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REPRESENTATION THEOREMS FOR PATH-INDEPENDENT CHOICE RULES

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Abstract. Path independence is arguably one of the most important choice rule properties in economic theory. We show that a choice rule is path independent if and only if it is rationalizable by a utility function satisfying ordinal concavity, a concept closely related to concavity notions in discrete mathematics. We also provide a representation result for choice rules that satisfy path independence and the law of aggregate demand.

1. Introduction

Plott (1973) introduced *path independence* as a property of social choice formalizing an idea in Arrow's book *Social Choice and Individual Values*. When a choice rule is path independent, any set of alternatives can be divided into segments, the rule applied first to each segment and then to the set of chosen alternatives from all segments without changing the final outcome. It is a desirable property in many contexts. For example, in college admissions, without path independence, the set of admitted students can depend on the order in which applications are reviewed, which can enable the malpractice of favoritism. Path independence is intimately related to other well-known properties: A choice rule is path independent

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if and only if it satisfies the *substitutes condition* and the *irrelevance of rejected contracts* (Aizerman and Malishevski, 1981).[1]

Although originating in social choice, path independence has found applications in different areas of economic theory, such as market design and decision theory. For example, in two-sided matching markets, when agents have path-independent choice rules, a *stable* matching exists (Blair, 1988).[2] In decision theory, path independence and its stochastic versions have been studied extensively (Kalai and Megiddo, 1980).[3] Path independence has also been studied in other fields, such as discrete mathematics, law, philosophy, and systems design.[4]

In economics, rational agents are usually modeled as utility maximizers: They have a utility function over sets of alternatives and, when they face a set of alternatives, they choose the subset with the highest utility over all subsets that can be selected. An alternative approach is to endow agents with choice rules, for example, when agents do not necessarily have a well-defined utility function or when the utility function is not observable.[5] A fundamental question linking these two approaches is whether a choice rule is *rationalizable* by a utility function so that the choice from any set of alternatives is the subset with the highest utility among all subsets. Results of this nature connecting choice rules with certain properties to utility functions with corresponding properties are called *representation theorems*.[6] Surprisingly, to the best of our knowledge, there is no representation theorem for path-independent choice rules, even though they have been studied in different

[1]The substitutes condition is also called Chernoff or Sen's α or heritage, and the irrelevance of rejected contracts is also called outcast or the independence of irrelevant alternatives.

[2]The substitutes condition is not sufficient for the existence of a stable matching when choice rules are the primitive of the model rather than utility functions or preferences. See the discussion in Aygün and Sönmez (2013) and Chambers and Yenmez (2017).

[3]See also Machina and Parks (1981) and a recent treatment by Ahn et al. (2018)

[4]For discrete mathematics see Gratzer and Wehrung (2016), for law Chapman (1997) and Hammond and Thomas (1989), for philosophy Rott (2001) and Stewart (2022), and for systems design Levin (1998).

[5]For example, choice rules are used to model diversity policies of schools (Hafalir et al., 2013; Ehlers et al., 2014; Echenique and Yenmez, 2015).

[6]See, for example, Chapter 1 of Mas-Colell et al. (1995).

areas of economics since their introduction five decades ago. For concreteness, following the market-design literature, we call alternatives as contracts in the rest of the paper.

In this paper, we provide two representation results for path-independent choice rules. First, we show that a choice rule is path independent if and only if it is rationalizable by a utility function satisfying *ordinal concavity* (Theorem 1). Roughly, ordinal concavity requires that when two sets of contracts are made closer to each other, either the utility function increases on at least one side or remains unchanged on both sides. In this context, getting closer may either mean adding or removing a contract that we start with or the existence of a second contract such that we add one of the contracts and remove the other one.

Ordinal concavity is weaker than *$M^\natural$-concavity* (Hafalir et al., 2022, Proposition 5), which is a standard notion of concavity used in the discrete convex analysis literature. Fujishige and Yang (2003) show that the *gross substitutes property* of Kelso and Crawford (1982) is equivalent to $M^\natural$-concavity. Therefore, one implication of our result is that the difference between the gross substitutes property and the substitutes condition (or path independence) can be attributed to the difference between $M^\natural$-concavity and ordinal concavity.[7] Submodularity is another well-known condition that is often associated with a variety of substitutability notions.[8] However, submodularity and ordinal concavity are logically unrelated, and in fact rationalizability by a submodular function does not imply path independence.[9] At a high level, one of our contributions is to identify an appropriate condition on utility functions that is tightly connected with path independence and the substitutes condition.

[7]To be more precise, this statement holds for rationalizable choice functions that satisfy the substitutes condition and for path-independent choice rules. In fact these two classes of choice rules are the same.

[8]For instance, $M^\natural$-concavity, or equivalently the gross substitutes condition of Kelso and Crawford (1982), implies submodularity (Murota and Shioura, 2001). See Lehmann et al. (2001) for the relationship between different properties including the gross substitutes and submodularity conditions.

[9]An example is available from the authors for each of these two claims.

In the market-design literature, another choice rule property that plays a crucial role is *the law of aggregate demand* (Hatfield and Milgrom, 2005). The law of aggregate demand states that when more contracts become available, the number of chosen contracts weakly increases. The law of aggregate demand, together with path independence, yields numerous results. It implies, for instance, the *rural hospitals theorem* in two-sided matching markets, which states that the number of contracts an agent gets is the same across all stable matchings. In addition, in the doctor-hospital matching problem, a generalization of the doctor-proposing deferred-acceptance mechanism of Gale and Shapley (1962) is *strategy-proof* for doctors (Hatfield and Milgrom, 2005).

In our second result, we show that a choice rule satisfies path independence and the law of aggregate demand if and only if it is rationalizable by a utility function that satisfies ordinal concavity and *size-restricted concavity* (Theorem 2). Size-restricted concavity can also be viewed as a version of discrete concavity with a quantifier such that the implication is required only for sets of contracts with different sizes.

The main difference between our work and the classical literature on social choice is that we assume the utility function (or the preference relation) is over sets of contracts, whereas in the classic setting, the utility function (or the preference relation) is over individual contracts (see Moulin (1985) for a summary). Likewise, our choice rule is combinatorial (see Echenique (2007)), whereas in the classical setting, multiple contracts represent indecision of the agent, and the agent is eventually assigned only one contract. Therefore, our results are independent of the social choice literature on rationalizability.

Ordinal concavity was recently introduced in Hafalir et al. (2022). Even though their formulation differs from ours, it is easy to check that their definition reduces to ours in the present paper's setting. In another recent work, Yang (2020) shows that path-independent choice rules are rationalizable, but he does not provide any representation results for path-independent choice rules. Chambers and Yenmez (2017) make a connection between the theory of path-independent choice rules and matching theory and utilize this connection to advance both fields.

One of the major contributions of our paper is to establish a close connection between choice rules in economics and concavity concepts in discrete mathematics. This connection allows us to shed light on economic problems with techniques of discrete optimization. For instance, in an abstract setting, Eguchi et al. (2003) and Murota and Yokoi (2015) show choice rules that are rationalizable by $M^\natural$-concave functions satisfy path independence and the law of aggregate demand. On an applied front, Kojima et al. (2018) build upon their results to find $M^\natural$-concave functions that rationalize a variety of practically relevant choice rules and establish their desirable properties including computational efficiency. In addition to introducing ordinal concavity, Hafalir et al. (2022) establish connections between ordinal concavity and choice rules in markets with dual objectives such as college admissions where diversity and meritocracy are typical goals. While advancing this research program further, the present paper is distinctive in that it provides conditions of discrete concavity that are *equivalent* to rationalizability of desirable choice rules, thus giving a final and complete answer to a foundational issue in this research agenda.

The rest of the paper is organized as follows. We define choice rules and their properties in Section 2, present our representation results in Section 3, discuss the relationship with rationalizability by an $M^\natural$-concave function in Section 4, and conclude in Section 5. We provide all proofs in the Appendix.

2. Preliminaries

Let $\mathcal{X}$ denote a finite set of **contracts**. A **choice rule** is a function $C : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ such that, for any $X \subseteq \mathcal{X}$, we have $C(X) \subseteq X$. We study two key properties of choice rules.

