



UTMD Working Paper

The University of Tokyo
Market Design Center

UTMD-046

Fair Matching under Constraints: Theory and Applications

Yuichiro Kamada
University of California, Berkeley
Fuhito Kojima
University of Tokyo

April 13, 2023

FAIR MATCHING UNDER CONSTRAINTS: THEORY AND APPLICATIONS

YUICHIRO KAMADA AND FUHITO KOJIMA

ABSTRACT. This paper studies a general model of matching with constraints. Observing that a stable matching typically does not exist, we focus on feasible, individually rational, and fair matchings. We characterize such matchings by fixed points of a certain function. Building on this result, we characterize the class of constraints on individual schools under which there exists a student-optimal fair matching (SOFM), the matching that is the most preferred by every student among those satisfying the three desirable properties. We study the numerical relevance of our theory using data on government-organized daycare allocation.

JEL Classification Numbers: C70, D47, D61, D63.

Keywords: matching with constraints, daycare allocation, school choice, fairness, stability, Tarski's fixed point theorem

Date: April 13, 2023.

Kamada: Haas School of Business, University of California, Berkeley, Berkeley, CA 94720, y.cam.24@gmail.com. Kojima: Department of Economics, Stanford University, Stanford, CA 94305, fkojima@stanford.edu. We thank the editor, Adam Szeidl, and anonymous referees for helpful comments. We are grateful to Mayor Takahiro Sato of Yamagata City and the officials at Yamagata's childcare department, especially Mr. Kazuhiro Sato, as well as Mayor Hironobu Narisawa of Bunkyo City and the officials at Bunkyo's childcare department, especially Mr. Hideki Okawa, Mr. Junya Takahashi, and Takahito Yokoyama, for providing data on their daycare seat allocation, as well as for answering numerous questions about their allocation mechanisms, on which our analysis in Section 5 is based. We benefited from an interview with Arisa Wada at Yamagata Mirai Lab. We are also grateful to Atila Abdulkadiroglu, Tommy Andersson, Eduardo Azevedo, Péter Biró, Gabriel Carroll, Yeon-Koo Che, Yan Chen, Federico Echenique, Haluk Ergin, Nima Haghpahan, John William Hatfield, Michihiro Kandori, Yusuke Kasuya, Thilo Klein, Ayako Kondo, Jacob Leshno, George Mailath, Mihai Manea, Megumi Murakami, Tomoko Murata, Aki Matsui, Yusuke Narita, Makiko Nishi, Shunya Noda, Yasunori Okumura, Arthur Robson, Al Roth, Larry Samuelson, Cameron Taylor, Kentaro Tomoeda, Pete Troyan, Utku Unver, Bob Wilson, Qingyun Wu, Yosuke Yasuda, and seminar participants at Chinese University of Hong Kong, Waseda, Osaka, Keio, Northwestern, Stanford, Universitat Autònoma de Barcelona, International Workshop on Economic Theory (Otaru), Conference on Economic Theory (Columbia), Matching in Practice (Cologne), Market Design Conference (Vanderbilt), Workshop on Mechanism Design and Behavioral Economics (Glasgow), SAET 2018 (Taipei), Asian Meeting of Econometric Society (Seoul), Workshop on Matching, Search and Market Design (National University of Singapore), and UBC-HKU summer theory conference (Vancouver) for comments. We gratefully acknowledge excellent research assistance of Omair Butt, Ashley Dreyer, Atharva Gupta, Ye Rin Kang, Kevin Li, Shivam Patel, Priyanka Shende, Lingxuan Wu, Yutong Zhang, and especially Yue Fan. Kojima's research was supported by the National Research Foundation through its Global Research Network Grant (NRF-2016S1A2A2912564).

“Didn’t Get a Slot in Day Care. Drop Dead, Japan!!!”

—An anonymous mother

1. INTRODUCTION

Daycare services for young children in Japan, as in many countries, are rationed by the government using matching mechanisms, and a large excess demand for daycare seats is a highly contentious political issue. A blog post by an anonymous mother who could not secure a slot in a daycare for her child became viral in 2016, leading to a large protest movement, a debate in Diet, and Prime Minister Abe’s promise to improve the situation (Osaki, 2016; Prime Minister’s Office, 2017). In the midst of such a heated political environment, increasing allocation of slots at daycares in a fair manner is now among the very top priorities for many politicians.

One notable complication of the daycare allocation problem is that the necessary teacher-child ratio varies across different ages. In the Japanese case, for instance, the national regulation requires at least one teacher for every three children of age 0 while the ratios are one teacher for every 6 children for ages 1 and 2, 20 children for age 3, and 30 children for ages 4 and 5 (Cabinet Office of Japan, 2017). This means that each daycare center’s feasibility constraint cannot be expressed by a mere capacity constraint. In fact, many matching markets are subject to constraints that are beyond the scope of the standard theory. In the medical match in the U.K. in the last century, for example, some hospitals required a cap on the number of women in their workforce (Roth, 1991). In school choice, a school may be subject to diversity requirements (Abdulkadiroğlu and Sönmez, 2003). As discussed in more detail later, many more matching markets with constraints have recently become the subject of research, including problems of refugee resettlement, college admissions with budget constraint, and school choice under bullying concerns.

The aim of this paper is to study a general model of matching with constraints. Framing the problem as matching between students and schools, we seek to understand the theoretical properties of this problem (with an eye on market design—we apply our theory to the daycare allocation problem based on real data from Japan). Instead of considering various special cases one by one, we begin with a very general model. In that model, each school is subject to an arbitrary constraint, which is represented simply by an arbitrary family of subsets of students that are feasible at that school. The standard model with capacity constraints is a special case in which a set of students is feasible at a school if

and only if its cardinality is no greater than the school’s capacity, but many more cases fall into the scope of our model.

Given the generality of the constraints, a stable matching does not necessarily exist. In fact, we find that each school’s constraint being a capacity constraint is a “necessary and sufficient” condition for guaranteeing existence (in a certain “maximal domain” sense). Faced with this impossibility result, we focus on a condition weaker than stability. More specifically, we consider a fairness notion which requires there be no justified envy, that is, there be no student who prefers a school to her outcome while another student with lower priority is matched to the school. We require this notion of fairness as well as feasibility and individual rationality as our basic requirements (by contrast, we do not require non-wastefulness, which together with the previous three conditions are equivalent to stability). Our first main result provides a characterization of matchings that satisfy these properties as fixed points of a certain mapping.

Equipped with the fixed-point characterization, we identify the class of constraints that guarantee the existence of a desirable matching. For this goal, we first define a student-optimal fair matching, SOFM. A matching is an SOFM if it satisfies fairness, feasibility, and individual rationality and, in addition, it is weakly preferred by every student to any matching that satisfies fairness, feasibility, and individual rationality. This is analogous to the well-known student-optimal stable matching but adapted to our more general environment where the latter may not exist. We identify a class of constraints on individual schools under which an SOFM exists. Specifically, if the constraint at each school is in a class that we call *general upper-bounds*, meaning that any subset of a feasible set of students is also feasible, then an SOFM exists. Moreover, if the constraint of even just one school is not a general upper-bound, then there exist student preferences and capacity constraints at other schools such that an SOFM fails to exist. In this sense, general upper-bound is the most permissive restriction on constraints that leads to the existence of an SOFM.

The family of general upper-bounds subsumes many constraints of interest. Standard cases such as capacities and type-specific quotas are examples of general upper-bounds, while floor (minimum) quotas and proportionality constraints are not. While excluding some cases, the family of general upper-bounds includes many less-known or more recent cases. Examples include college admissions with students with disabilities, refugee matching under multidimensional constraints, school choice under bullying concerns, and separation of conflicting groups in refugee matching.

The daycare allocation problem we discussed earlier provides an interesting example to which our theory can be applied. Daycare services are often highly subsidized and regulated in many countries such as most European and Asian countries. In Japan, the assignment of seats in daycare centers for children (of ages 0 to 5) is conducted by each municipality. It is a matching market: prices are set by the municipality, parents report ordinal preferences over daycare centers, and the municipality assigns seats following an algorithm based on reported preferences and pre-specified priorities over the applicants.¹

As discussed earlier, heterogeneity of teacher-child ratio across age groups implies that the constraint at each daycare center cannot be described by a capacity constraint. Most municipalities, however, treat the number of seats for each age group at each daycare center as fixed and rigid. This leads to an artificial constraint that is more restrictive than the original daycare constraint since the former fails to take into account the inherent flexibility in the latter. We establish that the original daycare constraint and the rigid constraint are both general upper-bounds, and that the SOFM under the former constraint is Pareto superior to the one under the latter.

We supplement our theoretical investigation by analyzing the data we obtained from two municipalities in Japan: Yamagata City and Bunkyo City, which are rural and urban cities in Japan, respectively. First, we compare SOFMs under the original daycare constraint and the rigid constraint. With 250 simulations for each mechanism, we find that the effect of allowing flexibility in constraints is substantial in our data from both municipalities. For example, the average fractions of children who are matched with a strictly preferred daycare center are 60.35% in Yamagata and 51.64% in Bunkyo, and the numbers of unallocated children decrease by 87.67% and 48.81%, respectively. Second, we compare the SOFMs with the real allocations in those municipalities. The real allocation mechanisms use rigid constraints while eliminating justified envy only within the same age group and tolerating existence of justified envy across different ages. Although theoretically no clear general relation exists between the SOFMs and the real outcomes, we find that the loss from the rigidity in the real allocations overwhelms the efficiency gains resulting from tolerating existence of certain justified envy. For example, the average fractions of children who are matched with a strictly preferred daycare center in the SOFM under flexible constraints are 16.56% in Yamagata and 25.30% in Bunkyo, while the corresponding numbers for the real allocation are merely 5.02% in Yamagata and

¹Priorities are based on characteristics of the children or their parents, such as whether parents have full-time jobs and whether the parent is a single parent. Most municipalities use either serial dictatorship or the “Boston” mechanism (also known as the immediate acceptance mechanism), with slight variations across municipalities. Almost all seats are allocated in the beginning of the academic year (April).

13.61% in Bunkyo. Moreover, the real allocation leaves a significant amount of justified envy: the proportions of applicants who have justified envy toward another applicant are 33.05% in Yamagata and 40.82% in Bunkyo. These results suggest that, relative to the mechanisms that are used in reality, our proposed mechanism may result in a nontrivial improvement in efficiency while eliminating justified envy completely.

In order to guarantee existence, we require fairness while dropping non-wastefulness. Moreover, the fairness notion is strong in a sense because even if a student and a school cannot feasibly match with each other, the student’s envy is still considered justifiable. An alternative would be to drop fairness while maintaining non-wastefulness, or to weaken the fairness notion so that a student’s envy is not justifiable if matching her with the corresponding school results in infeasibility. Which approach is more reasonable depends on applications. In the examples that we have in mind, some waste is tolerated while our fairness notion is considered primarily important. We provide real examples as well as in-depth discussions in, for instance, Remark 1 in Section 2 and Section 6.1.

We study a variety of further issues, including the following topics. First, we show that our theory readily generalizes to the case in which school priorities are weak, which extends the scope of applications to real-life cases such as disaster relief and centralized college admissions. We also study strategic issues. Although the SOFM mechanism is not strategy-proof unless all constraints are capacity constraints (in a maximal domain sense), we show that it is not the drawback of SOFM alone but it is, in fact, shared by every fair mechanism which satisfies feasibility and a mild efficiency requirement. Moreover, we show that the SOFM mechanism satisfies an incentive compatibility property called truncation-proofness, and that the mechanism is difficult to manipulate in large markets in a specific sense. In addition, we analyze the relationship between our general upper-bound and the “multidimensional constraints” studied by Delacrétaz, Kominers and Teytelboym (2016). We also study a relationship with Hatfield and Milgrom (2005) and establish a sense in which our results are not implied by theirs.

Related Literature. First and foremost, this paper contributes to the literature of matching with constraints. We discuss the most relevant related works in various parts of this paper. Other notable studies in this literature include Abdulkadiroğlu (2005), Ergin and Sönmez (2006), and Hafalir, Yenmez and Yildirim (2013) for school choice, Abraham, Irving and Manlove (2007) for a project allocation problem, and Westkamp (2013) and Aygun and Turhan (2016) for college admissions. Importantly, these papers study specific classes of constraints motivated by their intended applications. In a sharp contrast, the present paper starts with a fully general class of constraints and finds conditions on

constraints under which desirable matchings and mechanisms exist. These approaches are complementary to each other.

There is another strand of literature on matching with certain constraints that shares a broad motivation with our work while differing in several crucial aspects. Inspired by medical residency matching in Japan, Kamada and Kojima (2015) study a stable matching problem in which the number of doctors who can be matched to hospitals in each region is subject to a constraint. Alternative solution concepts are studied by Kojima, Tamura and Yokoo (2018) and Goto et al. (2014), while more general constraints are studied by Biro et al. (2010b), Kamada and Kojima (2017, 2018) and Goto et al. (2016). In particular, general upper-bound may be reminiscent of a condition called heredity studied by Goto et al. (2016) and Kamada and Kojima (2017). While sharing broad motivation, there are at least two major differences between these studies and ours. First, they study constraints imposed jointly on *subsets* of institutions (e.g., hospitals or schools) while our paper considers constraints imposed separately on individual institutions. Second, they restrict attention to constraints over the *numbers* of individuals in different institutions, while the present paper allows constraints to depend on the identity of the individuals. Due to these modeling differences, these two lines of works differ in many dimensions including scopes of applications, desirable properties considered, methods used, and obtained results. In particular, some of the papers cited here make use of the results from Hatfield and Milgrom (2005), while the results in the present paper cannot be obtained from that paper (see Appendix C.2).

Although not as closely related, there is also a recent literature on pure object allocation under constraints. Milgrom (2009) and Milgrom and Segal (2014) consider auction mechanisms under constraints, while Budish et al. (2013) analyze the problem of implementing lotteries for stochastic object allocation under constraints. The latter analysis has been extended in various directions by Che, Kim and Mierendorff (2013), Pycia and Ünver (2015), Akbarpour and Nikzad (2017), and Nguyen, Peivandi and Vohra (2016). These papers are different from ours in that they are primarily concerned with pure object allocation. Moreover, they do not consider elimination of justified envy, our central concern.

Our existence theorem relies on Tarski's fixed point theorem. This theorem has been used to find a stable matching in the literature such as Roth and Sotomayor (1988), Adachi (2000), Fleiner (2003), Echenique and Oviedo (2004, 2006), Hatfield and Milgrom (2005), Echenique and Yenmez (2007), Ostrovsky (2008), Hatfield and Kominers (2017), and Wu and Roth (2018). Given that our solution concept of fairness is different from

stability, the fixed-point mapping we employ is different from those used in the existing research, and existing approaches do not work.² In particular, as we mentioned earlier, Appendix C.2 establishes a precise sense in which the existing method of Hatfield and Milgrom (2005) cannot be used to obtain our results.

Elimination of justified envy is a standard fairness requirement in the literature of school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Tolerating some waste is less standard but becoming more common, perhaps due to difficulty (or even impossibility) of eliminating both waste and justified envy. Feasible, individually rational, and fair (but possibly wasteful) matchings are studied by Sotomayor (1996) and Blum, Roth and Rothblum (1997) for one-to-one matching, Wu and Roth (2018) for many-to-one matching, Delacrétaz, Kominers and Teytelboym (2016) for matching with multidimensional constraints, Kesten and Yazici (2012) for a setting in which all students are in the same priority class with one another at all schools, and Biró (2008), Fleiner and Jankó (2014), and Biró and Kiselgof (2015) for many-to-one matching with weak priorities. Our paper subsumes the settings of all these papers, so our results directly apply to their environments.

A small but rapidly growing literature has recently analyzed allocation of daycare seats, one of the applications of the present paper. Motivated by the practice in Denmark, Kennes, Monte and Tumennasan (2014) study the issue of dynamic stability arising from the overlapping-generations structure of children’s composition in daycare centers. Veski et al. (2017) study the effect of changes in priorities in the context of kindergarten allocation practices in Estonia. In a more descriptive work, Herzog and Klein (2018) discuss a variety of policy issues in childcare systems in several German municipalities. While sharing the interest in application of matching theory to childcare, the overlap with our paper is rather tangential; none of these papers analyzes the problem of constraints, the primary focus of the present work. A recent paper by Okumura (2018) is motivated by the daycare seat assignment in Japan and considers the issue of flexible allocation across ages, although his solution concept does not require our fairness concept and none of his results implies ours, and vice versa.³

²Wu and Roth (2018) study fair matchings, so one might think they are more closely related to ours. We note, however, that their operator characterizes stable matchings while our operator characterizes fair matchings. Moreover, they use their operator *only on the set of fair matchings* to begin with, while we *obtain fair matchings* as fixed points of our operator. Given these differences, the relation between their approach and ours is tangential at best.

³See also Delacrétaz (2019).

Lastly, this paper is part of the growing literature on matching theory and market design. Ever since the seminal contribution by Gale and Shapley (1962), matching theory proved to be a source of fruitful insights. What is especially remarkable is its use in applications to market design. Research in this field has been successfully applied to various problems such as medical match (Roth, 1984; Roth and Peranson, 1999), school choice (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003), organ donation (Roth, Sönmez and Ünver, 2004, 2005, 2007), and course allocation (Sönmez and Ünver, 2010; Budish and Cantillon, 2012), among others.

The remainder of this paper proceeds as follows. Section 2 introduces our model. In Section 3, we provide a fixed-point characterization of matchings that satisfy feasibility, individual rationality, and fairness. Based on that result, Section 4 identifies a necessary and sufficient condition for the existence of an SOFM. Turning to numerical analysis, in Section 5 we conduct simulations to quantify the welfare gains of relaxing constraints based on real data on the allocation of slots at daycare centers. Then, Section 6 provides a number of discussions. We conclude in Section 7. All proofs that are not in the main text are found in the Appendix. Information on the datasets is found in Yamagata City (2017) and Bunkyo City (2017), and details on the codes for simulations are found in Kamada and Kojima (2021).

