \setminus \{\check{e}\}$, by $f^{\triangledown}(e) = TTC(e)$ and $f^{\triangledown}(e^{i,j}) = TTC(e^{i,j})$, the pair has no incentive to collude. Hence, we consider the following two cases.

• **Case 1: $e = \check{e}$ and $e^{i,j} \neq \check{e}$.** Since each agent $i \in \{2, 3, 4\}$ receives his best object according to his preferences $\succ_i = \check{\succ}_i$, he has no incentive to collude with another agent at e. Thus, it suffices to consider the case where $\{i, j\} = \{1, 5\}$. Then, $\omega_1^{1,5} = \check{\omega}_1^{1,5} = \check{\omega}_5 = h_5$ and $f_1^{\triangledown}(e^{1,5}) = TTC_1(e^{1,5}) = h_5$. This means that agent 1 ends up with receiving his worst object according to his preferences $\succ_1 = \check{\succ}_1$. Hence, agent 1 has no incentive to collude with agent 5 at e.

• **Case 2: $e \neq \check{e}$ and $e^{i,j} = \check{e}$.** If $5 \in \{i, j\}$, by $f_5^{\triangledown}(e^{i,j})(= f_5^{\triangledown}(\check{e})) = h_5$, agent 5 ends up with receiving his worst object according to his preferences $\succ_5 = \check{\succ}_5$. Hence, agent 5 has no incentive to collude with another agent at e. We below consider the case where $\{i, j\} \subset \{1, 2, 3, 4\}$. Note that by $\omega^{i,j} = \check{\omega}$, $\omega = \check{\omega}^{i,j}$.

◦ *Subcase 2-1*: $\{i, j\} = \{1, 2\}$. Then, $\omega = (h_2, h_1, h_3, h_4, h_5)$ and $(f_1^{\triangledown}(e), f_2^{\triangledown}(e)) = (TTC_1(e), TTC_2(e)) = (h_2, h_1)$. This implies that both agents have already received their best objects according to their preferences $\succ_1 = \check{\succ}_1$ and $\succ_2 = \check{\succ}_2$. Hence, this pair has no incentive to collude at e.

◦ *Subcase 2-2*: $\{i, j\} = \{1, 3\}$. Then, $\omega = (h_3, h_2, h_1, h_4, h_5)$ and $(f_1^{\triangledown}(e), f_3^{\triangledown}(e)) = (TTC_1(e), TTC_3(e)) = (h_2, h_3)$. This implies that both agents have already received their best objects according to their preferences $\succ_1 = \check{\succ}_1$ and $\succ_3 = \check{\succ}_3$. Hence, this pair has no incentive to collude at e.

◦ *Subcase 2-3*: $\{i, j\} = \{1, 4\}$. Then, $\omega = (h_4, h_2, h_3, h_1, h_5)$ and $(f_1^{\triangledown}(e), f_4^{\triangledown}(e)) = (TTC_1(e), TTC_4(e)) = (h_4, h_2)$. This implies that agent 4 has already received his best object according to his preferences $\succ_4 = \check{\succ}_4$. Hence, agent 4 has no incentive to collude with agent 1 at e.

$\circ$ *Subcase 2-4*: $\{i, j\} = \{2, 3\}$. Then, $\omega = (h_1, h_3, h_2, h_4, h_5)$ and $(f_2^{\triangledown}(e), f_3^{\triangledown}(e)) = (TTC_2(e), TTC_3(e)) = (h_1, h_3)$. This implies that both agents have already received their best objects according to their preferences $\succ_2 = \check{\succ}_2$ and $\succ_3 = \check{\succ}_3$. Hence, this pair has no incentive to collude at e.

$\circ$ *Subcase 2-5*: $\{i, j\} = \{2, 4\}$. Then, $\omega = (h_1, h_4, h_3, h_2, h_5)$ and $(f_2^{\triangledown}(e), f_4^{\triangledown}(e)) = (TTC_2(e), TTC_4(e)) = (h_1, h_2)$. This implies that both agents have already received their best objects according to their preferences $\succ_2 = \check{\succ}_2$ and $\succ_4 = \check{\succ}_4$. Hence, this pair has no incentive to collude at e.

$\circ$ *Subcase 2-6*: $\{i, j\} = \{3, 4\}$. Then, $\omega = (h_1, h_2, h_4, h_3, h_5)$ and $(f_3^{\triangledown}(e), f_4^{\triangledown}(e)) = (TTC_3(e), TTC_4(e)) = (h_3, h_4)$. This implies that agent 3 has already received his best object according to his preferences $\succ_3 = \check{\succ}_3$, Hence, agent 3 has no incentive to collude with agent 4 at e. $\square$

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