Definition 1 (Plott (1973))**.** *A choice rule C satisfies **path independence** if, for any $X, X' \subseteq \mathcal{X}$,*

$$C(X \cup X') = C(C(X) \cup X').$$

Path independence plays a fundamental role in social choice, market design, and decision theory. It has also been used in different areas such as discrete mathematics, law, philosophy, and systems design.[10]

Definition 2 (Hatfield and Milgrom (2005))**.** *A choice rule C satisfies the* ***law of aggregate demand*** *if, for any* $X, X' \subseteq \mathcal{X}$*,*

$$X \supseteq X' \implies |C(X)| \geq |C(X')|.$$

The law of aggregate demand is a critical property in market design.[11] The law of aggregate demand, together with path independence, produces classic results such as the rural hospitals theorem (Fleiner, 2003) and strategy-proofness of a generalization of the doctor-proposing Gale-Shapley deferred-acceptance mechanism for doctors (Hatfield and Milgrom, 2005).

A **utility function** $u : 2^{\mathcal{X}} \to \mathbb{R}$ assigns a value to every set of contracts.[12] A choice rule C is **rationalizable** by a utility function u if, for any $X \subseteq \mathcal{X}$,

$$u(C(X)) > u(X') \text{ for every } X' \subseteq X \text{ with } X' \neq C(X).$$

In words, when a choice rule is rationalizable by a utility function, from any set of available contracts, the choice rule selects the unique subset with the highest utility.

3. Results

In this section, we provide two representation theorems using utility functions that satisfy notions of discrete concavity. To define these notions, we introduce some notation following the discrete convex analysis literature. For any $X \subseteq \mathcal{X}$ and $x \in \mathcal{X}$, let $X + x = X \cup \{x\}$ and $X - x = X \setminus \{x\}$. Likewise, for any $X \subseteq \mathcal{X}$, let $X + \emptyset = X$ and $X - \emptyset = X$.

Definition 3. *A utility function* u *satisfies* ***ordinal concavity*** *if, for any* $X, X' \subseteq \mathcal{X}$ *and* $x \in X \setminus X'$*, there exists* $x' \in (X' \setminus X) \cup \{\emptyset\}$ *such that*

(i) $u(X) < u(X - x + x')$*, or*

[10] See the references in the Introduction.

[11] Alkan and Gale (2003) call it *size monotonicity* in a matching context without contracts.

[12] $\mathbb{R}$ represents the set of real numbers.

(ii) $u(X') < u(X' + x - x')$, *or*

(iii) $u(X) = u(X - x + x')$ *and* $u(X') = u(X' + x - x')$.

In words, ordinal concavity requires that when X is made closer towards X' by removing x and adding x', and X' is made closer toward X by adding x and removing x', either at least one of the two function values strictly increases or both values remain unchanged.

Hafalir et al. (2022) study ordinal concavity in a market-design context to study agents with dual objectives such as administrators in college admissions who try to maximize both merit and diversity of admitted classes. Chen and Li (2021) study the same concavity notion in the context of operations research (calling it SSQM$^\natural$-concavity).[13] Ordinal concavity is a weaker notion than M$^\natural$-concavity (Hafalir et al., 2022, Proposition 5), which we define in Section 4 where we discuss their relationship in detail.

Our main result is a representation theorem for path-independent choice rules.

Theorem 1. *A choice rule is path independent if and only if it is rationalizable by a utility function satisfying ordinal concavity.*

The if direction of this result follows from our previous work (Hafalir et al., 2022, Theorem 2). The only-if direction is an existence result, and its proof is constructive. In what follows, we give the main idea in our construction. Given a path-independent choice rule C, our goal is to construct an ordinally concave utility function u such that, for any $X \subseteq \mathcal{X}$,

$$u(C(X)) > u(X') \text{ for every } X' \subseteq X \text{ with } X' \neq C(X).$$

By path independence, for every $X \subseteq \mathcal{X}$, $C(X) \supseteq X \cap C(\mathcal{X})$.[14] Therefore, from any set X, we must at least choose contracts that it has in $C(\mathcal{X})$ unless $X \cap C(\mathcal{X}) = \emptyset$. In the first step of our construction, we add one to the utility function for every contract in $C(\mathcal{X})$. In other words, our first step can be viewed as defining the following utility function: for any $X \subseteq \mathcal{X}$, $v(X) = |X \cap C(\mathcal{X})|$. Since $v(C(X)) = |C(X) \cap C(\mathcal{X})|$

[13]See Section 6.14 of Murota (2003) for other concavity notions with an ordinal content.

[14]This implication also follows from the substitutes condition, see Proposition 2 in Appendix A.

by definition and $|C(X) \cap C(\mathcal{X})| = |X \cap C(\mathcal{X})|$ by path independence, it follows that $v(C(X)) = |X \cap C(\mathcal{X})|$. Hence, for any $X' \subseteq X$, $v(C(X)) = |X \cap C(\mathcal{X})| \geq |X' \cap C(\mathcal{X})| = v(X')$. Therefore, we obtain

$$v(C(X)) \geq v(X') \text{ for every } X' \subseteq X.$$

This displayed inequality is required for rationalizability, but it is not enough. Specifically, we need the inequality to be strict when $X' \neq C(X)$. In the proof, for every $X \subseteq \mathcal{X}$, we construct a sequence of shrinking sets $\{E_X^k\}_{k \in \{1,\ldots\}}$ starting with $E_X^1 = \mathcal{X}$. We modify the utility function so that it increases linearly in the number of contracts that X has in $C(E_X^k)$ for each k. Our proof establishes that this function in fact makes the above inequality strict for $X' \neq C(X)$, thus rationalizing the given choice rule. This formulation is also useful for establishing ordinal concavity because the cardinality of the intersection of sets is tractable when we add or remove one contract from a set.

Next, we introduce another concavity notion that proves crucial for rationalizable choice rules to satisfy the law of aggregate demand.

Definition 4. *A utility function u satisfies **size-restricted concavity** if, for any $X, X' \subseteq \mathcal{X}$ with $|X| > |X'|$, there exists $x \in X \setminus X'$ such that*

(*i*) $u(X) < u(X - x)$, *or*

(*ii*) $u(X') < u(X' + x)$, *or*

(*iii*) $u(X) = u(X - x)$ *and* $u(X') = u(X' + x)$.

Like ordinal concavity, this condition states that either the function value strictly increases at least on one side or the function values remain unchanged on both sides when two sets move closer toward each other. Size-restricted concavity differs from ordinal concavity in that it requires X to have a strictly larger cardinality than X' and, furthermore, only one contract is added or removed when sets are made closer to each other. It is easy to see that size-restricted concavity is weaker than M$^\natural$-concavity while logically independent of ordinal concavity.[15]

[15] We define M$^\natural$-concavity in Section 4. If a utility function u satisfies M$^\natural$-concavity, then it satisfies the following property. For any $X, X' \subseteq \mathcal{X}$ with $|X| > |X'|$, there exists $x \in X \setminus X'$ such that $u(X) + u(X') \leq u(X - x) + u(X' + x)$. In fact, M$^\natural$-concavity is equivalent to this property in

Our second representation theorem uses size-restricted concavity.

Theorem 2. *A choice rule satisfies path independence and the law of aggregate demand if and only if it is rationalizable by a utility function that satisfies ordinal concavity and size-restricted concavity.*[16]

By Theorem 1, a choice rule that is rationalizable by an ordinally concave utility function satisfies path independence. To complete the if direction, we show that when the utility function also satisfies size-restricted concavity, the induced choice rule satisfies the law of aggregate demand as well. For the only-if direction, we show that the utility function we construct in the proof of Theorem 1 satisfies size-restricted concavity when the choice rule satisfies both path independence and the law of aggregate demand. This completes the proof using Theorem 1, which shows that the utility function satisfies ordinal concavity.

4. Rationalizability by an M$^\natural$-concave utility function

In this section, we explain the relationship between our results and the literature on choice rules that are rationalizable by utility functions satisfying *M$^\natural$-concavity*: A utility function u satisfies **M$^\natural$-concavity** if, for any $X, X' \subseteq \mathcal{X}$ and $x \in X \setminus X'$, there exists $x' \in (X' \setminus X) \cup \{\emptyset\}$ such that

$$u(X) + u(X') \leq u(X - x + x') + u(X' + x - x').$$

M$^\natural$-concavity implies ordinal concavity (Hafalir et al., 2022, Proposition 5). Moreover, when a choice rule is rationalizable by an M$^\natural$-concave function, the following result holds.

Proposition 1 (Eguchi et al. (2003) and Murota and Yokoi (2015))**.** *Let C be a choice rule that is rationalizable by an M$^\natural$-concave utility function. Then C satisfies path independence and the law of aggregate demand.*

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our setting (Murota and Shioura, 2018, Corollary 1.4). It can be easily verified that size-restricted concavity is an ordinal implication of this inequality. Therefore, M$^\natural$-concavity implies size-restricted concavity.