2. MODEL

Let there be a non-empty finite set of students I and a non-empty finite set of schools S . Each student i has a strict preference relation \succ_i over the set of schools and being unmatched (being unmatched is denoted by \emptyset). For any $s, s' \in S \cup \{\emptyset\}$, we write $s \succeq_i s'$ if and only if $s \succ_i s'$ or $s = s'$. Each school s has a strict priority order \succ_s over the set of students (note that we assume all schools regard all students as acceptable). For any $i, i' \in I$, we write $i \succeq_s i'$ if and only if $i \succ_s i'$ or $i = i'$. We denote by $\succ_I = (\succ_i)_{i \in I}$ the profile of all students' preferences, and by $\succ_S = (\succ_s)_{s \in S}$ the profile of all schools' priority orders. When there are three students i, i' , and i'' , for example, we write

$$\succ_s: i, i', i''$$

to mean that student i is of the highest priority, i' is of the second-highest priority, and i'' is of the lowest priority at s . School s is said to be **acceptable** to student i if $s \succ_i \emptyset$. We write, for example,

$$\succ_i: s, s'$$

to mean that school s is the most preferred, s' is the second most preferred, and s and s' are the only acceptable schools under preferences \succ_i of student i .

Each school s is subject to a *constraint*. A **constraint** at school s is a nonempty collection $\mathcal{F}_s \subseteq 2^I$ of sets of students. Denote $\mathcal{F}_S = (\mathcal{F}_s)_{s \in S}$. We say that a subset $I' \subseteq I$ is **feasible** at s if $I' \in \mathcal{F}_s$ and it is **infeasible** otherwise.

We refer to a tuple $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ as a **problem**.

A **matching** μ is a mapping that satisfies (i) $\mu_i \in S \cup \{\emptyset\}$ for all $i \in I$, (ii) $\mu_s \subseteq I$ for all $s \in S$, and (iii) for any $i \in I$ and $s \in S$, $\mu_i = s$ if and only if $i \in \mu_s$. That is, a matching simply specifies which student is assigned to which school (if any).

Let us define a few basic terms. First, a matching μ is **feasible** if $\mu_s \in \mathcal{F}_s$ for each $s \in S$. Second, a matching μ is **individually rational** if $\mu_i \succeq_i \emptyset$ for each $i \in I$. That is, no student is matched with an unacceptable school. Third, we say that i **has a justified envy toward** i' if there exists $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$. We say that a matching μ is **fair** if there exist no students i and i' such that i has a justified envy toward i' (see Remark 1 for a discussion). Fourth, a matching μ is **non-wasteful** if there is no pair $(i, s) \in I \times S$ such that $s \succ_i \mu_i$ and $\mu_s \cup \{i\}$ is feasible at s .⁴ Finally, a matching μ is said to be **stable** if it is feasible, individual rational, fair, and non-wasteful.

As we will see in Example 1, there may not exist a stable matching. For this reason, we will weaken our desiderata by dropping non-wastefulness, while still requiring fairness. The following concept will be of particular interest.

Definition 1. A matching μ is the **student-optimal fair matching (SOFM)** if (i) μ is feasible, individually rational, and fair, and (ii) $\mu_i \succeq_i \mu'_i$ for each $i \in I$ and every μ' that is feasible, individually rational, and fair.

Given (I, S, \mathcal{F}_S) , a **mechanism** φ is a function that maps preference profiles to matchings for each profile of priority orders. The matching under φ at students' preference profile \succ_I and priority profile \succ_S is denoted $\varphi^{\succ_S}(\succ_I)$, and student i 's match is denoted by $\varphi_i^{\succ_S}(\succ_I)$ for each $i \in I$.

In (I, S, \mathcal{F}_S) , a mechanism φ is said to be **strategy-proof** if there do not exist a profile of priority orders \succ_S , a profile of students' preferences \succ_I , a student $i \in I$, and preferences

⁴An alternative definition of non-wastefulness would require there be no pair $(i, s) \in I \times S$ such that $s \succ_i \mu_i$, $\mu_s \cup \{i\}$ is feasible at s , and $\mu_{\mu_i} \setminus \{i\}$ is feasible at μ_i if $\mu_i \neq \emptyset$. None of the results of this paper changes if we adopt this definition, so we do not consider it. Similarly, an alternative definition of fairness may require justified envy to satisfy feasibility of $\mu_{\mu_i} \setminus \{i\}$ at μ_i if $\mu_i \neq \emptyset$, but we do not consider such a definition.

\succ'_i of student i such that

$$\varphi_i^{\succ^S}(\succ'_i, \succ_{I \setminus \{i\}}) \succ_i \varphi_i^{\succ^S}(\succ_I).$$

An interpretation is that, for each realized preference profile for all students, each student simultaneously reports their preferences to the mechanism. Strategy-proofness requires that, in this game, reporting their true preferences is an optimal strategy for each student, irrespective of the reporting by the other students. In other words, no student has an incentive to misreport her preferences under the mechanism.⁵

3. FIXED-POINT CHARACTERIZATION OF FAIR MATCHINGS

This section studies matchings that satisfy the desirable properties introduced in the last section. More specifically, we characterize all the matchings that satisfy fairness, individual rationality, and feasibility by fixed points of a certain function on a finite (and hence a complete) lattice. We later use this result to study the existence and structure of matchings that satisfy these properties.

Our first observation is that there does not necessarily exist a stable matching.

Example 1 (Non-existence of a stable matching). Suppose that there are one school s and ten students i_1, i_2, \dots, i_{10} . Every student prefers to be matched to s rather than being unmatched. The school's priority is:

$$\succ_s: i_1, i_2, \dots, i_9, i_{10}.$$

Each student with an odd index costs the school 3 units of money while each student with an even index costs 4 units of money. School s is subject to a budget constraint of 20 in the sense that a set of students is feasible at s if and only if the sum of the costs associated with them does not exceed 20 units of money.⁶ Then, for example, the matching μ such that $\mu_s = \{i_1, i_2, i_3, i_4, i_5\}$ is fair but wasteful because $\mu_s \cup \{i_7\}$ is feasible and i_7 prefers to be matched to s rather than being unmatched. Meanwhile, the matching μ' such that $\mu'_s = \mu_s \cup \{i_7\}$ satisfies non-wastefulness but violates fairness because $s \succ_{i_6} \emptyset$, $i_7 \in \mu'_s$, and $i_6 \succ_s i_7$. In fact, there exists no stable matching in this example. To see this, first

⁵In this paper we do not consider incentive problems of the school side. This is because in the applications we have in mind, the priorities are exogenous to schools; for example, priorities are usually given by the local government in school choice and daycare allocation. Note also that Roth (1982) shows that there is no mechanism that produces a stable matching for all possible preference profiles and is strategy-proof for both students and schools even in a market in which all constraints are capacity constraints.

⁶This constraint is not the usual capacity constraint. It satisfies a restriction of “general upper-bound” that we introduce in Section 4.

note fairness requires that the set of students who are matched to s should be of the form $I^l := \bigcup_{k=1}^l \{i_k\}$ for some $l \in \{1, \dots, 10\}$, or $I^0 := \emptyset$. For any $l \leq 5$, the set I^l leads to wastefulness because $I^l \cup \{i_7\} \in \mathcal{F}_s$ and $s \succ_{i_7} \emptyset$. Meanwhile, for any $l \geq 6$, the set I^l is infeasible. \square

Remark 1. This non-existence example depends on our definition of fairness. Although our definition coincides with the standard one, in our general environment a matching can be deemed unfair even if there is no feasible way to satisfy students who have justified envy. In the above example, μ' is not fair even though i_6 cannot replace i_7 in a feasible manner ($(\mu' \cup \{i_6\}) \setminus \{i_7\}$ is infeasible at s). An alternative weaker definition would call μ' fair. Formally, we say that i **has a feasible justified envy toward** i' if there exists $s \in S$ such that (i) $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$ and (ii) $(\mu_s \cup \{i\}) \setminus \{i'\} \in \mathcal{F}_s$. We say that a matching μ is **weakly fair** if there exist no students i and i' such that i has a feasible justified envy toward i' . Note that the difference from the definition of fairness is the addition of condition (ii). Since we require a more stringent condition on the triples of the form (i, i', s) that cannot exist, the new notion is weaker than the original fairness concept.

Some readers have claimed that the fairness notion *must* be this weaker notion *as a matter of principle*. We, however, regard such a criticism as unjustified because we do offer real market examples in which our notion is practically relevant.

To elaborate, we first note that which definition is more reasonable depends on applications, just as most notions in economic theory. While there are cases in which the weaker notion may make more sense, there are a variety of markets in which our notion is more suitable. For example, universities around the world declare that they will not discriminate against students with disabilities in admissions.⁷ For instance, the University of Oxford's admissions website (University of Oxford, 2018) claims to “view applications from students with disabilities on the same grounds as those from other candidates, which are assessed purely on academic merit and potential.” At the same time, it is widely recognized that students with disabilities incur higher cost, which may result in a budget constraint similar to the one in Example 1. In this case, fairness requires that there be no situation in which a student is denied admission while a student with lower “academic merit and potential” is accepted, even if the former has disability and thus replacing the

⁷One might think disabilities represent only a minor fraction of the population, so any related design issue is unimportant in practice. We disagree with this view. For example, the U.S. Census Bureau reports that 17.6% of the U.S. population had a severe disability in 2014 (U.S. Census Bureau, Social Security Administration Supplement to the 2014 Panel of the Survey of Income and Program Participation).

latter student with the former violates the budget constraint. This appears to be the stated policy of universities such as Oxford. Similarly, in allocation of relief supplies in the wake of major earthquakes in Japan, organizers of some disaster shelters regarded fairness as so important that they refrained from assigning the resource to anyone when it is not feasible to satisfy everyone. Using a centralized mechanism in those kinds of markets is not even hypothetical. In fact, the nationwide college admission scheme in Hungary implements a matching which satisfies our fairness concept (Biró, 2008). We come back to these examples in Sections 4.1 and 6.1.

Even with an understanding of the above point, one might still regard our fairness notion as unreasonable in practice if it resulted in severe efficiency loss. For example, colleges such as Oxford would not commit to the non-discriminatory admission policy if they sometimes have to leave all slots empty to avoid justified envy. Although we acknowledge that there might also be real-life cases in which severe inefficiencies would be imposed, insisting on our fairness notion does not seem to cause major inefficiencies in our intended applications. In our simulations using real data from Japanese daycare matching, for instance, the mechanism that produces a fair matching (where the fairness is defined as ours) results in significant efficiency gains relative to the currently used mechanism. Furthermore, our fair mechanism turns out to be almost as efficient as a Pareto efficient mechanism, although the latter leaves many families with justified envy.

Lastly, the non-existence problem persists even if we weaken the fairness notion, and thus the problem is robust to the change of the fairness notion. Specifically, Appendix C.4 provides an example in which there exists no matching satisfying feasibility, individual rationality, non-wastefulness and weak fairness. \square

One might wonder if the non-existence as in Example 1 is far-fetched. We find that, quite the contrary, a stable matching does not exist in any of our 250 simulation runs using our daycare seat allocation data.⁸

Given the non-existence of a stable matching, our approach is to refrain from insisting on non-wastefulness while maintaining fairness as well as feasibility and individual rationality. One could consider an alternative approach in which non-wastefulness is required while fairness is not. As we discussed in the Introduction, which approach is more reasonable depends on applications. In the wide range of examples discussed in the Introduction, some waste is tolerated while fairness is considered primarily important.

⁸Since our data only specify part of the priority used in practice, we filled the gap by conducting 250 simulation runs.

Our approach is to characterize the matchings that satisfy fairness, individual rationality, and feasibility by fixed points of a function on a finite lattice. To establish this characterization, consider the space of cutoff profiles $P := \{1, \dots, |I|, |I| + 1\}^S$, endowed with a partial order \leq such that $p \leq p'$ if and only if p_s is weakly smaller than p'_s for all $s \in S$.⁹ This space is a finite (hence complete) lattice. For each school s , let $i^{(s,l)}$ be the student whose rank is the l^{th} from the bottom according to the priority of s (for example, the best student for s is $i^{(s,|I|)}$, and the worst is $i^{(s,1)}$). Also, consider a hypothetical student $i^* \notin I$ such that $i^* = i^{(s,|I|+1)}$ for all $s \in S$, and expand the domain of \succ_s for each $s \in S$ so that for any $i \in I$, $i^* \succ_s i$ holds. Given a cutoff profile $p \in P$, define the “demand” at each $s \in S$ as

$$D_s(p) := \{i \in I \mid i \succeq_s i^{(s,p_s)} \text{ and } s \succ_i \emptyset; i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'\}.$$

In this definition, the first part “ $i \succeq_s i^{(s,p_s)}$ and $s \succ_i \emptyset$ ” says that student i is as good as the cutoff student $i^{(s,p_s)}$ and finds s acceptable, while the second part “ $i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'$ ” says that s is the most preferred school among the ones at which i passes the cutoff.¹⁰ The demand for s is a collection of students who meet those two criteria.

We consider a mapping $T : P \rightarrow P$, called the **cutoff adjustment function**, defined as follows.

$$(3.1) \quad T_s(p) = \begin{cases} p_s + 1 & \text{if } D_s(p) \notin \mathcal{F}_s \\ p_s & \text{if } D_s(p) \in \mathcal{F}_s, \end{cases}$$

where we set $(|I| + 1) + 1 = 1$.¹¹ That is, for each s and cutoff profile, this mapping raises the cutoff at s by one if the demand for s is infeasible, and leaves the cutoff unchanged if it is feasible. This is reminiscent of the standard price-adjustment process in the general equilibrium theory except that the cutoff does not decrease even when s is “under-demanded.” For each $p \in P$, let μ^p be the matching such that

$$(3.2) \quad \mu_s^p = D_s(p) \text{ for each } s \in S.$$

That is, μ_s^p simply matches all students who demand s given cutoff profile p .

⁹The idea of using cutoffs per se is not novel. To our knowledge, Balinski and Sönmez (1999) is the first to use cutoffs for their study of fair matchings, and Azevedo and Leshno (2015) use cutoffs in their continuum population matching model. Two main differences of our approach from the literature are that we consider more general constraints than capacity constraints, and we provide a fixed-point characterization of fair matchings.

¹⁰In particular, if $p_s = |I| + 1$, then $i \succeq_s i^{(s,p_s)} = i^*$ (which is well defined as we expanded the domain of \succ_s) does not hold for any $i \in I$, so $D_s(p)$ is empty.

¹¹This definition ensures that the range of T is P .

Theorem 1. *If a cutoff profile $p \in P$ is a fixed point of the cutoff adjustment function T , then μ^p is feasible, individually rational, and fair. Moreover, if μ is a feasible, individually rational, and fair matching, then there exists a cutoff profile $p \in P$ with $\mu = \mu^p$ that is a fixed point of T .*

Proof. The first part: If p is a fixed point of T , then $T_s(p) = p_s$ must hold for all $s \in S$. The definition of T then implies that $D_s(p) \in \mathcal{F}_s$ must hold for each $s \in S$. Hence, feasibility follows. Individual rationality follows from the definition of $D_s(\cdot)$ for each $s \in S$.

To show fairness, assume for contradiction that μ^p is not fair. Then there exists a triple $(i, i', s) \in I^2 \times S$ such that $s \succ_i \mu_i^p$, $i' \in \mu_s^p$ and $i \succ_s i'$. Because $i' \in \mu_s^p$, by definition of $D_s(\cdot)$ and μ_s^p , it follows that $i' \preceq_s i^{(s, p_s)}$. Since $i \succ_s i'$, we have that $i \succ_s i^{(s, p_s)}$. This and the definition of $D_{\mu_i^p}(\cdot)$ imply $\mu_i^p \preceq_i s$, which is a contradiction.

The second part: Take a feasible, individually rational and fair matching μ . For each s , let $p_s = \min\{l | i^{(s, l)} \in \mu_s\}$ if $\mu_s \neq \emptyset$ (where the minimum exists because P is finite), and $p_s = |I| + 1$ otherwise. Then, individual rationality and fairness of μ and the definition of $D_s(\cdot)$ imply $\mu = \mu^p$. Also, feasibility of μ and the definition of T imply that $p = (p_s)_{s \in S}$ is a fixed point of T . \square

Roughly, the theorem characterizes the set of fair matchings (along with feasibility and individual rationality) as the set of fixed points of the mapping T . Note that the theorem does not impose any restriction on the form of constraints, besides non-emptiness. In particular, we do not assume that constraints are general upper-bounds, which is formally defined in the next section. We use this result in the next section to study existence of SOFM.

Remark 2. Recall that fair matchings may violate nonwastefulness. In fact, even the SOFM may feature some waste.¹² Given this fact, one may wonder if using fair matchings such as the SOFM is unappealing because of a potential blocking behavior.

We agree that existence of potential blocking behavior would be problematic if the government could not enforce a matching. Our paper considers, however, a situation in which the government has enforcement power. The reason that we still require axioms such as fairness despite government's ability to enforce a matching is that we view those axioms as normatively appealing, and not as positive requirement of preventing blocking behavior.

¹²In Example 1, matching μ is the SOFM and it violates nonwastefulness.

In our daycare allocation application discussed in the Introduction, for instance, the municipal government has control over all allocation decisions. In particular, even if an applicant wished that she could “block” a matching, she would need to formally apply to a regular matching process organized by the government together with all other applicants, so there would be no room for blocking as in labor markets. Other markets that our paper studies feature such enforcement power as well.