[16]We note that path independence of the choice rule or ordinal concavity of the rationalizing utility function is indispensable in this theorem. More precisely, without those assumptions, the law of aggregate demand is logically unrelated to rationalizability by a size-restricted concave function.

If a choice rule C is rationalizable by a utility function, then it is also rationalizable by any monotonic transformation of that utility function.[17] Therefore, even though M$^\natural$-concavity is a cardinal notion, rather than an ordinal one, Proposition 1 still holds when C is a choice rule that is rationalizable by a monotonic transformation of an M$^\natural$-concave utility function. In other words, if we replace M$^\natural$-concavity with "the ordinal content of M$^\natural$-concavity" in Proposition 1, the result continues to hold.[18]

One important implication of this discussion is that ordinal concavity is not equivalent to the ordinal content of M$^\natural$-concavity. This can be seen by noting that under ordinal concavity, the induced choice rule does not need to satisfy the law of aggregate demand, but under the ordinal content of M$^\natural$-concavity the law of aggregate demand holds. For example, consider a path-independent choice rule that does not satisfy the law of aggregate demand and the corresponding utility function that we construct in the proof of Theorem 1.[19] We know that the utility function satisfies ordinal concavity by Theorem 1. However, since the choice rule fails the law of aggregate demand, the utility function cannot be a monotonic transformation of an M$^\natural$-concave utility function by Proposition 1. In fact, even though both ordinal concavity and size-restricted concavity are implied by M$^\natural$-concavity, their conjunction is not equivalent to its ordinal content. To see why, we note that Kojima et al. (2018) provide a choice rule that satisfies path independence and the law of aggregate demand while not being rationalizable by an M$^\natural$-concave utility function.[20]

The preceding discussion sheds light on relationships between different concepts of substitutability. The substitutes condition of Hatfield and Milgrom (2005)

[17]A utility function u is a *monotonic transformation* of another utility function $\tilde{u}$ if there exists a strictly increasing function $g : \mathbb{R} \to \mathbb{R}$ such that $u(X) = g(\tilde{u}(X))$ for all $X \subseteq \mathcal{X}$.

[18]The ordinal content of cardinal notions has drawn some attention in the literature. For example, Chambers and Echenique (2009) study the ordinal content of *supermodularity* and Chambers and Echenique (2008) consider ordinal notions for *submodularity*.

[19]For example, let $\mathcal{X} = \{x, y, z\}$. Define choice rule C as, if $x \in X$, then $C(X) = \{x\}$, and $C(X) = X$, otherwise. It is easy to check that C is path independent but does not satisfy the law of aggregate demand.

[20]If the conjunction of ordinal concavity and size-restricted concavity were equivalent to the ordinal content of M$^\natural$-concavity, then the choice rule would also be rationalizable by an M$^\natural$-concave utility function. The example is on pages 811 and 812 of Kojima et al. (2018).

is closely related to the gross substitutes property of Kelso and Crawford (1982), which is the standard concept of substitutability in markets with continuous transfers. Those two conditions are often regarded as natural counterparts in markets with and without transfers, with the gross substitutes property being stronger than the substitutes condition. Our analysis precisely pins down how stronger the former is than the latter for rationalizable choice rules. To see this, we note that Fujishige and Yang (2003) show that the gross substitutes property is equivalent to $M^\natural$-concavity. Since Theorem 1 provides a representation for rationalizable choice rules satisfying the substitutes condition (rationalizability and the substitutes condition are jointly equivalent to path independence, see Proposition 2 below and Theorems 4 and 5 in Yang (2020)), one can attribute the difference between the two notions of substitutability to the difference between two kinds of discrete concavity, namely $M^\natural$-concavity and ordinal concavity. Similarly, Theorem 2 shows that size-restricted concavity is precisely the additional restriction on utility functions that corresponds to imposing the law of aggregate demand on choice rules in addition to path independence.

5. Concluding remarks

Concavity has been a central assumption in microeconomic analysis with divisible goods. Our analysis showed a sense in which it is essential in economies with indivisible goods as well. In particular, we showed that a path-independent choice rule is rationalizable by a utility function that satisfies a particular notion of discrete concavity, namely ordinal concavity. In fact, we showed that the relationship between path independence and rationalizability by a utility function with ordinal concavity is tight. In economies with divisible goods, it is often the case that maximization techniques of concave functions help us characterize equilibrium outcomes. Our representation theorems may prove useful in the analysis of economies with indivisible goods.

To our knowledge, the present paper is one of the first to provide representation theorems for combinatorial choice rules. One possible direction of future research is to provide representation results for other choice rules. It is not arduous to provide such results for canonical choice rules such as the responsive ones (Roth, 1985) as well as the q-acceptant and substitutable ones (Kojima and Manea, 2010).[21] Meanwhile, representation results are still open for most other choice rules such as those with type-specific quotas (Abdulkadiroğlu and Sönmez, 2003) and with reserves (Hafalir et al., 2013). More generally, it would be interesting to establish representation theorems for practically relevant choice rules.

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[21]The representation results are available from the authors upon request. They are not included in the present paper because the conditions on the utility functions are not naturally interpretable as concavity properties which are the main focus of the current paper.

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Appendix A. Proofs

In the appendix, we provide the proofs of our results.

A.1. The substitutes condition and the irrelevance of rejected contracts. We use the following choice rule properties in our proofs.

Definition 5 (Roth and Sotomayor (1990))**.** *A choice rule satisfies the* ***substitutes condition*** *if, for any $X, X' \subseteq \mathcal{X}$ with $X \subseteq X'$,*

$$C(X) \supseteq C(X') \cap X.$$

The substitutes condition is equivalent to the following condition: for any $X, X' \subseteq \mathcal{X}$ with $X \subseteq X'$, it holds that $X \backslash C(X) \subseteq X' \backslash C(X')$.

Definition 6 (Aygün and Sönmez (2013))**.** *A choice rule satisfies the* ***irrelevance of rejected contracts*** *if, for any $X, X' \subseteq \mathcal{X}$ with $X \subseteq X'$,*

$$C(X') \subseteq X \implies C(X) = C(X').$$

The following is a classical choice-theoretic result.

Proposition 2 (Aizerman and Malishevski (1981))**.** *A choice rule satisfies path independence if and only if it satisfies the irrelevance of rejected contracts and the substitutes condition.*

A.2. Proof of Theorem 1. The if direction is a special case of Theorem 2 of Hafalir et al. (2022). We prove the only-if direction. Let C be a choice rule that satisfies path independence. Then it also satisfies the irrelevance of rejected contracts and the substitutes condition (Proposition 2).

We proceed in three steps. In Section A.2.1, we construct a utility function $\tilde{u}$. In Section A.2.2, we prove that $\tilde{u}$ rationalizes C. In Section A.2.3, we prove that $\tilde{u}$ satisfies ordinal concavity.

A.2.1. *Construction of a utility function.* Let $n = |\mathcal{X}|$. For any $X \subseteq \mathcal{X}$, we define E_X^k for $k = 1, \ldots, n$, inductively as follows:

$$E_X^1 = \mathcal{X},$$
$$E_X^k = E_X^{k-1} \setminus (C(E_X^{k-1}) \setminus X) \text{ for } k = 2, \ldots, n.$$

We define α_k for $k = 1, \ldots, n$, inductively (from n to 1) as follows:

$$\alpha_n = 1,$$
$$\alpha_k = \max_{X \subseteq \mathcal{X}} \sum_{j=k+1}^{n} \alpha_j \cdot |X \cap C(E_X^j)| + 1 \text{ for } k = n-1, n-2, \ldots, 1.$$

We define $u : 2^{\mathcal{X}} \to \mathbb{R}$ by

$$u(X) = \sum_{k=1}^{n} \alpha_k \cdot |X \cap C(E_X^k)| \text{ for every } X \subseteq \mathcal{X}. \tag{1}$$

For any $X \subseteq \mathcal{X}$ and $k = 1, \ldots, n$, we define δ_X^k by

$$\delta_X^k = \begin{cases} \varepsilon & \text{if } C(E_X^k) \subseteq X \text{ and } C(E_X^j) \nsubseteq X \text{ for every } j \text{ with } j < k, \\ 0 & \text{otherwise,} \end{cases}$$

where ε is a sufficiently small number with $0 < \varepsilon < 1/n$. We define $\tilde{u} : 2^{\mathcal{X}} \to \mathbb{R}$ by

$$\tilde{u}(X) = u(X) - \sum_{k=1}^{n} \delta_X^k \cdot |X \setminus C(E_X^k)| \text{ for every } X \subseteq \mathcal{X}. \tag{2}$$

The following inequalities hold:

$$u(X) \geq \tilde{u}(X) \text{ and } \tilde{u}(X) > u(X) - 1 \text{ for every } X \subseteq \mathcal{X}, \tag{3}$$

where the strict inequality follows from $\sum_{k=1}^{n} \delta_X^k \cdot |X \setminus C(E_X^k)| \leq \varepsilon \cdot n < 1$.

Claim 1. *Let $X, X' \subseteq \mathcal{X}$ with $X \subseteq X'$. Then,*

$$E_X^j \subseteq E_{X'}^j, \text{ for every } j = 1, \ldots, n.$$

Proof. The proof is by mathematical induction. The claim trivially holds for $j = 1$ because $E_X^1 = E_{X'}^1 = \mathcal{X}$. Suppose that it holds for $j-1$. We show the claim for j.

By the definition of E, our goal is to prove that

$$\Big(E_X^{j-1} \setminus (C(E_X^{j-1}) \setminus X)\Big) \subseteq \Big(E_{X'}^{j-1} \setminus (C(E_{X'}^{j-1}) \setminus X')\Big). \tag{4}$$

Let $x \in E_X^{j-1} \setminus (C(E_X^{j-1}) \setminus X)$. By $x \in E_X^{j-1}$ and the induction hypothesis,

$$x \in E_{X'}^{j-1}. \tag{5}$$

By $x \notin C(E_X^{j-1}) \setminus X$, we have (i) $x \notin C(E_X^{j-1})$ or (ii) $x \in C(E_X^{j-1}) \cap X$. If (i) holds, then by $x \in E_X^{j-1}$, the induction hypothesis, and the substitutes condition, we have $x \notin C(E_{X'}^{j-1})$, which implies $x \notin C(E_{X'}^{j-1}) \setminus X'$. Together with (5), it implies that x is included in the right-hand side of (4). If (ii) holds, then $x \in X$, which implies $x \in X'$, and so $x \notin C(E_{X'}^{j-1}) \setminus X'$. Together with (5), it implies that x is included in the right-hand side of (4). □

A.2.2. *Proof of $\tilde{u}$ rationalizing C.* Fix an arbitrary $\bar{X} \subseteq \mathcal{X}$ with $\bar{X} \neq \emptyset$ and let $X^* = C(\bar{X})$. Our goal is to prove that X^* uniquely maximizes $\tilde{u}$ among all subsets of $\bar{X}$. The proof works in a number of steps. At each Step k, where $1 \leq k \leq n$, we provide two statements labeled as $(a|k)$ and $(b|k)$ below.

In Step 1, we show that either Claim $(a|1)$ or Claim $(b|1)$ holds. The proof is completed if $(a|1)$ holds. If $(b|1)$ holds, then we go to Step 2. In Step 2, we show that either Claim $(a|2)$ or Claim $(b|2)$ holds. Again, if $(a|2)$ holds then the proof is completed, and otherwise we go to Step 3. We continue this process until Claim $(a|k)$ holds for some $k \in \{1, \ldots, n\}$.

We define Ψ^k for $k = 0, \ldots, n$ inductively as follows:

$$\Psi^0 = \{X \subseteq \mathcal{X} \mid X \subseteq \bar{X}\},$$
$$\Psi^k = \left\{X \in \Psi^{k-1} \mid |X \cap C(E^k_{\bar{X}})| \geq |X' \cap C(E^k_{\bar{X}})| \text{ for every } X' \in \Psi^{k-1}\right\}$$
$$\text{for every } k = 1, \ldots, n.$$

Note that $\Psi^0 \supseteq \Psi^1 \supseteq \cdots \supseteq \Psi^n$.

Step k $(1 \leq k \leq n)$. Suppose that one of the following two conditions holds:

- $k = 1$, or
- $k \geq 2$ and $(b|j)$ holds in every Step $j = 1, \ldots, k-1$.

Then, one of the following two claims holds:

$(a|k)$**:** X^* uniquely maximizes $\tilde{u}$ among all elements in Ψ^0; or

$(b|k)$**:** $\delta^k_X = 0$ for every $X \in \Psi^{k-1}$, and Ψ^k satisfies the following four conditions:

(i) $X^* \in \Psi^k$,

(ii) $X \cap C(E^k_X) = \bar{X} \cap C(E^k_{\bar{X}})$ for every $X \in \Psi^k$,

(iii) $E^{k+1}_X = E^{k+1}_{\bar{X}} \subsetneq E^k_{\bar{X}}$ for every $X \in \Psi^k$, and

(iv) $u(X) > u(X')$ for every $X \in \Psi^k$ and $X' \in \Psi^{k-1} \setminus \Psi^k$.

Moreover, if $k = n$, then $(a|n)$ holds.

Proof of the statement for Step k. By the definition of E, $E^k_{\bar{X}} \supseteq \bar{X}$. Together with the substitutes condition, it implies that

$$X^* = C(\bar{X}) \supseteq \bar{X} \cap C(E^k_{\bar{X}}). \tag{6}$$

The following equality holds:

$$\Psi^k = \left\{X \in \Psi^{k-1} \mid X \supseteq \bar{X} \cap C(E^k_{\bar{X}})\right\}. \tag{7}$$

To see that (7) holds, let $X \in \Psi^{k-1}$ with $X \supseteq \bar{X} \cap C(E^k_{\bar{X}})$. Then, $X \cap C(E^k_{\bar{X}}) \supseteq \bar{X} \cap C(E^k_{\bar{X}})$, which implies $|X \cap C(E^k_{\bar{X}})| \geq |\bar{X} \cap C(E^k_{\bar{X}})|$. Since $|\bar{X} \cap C(E^k_{\bar{X}})| \geq |X' \cap C(E^k_{\bar{X}})|$ for every $X' \in \Psi^{k-1} \subseteq \Psi^0$, we obtain $X \in \Psi^k$. Conversely, let $X \in \Psi^{k-1}$ with

$X \not\supseteq \bar{X} \cap C(E^k_{\bar{X}})$. Then,

$$|X^* \cap C(E^k_{\bar{X}})| = |X^* \cap \big(\bar{X} \cap C(E^k_{\bar{X}})\big)| > |X \cap \big(\bar{X} \cap C(E^k_{\bar{X}})\big)| = |X \cap C(E^k_{\bar{X}})|,$$

where the first equality follows from $X^* \subseteq \bar{X}$, the strict inequality follows from (6) and $X \not\supseteq \bar{X} \cap C(E^k_{\bar{X}})$, and the last equality follows from $X \subseteq \bar{X}$. By the above displayed inequality and $X^* \in \Psi^{k-1}$ (which follows from $(b|k-1)$ (i)),[22] we get $X \notin \Psi^k$. It follows that (7) holds.

By (6), (7) and $X^* \in \Psi^{k-1}$,

$$X^* \in \Psi^k. \tag{8}$$

For any $X \in \Psi^k$ and $X' \in \Psi^{k-1} \setminus \Psi^k$,[23]

$$\begin{aligned}
&u(X) - u(X') \\
&= \Big\{\sum_{j=1}^{k} \alpha_j \cdot |X \cap C(E^j_X)| + \sum_{j=k+1}^{n} \alpha_j \cdot |X \cap C(E^j_X)|\Big\} \\
&- \Big\{\sum_{j=1}^{k} \alpha_j \cdot |X' \cap C(E^j_{X'})| + \sum_{j=k+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})|\Big\} \\
&\geq \sum_{j=1}^{k} \alpha_j \cdot |X \cap C(E^j_X)| - \Big\{\sum_{j=1}^{k} \alpha_j \cdot |X' \cap C(E^j_{X'})| + \sum_{j=k+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})|\Big\} \\
&= \alpha_k \cdot |X \cap C(E^k_X)| - \alpha_k \cdot |X' \cap C(E^k_{X'})| - \sum_{j=k+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \\
&= \alpha_k \cdot |X \cap C(E^k_{\bar{X}})| - \alpha_k \cdot |X' \cap C(E^k_{\bar{X}})| - \sum_{j=k+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \\
&\geq \alpha_k - \sum_{j=k+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \\
&\geq 1,
\end{aligned}$$

where

- the first inequality follows from $\sum_{j=k+1}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \geq 0$,

[22] If $k = 1$, then $X^* \in \Psi^0$ follows from $X^* = C(\bar{X}) \subseteq \bar{X}$.