Another question one might ask may be whether the government can make any matching “fair” by setting particular priorities. To answer this question, we note that any matching is fair for *some* priority profile, but priorities are *not* choice variables in our setting. Instead, we focus on applications in which priorities are primitives and come from normative considerations.

In the daycare application, for instance, priorities are based on characteristics such as whether there is one or two parents, their health status, whether they have cohabiting parents who can take care of the child, and so forth. These priorities convey certain normative considerations about who should be “prioritized” to receive childcare. Again, the same remark applies to other markets that we study as well. \square

4. GENERAL UPPER-BOUND AND SOFM

This section studies the existence and structure of matchings that satisfy fairness, individual rationality, and feasibility. More specifically, we establish a tight relationship between the existence problem and the nature of the constraints. We show that the class of general upper-bounds characterizes the situations in which an SOFM is guaranteed to exist.

4.1. General Upper-Bound.

Definition 2. A constraint \mathcal{F}_s is a **general upper-bound** if $I' \in \mathcal{F}_s$ and $I'' \subseteq I'$ imply $I'' \in \mathcal{F}_s$.

That is, a constraint at school s is a general upper-bound if, for any set of students that is feasible at s , every subset of it is also feasible. Note that if a constraint \mathcal{F}_s is a general upper-bound then $\emptyset \in \mathcal{F}_s$ must hold. This is because non-emptiness of \mathcal{F}_s implies existence of some $I' \subseteq I$ such that $I' \in \mathcal{F}_s$, and we have $\emptyset \subseteq I'$.

A special case of a general upper-bound is a *capacity constraint*: a constraint \mathcal{F}_s is a **capacity constraint** if there exists an integer $q \in \mathbb{N}$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \leq q$. Let us provide real-life examples of constraints to discuss the applicability of general upper-bound as well as its limitation.

(1) **Diversity in school choice (type-specific quotas):** Many school districts require certain diversity of the student body at each school. A common way in the literature to formalize this requirement is to impose type-specific quotas (Abdulkadiroğlu and Sönmez, 2003).¹³ Specifically, we require a constraint \mathcal{F}_s to satisfy the following: There exist a partition of the students $I := \bigcup_{t \in \mathcal{T}} I_t$ with an index set \mathcal{T} (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and integers $q \in \mathbb{N}$ and $q_t \in \mathbb{N}$ for every $t \in \mathcal{T}$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \leq q$ and $|I' \cap I_t| \leq q_t$ for every $t \in \mathcal{T}$. Because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply $|I''| \leq |I'| \leq q$ and $|I'' \cap I_t| \leq |I' \cap I_t| \leq q_t$ for every $t \in \mathcal{T}$, this constraint is a general upper-bound.

(2) **College admissions with students with disabilities (budget constraints):** In college admissions, it is widely recognized that students with disabilities incur more cost to the university. For example, The University of Oxford’s admissions website mentioned in Remark 1 describes a variety of accommodation and financial assistance they offer to students with disabilities. One way to model this setup is through a budget constraint. Formally, assume that each student i is associated with cost $c_i \in \mathbb{R}_+$ and say that constraint \mathcal{F}_s is a budget constraint if there exists $b \in \mathbb{R}_+$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $\sum_{i \in I'} c_i \leq b$. College admissions with students with disabilities could be modeled as a situation in which students with disabilities have higher cost than those without disabilities. Budget constraint is not necessarily a capacity constraint as seen in Example 1, but it is clearly a general upper-bound because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply $\sum_{i \in I''} c_i \leq \sum_{i \in I'} c_i \leq b$.

Cases of budget constraints in college admissions may also arise in different contexts (Biró et al., 2010a; Abizada, 2016). For example, a wide variety of universities commit to so-called “need-blind admission,” where admission decisions are made without regard to applicants’ financial situations. Yet, a college may be subject to a budget constraint. Because different students may need different amounts of financial aid, the constraint cannot be described as a capacity constraint in general, but it is a budget constraint and hence a general upper-bound.

(3) **Refugee match (multidimensional constraints):** There are many refugees seeking resettlement across the world. As of the end of 2016, there were 22.5 million refugees worldwide (United Nations High Commissioner for Refugees, 2017), and the number is growing. Given such an impending situation, authorities in many

¹³A cap on the number of women in medical match (Roth, 1991) can also be represented by type-specific quotas.

countries are faced with the task of matching refugee families to different localities. One of the requirements in this problem is to match all members of a family to the same place. In addition, different refugee families need different kinds of services such as job training, language class, and so on. Given the available resources of a locality, variations in family size and needs for different types of resources imply that the constraint is not necessarily a capacity constraint. This type of constraint is called multidimensional constraints and studied by Delacrétaz, Kominers and Teytelboym (2016).¹⁴ Appendix C.1 formally analyzes this model and shows that, among other things, this constraint is a general upper-bound.

- (4) **Anti-bullying school choice design:** Bullying has a negative effect on the victim academically as well as mentally. It is a concern in many countries, and governments take measures to address this issue. Motivated by such a concern, Kasuya (2016) considers anti-bullying policies in the context of school choice design. Specifically, he analyzes a requirement that, no pair of a bully and his or her victim be placed in the same school. To express this formally as a constraint, let $B \subseteq I \times I$ be the set of bullying incidents, meaning that $(i, j) \in B$ implies that “ i has bullied j .” The constraint at each school s (ignoring other types of constraints such as the school’s capacity for simplicity) can be expressed as $\mathcal{F}_s = \{I' \subseteq I \mid (i, j) \in B \Rightarrow \{i, j\} \not\subseteq I'\}$. This is a general upper-bound because $I'' \subseteq I'$ and $\{i, j\} \not\subseteq I'$ imply $\{i, j\} \not\subseteq I''$, hence $I' \in \mathcal{F}_s$ implies $(i, j) \in B \Rightarrow \{i, j\} \not\subseteq I''$.
- (5) **Separating conflicting groups in refugee match:** Separating different types of individuals may be important not only in the bullying context, but also in other applications. For example, in refugee match, authorities are concerned that refugees from conflicting religious or ethnic groups may fight with each other in refugee shelters if they live close to each other. The policy to separate them by placing them in different locations is used or being considered as a temporary, if not permanent, solution (Breitenbach, 2015). To model this policy formally as a constraint, assume that there exists a partition of the students (refugee families in this context) $I := \bigcup_{t \in \mathcal{T}} I_t$ with an index set \mathcal{T} (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if there exists $t \in \mathcal{T}$ such that $I' \subseteq I_t$. This is a general upper-bound because $I'' \subseteq I'$ and $I' \in \mathcal{F}_s$ imply there exists $t \in \mathcal{T}$ such that $I'' \subseteq I' \subseteq I_t$.

¹⁴Andersson and Ehlers (2016) also study the problem of refugee matching, but the model is different from the ones studied by Delacrétaz, Kominers and Teytelboym (2016) or us.

- (6) **Daycare allocation:** In many countries around the world, assignment of daycare seats for small children is organized by the (local) government through a matching mechanism. There are often legal restrictions on a teacher-child ratio at each daycare center. Since such a ratio varies across different ages of children, the constraints implied by these regulations cannot be described by capacity constraints. In Section 5 we show that they can be described as general upper-bounds and study the data of daycare allocation in Japan, where finding a desirable allocation in the face of this non-capacity general upper-bound has become a pressing practical issue (Okumura, 2018).

Of course, not all constraints of interest are general upper-bounds. The following are some of the examples.

- (1) **Floor constraints:** Floor constraints may appear in a variety of environments. For example, in school choice, a school may need at least a certain number of students in order to organize group activities, have students interact with one another, or simply to operate efficiently enough given the fixed costs. Formally, the constraint \mathcal{F}_s at each school s (ignoring other types of constraints such as the capacity of the school) is that there exists an integer $q \geq 1$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $|I'| \geq q$ (Ehlers et al., 2011; Fragiadakis et al., 2012; Fragiadakis and Troyan, 2016). This is not a general upper-bound because any $I' \subseteq I$ with $|I'| \geq q$ is in \mathcal{F}_s but, for instance, \emptyset is a subset of I' but $|\emptyset| = 0 < q$. Similarly, a type-specific floor constraint, i.e., a constraint in which at least a certain number of students of specific types are needed for feasibility, is not a general upper-bound.
- (2) **Proportionality constraints:** As mentioned earlier, a common requirement in matching markets is to achieve certain balance of workforce or student body. Such a requirement is sometimes expressed in terms of proportion. For instance, in the public school district of Cambridge, Massachusetts in 2003, the proportion of students from low socioeconomic status families was required to be within a range of 15% of the district-wide proportion (Nguyen and Vohra, 2017). A constraint \mathcal{F}_s is a proportionality constraint (Nguyen and Vohra, 2017) if there exist a partition of the students $I := \bigcup_{t \in \mathcal{T}} I_t$ with an index set \mathcal{T} (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) and numbers $\alpha_t, \beta_t \in [0, 1]$ for every $t \in \mathcal{T}$ with $\alpha_t \leq \beta_t$ and $\sum_{t \in \mathcal{T}} \alpha_t \leq 1 \leq \sum_{t \in \mathcal{T}} \beta_t$ such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if $\alpha_t |I'| \leq |I' \cap I_t| \leq \beta_t |I'|$ for every $t \in \mathcal{T}$. This is not necessarily a general upper-bound because any $I' \subseteq I$ with $\alpha_t |I'| \leq |I' \cap I_t| \leq \beta_t |I'|$ for every $t \in \mathcal{T}$ is in \mathcal{F}_s but, for instance, $I'' := I' \cap I_t$ for

any $t \in \mathcal{T}$ is a subset of I' while $|I'' \cap I_t| = |I''| > \beta_t |I''|$ if $\beta_t < 1$ and $I'' \neq \emptyset$, so $I'' \notin \mathcal{F}_s$.

We do not necessarily assert that real-market constraints are general upper-bounds in all or even most applications. In fact, the two examples of constraints shown just above are demonstrably *not* general upper-bounds. Instead, our study aims to characterize the situations, in terms of restrictions on constraints, under which a solution with desirable properties exists. As shown in Section 4.2, it turns out that general upper-bound is necessary and sufficient for guaranteeing the existence of such a solution.

4.2. Existence of SOFM. Equipped with Theorem 1, our approach is to study the desirable matchings by way of analyzing the fixed points of the cutoff adjustment function T . Our first observation is, though, that under general upper-bounds, the existence of a matching satisfying fairness as well as feasibility and individual rationality is trivial; an empty matching, i.e., a matching with $\mu_i = \emptyset$ for each $i \in I$, satisfies all the properties above.¹⁵ Since an empty matching is typically highly inefficient, this example also suggests that only requiring the above three conditions may not lead to a desirable outcome for students. To the extent that we care about student outcomes, an interesting question is whether one can identify a fair matching that is desirable in terms of welfare. The next theorem answers this question in the affirmative.

Theorem 2. *If the constraint at each school is a general upper-bound, then there exists an SOFM.*

Proof. Fix a general upper-bound for each school s , \mathcal{F}_s .

Claim 1. *The mapping T has the smallest fixed point, i.e., there exists $p \in P$ such that $T(p) = p$ and $p \leq p'$ for all $p' \in P$ with $T(p') = p'$.*

Proof of Claim 1. Tarski's fixed point theorem implies that if a function from a finite lattice into itself, $f : X \rightarrow X$, is weakly increasing (i.e., for any $x, x' \in X$, $x \leq x'$ implies $f(x) \leq f(x')$), then the set of the fixed points of f is a finite lattice, and in particular it has the smallest fixed point. Letting $f = T$ and $X = P$, we only need to show that, for any $p, p' \in P$, $p \leq p'$ implies $T(p) \leq T(p')$.

To see that this holds, fix $s \in S$. We shall show $T_s(p) \leq T_s(p')$. Note first that the conclusion holds if $p'_s = |I| + 1$. This is because, as \mathcal{F}_s is a general upper-bound,

¹⁵In addition, it is straightforward to show that $\emptyset \in \mathcal{F}_s$ for each $s \in S$ is necessary and sufficient to guarantee existence of a matching that is feasible, individually rational, and fair (in a maximal-domain sense).

$D_s(p') = \emptyset \in \mathcal{F}_s$, so $T_s(p) \leq |I| + 1 = T_s(p')$. Thus, in the remainder, we suppose $p'_s \neq |I| + 1$. In this case, $T_s(p) \leq T_s(p')$ is immediate if $p_s < p'_s$ by the definition of $T(\cdot)$. Hence we consider the case with $p_s = p'_s$. Then $p_{-s} \leq p'_{-s}$ implies that

$$D_s(p) = D_s(p_s, p_{-s}) \subseteq D_s(p_s, p'_{-s}) = D_s(p'_s, p'_{-s}) = D_s(p')$$

by the definition of $D_s(\cdot)$.¹⁶ Since \mathcal{F}_s is a general upper-bound, if $D_s(p)$ is infeasible at s then $D_s(p')$ is infeasible at s , too. This implies that, whenever $T_s(p) = p_s + 1$, we have $T_s(p') = p'_s + 1 = p_s + 1 = T_s(p)$. Finally, if $T_s(p) = p_s$, it is immediate from the definition of $T(\cdot)$ and $p_s = p'_s$ that $T_s(p') \geq T_s(p)$.

Hence we have that $T_s(p) \leq T_s(p')$ for any $p, p' \in P$ with $p \leq p'$. Since this argument holds for every $s \in S$, we have $T(p) \leq T(p')$. \square

To complete the proof of the theorem, let p^* be the smallest fixed point of the function T , whose existence is guaranteed by Claim 1. By the first part of Theorem 1, μ^{p^*} is a feasible, individually rational, and fair matching. Take an arbitrary matching μ that is feasible, individually rational, and fair. By the second part of Theorem 1, there exists a fixed point p of T such that $\mu = \mu^p$. By the relation $p^* \leq p$ and the definition of $D_s(\cdot)$ for each $s \in S$, the equality $\mu = \mu^p$ implies that $\mu_i^{p^*} \succeq_i \mu_i$ for each student $i \in I$. Hence we have shown that μ^{p^*} is an SOFM, completing the proof. \square

Remark 3 (Implication of Theorem 2). As illustrated in Section 4.1, the class of general upper-bounds subsumes many applications such as school choice (Abdulkadiroğlu and Sönmez, 2003), college admission with students with disability or budget constraints (Biró et al., 2010a; Abizada, 2016), refugee match (Delacrétaz, Kominers and Teytelboym, 2016), bullying in schools (Kasuya, 2016), and daycare allocation (Okumura (2018) and this paper). Theorem 2 shows that an SOFM exists in all of those settings. The existence of SOFM is especially appealing given that, as we have seen in Example 1, a stable matching does not always exist under general upper-bounds. In fact, Theorem 4 in Section 6.2 shows that the existence is guaranteed if *and only if* the constraints are capacity constraints, a condition violated by all the examples discussed here. \square

The proof of Theorem 2 is based on Tarski's fixed point theorem. This proof method implies that the set of fair matchings (along with feasibility and individual rationality) has a lattice structure under general upper-bounds. It also implies that a simple algorithm can be used to find the SOFM under such constraints, building on the cutoff adjustment function T . Consider the following algorithm, called the **cutoff adjustment algorithm**:

¹⁶The set inclusion relation follows because the property $i \succeq_{s'} i^{(s', p_{s'})}$ that appears in the definition of $D_s(\cdot)$ is implied by $i \succeq_{s'} i^{(s', p'_{s'})}$. All the equalities are straightforward.

- Step 0: Let $p^0 := (1, 1, \dots, 1)$.
- Step $t \geq 1$: Let $p^t = T(p^{t-1})$. If $p^t = p^{t-1}$, terminate the algorithm and define the outcome as the matching μ^{p^t} .¹⁷ Otherwise, go to step $t + 1$.

The cutoff adjustment algorithm is well-defined (i.e., it terminates in a finite number of steps). To see this, first note that since $p \leq T(p)$ for any $p \in P$ by the definition of T and the fact that $\emptyset \in \mathcal{F}_s$ for each $s \in S$ under general upper-bound, we have $p^{t-1} \leq p^t$ for every positive integer t . Because P is a finite set, $p^{t^*-1} = p^{t^*}$ for some finite t^* .¹⁸

To see that this algorithm produces the SOFM, note that $p^* := p^{t^*-1} = p^{t^*}$ satisfies $p^* = T(p^*)$, i.e., p^* is a fixed point of T . Moreover, for any fixed point p' of T , we have $p^0 \leq p'$ by definition of p^0 , so $p^* = T^{t^*}(p^0) \leq T^{t^*}(p') = p'$, implying that p^* is the smallest fixed point of T . Since the matching corresponding to the smallest fixed point of T is the SOFM (see the proof of Theorem 2 for details), the outcome produced by this algorithm is the SOFM. Thus, we have established the following result:

Proposition 1. *Suppose that the constraint at each school is a general upper-bound. Then, the outcome of the cutoff adjustment algorithm is the SOFM.*¹⁹

Under capacity constraints, a standard way to find a stable matching is the deferred acceptance algorithm. In Section 6.4, we present a generalization of the deferred acceptance algorithm which finds the SOFM under a general upper-bound and compare it with the cutoff adjustment algorithm.

As mentioned earlier, the class of general upper-bounds subsumes many practical cases, but there are some constraints that are not general upper-bounds, such as floor constraints. A question of interest, then, is whether the conclusion of Theorem 2 holds without the assumption of general upper-bound. The following result offers a sense in which the answer to this question is negative.

Theorem 3. *Fix a set of students I , a set of schools S with $|S| \geq 2$ and their priorities \succ_S , and a school $s \in S$ and its constraint \mathcal{F}_s . Suppose \mathcal{F}_s is not a general upper-bound.*²⁰

¹⁷For each $p \in P$, $T(p)$ and μ^p are as defined in Equations (3.1) and (3.2), respectively.