[23] If $k = n$, then the summation $\sum_{j=k+1}^{n} \alpha_j \cdot |X \cap C(E^j_X)|$ is defined to be 0.

- the second equality follows from $(b|j)$ (ii) for every j with $1 \leq j \leq k-1$,[24]
- the third equality follows from $E^1_X = E^1_{X'} = E^1_{\bar{X}} = \mathcal{X}$ (if $k=1$), and $X, X' \in \Psi^{k-1}$ and $(b|k-1)$ (iii) (if $k \geq 2$),
- the second inequality follows from $X \in \Psi^k$ and $X' \notin \Psi^k$, and
- the last inequality follows from the definition of α_k.

Hence,

$$u(X) - u(X') \geq 1. \tag{9}$$

We consider two cases.

Case 1: Suppose $\bar{X} \supseteq C(E^k_{\bar{X}})$. We prove that $(a|k)$ holds.

By $C(E^k_{\bar{X}}) \subseteq \bar{X} \subseteq E^k_{\bar{X}}$ and the irrelevance of rejected contracts,

$$X^* = C(\bar{X}) = C(E^k_{\bar{X}}). \tag{10}$$

Fix $X \in \Psi^k$ with $X \neq X^*$. The above equality and (7) imply

$$X \supseteq \bar{X} \cap C(E^k_{\bar{X}}) = \bar{X} \cap X^* = X^* = C(E^k_{\bar{X}}).$$

Together with (10) and $X \neq X^*$, it implies

$$X \supsetneq C(E^k_{\bar{X}}). \tag{11}$$

By (8), $X \in \Psi^k$, and $(b|k-1)$ (iii),[25] we have

$$E^k_{X^*} = E^k_X = E^k_{\bar{X}}. \tag{12}$$

Together with (10) and (11), it yields

$$X^* = C(E^k_{X^*}),\ X \supsetneq C(E^k_X), \tag{13}$$

which implies $C(E^k_{X^*}) \setminus X^* = \emptyset$ and $C(E^k_X) \setminus X = \emptyset$. By the definition of E,

$$E^k_{X^*} = E^{k+1}_{X^*} = \cdots = E^n_{X^*} \text{ and } E^k_X = E^{k+1}_X = \cdots = E^n_X.$$

Together with (12) and (13), it implies

$$X^* \cap C(E^j_{X^*}) = X \cap C(E^j_X) \text{ for every } j = k, \ldots, n. \tag{14}$$

[24] If $k=1$, then the second equality trivially holds.
[25] If $k=1$, then (12) follows from $E^1_{X^*} = E^1_X = E^1_{\bar{X}} = \mathcal{X}$.

By $(b|j)$ (ii) for every j with $1 \leq j \leq k-1$,[26]

$$X^* \cap C(E^j_{X^*}) = X \cap C(E^j_X) \text{ for every } j \text{ with } 1 \leq j \leq k-1. \tag{15}$$

By (14) and (15),

$$u(X^*) = u(X). \tag{16}$$

By the first claim of $(b|j)$ for every j with $1 \leq j \leq k-1$, (13), and the definition of δ,

$$\delta^k_{X^*} = \delta^k_X = \varepsilon \text{ and } \delta^j_{X^*} = \delta^j_X = 0 \text{ for every } j \in \{1,\dots,n\} \text{ with } j \neq k. \tag{17}$$

We obtain

$$\tilde{u}(X^*) = u(X^*) - \delta^k_{X^*} \cdot |X^* \setminus C(E^k_{X^*})| = u(X^*) \tag{18}$$

$$> u(X) - \delta^k_X \cdot |X \setminus C(E^k_X)|$$

$$= \tilde{u}(X), \tag{19}$$

where the first and last equalities follow from (17), the second equality follows from (13), and the strict inequality follows from (13) and (16).

Furthermore, for any $j = 0,\dots,k-1$ and any $X^j \in \Psi^j \backslash \Psi^{j+1}$,

$$u(X^*) > u(X^{k-1}) > \cdots > u(X^0), \tag{20}$$

where the first inequality follows from (8) and (9) and the other inequalities follow from $(b|j)$ (iv) for every j with $1 \leq j \leq k-1$. Hence, for any $X' \in \Psi^0 \setminus \Psi^k$, we have $\tilde{u}(X^*) = u(X^*) > u(X') \geq \tilde{u}(X')$, where the equality follows from (18), the strict inequality follows from (20), and the weak inequality follows from the former inequality of (3). Together with (19) for every $X \in \Psi^k$ with $X \neq X^*$, it yields $(a|k)$.

Case 2: Suppose $\bar{X} \not\supseteq C(E^k_{\bar{X}})$. We prove that $(b|k)$ holds. For any $X \in \Psi^{k-1}$, by the assumption of Case 2 and $X \subseteq \bar{X}$ (which follows from $X \in \Psi^{k-1} \subseteq \Psi^0$), we have $C(E^k_X) = C(E^k_{\bar{X}}) \nsubseteq X$ (where the equality follows from $(b|k-1)$ (iii)),[27] which implies $\delta^k_X = 0$. Thus, the first claim of $(b|k)$ holds.

[26]If $k=1$, then we do not need (15) in order to establish (16).

[27]If $k=1$, then the equality follows from $E^1_X = E^1_{\bar{X}} = \mathcal{X}$. The same comment applies to the equation $E^k_X = E^k_{\bar{X}}$ in the remaining part.

By (8), we obtain $(b|k)$ (i).

For any $X \in \Psi^k$, by (7), we have $X \supseteq \bar{X} \cap C(E^k_{\bar{X}})$. Together with $X \subseteq \bar{X}$ (which follows from $X \in \Psi^k \subseteq \Psi^0$), it yields

$$\bar{X} \cap C(E^k_{\bar{X}}) = X \cap C(E^k_{\bar{X}}). \tag{21}$$

Together with $E^k_X = E^k_{\bar{X}}$ (which follows from $(b|k-1)$ (iii)), it implies $(b|k)$ (ii).

For any $X \in \Psi^k$, we have

$$\begin{aligned} E^{k+1}_X &= E^k_X \setminus (C(E^k_X) \setminus X) \\ &= E^k_{\bar{X}} \setminus (C(E^k_{\bar{X}}) \setminus X) \\ &= E^k_{\bar{X}} \setminus \Big(C(E^k_{\bar{X}}) \setminus \big(X \cap C(E^k_{\bar{X}})\big)\Big) \\ &= E^k_{\bar{X}} \setminus \Big(C(E^k_{\bar{X}}) \setminus \big(\bar{X} \cap C(E^k_{\bar{X}})\big)\Big) \\ &= E^k_{\bar{X}} \setminus (C(E^k_{\bar{X}}) \setminus \bar{X}) \\ &= E^{k+1}_{\bar{X}}. \end{aligned}$$

where the second equality follows from $E^k_X = E^k_{\bar{X}}$ (which follows from $(b|k-1)$ (iii)) and the fourth equality follows from (21). Together with $C(E^k_{\bar{X}}) \setminus \bar{X} \neq \emptyset$ (which follows from the assumption of Case 2), it yields $(b|k)$ (iii). Finally, $(b|k)$ (iv) follows from (9). We conclude that $(b|k)$ holds.

Finally, we prove the claim that $(a|k)$ holds for $k = n$. By $(b|j)$ (iii) for j with $1 \le j \le n-1$, we get $|E^n_{\bar{X}}| \le 1$. Together with $\bar{X} \neq \emptyset$ and $\bar{X} \subseteq E^n_{\bar{X}}$ (which follows from the definition of E), we get $E^n_{\bar{X}} = \bar{X}$. Hence, $C(E^n_{\bar{X}}) \subseteq \bar{X}$. As proven in Case 1, $(a|n)$ holds. ■

By the statement for Step k ($1 \le k \le n$), there exists $j \in \{1, \dots, n\}$ such that $(a|j)$ holds. Therefore, we obtain the desired claim. □

A.2.3. *Proof of ordinal concavity of $\tilde{u}$.* We consider the following stronger notion than ordinal concavity. We say that a utility function u satisfies **ordinal concavity**$^+$ if, for any $X, X' \subseteq \mathcal{X}$ and $x \in X \setminus X'$, one of the following two conditions holds:

(i) there exists $x' \in (X' \setminus X) \cup \{\emptyset\}$ such that $u(X) < u(X - x + x')$, or
(ii) $u(X') < u(X' + x)$.

Let $X, X' \subseteq \mathcal{X}$ and $x \in X \setminus X'$. We show that $\tilde{u}$ defined by (2) satisfies (i) or (ii) in the above definition.

Let $\bar{X} = X \cup X'$ and $X^* = C(\bar{X})$. By following the same line of the proof in Section A.2.2, there exists $\ell \in \{0, \ldots, n-1\}$ such that

- For every j with $1 \leq j \leq \ell$, Case 2 holds in Step j and we obtain $(b|j)$, and
- Case 1 holds in Step $\ell + 1$ and we obtain $(a|\ell + 1)$.

As shown in the proof, there exists a sequence of collections of subsets of $\bar{X}$, $(\Psi^0, \ldots, \Psi^\ell)$, such that Ψ^j satisfies the conditions in $(b|j)$ for every j with $1 \leq j \leq \ell$. We define $k(X), k(X') \in \{0, \ldots, \ell\}$ by

$$k(X) = \max\bigl\{j \in \{0, \ldots, \ell\} \mid X \in \Psi^j\bigr\},$$
$$k(X') = \max\bigl\{j \in \{0, \ldots, \ell\} \mid X' \in \Psi^j\bigr\}.$$

Let $\bar{k} = \min\{k(X), k(X')\}$. By (7) and $X, X' \in \Psi^j$ for every j with $1 \leq j \leq \bar{k}$,

$$X \supseteq \bar{X} \cap C(E^j_{\bar{X}}) \text{ and } X' \supseteq \bar{X} \cap C(E^j_{\bar{X}}) \text{ for every } j \text{ with } 1 \leq j \leq \bar{k}.$$

Together with $x \in X \setminus X'$, it implies

$$X - x + x' \supseteq \bar{X} \cap C(E^j_{\bar{X}}) \text{ for every } x' \in (X' \setminus X) \cup \{\emptyset\} \text{ and } j \text{ with } 1 \leq j \leq \bar{k}, \text{ and}$$
$$X' + x \supseteq \bar{X} \cap C(E^j_{\bar{X}}) \text{ for every } j \text{ with } 1 \leq j \leq \bar{k}.$$

These conditions and (7) imply

$$X - x + x' \in \Psi^j \text{ for every } x' \in (X' \setminus X) \cup \{\emptyset\} \text{ and } j \text{ with } 1 \leq j \leq \bar{k}, \text{ and} \tag{22}$$
$$X' + x \in \Psi^j \text{ for every } j \text{ with } 1 \leq j \leq \bar{k}. \tag{23}$$

We consider two cases.

Case 1: Suppose $k(X') > k(X)$. By the definition of $k(X)$, we have $X \in \Psi^{k(X)}$ and $X \notin \Psi^{k(X)+1}$. Together with $X' \in \Psi^{k(X)+1}$ and (7), it yields

$$X' \supseteq \bar{X} \cap C(E^{k(X)+1}_{\bar{X}}) \text{ and } X \not\supseteq \bar{X} \cap C(E^{k(X)+1}_{\bar{X}}).$$

Since $\bar{X} = X \cup X'$, the above conditions have two implications. First, the former set-inclusion and $x \in X \setminus X'$ imply

$$x \notin C(E^{k(X)+1}_{\bar{X}}). \tag{24}$$

Second, there exists $x' \in X' \setminus X$ such that

$$x' \in C(E_{\bar{X}}^{k(X)+1}). \tag{25}$$

By (22) and $x' \in X' \setminus X$,

$$X - x + x' \in \Psi^j \text{ for every } j \text{ with } 1 \leq j \leq k(X). \tag{26}$$

We obtain

$$\begin{aligned}
&u(X - x + x') - u(X) \\
&= \Big\{ \sum_{j=1}^{k(X)+1} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| + \sum_{j=k(X)+2}^{n} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| \Big\} \\
&- \Big\{ \sum_{j=1}^{k(X)+1} \alpha_j \cdot |X \cap C(E^j_X)| + \sum_{j=k(X)+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \Big\} \\
&\geq \sum_{j=1}^{k(X)+1} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| \\
&- \Big\{ \sum_{j=1}^{k(X)+1} \alpha_j \cdot |X \cap C(E^j_X)| + \sum_{j=k(X)+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \Big\} \\
&= \alpha_{k(X)+1} \cdot |(X - x + x') \cap C(E^{k(X)+1}_{X-x+x'})| \\
&- \alpha_{k(X)+1} \cdot |X \cap C(E^{k(X)+1}_X)| - \sum_{j=k(X)+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \\
&= \alpha_{k(X)+1} \cdot |(X - x + x') \cap C(E^{k(X)+1}_{\bar{X}})| \\
&- \alpha_{k(X)+1} \cdot |X \cap C(E^{k(X)+1}_{\bar{X}})| - \sum_{j=k(X)+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \\
&= \alpha_{k(X)+1} - \sum_{j=k(X)+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \\
&\geq 1,
\end{aligned}$$

where

- the first inequality follows from $\sum_{j=k(X)+2}^{n} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| \geq 0$,

- the second equality follows from $X-x+x' \in \Psi^j$ (which is implied by (26)), $X \in \Psi^j$, and $(b|j)$ (ii) for every j with $1 \leq j \leq k(X)$,
- the third equality follows from $X - x + x' \in \Psi^{k(X)}$ (which is implied by (26)), $X \in \Psi^{k(X)}$, and $(b|k(X))$ (iii),[28]
- the fourth equality follows from (24) and (25), and
- the last inequality follows from the definition of $\alpha_{k(X)+1}$.

It follows that $u(X-x+x') \geq u(X)+1$, which together with (3) implies $\tilde{u}(X-x+x') > \tilde{u}(X)$. Hence, the conclusion of ordinal concavity$^+$ holds.

Case 2: Suppose $k(X') \leq k(X)(\leq n-1)$. We consider two subcases.

Subcase 2-1: Suppose that $x \notin C(E^k_{\bar{X}})$ for every $k \in \{k(X')+1, \ldots, n\}$.

By (7) and the definition of $k(X)$, we have $X \supseteq \bar{X} \cap C(E^j_{\bar{X}})$ for every j with $k(X')+1 \leq j \leq k(X)$. Together with the assumption of Subcase 2-1, it yields

$$X - x + x' \supseteq \bar{X} \cap C(E^j_{\bar{X}})$$
$$\text{for every } x' \in (X' \setminus X) \cup \{\emptyset\} \text{ and } j \text{ with } k(X')+1 \leq j \leq k(X).$$

This condition and (7) imply

$$X - x + x' \in \Psi^j \text{ for every } x' \in (X' \setminus X) \cup \{\emptyset\} \text{ and } j \text{ with } k(X')+1 \leq j \leq k(X).$$

These conditions, together with (22), yield

$$X - x + x' \in \Psi^j \text{ for every } x' \in (X' \setminus X) \cup \{\emptyset\} \text{ and } j \text{ with } 1 \leq j \leq k(X). \tag{27}$$

We consider two further subcases.

Subcase 2-1-1: Suppose $k(X) = \ell$. As we noted in the beginning of Section A.2.3 (after the definition of ordinal concavity$^+$), Case 1 holds in the proof of Step $\ell+1$ in Section A.2.2, which implies $\bar{X} \supseteq C(E^{\ell+1}_{\bar{X}})$ (see the first sentence of Case 1 after (9)). We consider two further subcases.

Subcase 2-1-1-1: Suppose $X \supseteq C(E^{\ell+1}_{\bar{X}})$. By $X - x \in \Psi^\ell$ (which follows from (27)), $X \in \Psi^\ell$, and $(b|\ell)$ (iii),[29] we have

$$E^{\ell+1}_{X-x} = E^{\ell+1}_{X} = E^{\ell+1}_{\bar{X}}. \tag{28}$$

[28]If $k(X)=0$, then the third equality follows from $E^1_{X-x+x'} = E^1_X = E^1_{\bar{X}} = \mathcal{X}$.
[29]If $\ell = 0$, then (28) follows from $E^1_{X-x} = E^1_X = E^1_{\bar{X}} = \mathcal{X}$.