¹⁸For the termination of the cutoff adjustment algorithm, the constraint need not be a general upper-bound but only has to satisfy $\emptyset \in \mathcal{F}_s$.

¹⁹The same conclusion can be obtained, following essentially the same proof, under a slight modification of the algorithm in which the cutoffs are adjusted at one school at a time.

²⁰Note that we are not explicitly making an assumption about the number of students, although the proof uses the fact that there are at least two students when the empty set is feasible. Such a bound obtains because the existence of a school whose constraint is not a general upper-bound while the empty set is feasible implies that there are at least two students.

Then there exist student preferences \succ_I and a profile \mathcal{F}_{-s} of capacity constraints such that an SOFM does not exist in the problem $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$.

This result shows that the class of general upper-bounds is a “maximal domain,” providing a sense in which Theorem 2 cannot be generalized further. In other words, general upper-bound is the most permissive restriction on constraints imposed on individual schools (as long as the capacity constraints are included) which guarantees the existence of an SOFM. Even if an SOFM is a desirable outcome, the policy maker cannot expect to always find an SOFM unless the constraints of each school is a general upper-bound. This finding may shed light on the non-existence and impossibility results in the literature. Settings with those negative results include floor constraints (Biro et al., 2010b; Ehlers et al., 2011; Fragiadakis et al., 2012; Fragiadakis and Troyan, 2016) and proportionality constraints (Nguyen and Vohra, 2017). These constraints are not general upper-bounds as pointed out in Section 4.1.

Remark 4. One may wonder why we consider SOFM instead of Pareto-undominated fair matchings. To provide a formal argument, let us define a few pieces of terminology. Given \succ_I , we say that a matching μ **Pareto-dominates** μ' if $\mu_i \succeq_i \mu'_i$ for all $i \in I$ and $\mu_i \succ_i \mu'_i$ for some $i \in I$. We call a matching μ **a Pareto-undominated fair matching** if (i) μ is fair, individually rational and feasible, and (ii) there is no matching that is fair, individually rational and feasible and Pareto-dominates μ . Note that the SOFM, when it exists, is a unique Pareto-undominated fair matching. Moreover, if there exists a unique Pareto-undominated fair matching, then it is the SOFM. In contrast to the SOFM, however, a Pareto-undominated fair matching exists whenever the set of fair, individually rational and feasible matchings is nonempty.

To answer the above question, we note that Theorems 2 and 3 can be interpreted in the context of Pareto-undominated fair matchings under general constraints. To begin, observe that when an SOFM does not exist, there are inevitably multiple Pareto-undominated fair matchings, so there is a tradeoff in that some students prefer one Pareto-undominated fair matching to another while others prefer the latter to the former. In applications such a tradeoff is often problematic as it may entail conflicts between different stakeholders. Our results identify all the cases in which an SOFM is guaranteed to exist, which in turn are exactly the cases in which the aforementioned tradeoff is guaranteed not to exist. In this sense, our analysis provides understanding of Pareto-undominated fair matchings in a broader context beyond merely general upper-bounds. \square

5. ALLOCATION OF DAYCARE CENTER SLOTS

This section describes the problem of allocating slots at daycare centers, “daycare allocation,” as an application of our theory. Then we conduct numerical analysis based on unique datasets on daycare allocation we obtained to study the numerical impact of allowing flexibility in constraints.

5.1. Background and Institutional Detail. Child care services are often highly subsidized and regulated in many countries. Some Scandinavian countries such as Sweden regard daycare service as parents’ right and guarantee a seat in some, albeit not necessarily the parents’ first choices, daycare center. At the other end of the spectrum are countries such as the United States, in which a typical daycare center is run by a private provider and financed by tuitions from parents, though with regulations. Some countries such as Japan lie between these extremes: daycare services are highly subsidized and regulated, but at the same time a slot at a daycare is not guaranteed for parents. In many of those countries, the supply of slots at daycares is very limited, leaving a large number of applicants unmatched and thus making allocation of slots at daycares an especially big concern for parents.

Japan provides a good example of the large excess demand for daycare seats. Recently, a blog post titled “Didn’t Get a Slot in Day Care. Drop Dead, Japan!!!” by an anonymous mother became viral, leading to a large protest movement. The blog post made its way into debates in the National Diet, forcing prime minister Abe to respond to the complaint (Osaki, 2016), with him later promoting various policies to increase child care capacity by more than 300,000 in 5 years (Prime Minister’s Office, 2017). The above blog title was chosen as one of the top ten “New Words and Buzzwords” of 2016, although the organizers admitted to uneasiness about selecting what is essentially a swear word (Kikuchi, 2016). In the midst of such a heated political environment, increasing the number of allocated slots at daycares in a fair manner is now among the very top agendas for many politicians; For instance, mayors of all ten most populous municipalities list daycare policy as one of their major political agendas. As we illustrate shortly, our theory may prove useful for daycare allocation problems with excess demand. For this purpose, we describe daycare allocation in Japan in some detail below.

In Japan, daycare centers serve children aged between 0 and 5 as of the beginning of a new academic year, and the assignment of seats in daycare centers is under the authority of each municipality.²¹ A number of features of this market make it an unusually good

²¹In this paper, we focus on *Ninka Hoiku En*, which could be translated into accredited daycare centers. There are also daycare centers that are not accredited. Since non-accredited centers are more expensive

subject of application of matching theory. First, daycares are heavily subsidized, with the tuitions and fees set by each municipal government at a low level, leading to excess demand in many municipalities.²² Given the excess demand, municipalities ration slots in daycare centers based on an algorithm. In fact, parents literally report ordinal preferences over daycare centers.²³ Moreover, while in principle a child could start attending a daycare at any time of the year, almost all seats are allocated for the beginning of the academic year, which is April in Japan. Thus, unlike in some other matching problems such as organ allocation, it appears to be a good approximation to treat each year’s allocation as a single static problem, just as is assumed in the present paper (as well as in most matching theory research in school choice and labor markets). Another aspect of daycare allocation in Japan is that, faced with high demand for daycare seats and resulting rationing, many municipalities follow formal assignment rules (the most popular are versions of serial dictatorship and the “Boston” mechanism).

For our purpose, three aspects of this market are especially important. First, the assignment rules are based on reported preferences as well as priorities, where the latter are determined by applicant characteristics such as whether parents have full-time jobs and whether the parent is a single parent, among others. Second, the national regulation requires the teacher-child ratio be at least one teacher for every three children of age 0 while the ratios are one teacher for every 6 children for ages 1 and 2, 20 children for age 3, and 30 children for ages 4 and 5 (Cabinet Office of Japan, 2017).²⁴ While such a constraint is clearly not the traditional capacity constraint as per-capita resources needed to care

(they are not subsidized by the central government) and considered to be of lower quality on average (Asai, Kambayashi and Yamaguchi, 2015), they are uncommon; of the children who are in daycare centers, as much as 93.5% go to accredited ones as of March 2016 (Ministry of Health, Labour and Welfare, 2017). Therefore, we focus on accredited daycare centers.

²²The average monthly fee per child is 20,491 Japanese yen (about 200 U.S. dollars) as of 2012 (Ministry of Health, Labour and Welfare, 2014). Subsequently, the government made daycare services free of charge for certain types of families.

²³The application form for Yamagata City (Yamagata City, 2019a), for instance, offers space for listing a ranking of at most 5 desired daycare centers as a default, but emphasizes that the applicant may list as many centers as she wants. All the municipalities that we are aware of provide application forms that are similar in that they ask a ranking over the applicants’ desired daycare centers, although the number of centers that can be listed vary across municipalities.

²⁴The national regulation also imposes certain age-dependent space-child ratio requirements, and there are other exceptions and variations in rules. However, we ignore these in our analysis. While municipalities are allowed to place more stringent regulations than those imposed by the national government, many municipalities including Yamagata, on which our simulations in Section 5.3 are based, follow the national regulation on teacher-child ratios. We conducted simulations under specifications including both

for a child varies across the child’s age, it is a general upper-bound and hence our theory implies an SOFM exists in this problem. Third, even though the true constraint itself is not a capacity constraint, most municipalities treat the number of seats at each daycare center for each age as fixed and non-transferable across different ages, thus effectively setting an artificial capacity constraint.

One might wonder if teachers specialize in certain ages and cannot be moved across different age groups. In Japan, however, the license to work as a daycare teacher is not associated with any specific age groups. Furthermore, as illustrated in the next paragraph, municipalities indeed treat teachers as transferrable across different age groups.

One might also think that implementing non-capacity constraints is unrealistic from the administrative point of view. Quite the contrary, faced with great pressure to improve daycare allocation, some municipalities have been experimenting with making the allocation “more flexible” by relaxing the artificial capacity constraint in one way or another. Specifically, they have introduced policies to have daycare centers with excess supply of seats for certain ages admit more children of ages with excess demand, although in ad hoc manners.²⁵ Such policies could be quite effective if designed and implemented well. In fact, due to a variety of institutional features, it turns out that even highly demanded daycare centers often have some vacancies for older children while having excess demand for younger children. For example, Suzuki (2018) points out that about 190,000 seats (across all ages and including both accredited and non-accredited daycares) were vacant in Japan as of 2017, and attributes a significant part of this figure to the excess supply of seats for older children.

To proceed formally, let there be a set of ages \mathcal{T} and for each $t \in \mathcal{T}$, let there be a teacher-child ratio r_t . The interpretation is that if one teacher can watch up to n children of age t , then $r_t = 1/n$. A constraint \mathcal{F}_s is a **daycare constraint** if there are a number m_s (representing the number of teachers) and a partition of the students (children in this context) $I := \bigcup_{t \in \mathcal{T}} I_t$ (with $I_t \cap I_{t'} = \emptyset$ if $t \neq t'$) such that, for any $I' \subseteq I$, $I' \in \mathcal{F}_s$ if and only if

$$(5.1) \quad \sum_{t \in \mathcal{T}} r_t \cdot |I' \cap I_t| \leq m_s.$$

teacher-child ratio and space-child ratio and verified that the main conclusions we report in this draft are robust.

²⁵Municipalities with such policies include populous cities such as Yokohama, Kawasaki, Saitama, and Sendai. See Okumura (2018) for details.

Because $I'' \subseteq I'$ implies $|I'' \cap I_t| \leq |I' \cap I_t|$ for every $t \in \mathcal{T}$, this constraint is a general upper-bound. In our personal communication, a city official verified that this inequality is the correct way to interpret the legal requirement.

Despite the fact that the constraint is as described in inequality (5.1), municipalities often operationalize it by having their assignment based on a more restrictive constraint. We say that a constraint \mathcal{F}'_s is a **rigid constraint** associated with \mathcal{F}_s if there exists a number q^t for each age $t \in \mathcal{T}$ such that $\sum_{t \in \mathcal{T}} r_t q^t \leq m_s$, where for any $I' \subseteq I$, $I' \in \mathcal{F}'_s$ if and only if $|I' \cap I_t| \leq q^t$ for every $t \in \mathcal{T}$. Note that the latter is a special case of the constraint of type-specific quotas discussed in Section 4.1, with an additional property that the capacity for the number of students in the entire school s is not binding.²⁶

5.2. Theoretical Preparation. We are interested in the impact of applying SOFM in practice. In preparation for numerical analysis, this section provides theoretical results (which we will later quantify with data).

We first analyze the effect of allowing flexibility in seat allocation across different ages. To isolate the effect of flexibility, for this exercise we fix the studied mechanism to be the SOFM. The following observation is of interest even though it is straightforward to show:

Proposition 2. *Fix a set of students I , a set of schools S , student preferences \succ_I , and school priorities \succ_S . Let $\mathcal{F}_S := (\mathcal{F}_s)_{s \in S}$ and $\mathcal{F}'_S := (\mathcal{F}'_s)_{s \in S}$ be profiles of general upper-bounds such that $\mathcal{F}_s \subseteq \mathcal{F}'_s$ for every $s \in S$, and μ and μ' be the SOFM in the problems $(I, S, \succ_I, \succ_S, \mathcal{F}_S)$ and $(I, S, \succ_I, \succ_S, \mathcal{F}'_S)$, respectively. Then, $\mu'_i \succeq_i \mu_i$ for every $i \in I$, and $\mu_i \in S$ implies $\mu'_i \in S$.*

In words, the first part of this proposition shows that relaxing constraints leads to a Pareto improvement of student outcomes. The second part shows that, in particular, the set of unmatched students becomes weakly smaller in the set inclusion sense as the constraints are relaxed. This proposition, while being a very simple (perhaps trivial) observation, may be of interest in practice. This shows that a policy maker may be able to improve the welfare of the market participants if she relaxes the constraints, say by devoting more resources or simply by lifting some regulations. While this seems intuitive and hardly surprising, it is worth noting that it is a *Pareto improvement* for students. In other words, relaxing constraints does not create any distributional tradeoffs and makes *every* student weakly better off.

²⁶Under the notations in Section 4.1, this corresponds to setting $q \geq \sum_{t \in \mathcal{T}} q_t$.

To put this proposition into perspective, let us mention a comparative statics result due to Konishi and Ünver (2006).²⁷ The latter result states that, under capacity constraints, when the capacity of each school increases, every student becomes weakly better off at the student-optimal stable matching. Because the SOFM reduces to the student-optimal stable matching under capacity constraints (see Section 6.5 for a formal definition and analysis), Proposition 2 is a generalization of the result of Konishi and Ünver (2006) to the cases in which constraints are not necessarily capacity constraints.

To apply Proposition 2 to our daycare allocation problem, we compare the SOFM under the daycare constraint (henceforth “flexible SOFM”) and the SOFM under an arbitrary rigid constraint associated with that daycare constraint (henceforth “rigid SOFM”). Since the relaxation from “rigid” constraints to “flexible” ones satisfies the hypothesis of Proposition 2 in the context of daycare allocation, we obtain the following corollary.

Corollary 1. *In the daycare allocation problem, fix a set of children I , a set of daycares S , child preferences \succ_I , and daycare priorities \succ_S . Then, every child is weakly better off under the flexible SOFM than under the rigid SOFM. The set of children who are unmatched at the flexible SOFM is a subset of those at the rigid SOFM.*

Another theoretical issue we study involves an efficiency property of SOFM. Due to the insistence on fairness, one may wonder if the SOFM results in severe inefficiency. We next address this concern by providing upper bounds on a measure of inefficiency.

Formally, given \succ_I , let \mathcal{M}^{IR} be the set of individually rational matchings and, for each matching μ , let $|\mu| = \sum_{s \in S} |\mu_s|$. We define the **waste** of a feasible matching $\mu \in \mathcal{M}^{\text{IR}}$ under (\succ_I, \mathcal{F}_S) by

$$(5.2) \quad \max_{\mu' \in \mathcal{M}^{\text{IR}}} \{|\mu'| : \mu'_s \in \mathcal{F}_s \text{ and } \mu_s \subseteq \mu'_s \text{ for all } s \in S\} - |\mu|.$$

In words, the waste of a matching is the maximum number of additional students who can be matched to schools without displacing any existing students or violating feasibility or individual rationality.

Let $\bar{W}(\mathcal{F}_S) = \sum_{s \in S} \bar{w}_s(\mathcal{F}_s)$, where for each $s \in S$,

$$\bar{w}_s(\mathcal{F}_s) := \max_{I', I'' \subseteq I, j \in I} \{ |I''| : I' \cup \{j\} \notin \mathcal{F}_s, I' \cup I'' \in \mathcal{F}_s, I' \cap I'' = \emptyset \},$$

when the set on the right hand side is nonempty, and 0 otherwise. To illustrate this notion, recall that we allow for general constraints beyond capacity constraints. Thus, it is possible that one student cannot be feasibly matched to the school s in addition to

²⁷See also an earlier result by Crawford (1991) who considers comparative statics with respect to adding an agent.

existing students I' while some other n students can. The measure \bar{w}_s is the maximum of n across all I' . The measure \bar{W} is the sum of \bar{w}_s for all schools. The following proposition demonstrates that the function \bar{W} provides an upper bound of the waste.

Proposition 3. *Assume that \mathcal{F}_s is a general upper-bound for each $s \in S$ and let μ be the SOFM at $(\succ_I, \succ_S, \mathcal{F}_S)$. The waste of μ under (\succ_I, \mathcal{F}_S) is at most $\bar{W}(\mathcal{F}_S)$.*

We note that there is a sense in which the bound in Proposition 3 is tight. To see this, consider a problem with one school s . For any general upper-bound constraint \mathcal{F}_s , there exist a priority and a preference profile such that the waste of SOFM is exactly equal to the bound $\bar{w}_s(\mathcal{F}_s)$.²⁸

In the daycare matching application, it is theoretically possible—in the worst case—to have large waste under the SOFM mechanism.²⁹ This worst case, however, is based on a particular preference and priority structure such that, at each school s , a 0-year-old child is the highest in the priority among those who are rejected, and there is a set \tilde{I}_s of nine (or more) 4- or 5-year-old children who are also rejected, where \tilde{I}_s and $\tilde{I}_{s'}$ are disjoint if $s \neq s'$.

Such a preference and priority structure strikes us as highly unlikely, and thus we present an upper bound for the waste for each given preference and priority structure.³⁰ Formally, let $\hat{W}(\succ_I, \succ_S, \mathcal{F}_S) := \sum_{s \in S} \hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S)$, where for each $s \in S$,

$$(5.3) \quad \hat{w}_s(\succ_I, \succ_S, \mathcal{F}_S) := \max_{\bar{I} \subseteq I} \{|\bar{I}| : \text{For the SOFM } \mu \text{ at } (\succ_I, \succ_S, \mathcal{F}_S), \mu_s \cup \bar{I} \in \mathcal{F}_s, \mu_i = \emptyset, s \succ_i \emptyset \text{ for all } i \in \bar{I}\}.$$

To explain, \hat{w}_s is the highest number of unmatched students at the SOFM who can be profitably added to school s without violating the feasibility constraint. The measure \hat{W} is the sum of \hat{w}_s for all schools. The next proposition establishes that \hat{W} provides an upper bound of the waste and finds its relation with the other bound \bar{W} .

Proposition 4. *Assume that \mathcal{F}_s is a general upper-bound for each $s \in S$.*

- (1) *Let μ be the SOFM at $(\succ_I, \succ_S, \mathcal{F}_S)$. The waste of μ under (\succ_I, \mathcal{F}_S) is at most $\hat{W}(\succ_I, \succ_S, \mathcal{F}_S)$.*

²⁸Specifically, fix (arbitrarily) I' , I'' , and j that achieve the maximum in the definition of $\bar{w}_s(\mathcal{F}_s)$; The priority of s ranks all students in I' highest, j next, then I'' , and finally all other students; All students prefer s to \emptyset . One can verify that the waste of the SOFM under \mathcal{F}_s is $\bar{w}_s(\mathcal{F}_s)$.

²⁹In the simulations we explain later, this upper bound of the waste is 657 where there are 1437 applicants.

³⁰With this information, it is in principle possible to count the waste itself, but it turns out to be computationally difficult. The bound we present here, in contrast, is easier to compute.

$$(2) \hat{W}(\succ_I, \succ_S, \mathcal{F}_S) \leq \bar{W}(\mathcal{F}_S).$$

The first part implies that \hat{W} serves as an upper bound for the waste under the SOFM. To obtain the intuition for this result, we note that the only difference between the waste at the SOFM and \hat{W} is that the latter may “count” the same unmatched student multiple times if she regards multiple schools with vacancy as acceptable. The second part compares the two bounds, where we show the bound \bar{W} is looser than \hat{W} . This is because \bar{W} does not use any information about priority or preferences. We note that Proposition 3 is a corollary of these two results.

While Corollary 1 is unambiguous about the direction of welfare change for students when the constraints change, it is silent about the magnitude of the welfare improvement. Similarly, Proposition 3 does not provide the actual waste in a given application, merely offering its bounds. Proposition 4 refines the bound, but whether that bound is large or small in applications is not clear from the general expression (5.3). For better quantification, we next study real-life data of daycare allocation.

5.3. Data and Simulations. We use administrative data on daycare seat allocation obtained from Yamagata City and Bunkyo City in Japan. Yamagata City is the prefectural capital of Yamagata prefecture in the northeastern part of Japan, with about 250,000 residents as of 2018. Bunkyo City is one of the 23 special districts of Tokyo. It has about 230,000 residents as of 2018 and has a population density 30 times larger than that of Yamagata City. Compared with Yamagata, Bunkyo City is much more urban, has a higher concentration of educational institutions, and attracts many more dual-income families investing heavily in education and demanding childcare, which seems to make its daycare allocation problem more pressing. Despite such a difference, it turned out that the results of the simulations for these two municipalities resemble each other. For this reason, we only detail the result from Yamagata City here, while providing the results for Bunkyo City in the Appendix B.

Our data involve applicants (who are anonymized), usually parents, representing children who would begin attending the daycare in April of 2018. There were 1437 applicants aged between 0 to 5 as of April 1, 2018 on which they would begin attending the daycare. There were 442, 469, 195, 211, 63, and 57 children in ages 0, 1, 2, 3, 4, and 5, respectively. For each applicant, the data show her reported preferences over the daycare centers and priority ranking (the priorities are common across daycare centers). The average length of the reported preference ranking is 4.52. The priority order is based on the applicant characteristics such as parents’ job status and the number of adults available for care at home. In fact, Yamagata City (2019b) discloses the explicit formula that converts

relevant characteristics of each family to the (common) priority ranking. There are 93 daycare centers in our dataset. For each daycare center, the data show how many seats are supplied for each age (under the rigid constraint).³¹ The total number of seats supplied is 1682. Finally, the dataset contains the matching produced by Yamagata’s mechanism, which we call the “actual allocation.”

Regarding reported preferences, we note that the mechanism in Yamagata is based on serial dictatorship (with no restriction on the number of daycares that can be listed).³² Strategy-proofness of serial dictatorship appears to be fairly well understood in practice, which gives some justification for treating reported preferences as true preferences. A popular how-to book for parents (Habu, 2016), for example, compares strategic properties of serial dictatorship with those of Boston mechanism as follows (these two mechanisms are the most popular mechanisms for daycare allocation in Japan). “In the first mechanism [serial dictatorship], ... there is no advantage or disadvantage associated with your stated first choice, while in the second mechanism [Boston mechanism], ... if you list a competitive daycare center as your first choice, the probability that you will be admitted by no daycare center can increase.”

In our simulation, we made several modeling choices given data limitations. First, the data we have involve ties although the actual priority order is strict. This is because our data lack information on some characteristics used by Yamagata to determine the strict order, such as whether the child is currently in an alternative form of childcare and whether the family has a member with disability. Given the absence of further information, we randomly break ties using a single tie-breaking rule according to the uniform distribution; Note that priorities are common across daycare centers in practice in Yamagata, and our single tie-breaking rule replicates such a property. For each mechanism that we consider, we conducted 250 runs of simulations, with each run corresponding to a realization of the tie-broken priorities.

³¹At some daycare centers in the data, the number of allocated children for a given age under the actual allocation exceeded the supply of seats reported to us. According to the officials at Yamagata City, such instances are due to new supplies of capacities that arose after the disclosed data were compiled. For such cases, we used the number of seats actually allocated as the capacity for the given age.

³²The mechanism is slightly different from pure serial dictatorship, i.e., there are a few special rules, mainly regarding children with siblings. In our numerical analysis, however, this difference causes only a minor difference between the assignments from pure serial dictatorship and the actual one. Indeed, a webpage of Yamagata City (Yamagata City, 2019b) explains their mechanism to parents as a pure serial dictatorship.

The second limitation involves constraints. For daycare centers, our dataset does not show the entire collection of feasible sets of children or the number of teachers corresponding to the flexible constraints. Instead, it only shows the number of advertised seats at each daycare center for each age, which is exactly enough to specify the rigid constraints. To overcome this limitation, we define m_s for each s in the daycare constraints (Equation (5.1)) by $m_s := \sum_{t \in \mathcal{T}} r_t \cdot q^t$, where r_t and q^t are those in the data (recall that r_t is the teacher-child ratio under the national regulation, and q^t is the number of advertised seats for age t at daycare center s). That is, m_s is the smallest possible number of teachers such that the constraint implied by the number of advertised seats in data is a rigid constraint associated with our daycare constraint.³³

Third, we assume that our data of reported preferences are true preferences, and that the same preferences would be reported in all mechanisms. These assumptions are justified by the fact that all the mechanisms we study in this section are strategy-proof. In particular, as mentioned earlier, the reported preferences in our data are based on a version of serial dictatorship whose strategy-proofness is well understood. In addition, while the SOFM mechanism is not strategy-proof in general, in Section 6.3 we observe that it is strategy-proof under common priority across daycare centers (a feature in our dataset). We note that the recent literature provides empirical evidence that agents may misreport their preferences even under strategy-proof mechanisms.³⁴ Our results are valid up to the assumption that families truthfully report their preferences.

We find that the effect of allowing flexibility in constraints under SOFM is substantial in our data: the average number of children who are matched with a strictly preferred daycare center in the flexible SOFM compared to the rigid SOFM is 867.27, which amounts to 60.35% of all applicants (Table 1).³⁵ By contrast, no applicant is made worse off, as implied by Proposition 2. The number of children who are unallocated changes from 713.79 to 88.02, a 87.67% decrease (Figure 1). The average numbers of children who are matched to their first choice, first two choices, and the first three choices increase

³³We set m_s as the bare minimum that is consistent with the data on advertised seats so that we do not overstate our estimate of the gains from removing the rigid constraint. In a similar spirit, we allow for non-integral values of m_s although the number of teachers is an integer in practice. With an alternative specification setting m_s to be the integer rounded up from our present definition, for instance, our estimate of the gains from removing rigid constraints would be larger (see Proposition 2).

³⁴Papers in this literature includes Rees-Jones (2017, 2018); Fack, Grenet and He (2019); Hassidim, Romm and Shorrer (2019); Artemov, Che and He (2019).

³⁵This table as well as others also report simulations of other mechanisms we discuss below.

From/To	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
rigid SOFM	—	867.27 (60.35%) (SE= 5.66)	658.46 (45.82%) (SE= 5.63)	881.94 (61.37%) (SE= 5.60)
flexible SOFM	0	—	72.13 (5.02%) (SE= 2.94)	49.78 (3.46%) (SE= 3.97)
actual allocation	13.19 (0.92%) (SE= 1.91)	237.94 (16.56%) (SE= 2.40)	—	248.68 (17.31%) (SE= 2.63)
flexible ETSD	0	0	62.88 (4.38%) (SE= 2.40)	—

TABLE 1. The number of applicants who are made strictly better off by a change of a mechanism. For each mechanism that we consider, we conducted 250 runs of simulations, with each run corresponding to a realization of the tie-broken priorities. The percentage is out of all the 1437 applicants. The “SE” stands for the standard error of the raw number.

by 124.04%, 75.84%, and 59.29%, respectively (Figure 2).³⁶ Our analysis suggests that utilizing the flexible nature of the constraints can be substantial in some environments.

Next, we compare the rigid and flexible SOFMs with Yamagata’s actual assignment. Yamagata’s mechanism is based on what we call **rigid (justified) envy-tolerating serial dictatorship (rigid ETSD)**. Rigid ETSD runs serial dictatorship, treating the problem for each age as separate from others. That is, for each age, it runs serial dictatorship for children of that age and the number of seats committed to that age in advance. This means that, among other things, there may remain justified envy between two children i and i' if they are of different ages, while by construction there is no justified envy between children of the same age.³⁷ Yamagata’s assignment is expected to have some efficiency advantage over the rigid SOFM since justified envy is tolerated across different ages. Meanwhile, the comparison between Yamagata’s assignment and the flexible SOFM is theoretically indeterminate. This is because Yamagata’s assignment is based on the rigid constraint, which may or may not overwhelm the efficiency gains from tolerating justified envy across different ages.

³⁶If an applicant lists k daycare centers in her reported preferences and gets unassigned to any of them, then we list her as being assigned to her $(k + 1)$ st choice.

³⁷This feature makes rigid ETSD similar in spirit to a mechanism proposed by Okumura (2018) in that both mechanisms require elimination of justified envy between children of the same age only. The same comment applies to the “flexible ETSD” discussed below.

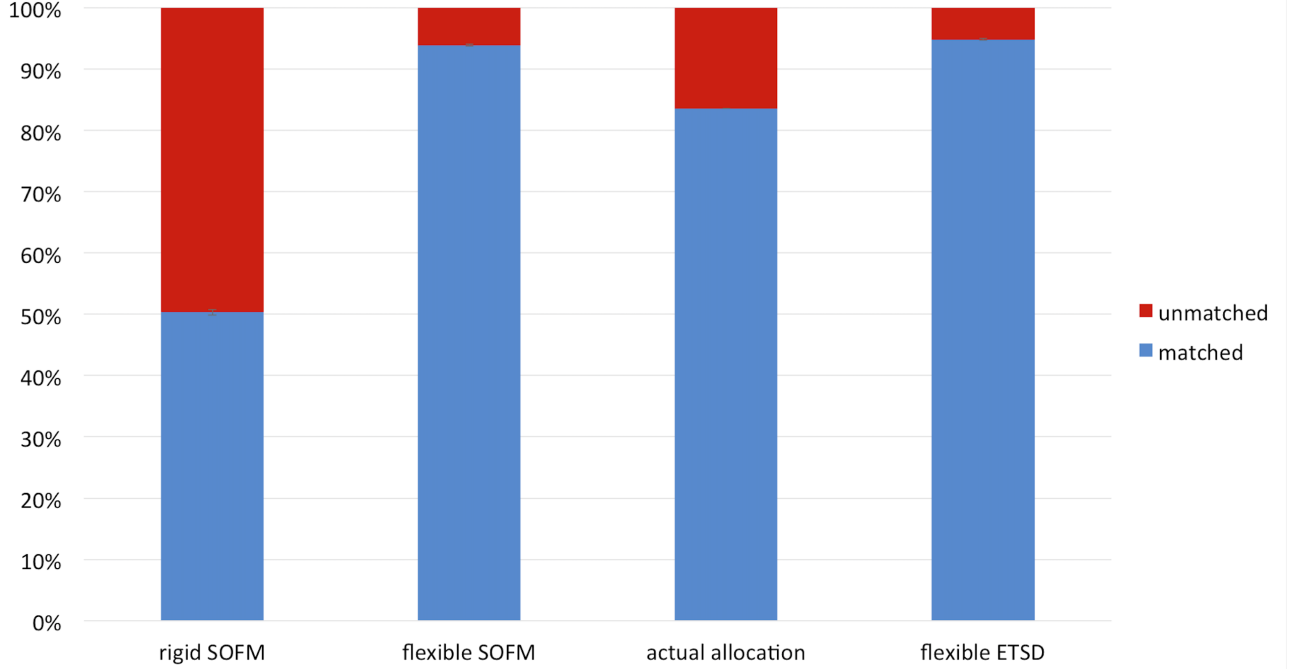


FIGURE 1. The fractions of matched applicants under different mechanisms. The error bars represent the standard errors.

Remark 5. One may question the relevance of justified envy across different age groups. However, parents regard the existence of such justified envy as problematic. An example of complaints is that it is more difficult for a 1 year old child to get into a daycare than a 0 year old, even if the former has a higher priority. When we had a meeting with city officials, they also expressed the same concern. \square

Our simulations reveal that the flexible SOFM outperforms Yamagata’s assignment not only in terms of fairness but also in terms of efficiency. Regarding efficiency, all of our efficiency measures favor the flexible SOFM; The average fraction of unmatched children decreased by 62.71%, and 16.56% of children are matched with strictly preferred daycare under the flexible SOFM while only 5.02% are matched with strictly preferred daycare under the actual allocation. Turning our focus to fairness, Table 2 provides several measures of justified envy for Yamagata’s assignment (note that all measures of justified envy are zero for the rigid and flexible SOFM). There are 989 pairs (i, s) such that i has a justified envy toward someone matched to s under the actual allocations, which amounts to 15.24% of all pairs (i, s) such that s is acceptable to i . Also, students involved in at least one of such pairs and daycares involved are 33.05% and 66.67% of the respective total numbers. The amount of justified envy for Yamagata’s actual assignment seems

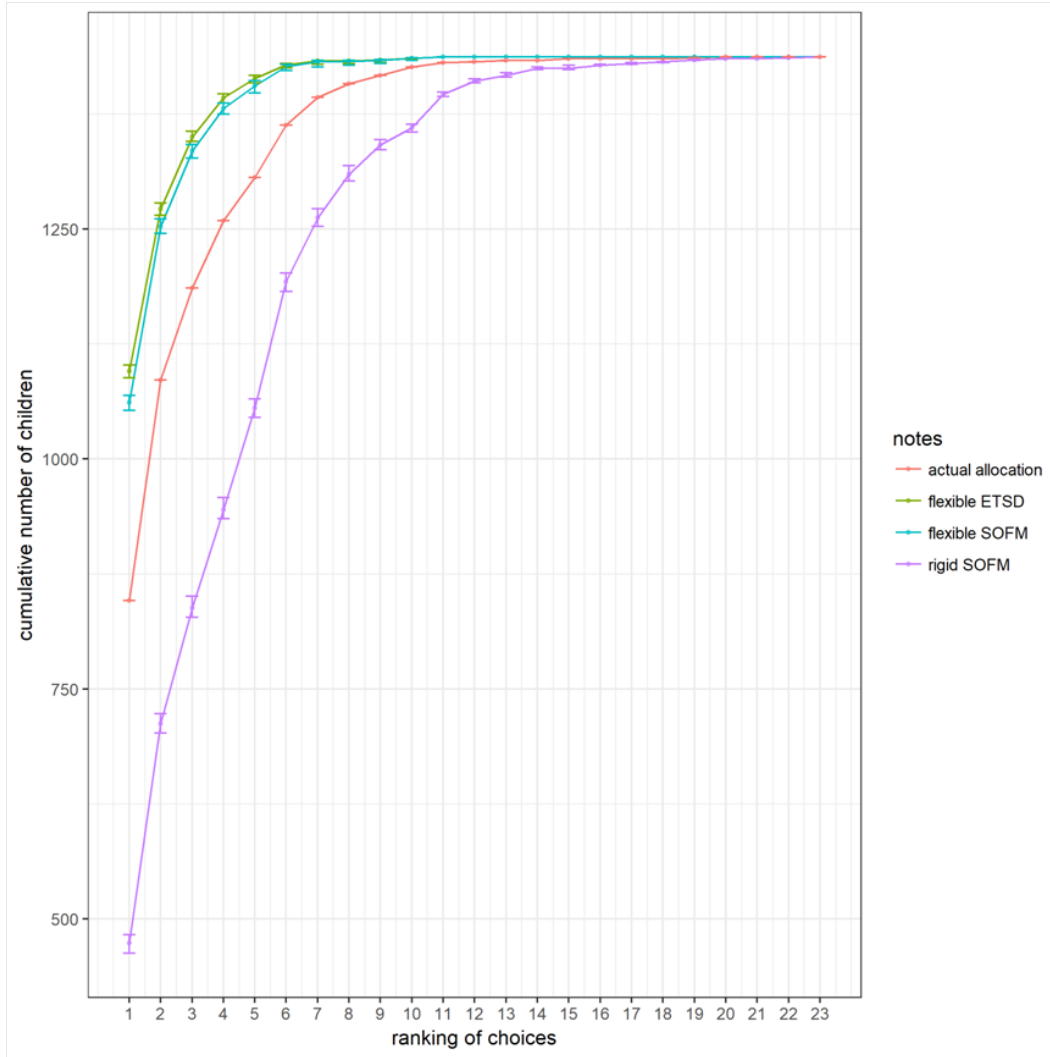


FIGURE 2. Rank distributions under different mechanisms: The graph reports the average cumulative number of children at each rank, as well as its range (shown by the error bars) across all 250 simulation runs.

broadly comparable to those in TTC on Boston and New Orleans data (Abdulkadiroglu et al., 2017).