Together with the assumptions of Subcases 2-1 and 2-1-1-1, these equations imply

$$X - x \supseteq C(E^{\ell+1}_{X-x}) \text{ and } X \supseteq C(E^{\ell+1}_{X}). \tag{29}$$

If $\ell + 1 < n$, then by the definition of E and (29), we get

$$E^{\ell+1}_{X-x} = E^{\ell+2}_{X-x} \text{ and } E^{\ell+1}_{X} = E^{\ell+2}_{X},$$

which together with (29) imply $X - x \supseteq C(E^{\ell+2}_{X-x})$ and $X \supseteq C(E^{\ell+2}_{X})$. Repeating this procedure,

$$\begin{aligned} &E^{j}_{X-x} = E^{j}_{X} \text{ for every } j = \ell+1, \dots, n, \text{ and} \\ &X - x \supseteq C(E^{j}_{X-x}) \text{ and } X \supseteq C(E^{j}_{X}) \text{ for every } j = \ell+1, \dots, n. \end{aligned} \tag{30}$$

We obtain the following:[30]

$$\begin{aligned} &u(X-x) - u(X) \\ &= \Big\{ \sum_{j=1}^{\ell} \alpha_j \cdot |(X-x) \cap C(E^{j}_{X-x})| + \sum_{j=\ell+1}^{n} \alpha_j \cdot |(X-x) \cap C(E^{j}_{X-x})| \Big\} \\ &- \Big\{ \sum_{j=1}^{\ell} \alpha_j \cdot |X \cap C(E^{j}_{X})| + \sum_{j=\ell+1}^{n} \alpha_j \cdot |X \cap C(E^{j}_{X})| \Big\} \\ &= \sum_{j=\ell+1}^{n} \alpha_j \cdot |(X-x) \cap C(E^{j}_{X-x})| - \sum_{j=\ell+1}^{n} \alpha_j \cdot |X \cap C(E^{j}_{X})| \\ &= 0, \end{aligned}$$

where

- the second equality follows from $X - x \in \Psi^j$ (which is implied by (27)), $X \in \Psi^j$ and $(b|j)$ (ii) for every j with $1 \leq j \leq \ell$, and
- the third equality follows from (30).

Hence,

$$u(X-x) - u(X) = 0 \tag{31}$$

[30]If $\ell = 0$, then the summation $\sum_{j=1}^{\ell} \alpha_j \cdot |(X-x) \cap C(E^{j}_{X-x})|$ is defined to be 0.

By $X - x \in \Psi^j$, $X \in \Psi^j$ and the first statement of $(b|j)$ for every j with $1 \leq j \leq \ell$, we have

$$\delta^j_X = 0 \text{ and } \delta^j_{X-x} = 0 \text{ for every } j \text{ with } 1 \leq j \leq \ell. \tag{32}$$

We obtain

$$\begin{aligned}\sum_{k=1}^{n} \delta^k_X \cdot |X \setminus C(E^k_X)| &= \delta^{\ell+1}_X \cdot |X \setminus C(E^{\ell+1}_X)| \\ &> \delta^{\ell+1}_{X-x} \cdot |(X - x) \setminus C(E^{\ell+1}_{X-x})| \\ &= \sum_{k=1}^{n} \delta^k_{X-x} \cdot |(X - x) \setminus C(E^k_{X-x})|,\end{aligned}$$

where the two equalities follow from (29), (32), and the definition of δ, and the strict inequality follows from (28) and (29). This inequality and (31) imply

$$\tilde{u}(X - x) > \tilde{u}(X).$$

Hence, the conclusion of ordinal concavity$^+$ holds.

Subcase 2-1-1-2: Suppose $X \not\supseteq C(E^{\ell+1}_{\bar{X}})$. Since $\bar{X} = X \cup X'$ and $\bar{X} \supseteq C(E^{\ell+1}_{\bar{X}})$ (which follows from the assumption of Subcase 2-1-1), there exists $x' \in X' \setminus X$ such that $x' \in C(E^{\ell+1}_{\bar{X}})$. Then,

$$\begin{aligned}& u(X - x + x') - u(X) \\ &= \Big\{ \sum_{j=1}^{\ell+1} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| + \sum_{j=\ell+2}^{n} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| \Big\} \\ &- \Big\{ \sum_{j=1}^{\ell+1} \alpha_j \cdot |X \cap C(E^j_X)| + \sum_{j=\ell+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \Big\} \\ &\geq \sum_{j=1}^{\ell+1} \alpha_j \cdot |(X - x + x') \cap C(E^j_{X-x+x'})| \\ &- \Big\{ \sum_{j=1}^{\ell+1} \alpha_j \cdot |X \cap C(E^j_X)| + \sum_{j=\ell+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)| \Big\} \\ &= \alpha_{\ell+1} \cdot |(X - x + x') \cap C(E^{\ell+1}_{X-x+x'})| \\ &- \alpha_{\ell+1} \cdot |X \cap C(E^{\ell+1}_X)| - \sum_{j=\ell+2}^{n} \alpha_j \cdot |X \cap C(E^j_X)|\end{aligned}$$

$$= \alpha_{\ell+1} \cdot |(X - x + x') \cap C(E_{\bar{X}}^{\ell+1})|$$
$$- \alpha_{\ell+1} \cdot |X \cap C(E_{\bar{X}}^{\ell+1})| - \sum_{j=\ell+2}^{n} \alpha_j \cdot |X \cap C(E_X^j)|$$
$$= \alpha_{\ell+1} - \sum_{j=\ell+2}^{n} \alpha_j \cdot |X \cap C(E_X^j)|$$
$$\geq 1,$$

where

- the first inequality follows from $\sum_{j=\ell+2}^{n} \alpha_j \cdot |(X - x + x') \cap C(E_{X-x+x'}^j)| \geq 0$,
- the second equality follows from $X - x + x' \in \Psi^j$ (which is implied by (27)), $X \in \Psi^j$, and $(b|j)$ (ii) for every j with $1 \leq j \leq \ell$,
- the third equality follows from $X - x + x' \in \Psi^\ell$ (which is implied by (27)), $X \in \Psi^\ell$, and $(b|\ell)$ (iii),[31]
- the fourth equality follows from the assumption of Subcase 2-1 and $x' \in C(E_{\bar{X}}^{\ell+1})$, and
- the last inequality follows from the definition of $\alpha_{\ell+1}$.

It follows that $u(X - x + x') \geq u(X) + 1$, which together with (3) implies $\tilde{u}(X - x + x') > \tilde{u}(X)$. Hence, the conclusion of ordinal concavity$^+$ holds.

Subcase 2-1-2 Suppose $k(X) < \ell$, which implies $X \notin \Psi^{k(X)+1}$. By (7), we have $X \not\supseteq \bar{X} \cap C(E_{\bar{X}}^{k(X)+1})$. By following the same line of the proof of Subcase 2-1-1-2 (with $k(X)$ playing the role of ℓ), there exists $x' \in X' \setminus X$ with $\tilde{u}(X - x + x') > \tilde{u}(X)$. Hence, the conclusion of ordinal concavity$^+$ holds.

Subcase 2-2: Suppose that there exists $k \in \{k(X') + 1, \ldots, n\}$ such that $x \in C(E_{\bar{X}}^k)$. Together with

- $E_{X'+x}^j \subseteq E_{\bar{X}}^j$ for every $j = 1, \ldots, n$ (which follows from $X' + x \subseteq \bar{X}$ and Claim 1),
- $x \in X' + x \subseteq E_{X'+x}^j$ (where the set-inclusion follows from the definition of E) for every $j = 1, \ldots, n$, and
- the substitutes condition,

[31] If $\ell = 0$, then the third equality follows from $E_{X-x+x'}^1 = E_X^1 = E_{\bar{X}}^1 = \mathcal{X}$.

it implies that there exists $k' \in \{k(X')+1,\dots,n\}$ with $x \in C(E^{k'}_{X'+x})$. Let $k^* \in \{k(X')+1,\dots,n\}$ denote the minimum index such that $x \in C(E^{k^*}_{X'+x})$.