Another natural question is to see what happens in serial dictatorship if the rigid constraint is removed so that it is only subject to the daycare constraint. In the induced mechanism, called **flexible (justified) envy-tolerating serial dictatorship (flexible ETSD)**, the current dictator receives her most preferred daycare such that adding her to the current match does not lead to infeasibility.

Since some justified envy is tolerated while the constraint is flexible in this mechanism, its efficiency is expected to be even higher than both Yamagata’s actual assignment and

	rigid SOFM	flexible SOFM	actual allocation	flexible ETSD
pairs with justified envy	0	0	989 (15.24%)	157.19 (2.42%) (SE= 10.84)
students with justified envy	0	0	475 (33.05%)	129.96 (9.04%) (SE= 6.71)
daycares with justified envy	0	0	62 (66.67%)	22.16 (23.83%) (SE= 1.64)

TABLE 2. Measures of justified envy under different mechanisms. The percentages for pairs with justified envy divide the numbers of pairs with justified envy by the numbers of pairs (i, s) such that s is acceptable to i . The “SE” stands for the standard error of the raw number.

the flexible SOFM (in fact, it is easy to verify that the flexible ETSD is Pareto efficient for applicants and that it does not allow for any waste). Perhaps surprisingly, however, the magnitude of the improvement of this mechanism over the flexible SOFM seems rather small; the average number of unmatched children decreases only by 13.83 (15.72%) and the average number of children who become strictly better off under the flexible ETSD is 49.78 (3.46%). This difference is much smaller than the improvement of the flexible SOFM over Yamagata’s assignment, whose corresponding numbers are 174.98 (62.71%) and 237.94 (16.56%), respectively. Moreover, the average of the upper bounds of the waste under the flexible SOFM, $\hat{W}(\succ_I, \succ_S, \mathcal{F}_S)$, turned out to be 24.5 slots. Hence, the waste under the flexible SOFM does not seem to be significant. Meanwhile, a significant amount of justified envy still remains under flexible ETSD, although the numbers are less than under Yamagata’s actual assignment. These numbers suggest that the flexible SOFM may be a potentially useful mechanism in daycare allocation.

One question of interest is whether the numerical patterns we find for Yamagata are generalizable outside this specific case. To answer this question, we conduct the same set of simulations based on the data we obtained from Bunkyo City, one of the 23 special districts of Tokyo. It is much more urban than Yamagata and also has many educational institutions across children’s ages including the University of Tokyo. It also features much severer shortage of daycare seats; almost half of the applicants are unassigned in the actual allocation. Despite these differences, Appendix B reports that our simulations on

Bunkyo’s data finds efficiency gains from removing rigid constraints in daycare assignments comparable to those found in Yamagata.³⁸

Summary. The key findings from our simulations are as follows:

- (1) Relaxing the constraint from the rigid constraint to the daycare constraint results in a significant Pareto improvement.
- (2) Although insisting on our fairness notion may in principle cause waste, it is small in data. Moreover, the SOFM under the daycare constraint is almost as efficient as a Pareto-efficient alternative, the flexible ETSD. Meanwhile, the latter mechanism leaves a significant number of parents with justified envy toward others while our mechanism completely eliminates justified envy.
- (3) SOFM under the daycare constraint outperforms the actual allocation in efficiency, while the latter leaves a significant number of parents with justified envy.
- (4) The findings are not specific to Yamagata, and are confirmed by data in another city as well.

6. DISCUSSION

This section provides discussions. Section 6.1 extends our analysis to the cases with weak priority. In Section 6.2, we show that capacity constraints are necessary and sufficient for guaranteeing the existence of a stable matching in the “maximal domain” sense. Section 6.3 addresses strategic issues under general upper-bounds. Section 6.4 provides an alternative algorithm that finds the SOFM. In Section 6.5, we examine the difference between the SOFM and the student-optimal stable matching. Appendix C.1 compares our model with a model of multidimensional constraints. In Appendix C.2, we provide a sense in which our results cannot be obtained from the existing results in the “matching with contracts” model.

6.1. Weak Priority Orders. As we have argued, there are practical problems in which fairness is so important that even some inefficiencies are tolerated. This is perhaps most vividly seen in the context of natural disasters, where the organizer of a disaster shelter needs to allocate the relief supplies such as food. For example, in the wake of the Great Hanshin Earthquake in 1995 which killed more than 6400 people in Japan, the organizers of a disaster shelter who had 150 lunch boxes and two boxes of apples refrained from

³⁸ Bunkyo’s mechanism is based on the serial dictatorship. We note, however, that it restricts applicants to list at most five daycare centers in their ranking, so care is warranted when interpreting the results.

allocating them because they were not sufficient to allocate to everyone. In another shelter, the government instructed not to allocate relief supplies until there were enough to allocate to everyone in the shelter.³⁹ Even without such an instruction by the government, the organizers of yet another shelter made the same rule on their own (Hayashi, 2003).

Similar cases are repeatedly reported during natural disasters. For example, in the aftermath of Tohoku earthquake of 2011 which killed more than 15800 people in Japan, the organizers of a disaster shelter in Fukushima refrained from allocating relief supplies, and they attributed their decision to fairness concerns, saying they “were worried about conflicts among disaster victims” (Town of Tomioka, 2015).

Remark 6. One might suspect that cases like the ones described above are outliers. It turns out, however, those are not isolated cases. In fact, Board of Education of Hyogo Prefecture (1996) reports the decisions of multiple disaster shelters to tolerate waste in favor of fairness were made under the direction of the government at the time of Great Hanshin Earthquake in 1995. Furthermore, as detailed later, there are large-scale allocation problems in practice in which waste is tolerated in favor of fairness under weak priority such as the nationwide college admission in Hungary. \square

One of the key features of these problems is that priorities are weak. For instance, in the first of the aforementioned shelters in the Great Hanshin Earthquake, priority is given to the elderly and children before everyone else, but there were a large number of people in the same priority class (there were more than 1000 people in total). In the third shelter, there were only four priority classes (for instance, individuals whose houses were destroyed by the earthquake are given higher priority than those whose houses were not and who only need relief supplies).

Motivated by these real cases, in this subsection we study fair matching under weak priorities. To do so, we generalize the model of Section 2 by assuming each school s has a weak priority order over the set of students. Specifically, the weak priority \succeq_s is not required to have the property that $i \succeq_s i'$ and $i' \succeq_s i$ imply $i = i'$. In this context, we say that i has a justified envy toward i' if there exists $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succeq_s i'$. Note that this condition reduces to the earlier definition when priorities are strict. Fairness and stability are analogously defined, while all other concepts are unchanged.

The following example shows a stable matching fails to exist even if the constraints are capacity constraints.

³⁹Cases reported here are from Board of Education of Hyogo Prefecture (1996).

Example 2 (Non-existence of a stable matching under weak priorities). There is a disaster shelter which is endowed with 150 lunch boxes. In this shelter, there are three groups of individuals: 70 children, 70 elderly individuals, and 70 adults. Priorities are weak; children have the highest priority, elderly individuals have the next priority, and adults have the lowest priority, and individuals in the same group have the same priority as one another. Every individual can consume at most one lunch box and prefers to receive a lunch box than not.

In this problem, a stable matching does not exist. To see this, first note that non-wastefulness implies all lunch boxes are allocated. This fact and fairness imply that the only candidate for a stable matching assigns the lunch boxes to all children and elderly individuals while assigning the remaining 10 to adults. This matching, however, is not fair because an adult without a lunch box has justified envy toward an adult with a lunch box. \square

The above example shows that a stable matching based on fairness defined in this subsection does not necessarily exist, even if all schools have capacity constraints.⁴⁰ In fact, it is possible to show that strictness of the priorities is “necessary” in a maximal-domain sense. This suggests that requiring stability is too demanding.

The good news is that there exists an SOFM even under weak priorities if the constraint of each school is a general upper-bound. We can show this result by largely following the proof of Theorem 2, though with some care that we detail shortly: As in the proof of Theorem 2, consider the space of cutoff profiles $P = \{1, \dots, |I|, |I| + 1\}^S$. For each $s \in S$, assign indices $1, 2, \dots, |I|$ to all the students, one index for each student, with the restrictions that (i) each index is assigned to exactly one student and (ii) if $i \succ_s i'$, then i has a higher index than i' . For each school s , let $i^{(s,l)}$ be the student whose index is l .⁴¹ Intuitively, the indices represent a tie-breaking of the given weak priority, where 1 represents the lowest and $|I|$ the highest ranks among all the students in I breaking the ties.

With this modification, the rest of the proof for Theorem 2 needs little change. In particular, as in the original proof, we can show that if a cutoff profile p is a fixed point of

⁴⁰The existing literature often uses an alternative (weaker) notion of fairness. Specifically, it requires there be no $i, i' \in I$ and $s \in S$ such that $s \succ_i \mu_i$, $i' \in \mu_s$ and $i \succ_s i'$ (instead of $i \succeq_s i'$). Under this definition of fairness, it is well-known that a stable matching exists. In applications we have in mind such as allocation of disaster relief supplies, however, our notion of fairness is more appropriate. In fact, as we discuss below, the actual Hungarian college admissions use our stronger notion of fairness.

⁴¹As in the proof of Theorem 2, we also consider a hypothetical student $i^* \notin I$ such that $i^* = i^{(s,|I|+1)}$ for all $s \in S$.

T , then matching μ^p such that $\mu_s^p = D_s(p)$ for each $s \in S$ is fair. One might wonder why such a proof works; we “break ties” when defining the indices of students while fairness requires there be no envy even between students with the same rank *according to the original weak priority*.⁴²

To see the reason, recall the definition of the demand function $D_s(\cdot)$,

$$D_s(p) := \{i \in I \mid i \succeq_s i^{(s,p_s)} \text{ and } s \succ_i \emptyset; i \succeq_{s'} i^{(s',p_{s'})} \Rightarrow s \succeq_i s'\}.$$

Here, the key is that all the relationships in $D_s(\cdot)$ involving school priorities are specified with respect to the original weak priority, not the one after breaking ties. This allows that, for any given cutoff profile p , if multiple students have the same rank at a school, then either all of them can demand that school or none of them can, ensuring there be no envy among them.

For concreteness, consider an example with three students i_1 , i_2 , and i_3 and one school s , where i_3 has the highest priority while i_1 and i_2 are tied and below i_3 . Assume that s is acceptable to all students. Then $D_s(p)$ reduces to $D_s(p) := \{i \in I \mid i \succeq_s i^{(s,p_s)}\}$. The indexing procedure would assign either 1 or 2 to i_1 , the other index to i_2 , and 3 to i_3 . Suppose we assign 1, 2, and 3 to i_1, i_2 , and i_3 , respectively. In constructing $D_s(p)$, the only subtle case is when $p_s = 2$.⁴³ In that case, one might think that $D_s(p) = \{i_2, i_3\}$ holds because i_2 and i_3 have indices no smaller than 2 while i_1 's index is 1, and it might interfere with fairness because i_1 would have a justified envy toward i_2 . Fortunately, this concern is unfounded because $i^{(s,2)} = i_2$ and i_1 and i_2 have the same priority at s , so $i_1 \succeq_s i^{(s,2)}$ holds, and thus $D_s(p) = \{i_1, i_2, i_3\}$. The point is that even though the index of i_1 is strictly lower than that of i_2 , that is only for convenience, and our construction of the demand $D_s(p)$ relies on the original weak priority, guaranteeing fairness of the induced matching.

Finally, let us mention that, although fairness concerns may be the most salient in allocation during disasters such as the major earthquakes discussed earlier, there are other examples in which fairness issues are present and weak priority orders are involved. For example, Hungarian college admissions are conducted by a central clearinghouse and a fair matching is produced, where the fairness notion follows our definition (so Hungarian college admissions tolerate waste in practice). In their system, priorities at each college

⁴²Such a concern would not arise if fairness did not require nonexistence of envy toward a student with the same rank, as defined in e.g., Abdulkadiroğlu and Sönmez (2003) and Abdulkadiroğlu, Pathak and Roth (2009).

⁴³Just as in the case with strict priority, we have $i \succeq_s i^{(s,1)}$ for $i = i_1, i_2, i_3$ and $i \succeq_s i^{(s,3)}$ only for $i = i_3$, implying $D_s(p) = \{i_1, i_2, i_3\}$ if $p_s = 1$ and $D_s(p) = \{i_3\}$ if $p_s = 3$.

are based on test scores, and the applicants with the same score are ranked equally (Biró, 2008). Chilean college admissions share these features, although the mechanism in use differs from the one in Hungary (Ríos et al., 2014). Similarly, fairness is explicitly taken into account in Turkish Navy, where officers in charge of logistical support are instructed to distribute resources homogeneously across units (Kesten and Yazici, 2012). Our analysis may be applied to each of these applications in practice.

6.2. Nonexistence of a Stable Matching. In Section 4 we showed that there is an environment in which no stable matching exists under a general upper-bound. In this subsection we provide a theorem clarifying the extent to which this impossibility result holds.

Theorem 4. *Fix a set of students I , a set of schools S , and a school $s \in S$ and its constraint \mathcal{F}_s . Suppose that \mathcal{F}_s is not a capacity constraint while being a general upper-bound. Then there exist a school priority \succ_s and student preferences \succ_I such that, for any constraint profile \mathcal{F}_{-s} and priority profile \succ_{-s} , there exists no stable matching in the problem $(I, S, \succ_I, \succ_s, \mathcal{F}_S)$.*

Since a stable matching is guaranteed to exist if each school has a capacity constraint (Gale and Shapley, 1962; Roth and Sotomayor, 1990), Theorem 4 provides a tight characterization for the existence of a stable matching. Roughly, capacity constraints are the maximal domain for the existence of a stable matching, that is, a stable matching is guaranteed to exist if and only if each school has a capacity constraint. This result provides a justification for market designers faced with non-capacity constraints to seek a solution that may be unstable. The SOFM may be an appealing alternative because it always exists under general upper-bound and, while possibly unstable, it is most preferred by every student among all matchings that are fair, individually rational and feasible.

One might wonder why this impossibility result does not contradict the existence of a stable matching with substitutable preferences (e.g., responsive preferences with type-specific quota) which may not be associated with any capacity constraint. One major difference is that our definition of stability requires that there exist no justified envy even if satisfying such justified envy is infeasible, which fits the applications we have in mind (see Remark 1 for a discussion of different fairness notions).

6.3. Strategic Issue. Assume that the constraint at each school is a general upper-bound. Consider a mechanism, called **the SOFM mechanism** which, for every input, produces the SOFM; of course this mechanism is well-defined by Theorem 2 because we assume each school's constraint is a general upper-bound. One question of interest is

whether the SOFM mechanism is strategy-proof. To answer this question, we begin by noting that the SOFM mechanism is strategy-proof if the constraint of each school is a capacity constraint. This is because the SOFM and the student-optimal stable matching coincide under capacity constraints (Balinski and Sonmez 1999), and the mechanism that produces the student-optimal stable matching is strategy-proof in that setting (Roth 1982, Dubins and Freedman 1981).

Given this observation, the remaining question is whether strategy-proofness holds more generally. The following result offers a sense in which such a generalization is impossible.

Theorem 5. *Fix a set of students I , a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint \mathcal{F}_s . Suppose that \mathcal{F}_s is not a capacity constraint while being a general upper-bound. Then there exists a profile \mathcal{F}_{-s} of capacity constraints such that the SOFM mechanism is not strategy-proof in (I, S, \mathcal{F}_S) .*

Theorem 5 only relaxes \mathcal{F}_s while keeping \mathcal{F}_{-s} to be capacity constraints, and hence this is a maximal domain result. An implication is that the SOFM mechanism is no longer guaranteed to be strategy-proof once we go beyond the class of capacity constraints. Given this impossibility, one question of interest is whether any other reasonable mechanism satisfies strategy-proofness along with fairness, individual rationality, and feasibility. Of course, the mechanism that always produces an empty matching is strategy-proof and satisfies the other three properties, but that mechanism is hardly justifiable as it is extremely inefficient. Thus, the relevant question is whether there is a mechanism satisfying strategy-proofness and the other desiderata, while also satisfying at least some desirable efficiency property.

As it turns out, the lack of strategy-proofness is not the drawback of the SOFM mechanism alone, but is shared by a broad class of mechanisms that satisfy a very mild efficiency property. To state this finding formally, we say that mechanism φ satisfies **unanimity** if, for any \succ_I and \succ_S , if a matching μ such that μ_i is the most preferred outcome for every $i \in I$ at \succ_i is feasible, then $\varphi^{\succ_S}(\succ_I) = \mu$. In words, unanimity requires that if a matching in which every student is matched to her first choice is feasible, then the mechanism should produce that matching. This is arguably a very mild requirement and is satisfied by the SOFM mechanism and many other mechanisms. It turns out that this mild requirement is incompatible with strategy-proofness and other properties, as stated in the following generalization of Theorem 5.

Theorem 6. *Fix a set of students I , a set of schools S with $|S| \geq 2$, and a school $s \in S$ and its constraint \mathcal{F}_s . Suppose that \mathcal{F}_s is not a capacity constraint while being a*

general upper-bound. Then there exists a profile \mathcal{F}_{-s} of capacity constraints such that there exists no mechanism that satisfies feasibility, fairness, unanimity, and strategy-proofness in (I, S, \mathcal{F}_S) .⁴⁴

This result shows that the lack of strategy-proofness is not a deficiency specific to the SOFM mechanism. Rather, this theorem establishes that there is a more fundamental incompatibility between strategy-proofness and other requirements once we go beyond the restrictive domain of capacity constraints.

Although the lack of strategy-proofness is inherent in the SOFM mechanism and even other mechanisms satisfying fairness, it does not necessarily imply that misreporting happens in practice.

To see this, first suppose that the priority over the students is common across all schools. In that case, one can show that the SOFM outcome can be produced by a variant of a serial dictatorship with respect to the common priority. Based on this observation, it is straightforward to show that the SOFM under common priority is strategy-proof. While common priority is fairly restrictive, it is satisfied in some practical applications such as daycare allocation in Japan (see Section 5).

Second, even if the priority is not common across schools, it may be difficult for students to precisely identify the case in which her misreporting is profitable, let alone what particular preferences to use for misreporting. We provide two such senses.

The first sense is that a particularly simple and canonical class of misreporting is never profitable under the SOFM mechanism. Formally, we say a preference relation \succ'_i is a **truncation** of preference relation \succ_i if $s \succ'_i s'$ if and only if $s \succ_i s'$ for every $s, s' \in S$, and $s \succ'_i \emptyset$ implies $s \succ_i \emptyset$ for every $s \in S$. In (I, S, \mathcal{F}_S) , a mechanism φ is said to be **truncation-proof** if there do not exist a profile of priority orders \succ_S , a profile of students' preferences \succ_I , a student $i \in I$, and (reported) preference relation \succ'_i of student i that is a truncation of \succ_i such that

$$\varphi_i^{\succ_S}(\succ'_i, \succ_{I \setminus \{i\}}) \succ_i \varphi_i^{\succ_S}(\succ_I).$$

Truncations are simple and commonly studied in the literature on matching (see e.g., Roth and Vande Vate (1991)).

⁴⁴This result is a generalization of Theorem 5 because the SOFM mechanism satisfies feasibility, fairness and unanimity. We do not require individual rationality in Theorem 6 because it turns out it is not necessary for establishing the result. Given that this is an impossibility result, not requiring a condition makes the claim stronger.

Proposition 5. *Suppose that the constraint at each school is a general upper-bound. Then, the SOFM mechanism is truncation-proof.*

We note that an analogous result does not hold for more general constraints. More specifically, in Appendix C.3 we consider an example in which school constraints are not general upper-bounds. In that example, we show there is no truncation-proof mechanism that always produces a Pareto-undominated fair matching.

The second sense in which manipulations may be difficult can be seen through the cutoff adjustment algorithm which by Proposition 1 implements the SOFM mechanism.

Proposition 6. *Fix a set of students I , a set of schools S , general upper-bound constraints $(\mathcal{F}_s)_{s \in S}$, student preferences \succ_I , school priorities \succ_S , and a misreported preference \succ'_i of a student $i \in I$. Let p and p' be the cutoff profiles produced at the end of the cutoff adjustment algorithm under \succ_I and $(\succ'_i, \succ_{I \setminus \{i\}})$, respectively. If i is matched to s under $(\succ'_i, \succ_{I \setminus \{i\}})$ and i prefers s to the outcome under \succ_I , then $s \in S$ and $p_s > p'_s$.*

That is, if a student's manipulation leads her to match with a more preferred school, then that school's cutoff has to be strictly lower under the misreported preferences. Straightforward as this observation may be, it helps identify cases in which strategic manipulation is not profitable. For instance, in large markets, the cutoff of a school is determined by the *highest-priority applicant* who is rejected from it, and this depends on the entire distribution of students' reported preferences as well as school priorities. Thus, it appears unlikely that any one particular student is in a position to influence the cutoff in any significant manner.

Based on the above discussion, we aim to prove that the SOFM mechanism has an approximate incentive compatibility property in large markets. Specifically, we consider a model with tiers of students (see e.g., Kesten (2009) and Che and Tercieux (2019) for matching models with tiers). The tier structure is a generalization of the priority structure in the Japanese daycare allocation problem for which we conducted simulations (Section 5), and represents cases in which schools' priorities may be different but have certain commonality.

Formally, we consider a *sequence of problems* (Π^1, Π^2, \dots) as follows. The n^{th} problem is denoted $\Pi^n = (I^n, S^n, \succ_{I^n}, \succ_{S^n}, \mathcal{F}_{S^n})$, and there are constants $L, L' < \infty$ such that, for each $n \in \mathbb{N}$, the following hold.

- The number of students is n , i.e., $|I^n| = n$.
- The number of schools is bounded, i.e., $|S^n| < L$.

- The set of students I^n is partitioned into K^n “tiers,” i.e., $\bigcup_{k=1}^{K^n} T_k^n = I^n$, $T_k^n \cap T_{k'}^n = \emptyset$ for all k and k' with $k \neq k'$.
- For each $s \in S^n$, $i \succ_s i'$ holds if $i \in T_k^n$ and $i' \in T_{k+1}^n$ for any k .
- The size of each tier is bounded, i.e., $|T_k^n| < L'$ for each k .
- The constraint \mathcal{F}_s is a general upper-bound for each $s \in S^n$.

Let the SOFM mechanism be denoted by ϕ . Given a problem $\Pi = (I, S, \succ_I, \succ_S, \mathcal{F}_S)$, let

$$D(\Pi) = \{i \in I \mid \exists \succ'_i \text{ s.t. } \phi_i^{\succ_S}(\succ'_i, \succ_{I \setminus \{i\}}) \succ_i \phi_i^{\succ_S}(\succ_I)\}$$

be the set of students who have incentives to misreport their preferences.

Proposition 7. *In any sequence of problems (Π^1, Π^2, \dots) , $\lim_{n \rightarrow \infty} \frac{|D(\Pi^n)|}{|I^n|}$ exists and is equal to 0.*

That is, the fraction of students who have a profitable misreporting strategy tends to zero in the limit as the market becomes large. The result is rather strong for at least two reasons. First, the definition of $D(\Pi)$ is permissive in the sense that the mere existence of a profitable misreporting strategy is sufficient to count a student as having an incentive for misreporting, irrespective of, for example, the number of such misreporting strategies or difficulty of finding them. Second, for each problem Π^n in the sequence, the preferences of students are arbitrary, and in particular, it can be chosen so that $D(\Pi^n)$ takes the maximum size. The result states that even if the preferences in Π^n for each n is replaced with such preferences, the limit of the fraction $\frac{|D(\Pi^n)|}{|I^n|}$ is still zero.

Remark 7. For Proposition 7, we highlight two assumptions that we impose when the market becomes large. First, the number of schools remains bounded by a constant L . Second, the number of tiers increases while the size of each tier remains bounded by a constant L' , which could be interpreted as making the priority structure close to common priorities across schools. It is easy to verify that the conclusion of the proposition does not necessarily hold if either of these assumptions is dropped.

With that being said, we can prove a more general result in which we allow for the possibility that the number of schools and/or the number of students in a tier become unboundedly large as n grows. More specifically, we suppose that, for all n , there are $L(n) < \infty$ such that $|S^n| < L(n)$ (instead of a uniform bound L) and $L'(n) < \infty$ such that $|T_k^n| < L'(n)$ (instead of a uniform bound L'). The proof in the Appendix shows that, if

$$\lim_{n \rightarrow \infty} \frac{L(n) \cdot L'(n)}{|I^n|} = 0,$$

then $\lim_{n \rightarrow \infty} \frac{|D(\Pi^n)|}{|I^n|}$ exists and is equal to 0. This result implies Proposition 7. \square

6.4. An Alternative Algorithm. Suppose that the constraint at each school is a general upper-bound. Consider the following generalization of the deferred acceptance algorithm, called the **cumulative offer algorithm**:

- Step $t \geq 1$: Each student applies to her first choice school among those that have never rejected anyone whose priority is weakly higher than her if it is acceptable, while making no application otherwise. For each school s , let $\{i_1, i_2, \dots, i_k\}$ be the set of students who have ever applied to it, with $i_1 \succ_s i_2 \succ_s \dots \succ_s i_k$. If $\{i_1, i_2, \dots, i_k\} \in \mathcal{F}_s$, then let s temporarily keep $\{i_1, i_2, \dots, i_k\}$; otherwise, let s temporarily keep the set of students of the form $\{i_1, i_2, \dots, i_{k'}\}$ such that $\{i_1, i_2, \dots, i_{k'}\} \in \mathcal{F}_s$ and $\{i_1, i_2, \dots, i_{k'+1}\} \notin \mathcal{F}_s$.⁴⁵ School s rejects all the remaining students who have ever applied to it, $\{i_{k'+1}, \dots, i_k\}$. If no student is rejected by a new school, then terminate the algorithm and define the outcome as the matching in which each school is matched to the set of students who it currently keeps.⁴⁶ Otherwise, go to Step $t + 1$.

The algorithm terminates in a finite number of steps because there is a new rejection in every step that is not terminal and there are only a finite number of student-school pairs. Therefore, the outcome of this algorithm is well-defined.

Note that this algorithm has a “cumulative” feature, that is, when a school temporarily keeps students, it considers all the students who have ever applied to it, even if they were rejected in an earlier step. Moreover, the school never rejects an applicant i while keeping another student i' with lower priority, even if keeping i is infeasible and keeping i' is feasible. These features are important for guaranteeing fairness of the resulting matching.

Proposition 8. *Suppose that the constraint at each school is a general upper-bound. Then, the outcome of the cumulative offer algorithm is the SOFM.*

A corollary of Propositions 1 and 8 is that the outcomes of the cumulative offer algorithm and the cutoff adjustment algorithm coincide with each other. Given this fact, one might question the merit of our approach based on Tarski’s fixed point theorem on the space of cutoffs. Our response is that there are at least two reasons to favor our approach. First, with our approach, it is easy to show that there exists an SOFM; this result is based on the well understood structure of fixed points of an increasing function on a complete lattice. By contrast, while it is possible to establish that the outcome of

⁴⁵Because \mathcal{F}_s is a general upper-bound, k' is uniquely defined.

⁴⁶As will be seen in Proposition 8, the outcome is indeed a matching under general upper-bounds.

the cumulative offer algorithm is the SOFM without reference to this lattice structure, it would need a specific argument tailored to that particular algorithm.⁴⁷ Second, and related, the “right” way to generalize the deferred acceptance algorithm is not clear in the absence of the insight from our approach. For example, the “cumulative” nature of our cumulative offer algorithm, i.e., that each school considers the current *and previous* applicants, is not an ex ante obvious feature for the right generalization of the deferred acceptance algorithm.

6.5. SOSM and SOFM. One may wonder whether the following claim might hold: If there exists a stable matching μ such that $\mu_i \succeq_i \mu'_i$ holds for each $i \in I$ and any stable matching μ' (the student-optimal stable matching, or the SOSM) in that market, it is the same as the SOFM. This claim turns out to be false. To see this, consider the following example.

Example 3. Suppose that there are four students i_1, i_2, i_3 , and i_4 , and two schools, s_1 and s_2 . Let preferences and priorities be as follows:

$$\begin{aligned} \succ_{i_1} &: s_2, s_1 & \succ_{s_1} &: i_1, i_2, i_3, i_4 \\ \succ_{i_2} &: s_1, s_2 & \succ_{s_2} &: i_2, i_1, i_3, i_4 \\ \succ_{i_3} &: s_1, s_2 \\ \succ_{i_4} &: s_1, s_2 \end{aligned}$$

The constraints are: $\mathcal{F}_{s_1} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}, \{i_2, i_4\}\}$ and $\mathcal{F}_{s_2} = \{\emptyset, \{i_1\}, \{i_2\}, \{i_3\}, \{i_4\}\}$. Note that the constraint of school s_2 is a capacity constraint while the constraint of school s_1 is not, and both are general upper-bounds.

Now, consider the following two matchings.

$$\mu = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_1 & i_2 & i_3, i_4 \end{pmatrix}, \quad \mu' = \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2 & i_1 & i_3, i_4 \end{pmatrix}.$$

By inspection, one can verify that μ is the SOSM while μ' is the SOFM. Note that μ' is unstable because i_4 can be feasibly matched to s_1 . Since $\mu' \neq \mu$, the SOFM and the SOSM are different.⁴⁸ \square

⁴⁷Delacrétaz, Kominers and Teytelboym (2016) construct an equivalent algorithm to the cumulative offer algorithm using the language of multidimensional services. See Appendix C.1 for details.

⁴⁸We note that the existence of i_3 plays a major role in this example. For example, consider

$$\mu'' := \begin{pmatrix} s_1 & s_2 & \emptyset \\ i_2, i_4 & i_1 & i_3 \end{pmatrix}.$$

- Remark 8.** (1) The difference between the SOSM and the SOFM is caused by the generality of general upper-bound. To see this, recall that, if each school's constraint is a capacity constraint, then the SOSM and the SOFM are identical to each other (Theorem 2 of Balinski and Sonmez (1999)). Combined with Theorem 3, this implies that, whenever we impose a condition on constraints over individual schools which guarantees the existence of a SOSM, it is identical to the SOFM.
- (2) Since any stable matching is fair (by definition), for any problem with general upper-bounds, the SOFM is weakly preferred by every student to any stable matching (if the latter exists). Therefore, whenever there exists a SOSM and it is different from the SOFM, the SOFM is strictly preferred to the SOSM by some students while weakly preferred by every student. Example 3 shows that this can actually happen. \square

7. CONCLUSION

This paper studied a matching problem where institutions are subject to general constraints. Observing that a stable matching typically does not exist, our approach is to tolerate some waste while requiring fairness. Our first main result characterizes feasible, individually rational, and fair matchings by fixed points of a certain mapping on the space of cutoff profiles. Building upon this result, we find a necessary and sufficient condition for guaranteeing the existence of a student-optimal fair matching (SOFM). The condition is that the constraint of each school is a general upper-bound. Then we provide a constructive algorithm to find an SOFM based on our fixed-point mapping. Furthermore, we apply our findings to centralized allocation of daycare seats and find that the SOFM mechanism under flexible constraints performs substantially better than an alternative algorithm that treats age-specific capacities as rigid constraints.

This paper leaves various important questions for future research. For instance, our result shows the general upper-bound is exactly the condition that guarantees the existence of an unambiguously desirable fair matching, i.e., an SOFM. In particular, when the constraint of even one school fails to be a general upper-bound, there is no guarantee that an SOFM exists. In such a case, a natural question for researchers and policymakers alike is what kind of property to aim for. One possibility may be to find a Pareto-undominated fair matching as defined in Remark 4 (Section 4.2). Although the existence of such a

We can see that μ'' is not fair (hence unstable), because i_3 likes s_1 better than \emptyset and i_3 has higher priority than i_4 at s_1 . As Remark 1 illustrates in detail, μ'' does not satisfy a desired fairness criterion in our applications.

matching is rather straightforward given the finiteness of the environment, a constructive algorithm may be nontrivial and interesting. For example, the cutoff adjustment algorithm does not always find a Pareto-undominated fair matching (see Appendix C.3). Moreover, as pointed out in Section 6.3, there exists no truncation-proof mechanism that produces a Pareto-undominated fair matching under general constraints (recall that the SOFM mechanism is truncation-proof under general upper-bounds).

Also, a more in-depth study of the tradeoff between fairness and non-wastefulness may be interesting. At various places of the present paper (such as the Introduction, Remark 1, and Section 6.1), we have been making clear that our fairness notion is the appropriate one in the applications that we have in mind. Moreover, the theoretical and empirical analyses in Section 5 show that the efficiency loss from requiring our strong notion of fairness is not significant in our daycare application. We acknowledge, however, that there can be other applications in which efficiency loss can be significant. For such applications, it may be of interest to quantify the efficiency loss as well as “fairness loss” of a non-wasteful solution to appropriately trade-off these two. Such a study, of course, is beyond the scope of the present paper but could be a fruitful direction of research.

Another direction for future research would involve data. For example, we conducted numerical analysis of datasets on daycare allocation in two municipalities and found large welfare gains in both cases. To what extent is such a finding generalizable to daycare allocation elsewhere? How about other applications such as school choice with diversity constraints, college admissions involving students with disabilities, and refugee matching? These questions are beyond the scope of our paper, but they seem to be important questions for future research.

Finally, it would be interesting to use our findings for design in practice. Such an experience may not only improve outcomes in real problems like daycare seat allocation, but it may also provide more insight about possible directions for future research. For instance, is the lack of exact strategy-proofness for students a major drawback in practice, or is an approximate incentive compatibility as shown in our analysis sufficient to eliminate strategic behavior? Is the SOFM mechanism transparent enough for applicants to understand or trust? Are there any unintended consequences of the change in the mechanism? These are just a few of many possible questions that one may be able to investigate with the feedback from policy experience. We wait for future research to answer these questions.

Data Availability Statement The data underlying this article were provided by Yamagata City and Bunkyo City by permission. The code is available on Zenodo at <https://doi.org/10.5281/zenodo.7690023>.