By (7) and $X' \in \Psi^j$ for every j with $1 \leq j \leq k(X')$,

$$X'+x \in \Psi^j \text{ for every } j \text{ with } 1 \leq j \leq k(X').$$

Together with $(b|j)$ (ii) for every j with $1 \leq j \leq k(X')$, it implies

$$(X'+x) \cap C(E^j_{X'+x}) = X' \cap C(E^j_{X'}) \text{ for every } j \text{ with } 1 \leq j \leq k(X'). \tag{33}$$

By $E^1_{X'} = E^1_{X'+x} = \mathcal{X}$ and $(b|j)$ (iii) for every j with $1 \leq j \leq k(X')$, we have

$$E^j_{X'} = E^j_{X'+x} \text{ for every } j = 1,\dots,k(X')+1. \tag{34}$$

If $k(X')+1 < k^*$, then

$$\begin{aligned} E^{k(X')+2}_{X'+x} &= E^{k(X')+1}_{X'+x} \setminus \Big(C(E^{k(X')+1}_{X'+x}) \setminus (X'+x) \Big) \\ &= E^{k(X')+1}_{X'+x} \setminus \Big(C(E^{k(X')+1}_{X'+x}) \setminus X' \Big) \\ &= E^{k(X')+1}_{X'} \setminus \Big(C(E^{k(X')+1}_{X'}) \setminus X' \Big) \\ &= E^{k(X')+2}_{X'}, \end{aligned}$$

where the second equality follows from $x \notin C(E^{k(X')+1}_{X'+x})$ (which follows from $k(X')+1 < k^*$ and the minimality of k^*) and the third equality follows from (34). If $k(X')+2 < k^*$, then by the same argument as above, we obtain $E^{k(X')+3}_{X'+x} = E^{k(X')+3}_{X'}$. Repeating this procedure, we obtain

$$E^j_{X'} = E^j_{X'+x} \text{ for every } j \text{ with } k(X')+2 \leq j \leq k^*.$$

This condition and (34) imply

$$E^j_{X'} = E^j_{X'+x} \text{ for every } j = 1,\dots,k^*. \tag{35}$$

Together with the minimality of k^*, it yields

$$(X'+x) \cap C(E^j_{X'+x}) = X' \cap C(E^j_{X'}) \text{ for every } j \text{ with } k(X')+1 \leq j \leq k^*-1. \tag{36}$$

We obtain

$$u(X'+x) - u(X')$$

$$
\begin{aligned}
&= \Big\{ \sum_{j=1}^{k^*} \alpha_j \cdot |(X'+x) \cap C(E^j_{X'+x})| + \sum_{j=k^*+1}^{n} \alpha_j \cdot |(X'+x) \cap C(E^j_{X'+x})| \Big\} \\
&- \Big\{ \sum_{j=1}^{k^*} \alpha_j \cdot |X' \cap C(E^j_{X'})| + \sum_{j=k^*+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \Big\} \\
&\geq \sum_{j=1}^{k^*} \alpha_j \cdot |(X'+x) \cap C(E^j_{X'+x})| \\
&- \Big\{ \sum_{j=1}^{k^*} \alpha_j \cdot |X' \cap C(E^j_{X'})| + \sum_{j=k^*+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \Big\} \\
&= \alpha_{k^*} \cdot |(X'+x) \cap C(E^{k^*}_{X'+x})| \\
&- \alpha_{k^*} \cdot |X' \cap C(E^{k^*}_{X'})| - \sum_{j=k^*+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \\
&= \alpha_{k^*} - \sum_{j=k^*+1}^{n} \alpha_j \cdot |X' \cap C(E^j_{X'})| \\
&\geq 1,
\end{aligned}
$$

where

- the first inequality follows from $\sum_{j=k^*+1}^{n} \alpha_j \cdot |(X'+x) \cap C(E^j_{X'+x})| \geq 0$,
- the second equality follows from (33) and (36),
- the third equality follows from (35) and $x \in C(E^{k^*}_{X'+x})$, and
- the last inequality follows from the definition of α_{k^*}.

It follows that $u(X'+x) \geq u(X')+1$, which together with (3) implies $\tilde{u}(X'+x) > \tilde{u}(X')$. Hence, the conclusion of ordinal concavity$^+$ holds. □

A.3. Proof of Theorem 2.

A.3.1. *Proof of the if direction.* Let C be a choice rule and u be a utility function that rationalizes C and satisfies ordinal concavity and size-restricted concavity. By the if direction of Theorem 1, C satisfies path independence. It remains to prove that C satisfies the law of aggregate demand. The proof is similar to that of Theorem 3.10 in Murota (2016). Suppose, for contradiction, that the law of aggregate demand is violated, i.e., there exist X and X' such that $X \supseteq X'$ and $|C(X')| > |C(X)|$. By size-restricted concavity of u, there exists $x \in C(X') \setminus C(X)$ such that

(i) $u(C(X')) < u(C(X') - x)$, or
(ii) $u(C(X)) < u(C(X) + x)$, or
(iii) $u(C(X')) = u(C(X') - x)$ and $u(C(X)) = u(C(X) + x)$.

If (i) or the first equality of (iii) holds, then we obtain a contradiction to $C(X')$ uniquely maximizing u among all subsets of X'. If (ii) or the second equality of (iii) holds, then together with $x \in C(X') \subseteq X' \subseteq X$, we obtain a contradiction to $C(X)$ uniquely maximizing u among all subsets of X. We conclude that C satisfies the law of aggregate demand.

A.3.2. *Proof of the only-if direction.* Let C satisfy path independence and the law of aggregate demand. We define $\tilde{u}$ as in (2). By the only-if direction of Theorem 1, $\tilde{u}$ rationalizes C and satisfies ordinal concavity. It remains to prove that $\tilde{u}$ satisfies size-restricted concavity. Let $X, X' \subseteq \mathcal{X}$ with $|X| > |X'|$ and $X^* = C(X \cup X')$. Recall that $\tilde{u}$ satisfies ordinal concavity$^+$ (see the first paragraph of Section A.2.3).

Case 1: Suppose $X^* \setminus X' \neq \emptyset$. By ordinal concavity$^+$ applied to X^*, X', and an arbitrarily chosen $x \in X^* \setminus X'$, we have

(i) there exists $x' \in (X' \setminus X^*) \cup \{\emptyset\}$ such that $\tilde{u}(X^*) < \tilde{u}(X^* - x + x')$, or
(ii) $\tilde{u}(X') < \tilde{u}(X' + x)$.

If (i) holds, since x' satisfies $x' = \emptyset$ or $x' \in X' \subseteq X \cup X'$, we obtain a contradiction to X^* maximizing $\tilde{u}$ among all subsets of $X \cup X'$. Hence, (ii) must hold. Since $x \in X^* \setminus X' \subseteq (X \cup X') \setminus X' = X \setminus X'$, the conclusion of size-restricted concavity holds.

Case 2: Suppose $X^* \setminus X' = \emptyset$, i.e., $X^* \subseteq X'$. By the law of aggregate demand, letting $X^{**} = C(X)$,

$$|X^{**}| = |C(X)| \leq |C(X \cup X')| = |X^*|. \tag{37}$$

Subcase 2-1: Suppose $X^{**} \supseteq (X \setminus X')$. Since C satisfies path independence, by Proposition 2, C satisfies the substitutes condition. Hence, the following set-inclusion holds:[32]

$$X^* \setminus X^{**} \subseteq X' \setminus X. \tag{38}$$

Since $|X| > |X'|$,

$$|X \setminus X'| > |X' \setminus X|. \tag{39}$$

Then,

$$|X^* \setminus X^{**}| \leq |X' \setminus X| < |X \setminus X'| \leq |X^{**} \setminus X^*|,$$

where the first inequality follows from (38), the second inequality follows from (39), and the last inequality follows from $X \setminus X' \subseteq X^{**} \setminus X' \subseteq X^{**} \setminus X^*$, where

- the first set-inclusion follows from the assumption of Subcase 2-1, and
- the second set-inclusion follows from the assumption of Case 2.

The above displayed inequality implies $|X^{**}| > |X^*|$. We obtain a contradiction to (37). Hence, Subcase 2-1 is not possible.

Subcase 2-2: The remaining possibility is $X^{**} \not\supseteq (X \setminus X')$, i.e., there exists $x \in X \setminus X'$ with $x \notin X^{**}$. By ordinal concavity$^+$ applied to X, X^{**} and $x \in X \setminus X^{**}$, together with $X^{**} \subseteq X$, we have

(i) $\tilde{u}(X) < \tilde{u}(X - x)$, or
(ii) $\tilde{u}(X^{**}) < \tilde{u}(X^{**} + x)$.

If (ii) holds, then together with $x \in X$, we obtain a contradiction to X^{**} maximizing $\tilde{u}$ among all subsets of X. Hence, (i) holds. Since $x \in X \setminus X'$, the conclusion of size-restricted concavity holds.

[32]To see that (38) holds, suppose that there exists $x \in X^* \setminus X^{**}$ with $x \notin X' \setminus X$. By $x \in X^*$ and $X^* \subseteq X'$ (which follows from the assumption of Case 2), we have $x \in X'$. Together with $x \notin X' \setminus X$, it implies $x \in X \cap X'$. By combining $x \notin C(X) = X^{**}$, $x \in X$, and $x \in X^* = C(X \cup X')$, we obtain a contradiction to the substitutes condition.