REFERENCES

- Abdulkadiroğlu, A.** 2005. “College admissions with affirmative action.” *International Journal of Game Theory*, 33(4): 535–549.
- Abdulkadiroğlu, Atila, and Tayfun Sönmez.** 2003. “School Choice: A Mechanism Design Approach.” *American Economic Review*, 93: 729–747.
- Abdulkadiroğlu, Atila, Parag A. Pathak, and Alvin E. Roth.** 2009. “Strategy-proofness versus Efficiency in Matching with Indifferences: Redesigning the NYC High School Match.” *American Economic Review*, 99: 1954–1978.
- Abdulkadiroğlu, Atila, Yeon-Koo Che, Parag A Pathak, Alvin E Roth, and Olivier Tercieux.** 2017. “Minimizing Justified Envy in School Choice: The Design of New Orleans’ OneApp.” working paper, National Bureau of Economic Research.
- Abizada, Azar.** 2016. “Stability and incentives for college admissions with budget constraints.” *Theoretical Economics*, 11(2): 735–756.
- Abraham, D. J., R.W. Irving, and D.F. Manlove.** 2007. “Two algorithms for the student-project allocation problem.” *Journal of Discrete Algorithms*, 5: 73–90.
- Adachi, Hiroyuki.** 2000. “On a characterization of stable matchings.” *Economics Letters*, 68(1): 43–49.
- Akbarpour, Mohammad, and Afshin Nikzad.** 2017. “Approximate random allocation mechanisms.”
- Andersson, Tommy, and Lars Ehlers.** 2016. “Assigning refugees to landlords in Sweden: Stable maximum matchings.” mimeo.
- Artemov, Georgy, Yeon-Koo Che, and Yinghua He.** 2019. “Strategic ‘Mistakes’: Implications for Market Design Research.” mimeo.
- Asai, Yukiko, Ryo Kambayashi, and Shintaro Yamaguchi.** 2015. “Childcare availability, household structure, and maternal employment.” *Journal of the Japanese and International Economies*, 38: 172–192.
- Aygun, Orhan, and Bertan Turhan.** 2016. “Dynamic reserves in matching markets: Theory and applications.”
- Aygun, Orhan, and Tayfun Sönmez.** 2013. “Matching with contracts: Comment.” *American Economic Review*, 103(5): 2050–51.

- Azevedo, Eduardo M, and Jacob D Leshno.** 2015. “A supply and demand framework for two-sided matching markets.” *Journal of Political Economy*, *forthcoming*.
- Balinski, Michel, and Tayfun Sönmez.** 1999. “A tale of two mechanisms: student placement.” *Journal of Economic theory*, 84(1): 73–94.
- Biró, Péter.** 2008. “Student admissions in Hungary as Gale and Shapley envisaged.” *University of Glasgow Technical Report TR-2008-291*.
- Biró, Péter, and Sofya Kiselgof.** 2015. “College admissions with stable score-limits.” *Central European Journal of Operations Research*, 23(4): 727–741.
- Biró, Petér, Tamás Fleiner, Robert W. Irving, and David F. Manlove.** 2010*a*. “The College Admissions problem with lower and common quotas.” *Theoretical Computer Science*, 411: 3136–3153.
- Biro, P., T. Fleiner, R.W. Irving, and D.F. Manlove.** 2010*b*. “The College Admissions Problem with Lower and Common Quotas.” *Theoretical Computer Science*, 411(34-36): 3136–3153.
- Blum, Yosef, Alvin E. Roth, and Uriel Rothblum.** 1997. “Vacancy Chains and Equilibration in Senior-Level Labor Markets.” *Journal of Economic Theory*, 76: 362–411.
- Board of Education of Hyogo Prefecture.** 1996. “Surviving the earthquake: Education in Hyogo in the aftermath of the great earthquake (in Japanese).” <http://www.hyogo-c.ed.jp/somu-bo/ikite/ikite.html>.
- Breitenbach, Dagmar.** 2015. “Refugees don’t leave their conflicts behind.” <http://www.dw.com/en/refugees-dont-leave-their-conflicts-behind/a-18746390>.
- Budish, E., and E. Cantillon.** 2012. “The Multi-unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard.” *American Economic Review*, 102: 2237–2271.
- Budish, Eric, Yeon-Koo Che, Fuhito Kojima, and Paul R. Milgrom.** 2013. “Designing Random Allocation Mechanisms: Theory and Applications.” *American Economic Review*, 103(2): 585–623.
- Bunkyo City.** 2017. “Daycare Allocation Data.” Details are available at <https://zenodo.org/record/4516857>.
- Bunkyo City.** 2018. “On daycare allocation decision for 2018.” <http://www.city.bunkyo.lg.jp/kyoiku/kosodate/okosan/nicchu/ninka/30nen/sennkouhouhou.html>, Accessed on 2020-01-29.
- Cabinet Office of Japan.** 2017. “On the new child- and childbearing support policy (in

- Japanese)." <http://www8.cao.go.jp/shoushi/shinseido/outline/pdf/setsume.pdf>, Accessed on 2018-03-15.
- Che, Yeon-Koo, and Olivier Tercieux.** 2019. "Efficiency and stability in large matching markets." *Journal of Political Economy*, 127(5): 2301–2342.
- Che, Yeon-Koo, Jinwoo Kim, and Konrad Mierendorff.** 2013. "Generalized Reduced-Form Auctions: A Network-Flow Approach." *Econometrica*, 81(6): 2487–2520.
- Crawford, V.P.** 1991. "Comparative statics in matching markets." *Journal of Economic Theory*, 54(2): 389–400.
- Delacrétaz, David.** 2019. "Stability in matching markets with sizes."
- Delacrétaz, David, Scott Duke Kominers, and Alexander Teytelboym.** 2016. "Refugee resettlement." Working Paper.
- Echenique, Federico, and Bumin Yenmez.** 2007. "A Solution to Matching with Preferences Over Colleagues." *Games and Economic Behavior*, 59: 46–71.
- Echenique, Federico, and Jorge Oviedo.** 2004. "Core many-to-one matchings by fixed-point methods." *Journal of Economic Theory*, 2(115): 358–376.
- Echenique, Federico, and Jorge Oviedo.** 2006. "A Theory of Stability in Many-to-Many Matching." *Theoretical Economics*, 1: 233–273.
- Ehlers, L., I. Hafalir, M.B. Yenmez, and M. Yildirim.** 2011. "School Choice with Controlled Choice Constraints: Hard Bounds versus Soft Bounds." *mimeo*.
- Ergin, Haluk, and Tayfun Sönmez.** 2006. "Games of School Choice under the Boston Mechanism." *Journal of Public Economics*, 90: 215–237.
- Fack, Gabrielle, Julien Grenet, and Yinghua He.** 2019. "Beyond Truth-Telling: Preference Estimation with Centralized School Choice and College Admissions." *American Economic Review*, 109(4): 1486–1529.
- Fleiner, Tamás, and Zsuzsanna Jankó.** 2014. "Choice function-based two-sided markets: stability, lattice property, path independence and algorithms." *Algorithms*, 7(1): 32–59.
- Fleiner, Thomas.** 2003. "A Fixed-Point Approach to Stable Matchings and Some Applications." *Mathematics of Operations Research*, 28: 103–126.
- Fragiadakis, Daniel, and Peter Troyan.** 2016. "Improving Matching under Hard Distributional Constraints." forthcoming, *Theoretical Economics*.
- Fragiadakis, D.E., A. Iwasaki, P. Troyan, S. Ueda, and M. Yokoo.** 2012. "Strategyproof Matching with Minimum Quotas." *mimeo*.
- Gale, David, and Lloyd S. Shapley.** 1962. "College Admissions and the Stability of Marriage." *American Mathematical Monthly*, 69: 9–15.

- Goto, Masahiro, Atsushi Iwasaki, Yujiro Kawasaki, Yosuke Yasuda, and Makoto Yokoo.** 2014. “Improving Fairness and Efficiency in Matching with Distributional Constraints: An Alternative Solution for the Japanese Medical Residency Match.” https://mpra.ub.uni-muenchen.de/53409/1/MPRA_paper_53409.pdf, last accessed on March 9th, 2018.
- Goto, Masahiro, Fuhito Kojima, Ryoji Kurata, Akihisa Tamura, and Makoto Yokoo.** 2016. “Designing Matching Mechanisms under General Distributional Constraints.” *American Economic Journal: Microeconomics*, forthcoming.
- Habu, Sachiko.** 2016. *Hoiku en ni hairi tai! (We want a daycare slot!) (in Japanese)*. Nikkei Business Publishing.
- Hafalir, Isa E, M Bumin Yenmez, and Muhammed A Yildirim.** 2013. “Effective affirmative action in school choice.” *Theoretical Economics*, 8(2): 325–363.
- Hassidim, Avidan, Assaf Romm, and Ran I. Shorrer.** 2019. “Strategic Behavior in a Strategy-Proof Environment.” mimeo.
- Hatfield, John William, and Paul Milgrom.** 2005. “Matching with Contracts.” *American Economic Review*, 95: 913–935.
- Hatfield, John William, and Scott Duke Kominers.** 2013. “Hidden Substitutes.” Working paper, Harvard Univ.
- Hatfield, John William, and Scott Duke Kominers.** 2017. “Contract design and stability in many-to-many matching.” *Games and Economic Behavior*, 101: 78–97.
- Hayashi, Haruo.** 2003. *Earthquake disaster prevention studies to protect lives(in Japanese)*. Tokyo:Iwanami Publishing.
- Herzog, Sabrina, and Thilo Klein.** 2018. “Matching Practices for Childcare—Germany.” <http://www.matching-in-practice.eu/matching-practices-for-childcare-germany/>.
- Kamada, Yuichiro, and Fuhito Kojima.** 2015. “Efficient Matching under Distributional Constraints: Theory and Applications.” *American Economic Review*, 105(1): 67–99.
- Kamada, Yuichiro, and Fuhito Kojima.** 2017. “Stability concepts in matching with distributional constraints.” *Journal of Economic Theory*, 168: 107–142.
- Kamada, Yuichiro, and Fuhito Kojima.** 2018. “Stability and Strategy-Proofness for Matching with Constraints: A Necessary and Sufficient Condition.” *Theoretical Economics*, 13: 761–794.
- Kamada, Yuichiro, and Fuhito Kojima.** 2021. “Replication package for: Fair Matching under Constraints: Theory and Applications.”

<https://doi.org/10.5281/zenodo.4516857>.

- Kasuya, Yusuke.** 2016. “Anti-bullying School Choice Mechanism Design.” mimeo.
- Kennes, John, Daniel Monte, and Norovsambuu Tumennasan.** 2014. “The day care assignment: A dynamic matching problem.” *American Economic Journal: Microeconomics*, 6(4): 362–406.
- Kesten, Onur.** 2009. “School Choice with Consent.” *Quarterly Journal of Economics*, forthcoming.
- Kesten, Onur, and Ayse Yazici.** 2012. “The Pareto-dominant strategy-proof and fair rule for problems with indivisible goods.” *Economic Theory*, 50(2): 463–488.
- Kikuchi, Daisuke.** 2016. “‘Superhuman’ slugger tops year’s list of buzzwords.” *the Japan Times*,
https://www.japantimes.co.jp/author/int-daisuke_kikuchi/.
- Kojima, Fuhito, Akhisa Tamura, and Makoto Yokoo.** 2018. “Designing Matching Mechanisms under Constraints: An Approach from Discrete Convex Analysis.” *Journal of Economic Theory*, 176: 803–833.
- Konishi, Hideo, and M. Utku Ünver.** 2006. “Games of Capacity Manipulation in the Hospital-Intern Market.” *Social Choice and Welfare*, 27: 3–24.
- Milgrom, P.** 2009. “Assignment messages and exchanges.” *American Economic Journal: Microeconomics*, 1(2): 95–113.
- Milgrom, Paul, and Ilya Segal.** 2014. “Deferred-acceptance auctions and radio spectrum reallocation.” 185–186.
- Ministry of Health, Labour and Welfare.** 2014. “Summary of the Survey on the Regional Child Welfare Project (in Japanese).”
<http://www.mhlw.go.jp/toukei/saikin/hw/jidou/12/dl/kekka-01.pdf>.
- Ministry of Health, Labour and Welfare.** 2017. “Summary of the Current Status of Non-Publicly Certified Daycare Centers, Fiscal Year 2015 (in Japanese).”
<http://www.mhlw.go.jp/file/04-Houdouhappyou-11907000-Koyoukintoujidoukateikyoku-Hoikuka/0000112872.1.pdf>.
- Nguyen, Thanh, Ahmad Peivandi, and Rakesh Vohra.** 2016. “Assignment problems with complementarities.” *Journal of Economic Theory*, 165: 209–241.
- Nguyen, Thành, and Rakesh Vohra.** 2017. “Stable Matching with Proportionality Constraints.”
- Okumura, Yasunori.** 2018. “School choice with general constraints: a market design approach for nursery school waiting lists problem in Japan.” forthcoming, Japanese Economic Review.

- Osaki, Tomohiro.** 2016. “Angry blog post sparks movement for improved day care.” *the Japan Times*, <https://www.japantimes.co.jp/news/2016/03/07/national/angry-blog-post-sparks-movement-for-improved-day-care/>.
- Ostrovsky, Michael.** 2008. “Stability in supply chain networks.” *American Economic Review*, 98(3): 897–923.
- Pathak, Parag A, and Tayfun Sönmez.** 2013. “School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation.” *American Economic Review*, 103(1): 80–106.
- Prime Minister’s Office.** 2017. “Measures for Waitlisted Children (in Japanese).” <https://www.kantei.go.jp/jp/headline/taikijido/index.html>.
- Pycia, Marek, and M Utku Ünver.** 2015. “Decomposing random mechanisms.” *Journal of Mathematical Economics*, 61: 21–33.
- Rees-Jones, Alex.** 2017. “Mistaken play in the deferred acceptance algorithm: Implications for positive assortative matching.” *American Economic Review*, 107(5): 225–29.
- Rees-Jones, Alex.** 2018. “Suboptimal behavior in strategy-proof mechanisms: Evidence from the residency match.” *Games and Economic Behavior*, 108: 317–330.
- Ríos, Ignacio, Tomás Larroucau, Giorgiogiulio Parra, and Roberto Cominetti.** 2014. “College admissions problem with ties and flexible quotas.”
- Roth, Alvin E.** 1982. “The Economics of Matching: Stability and Incentives.” *Mathematics of Operations Research*, 7: 617–628.
- Roth, Alvin E.** 1984. “The Evolution of the Labor Market for Medical Interns and Residents: A Case Study in Game Theory.” *Journal of Political Economy*, 92: 991–1016.
- Roth, Alvin E.** 1991. “A Natural Experiment in the Organization of Entry Level Labor Markets: Regional Markets for New Physicians and Surgeons in the U.K.” *American Economic Review*, 81: 415–440.
- Roth, Alvin E., and Elliott Peranson.** 1999. “The Redesign of the Matching Market for American Physicians: Some Engineering Aspects of Economic Design.” *American Economic Review*, 89: 748–780.
- Roth, Alvin E., and John Vande Vate.** 1991. “Incentives in Two-sided Matching with Random Stable Mechanisms.” *Economic Theory*, 1: 31–44.
- Roth, Alvin E., and Marilda A. O. Sotomayor.** 1990. *Two-sided Matching: a Study in Game-theoretic Modeling and Analysis*. Cambridge:Econometric Society monographs.

- Roth, Alvin E., and Marilda Sotomayor.** 1988. “Interior points in the core of two-sided matching markets.” *Journal of Economic Theory*, 45(1): 85–101.
- Roth, Alvin E., Tayfun Sönmez, and Utku Ünver.** 2004. “Kidney Exchange.” *Quarterly Journal of Economics*, 119: 457–488.
- Roth, Alvin E., Tayfun Sönmez, and Utku Ünver.** 2005. “Pairwise Kidney Exchange.” *Journal of Economic Theory*, 125: 151–188.
- Roth, Alvin E., Tayfun Sönmez, and Utku Ünver.** 2007. “Efficient Kidney Exchange: Coincidence of Wants in Markets with Compatibility-Based Preferences.” *American Economic Review*, 97: 828–851.
- Sönmez, Tayfun, and Utku Ünver.** 2010. “Course Bidding at Business Schools.” *International Economic Review*, 51(1): 99–123.
- Sotomayor, Marilda A. O.** 1996. “A Non-constructive Elementary Proof of the Existence of Stable Marriages.” *Games and Economic Behavior*, 13: 135–137.
- Suzuki, Wataru.** 2018. *An economist tries to reduce the daycare waitlist to zero (in Japanese)*. Shincho-sha Publishing Company.
- Town of Tomioka.** 2015. “Town of Tomioka; Record of Tohoku Earthquake and the Nuclear Disaster (in Japanese).” <http://www.tomioka-town.jp/living/Files/2015/06/17/1.pdf>.
- United Nations High Commissioner for Refugees.** 2017. “Global trends – Forced Displacement in 2016.” <http://www.unhcr.org/en-us/statistics/unhcrstats/5943e8a34/global-trends-forced-displacement-2016.html>, Accessed on 2018-03-09.
- University of Oxford.** 2018. “Applicants with disabilities.” <https://www.ox.ac.uk/admissions/graduate/applying-to-oxford/applicants-with-disabilities?wssl=1>, Accessed on 2018-01-10.
- Veski, André, Péter Biró, Kaire Põder, Triin Lauri, et al.** 2017. “Efficiency and fair access in kindergarten allocation policy design.” *The Journal of Mechanism and Institution Design*, 2(1): 57–104.
- Westkamp, Alexander.** 2013. “An Analysis of the German University Admissions System.” *Economic Theory*, 53(3): 561–589.
- Wu, Qingyun, and Alvin E Roth.** 2018. “The lattice of envy-free matchings.” *Games and Economic Behavior*.
- Yamagata City.** 2017. “Daycare Allocation Data.” Details are available at <https://zenodo.org/record/4516857>.

Yamagata City. 2019*a*. “The application form.” <https://www.city.yamagata-yamagata.lg.jp/kakuka/kosodate/hoikuikusei/sogo/R2nendo/2.pdf>, Accessed on 2019-12-16.

Yamagata City. 2019*b*. “On daycare allocation.” <https://www.city.yamagata-yamagata.lg.jp/kosodate/sub2/hoikuen/f38e47df45d2c444.html>, Accessed on 2019-12-16.